UNDER REVIEW

RESCIENCE C

Replication / Ecology

[Re] Reproductive pair correlations and the clustering of organisms

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Introduction

This article is a reproduction of [1].

Methods

Analytical solution of G

Derivation of G(r,t) – Finding back Eq. (2) in the original paper?

$$\frac{\partial G}{\partial t} = 2Dr^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial G}{\partial r}\right) + 2(\lambda-\mu)G + \gamma r^{1-d}\frac{\partial}{\partial r}\left(r^{d+1}\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \quad (1)$$

We will focus on the case $d = 2$ and $\lambda = \mu$, which means Eq. (1) can be reduced to

$$\frac{\partial G}{\partial t} = \frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \quad (2)$$

Analytical solution with advection – In the presence of advection ($\gamma \neq 0$), a steady-state solution can be found.

$$\begin{aligned}
0 &= \frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \\
\Leftrightarrow 0 &= 2\pi r\left(\frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r})\right) \\
\Leftrightarrow 0 &= 2\pi\left(2D\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \gamma\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right)\right) + 2\pi r 2\lambda C\delta(\boldsymbol{r}) \quad (3)
\end{aligned}$$

We can then integrate Eq. (2) over a small area centered on a particle, with radius ρ. Let us first note that

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Code is available at https://github.com/CoraliePicoche/brownian_bug_fluid/code..

$$
\begin{aligned}
\int_{R^2} \delta(\boldsymbol{r}) d^2\boldsymbol{r} &= 1 \\
\Leftrightarrow \int_0^{2\pi} \int_0^{\rho} \delta(\boldsymbol{r'}) r' dr' d\theta &= 1 \\
\Leftrightarrow 2\pi \int_0^{\rho} \delta(\boldsymbol{r'}) r' dr' &= 1
\end{aligned} \tag{4}
$$

Using Eq. (3) and (4), we can integrate between 0 and ρ,

$$
\begin{aligned}
0 &= 2\pi \left(2D\rho \frac{\partial G}{\partial r} + \gamma \rho^3 \frac{\partial G}{\partial r} \right) + 2\lambda C \\
\Leftrightarrow \frac{\partial G}{\partial r} &= -\frac{1}{2\pi} \frac{2\lambda C}{2D\rho + \gamma \rho^3}
\end{aligned} \tag{5}
$$

We can integrate between ρ and ∞, knowing that $G(\infty) = C^2$.

$$
C^2 - G(\rho) = -\frac{1}{2\pi} \int_{\rho}^{\infty} \frac{2\lambda C}{2Dr + \gamma r^3} dr \tag{6}
$$

Using the variable change $u = 2Dr + \gamma r^3$, the integral is equivalent to $\int \frac{u'}{u} du$

$$
C^2 - G(\rho) = -\frac{\lambda C}{2\pi} \frac{1}{4D} [\log(\gamma) - \log(\frac{2D}{r^2} + \gamma)] \tag{7}
$$

$$
\Leftrightarrow G(\rho) = \frac{\lambda C}{8\pi D} \log \left(\frac{2D + \gamma r^2}{\gamma r^2} \right) + C^2 \tag{8}
$$

Finally, the pair correlation function $g = G/C^2$ is defined as

$$
g = \frac{\lambda}{8\pi DC} \log \left(\frac{2D + \gamma r^2}{\gamma r^2} \right) + 1 \tag{9}
$$

Analytical solution without advection — When $U = 0$, $\gamma = 0$ and there is no steady solution. We can get back to Eq. (2).

$$
\frac{\partial G}{\partial t} = \frac{2D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + 2\lambda C \delta(\boldsymbol{r}) \tag{10}
$$

Assuming an isotropic environment, this means

$$
\frac{\partial G}{\partial t} - 2D\Delta G = 2\lambda C \delta(\boldsymbol{r}) \tag{11}
$$

where $\Delta = \nabla^2$ is the Laplacian operator.
We therefore have

$$
\mathcal{L} G(\boldsymbol{r}, t) = 2\lambda C \delta(\boldsymbol{r}) \tag{12}
$$

where $\mathcal{L}$ is the linear differential operator $\partial_t - 2D\Delta$.
Let's use the Green's function H, defined with $\mathcal{L}H = \delta(\boldsymbol{r}, t) = \delta(\boldsymbol{r})\delta(t)$.
By definition, we know that $G(y) = \int H(y, s) 2\lambda C \delta(s) ds$ (where $y = (\boldsymbol{r}, t)$) is a solution to Eq.(12).

$$
\begin{aligned}
G(\boldsymbol{r},t) &= 2\lambda C \int_{R^2}\int_0^t H(\boldsymbol{r}-\boldsymbol{r'},t')\delta(\boldsymbol{r'})d\boldsymbol{r}'dt' \\
\Leftrightarrow &= 2\lambda C \int_0^t H(\boldsymbol{r},t')dt' \qquad (13)
\end{aligned}
$$

Let's substitute Eq.(13) in Eq. (10):

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(2\lambda C\int_0^t H(\boldsymbol{r},t')dt'\right) &= 2D2\lambda C\Delta\int_0^t H(\boldsymbol{r},t')dt' + 2\lambda C\delta(\boldsymbol{r}) \\
\Leftrightarrow \int_0^t \left(\frac{\partial H(\boldsymbol{r},t')}{\partial t'} - 2D\Delta H(\boldsymbol{r},t')\right)dt' &= \delta(\boldsymbol{r}) \\
\Leftrightarrow \int_0^t \delta(\boldsymbol{r})\delta(t')dt' &= \delta(\boldsymbol{r})
\end{aligned}
$$

which is true.
A solution for the Green's function using $\mathcal{L} = \partial_t - 2D\Delta$ in 2 dimensions is $H(r,t) = \frac{1}{4\pi 2Dt}\exp(\frac{-r^2}{4\times 2Dt})$.
$G(r,t)$ can then be computed:

$$
G(r,t) = 2\lambda C\left[\frac{E1\left(\frac{r^2}{8Dt'}\right)}{8D\pi}\right]_0^t \qquad (14)
$$

where $E1$ is the exponential integral. Using $G(r,0) = C^2$ and $\lim_{x\to+\infty} E1 = 0$ in Eq. (14),

$$
G(r,t) = 2\lambda C\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + C^2 \qquad (15)
$$

$$
\Leftrightarrow g(r,t) = \frac{2\lambda}{C}\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + 1 \qquad (16)
$$

Results

Discussion

References

1. W. R. Young, A. J. Roberts, and G. Stuhne. "Reproductive pair correlations and the clustering of organisms." In: **Nature** 412.6844 (2001), pp. 328–331.