UNDER REVIEW

RESCIENCE C

Replication / Ecology

[Re] Reproductive pair correlations and the clustering of organisms

Coralie Picoche[1,2] and Frederic Barraquand[1,2]

[1]Institute of Mathematics of Bordeaux, CNRS University of Bordeaux, Talence, France – [2]Integrative and Theoretical Ecology, LabEx COTE, University of Bordeaux, Pessac, France

Edited by
(Editor)

Received
01 November 2018

Published
19 February 2021

DOI
–

Introduction

This article is a reproduction of [1].

Methods

Analytical solution of G

Derivation of G(r,t) — Finding back Eq. (2) in the original paper?

$$\frac{\partial G}{\partial t} = 2Dr^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial G}{\partial r}\right) + 2(\lambda-\mu)G + \gamma r^{1-d}\frac{\partial}{\partial r}\left(r^{d+1}\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{x}) \quad (1)$$

where $\boldsymbol{x}$ is the position of the particle.
We will focus on the case $d = 2$ and $\lambda = \mu$, which means Eq. (1) can be reduced to

$$\frac{\partial G}{\partial t} = \frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{x}) \quad (2)$$

Analytical solution with advection — In the presence of advection ($\gamma \neq 0$), a steady-state solution can be found.

$$\begin{aligned}
\frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{x}) &= 0 \\
\Leftrightarrow 2\pi r\left(\frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{x})\right) &= 0 \\
\Leftrightarrow 2\pi\left(2D\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \gamma\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right)\right) + 2\pi r 2\lambda C\delta(\boldsymbol{x}) &= 0 \quad (3)
\end{aligned}$$

We can then integrate Eq. (3) over a small area centered on a particle, with radius ρ. Let us first note that

Correspondence should be addressed to Coralie Picoche (coralie.picoche@u-bordeaux.fr)
The authors have declared that no competing interests exist.
Code is available at https://github.com/CoraliePicoche/brownian_bug_fluid/code..

$$\begin{aligned} \int_{R^2} \delta(\boldsymbol{x}) d^2\boldsymbol{x} &= 1 \\ \Leftrightarrow \int_0^{2\pi} \int_0^{\rho} \delta(r')\delta(\theta) r' dr' d\theta &= 1 \\ \Leftrightarrow 2\pi \int_0^{\rho} \delta(\boldsymbol{x'}) r' dr' &= 1 \end{aligned} \tag{4}$$

Using Eq. (3) and (4), we can integrate between 0 and ρ,

$$\begin{aligned} 0 &= 2\pi \left(2D\rho \frac{\partial G}{\partial r} + \gamma \rho^3 \frac{\partial G}{\partial r} \right) + 2\lambda C \\ \Leftrightarrow \quad \frac{\partial G}{\partial r} &= -\frac{1}{2\pi} \frac{2\lambda C}{2D\rho + \gamma\rho^3} \end{aligned} \tag{5}$$

Eq. (5) can now be integrated between ρ and ∞, knowing that $G(\infty) = C^2$.

$$C^2 - G(\rho) = -\frac{1}{2\pi} \int_{\rho}^{\infty} \frac{2\lambda C}{2Dr + \gamma r^3} dr \tag{6}$$

Using the variable change $u = 2Dr + \gamma r^3$, the integral is equivalent to $\int \frac{u'}{u} du$.

$$C^2 - G(\rho) = -\frac{\lambda C}{2\pi} \frac{1}{4D} [\log(\gamma) - \log(\frac{2D}{r^2} + \gamma)] \tag{7}$$

$$\Leftrightarrow G(\rho) = \frac{\lambda C}{8\pi D} \log \left(\frac{2D + \gamma r^2}{\gamma r^2} \right) + C^2 \tag{8}$$

Finally, the pair correlation function $g = G/C^2$ is defined as

$$g = \frac{\lambda}{8\pi DC} \log \left(\frac{2D + \gamma r^2}{\gamma r^2} \right) + 1 \tag{9}$$

Analytical solution without advection — When $U = 0$, $\gamma = 0$ and there is no steady solution. We can get back to Eq. (2).

$$\frac{\partial G}{\partial t} = \frac{2D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + 2\lambda C \delta(\boldsymbol{x}) \tag{10}$$

Assuming an isotropic environment (and switching to the polar coordinate system), this means

$$\frac{\partial G}{\partial t} - 2D\Delta G = 2\lambda C \delta(\boldsymbol{x}) \tag{11}$$

where $\Delta = \nabla^2$ is the Laplacian operator.

We therefore have

$$\mathcal{L} G(\boldsymbol{x}, t) = 2\lambda C \delta(\boldsymbol{x}) \tag{12}$$

where $\mathcal{L}$ is the linear differential operator $\partial_t - 2D\Delta$.

We can use the Green's function H, defined with $\mathcal{L}H = \delta(\boldsymbol{x}, t) = \delta(\boldsymbol{x})\delta(t)$.

By definition, we know that $G(y) = \int H(y, s) 2\lambda C \delta(s) ds$ (where $y = (\boldsymbol{x}, t)$) is a solution to Eq.(12).

$$
\begin{aligned}
G(\boldsymbol{x},t) &= 2\lambda C \int_{R^2} \int_0^t H(\boldsymbol{x}-\boldsymbol{x}',t')\delta(\boldsymbol{x}')d^2\boldsymbol{x}'dt' \\
\Leftrightarrow &= 2\lambda C \int_0^t H(\boldsymbol{x},t')dt'
\end{aligned} \tag{13}
$$

Eq.(13) can be used in Eq. (10):

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(2\lambda C \int_0^t H(\boldsymbol{x},t')dt'\right) &= 2D2\lambda C\Delta \int_0^t H(\boldsymbol{x},t')dt' + 2\lambda C\delta(\boldsymbol{x}) \\
\Leftrightarrow \int_0^t \left(\frac{\partial H(\boldsymbol{x},t')}{\partial t'} - 2D\Delta H(\boldsymbol{x},t')\right) dt' &= \delta(\boldsymbol{x}) \\
\Leftrightarrow \int_0^t \delta(\boldsymbol{x})\delta(t')dt' &= \delta(\boldsymbol{x})
\end{aligned}
$$

which is true.

A solution for the Green's function using $\mathcal{L} = \partial_t - 2D\Delta$ in 2 dimensions is

$$
H(r,t) = \frac{1}{4\pi 2Dt}\exp\left(\frac{-r^2}{4\times 2Dt}\right)
$$

.
$G(r,t)$ can then be computed:

$$
G(r,t) = 2\lambda C \left[\frac{E1\left(\frac{r^2}{8Dt'}\right)}{8D\pi}\right]_0^t \tag{14}
$$

where $E1$ is the exponential integral. Using $G(r,0) = C^2$ and $\lim_{x\to+\infty} E1 = 0$ in Eq. (14),

$$
G(r,t) = 2\lambda C\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + C^2 \tag{15}
$$

$$
\Leftrightarrow g(r,t) = \frac{2\lambda}{C}\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + 1 \tag{16}
$$

Results

Discussion

References

1. W. R. Young, A. J. Roberts, and G. Stuhne. "Reproductive pair correlations and the clustering of organisms." In: **Nature** 412.6844 (2001), pp. 328–331.

Supplementary Material

Stretching parameter γ

γ is computed with simulations, with the formula $r(t) \propto exp(\gamma dt) \rightarrow \frac{1}{2} ln(r(t)) = \gamma t$ if $d = 2$ with r being the separation between pairs of particles. γ is estimated as the slope of

$$\frac{1}{2} \langle ln(r(t)) \rangle = f(t)$$

with $\langle ln(r(t)) \rangle$ being the average obtained from 800 pairs of particles.

$$\forall t, \langle ln(r(t)) \rangle = \frac{1}{800} \sum_{p=1}^{800} ln(r(\boldsymbol{x}_{1,p}(t) - \boldsymbol{x}_{2,p}(t)))$$

where $r(\boldsymbol{x}_{1p}(t) - \boldsymbol{x}_{2p}(t))$ is the distance between a particle $1p$ at position $\boldsymbol{x}_{1p}$ and its counterpart $2p$, initialized with $r(0) = 10^{-7} \forall p$ (see Fig. 1 for γ estimates).

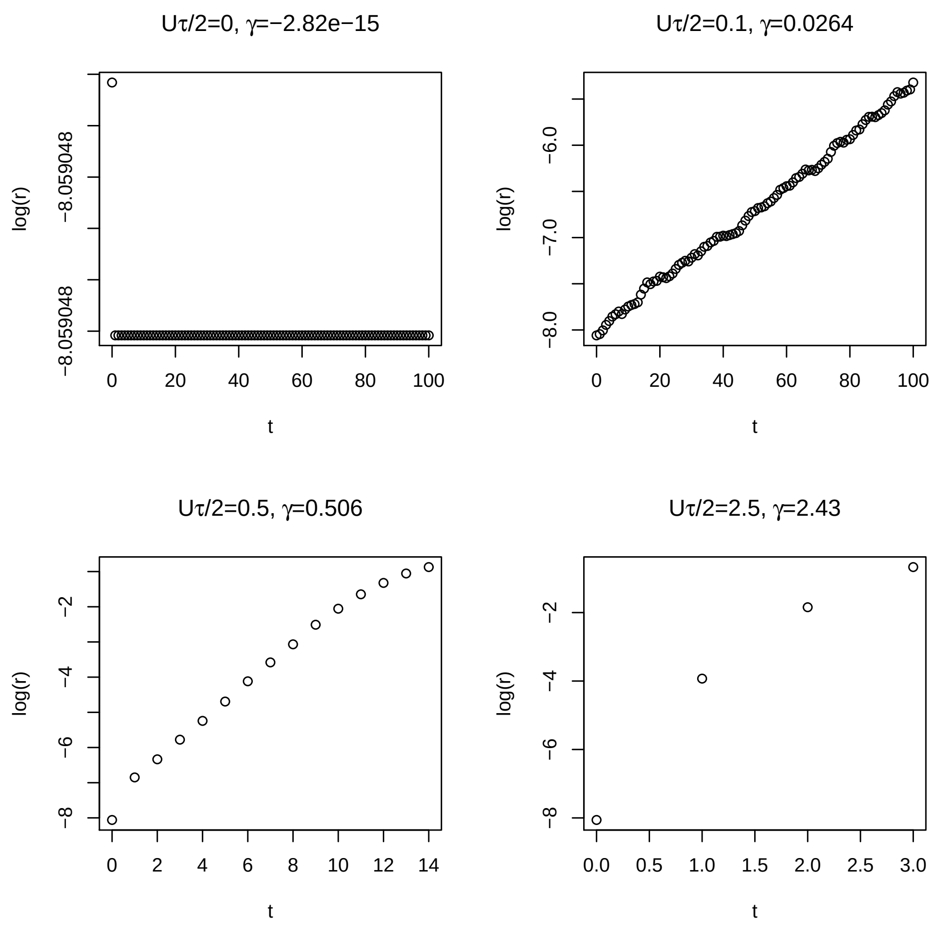

Figure 1. Estimates of γ for different $U\tau/2$