Replication / Ecology

[Re] Reproductive pair correlations and the clustering of organisms

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RESCIENCE C

Introduction

This article is a reproduction of [1].

Methods

Analytical solution of G

Derivation of G(r,t) – Finding back Eq. (2) in the original paper?

$$\frac{\partial G}{\partial t} = 2Dr^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial G}{\partial r}\right) + 2(\lambda - \mu)G + \gamma r^{1-d}\frac{\partial}{\partial r}\left(r^{d+1}\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \quad (1)$$

We will focus on the case $d = 2$ and $\lambda = \mu$, which means Eq. (1) can be reduced to

$$\frac{\partial G}{\partial t} = \frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \quad (2)$$

Analytical solution with advection – In the presence of advection ($\gamma \neq 0$), a steady-state solution can be found.

$$\frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \quad = \quad 0$$

$$\Leftrightarrow 2\pi r\left(\frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r})\right) \quad = \quad 0$$

$$\Leftrightarrow 2\pi\left(2D\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \gamma\frac{\partial}{\partial r}\left(r^3\frac{\partial G}{\partial r}\right)\right) + 2\pi r 2\lambda C\delta(\boldsymbol{r}) \quad = \quad 0 \quad (3)$$

We can then integrate Eq. (3) over a small area centered on a particle, with radius ρ. Let us first note that

$$\int_{R^2} \delta(\boldsymbol{r})d^2\boldsymbol{r} = 1$$

$$\Leftrightarrow \int_0^{2\pi} \int_0^{\rho} \delta(\boldsymbol{r}')r'dr'd\theta = 1$$

$$\Leftrightarrow 2\pi \int_0^{\rho} \delta(\boldsymbol{r}')r'dr' = 1 \tag{4}$$

Using Eq. (3) and (4), we can integrate between 0 and ρ,

$$0 = 2\pi \left(2D\rho\frac{\partial G}{\partial r} + \gamma\rho^3\frac{\partial G}{\partial r} \right) + 2\lambda C$$

$$\Leftrightarrow \quad \frac{\partial G}{\partial r} = -\frac{1}{2\pi}\frac{2\lambda C}{2D\rho + \gamma\rho^3} \tag{5}$$

Eq. (5) can now be integrated between ρ and ∞, knowing that $G(\infty) = C^2$.

$$C^2 - G(\rho) = -\frac{1}{2\pi}\int_\rho^\infty \frac{2\lambda C}{2Dr + \gamma r^3}dr \tag{6}$$

Using the variable change $u = 2Dr + \gamma r^3$, the integral is equivalent to $\int \frac{u'}{u}du$.

$$C^2 - G(\rho) = -\frac{\lambda C}{2\pi}\frac{1}{4D}[\log(\gamma) - \log(\frac{2D}{r^2} + \gamma)] \tag{7}$$

$$\Leftrightarrow G(\rho) = \frac{\lambda C}{8\pi D}\log\left(\frac{2D + \gamma r^2}{\gamma r^2}\right) + C^2 \tag{8}$$

Finally, the pair correlation function $g = G/C^2$ is defined as

$$g = \frac{\lambda}{8\pi DC}\log\left(\frac{2D + \gamma r^2}{\gamma r^2}\right) + 1 \tag{9}$$

Analytical solution without advection — When $U = 0$, $\gamma = 0$ and there is no steady solution. We can get back to Eq. (2).

$$\frac{\partial G}{\partial t} = \frac{2D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + 2\lambda C\delta(\boldsymbol{r}) \tag{10}$$

Assuming an isotropic environment (and switching to the polar coordinate system), this means

$$\frac{\partial G}{\partial t} - 2D\Delta G = 2\lambda C\delta(\boldsymbol{r}) \tag{11}$$

where $\Delta = \nabla^2$ is the Laplacian operator.

We therefore have

$$\mathcal{L}G(\boldsymbol{r}, t) = 2\lambda C\delta(\boldsymbol{r}) \tag{12}$$

where $\mathcal{L}$ is the linear differential operator $\partial_t - 2D\Delta$.

We can use the Green's function H, defined with $\mathcal{L}H = \delta(\boldsymbol{r}, t) = \delta(\boldsymbol{r})\delta(t)$.

By definition, we know that $G(y) = \int H(y, s)2\lambda C\delta(s)ds$ (where $y = (\boldsymbol{r}, t)$) is a solution to Eq.(12).

$$G(\boldsymbol{r},t) \qquad = \qquad 2\lambda C \int_{R^2} \int_0^t H(\boldsymbol{r}-\boldsymbol{r}',t')\delta(\boldsymbol{r}')d^2\boldsymbol{r}'dt'$$

$$\Leftrightarrow \qquad = \qquad 2\lambda C \int_0^t H(\boldsymbol{r},t')dt' \qquad (13)$$

Eq.(13) can be used in Eq. (10):

$$\frac{\partial}{\partial t}\left(2\lambda C \int_0^t H(\boldsymbol{r},t')dt'\right) \qquad = 2D2\lambda C\Delta \int_0^t H(\boldsymbol{r},t')dt' + 2\lambda C\delta(\boldsymbol{r})$$

$$\Leftrightarrow \int_0^t \left(\frac{\partial H(\boldsymbol{r},t')}{\partial t'} - 2D\Delta H(\boldsymbol{r},t')\right)dt' = \delta(\boldsymbol{r})$$

$$\Leftrightarrow \int_0^t \delta(\boldsymbol{r})\delta(t')dt' \qquad = \delta(\boldsymbol{r})$$

which is true.

A solution for the Green's function using $\mathcal{L} = \partial_t - 2D\Delta$ in 2 dimensions is

$$H(r,t) = \frac{1}{4\pi 2Dt}\exp(\frac{-r^2}{4 \times 2Dt})$$

.
$G(r,t)$ can then be computed:

$$G(r,t) = 2\lambda C \left[\frac{E1\left(\frac{r^2}{8Dt'}\right)}{8D\pi}\right]_0^t \qquad (14)$$

where $E1$ is the exponential integral. Using $G(r,0) = C^2$ and $\lim_{x\to+\infty} E1 = 0$ in Eq. (14),

$$G(r,t) = 2\lambda C\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + C^2 \qquad (15)$$

$$\Leftrightarrow g(r,t) = \frac{2\lambda}{C}\frac{E1\left(\frac{r^2}{8Dt}\right)}{8D\pi} + 1 \qquad (16)$$

Results

Discussion

References

1. W. R. Young, A. J. Roberts, and G. Stuhne. "Reproductive pair correlations and the clustering of organisms." In: **Nature** 412.6844 (2001), pp. 328–331.