Universidade Estadual de Londrina
Centro de Ciências Exatas
Departamento de Física

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Técnicas de infêrencia estatística aplicadas a cosmologia

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Técnicas de infêrencia estatística aplicadas a cosmologia

Trabalho de Conclusão de Curso apresentado ao Curso de Física Bacharelado da Universidade Estadual de Londrina, como requisito parcial para a obtenção do título de Bacharel em Física Bacharelado.

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Dedico este trabalho a todos aqueles que, de alguma forma,
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Abstract

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Lista de Siglas e Abreviaturas

FLRW *Firedmann-Lemaître-Robertson-Walker*

Sumário

1 General Relativity

In the second half of the XIX century, physicists were avidly searching for proof of the existence of the *luminiferous ether*. The *ether* was theorized as a required propagation medium and privileged inertial frame for electromagnetic waves. The observation of this entity could bring the unification of magnetism, electricity, and light to a close. One possible method of measurement of the effect of such an absolute entity – which was the approach of the Michelson-Morley experiment – was the velocity of light relative to its flow. This experiment "undermined the whole structure of the old ether theory and thus served to introduce the new theory of relativity".

Einstein's acumen lay in the physical interpretation of the result of the Michelson-Morley experiment. Physical laws should be independent of the absolute velocity of an observer. The validity of the Maxwell equations, especially the velocity of light in a vacuum, and the relativity principle are foundations for the Special Theory of Relativity. This first theory is restricted to a specific set of reference frames. The Lorentz Transformations are particular to inertial frames, particles in rectilinear uniform motion.

This restrictive characteristic gives the impression that the Galilean coordinates are a fundamental referential, even though there is no reason why to believe it is so. It is easy to see that from a Non-Lorentzian change of coordinates geometrical forces could arise. But this same phenomenon can be used to "transform away" gravitational forces. The Principle of Equivalence states that a gravitational field of force is strictly equivalent to one introduced by a transformation of coordinates and no possible experiment can distinguish between the two.

Einstein's second theory is a generalization to all these possible reference frames, even though we may need to look into complex geometries and non-euclidean coordinate systems. As Einstein says in his 1916 work "The mathematical apparatus useful for the general relativity theory, lay already complete in the Absolute Differential Calculus".

1.1 Differential Geometry

Our minimal mathematical representation of spacetime is a given set of spacetime points, $\mathcal{M}$, which locally resembles Euclidean space. Furthermore, this space can be parametrized by a set of real numbers, with which we can describe the dynamical evolution of a system using differential equations. The set of spacetime points with these two characteristics is called a differentiable manifold. (Here I'm skipping the definition of coordinate charts, atlas, tangent/cotangent spaces, *etc.*)

In a single point of the manifold, we can describe two vector spaces. First, $\mathcal{T}_p\mathcal{M}$, the tangent space of $\mathcal{M}$ at the point p. $\mathcal{T}_p\mathcal{M}$ is a set of vectors

$$V = V^\mu \partial_\mu, \tag{1.1}$$

where V^μ are the components and ∂_μ are the basis. These objects take a function defined on $\mathcal{M}$ to a real number on $\mathcal{R}$. The vector components transform according to the following law:

$$\widetilde{V}^\nu = \frac{\partial \widetilde{x}^\nu}{\partial x^\mu} V \mu. \tag{1.2}$$

And, $\mathcal{T}_p^*\mathcal{M}$, the cotangent space of $\mathcal{M}$ at the point p. This space consists of a set of forms

$$w = w_\mu dx^\mu, \tag{1.3}$$

which in turn take a vector from $\mathcal{T}_p\mathcal{M}$ to a real number in $\mathcal{R}$. Forms components transform accordingly:

$$\widetilde{w}_\nu = \frac{\partial x_\mu}{\partial \widetilde{x}^\nu} w_\mu. \tag{1.4}$$

Knowing a set of vectors $(V^{\mu_1}, V^{\mu_2}, \ldots, V^{\mu_n})$ and a set of form $(w_{\nu_1}, w_{\nu_2}, \ldots, w_{\nu_m})$, their product will transform as:

$$\widetilde{V}^{\alpha_1}\widetilde{V}^{\alpha_2}\ldots\widetilde{V}^{\alpha_n}\widetilde{w}_{\beta_1}\widetilde{w}_{\beta_2}\ldots\widetilde{w}_{\beta_m} = \frac{\partial \widetilde{x}^{\alpha_1}}{\partial x^{\mu_1}}\frac{\partial \widetilde{x}^{\alpha_2}}{\partial x^{\mu_2}}\cdots\frac{\partial \widetilde{x}^{\alpha_n}}{\partial x^{\mu_n}}\frac{\partial x_{\nu_1}}{\partial \widetilde{x}^{\beta_1}}\frac{\partial x_{\nu_2}}{\partial \widetilde{x}^{\beta_2}}\cdots\frac{\partial x_{\nu_m}}{\partial \widetilde{x}^{\beta_m}} V^{\mu_1}V^{\mu_2}\ldots V^{\mu_n} w_{\nu_1}w_{\nu_2}\ldots w_{\nu_m}. \tag{1.5}$$

Objects that respect the transformation law above are called tensors and can be thought of as the product of vectors and forms;

$$A^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} = V^{\mu_1}V^{\mu_2}\ldots V^{\mu_n} w_{\nu_1}w_{\nu_2}\ldots w_{\nu_m}. \tag{1.6}$$

We say the order of a tensor is (n, m) when it is constructed from n vectors and m forms.

Moreover, following the same line of thought, one can create a different ranked tensor from the product of two other tensors.

$$T^{\mu\nu}{}_{\sigma\lambda} = A^{\mu\nu} B_{\sigma\lambda}. \tag{1.7}$$

Notice how we have combined two lower-ranked tensors, $(2, 0)$ and $(0, 2)$, to create a third with order $(2 + 0, 0 + 2)$.

Another possible operation is the rank reduction of a tensor. This can be done by simply contracting two indexes of a tensor.

$$T^\mu{}_\nu = T^{\mu\sigma}{}_{\sigma\nu}. \tag{1.8}$$

The contraction between two indexes transformed a $(2, 2)$ tensor into a $(1, 1)$ tensor.

An important feature of tensors is their symmetries. We say a tensor is symmetric if

$$A_{\mu\nu} = A_{\nu\mu}, \tag{1.9}$$

and antisymmetric if

$$A_{\mu\nu} = -A_{\nu\mu}. \tag{1.10}$$

A vital remark regarding the dynamic of an entity. These mathematical objects are defined in a point of our differentiable manifold. For instance, comparing vectors from two distinct points is, so far, a fool's errand. They exist in entirely different mathematical spaces and are, therefore, incomparable. We will soon explore how one might connect such particular spaces.

We have hitherto described arbitrary tensors without any specific usage for the understanding of the general theory. First and foremost, we shall do a brief study of the metric tensor, or as Einstein called it, the covariant fundamental tensor.

The metric tensor is a bilinear map from vectors on the tangent space to a real number. The line element is the distance between two points in a given space and is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.11}$$

The metric is a symmetric tensor. Moreover, it is an orthogonal tensor, *i.e.*, $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu{}^\nu$. For this reason, it is conventional to use the metric to raise and lower indexes;

$$g_{\mu\nu} A^\nu = A_\mu. \tag{1.12}$$

Let's go back to the remark made about the connectivity of points on a manifold. We aim to create an entirely covariant theory; therefore, we expect that our mathematical apparatus should be invariant under the tensor transformation law. It is easy to see that the partial derivative – a primary candidate for our connectivity problem – does not transform accordingly (Leibniz Rule). Hence, we must define a covariant derivative. Caring only about the free indexes, we may add the following correction to the partial derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma. \tag{1.13}$$

This factor is called connection. Furthermore, we impose that

$$\widetilde{\nabla}_\mu \widetilde{V}^\nu = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu} \frac{\partial \widetilde{x}^\nu}{\partial x^\beta} \nabla_\alpha V^\beta, \tag{1.14}$$

which makes the connection transform in such a way that it "cancels out" the irregularities in the partial derivative transformation. The covariant derivative of a form is

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\sigma_{\mu\nu} w_\sigma, \tag{1.15}$$

and its effect on a scalar is simply the partial derivative.

Finally, we also expect the connection to satisfy two other characteristics. First, it should be torsion-free, *i.e.*, symmetric in its two lower indexes. Secondly, it should be metric-compatible, meaning the covariant derivative of the metric is null. Following all these rules, we can uniquely define a connection. The Christoffel Symbol,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right], \tag{1.16}$$

is the only possible torsion-free, metric-compatible connection. Einstein's general theory is based on this connection.

We can define a directional covariant derivative

$$\frac{D}{d\lambda} V^{\nu} = \frac{dx^{\mu}}{d\lambda} \nabla_{\mu} V^{\nu}, \tag{1.17}$$

over which we bring attention to a special case. When the above derivative is null we say that the vector V^{ν} is being parallel transported along the path $dx^{\mu}/d\lambda$. Such requirements give rise to

$$\frac{d}{d\lambda} V^{\nu} + \Gamma^{\nu}_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} V^{\beta} = 0, \tag{1.18}$$

known as the equation of parallel transport.

With the knowledge we gathered so far, we are now able to derive the geodesic equation of a timelike path. From (1.11), we shall take ds as the proper time $d\tau$ and parametrize the equation to a real value λ. Furthermore, we can integrate the expression to find τ. This leaves us with

$$\tau = \int \sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}} d\lambda. \tag{1.19}$$

We added a minus sign to turn the interval positive. For simplicity, we rename the term inside the square root as f. Additionally, we take an infinitesimal functional variation of the equation, therefore,

$$\delta\tau = \int \delta\sqrt{-f} d\lambda = \int \frac{1}{2} (-f)^{-\frac{1}{2}} \delta(-f) d\lambda. \tag{1.20}$$

Using the identity in terms of the four-velocity $U^{\mu} = dx^{\mu}/d\tau$,

$$g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = g_{\mu\nu} U^{\mu} U^{\nu} = -1, \tag{1.21}$$

and parametrizing λ to τ we can rewrite (1.20) as

$$\delta\tau = -\frac{1}{2} \int \delta f d\tau. \tag{1.22}$$

By adding small variations to the metric and the path,

$$x^\mu \to x^\mu + \delta x^\mu, \tag{1.23}$$

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{1.24}$$

we can solve to the first order and analyze the integrand at the boundary using the Leibniz Rule. The resultant integrand should be null by imposing no variations of the proper time.

Finally, the following is true:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{1.25}$$

The equation above represents the path that extremizes the proper time and is called the geodesic equation. Once we are set on a particular metric we can use this equation to find the geodesic paths. It is essential to point out that any parameter linearly related to τ will leave (1.25) unchanged. These are called affine parameters.

Furthermore, by identifying $dx^\mu/d\tau = V^\mu$, we can see that (1.25) becomes (1.18);

$$\frac{D}{d\tau}V^\mu = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}V^\alpha V^\beta = 0. \tag{1.26}$$

This means that the geodesic is a path in which a vector is parallel transported along itself, *i.e.*, a straight line is a path that parallel transport its tangent vector.

Using the symmetry of $V^\alpha V^\beta$ one can rewrite (1.18), yielding

$$\frac{dV_\mu}{d\tau} = \frac{1}{2}\partial_\mu g_{\alpha\beta}V^\alpha V^\beta. \tag{1.27}$$

If the metric is independent of some particular coordinate x^μ, then the form V_μ is invariant. In other words, there is an isometry of the metric of which V^μ is a conserved quantity. This construction is highly dependent on coordinates, whereas the following is true in a covariant manner:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{1.28}$$

(1.28) is called Killing Equation, and $\vec{K}$ is the Killing Vector. Carrying the metric along the field of vectors $\vec{K}$ leaves it unchanged. Each Killing Vector Field in an n-dimensional manifold represents a unique isometry.

The word curvature is expected to come up when reading about general relativity and differential geometry. In our case, we shall study the intrinsic curvature, a quantity that can be measured within the geometry. This intrinsicality does not require our manifold to be embedded in a higher-dimensional space.

But how does one measure such a thing? The answer lies in the effect curvature has on vectors' cyclic transportation along our geometry. Let's take the commutator of the covariant derivative over a test vector;

$$\left[\nabla_\mu, \nabla_\nu\right] V^\sigma = \nabla_\mu\nabla_\nu V^\sigma - \nabla_\nu\nabla_\mu V^\sigma. \tag{1.29}$$

Using (1.13) and some algebraic tricks, we end up with:

$$[\nabla_\mu, \nabla_\nu] V^\sigma = \left(\partial_\mu \Gamma^\sigma_{\nu\gamma} - \partial_\nu \Gamma^\sigma_{\mu\gamma} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\gamma} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\gamma}\right) V^\gamma. \tag{1.30}$$

The expression inside the parenthesis is the transformation acting on the vector after the cycle. The Riemann Tensor is defined as

$$R^\sigma{}_{\gamma\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\gamma} - \partial_\nu \Gamma^\sigma_{\mu\gamma} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\gamma} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\gamma}, \tag{1.31}$$

and it is used to measure the intrinsic curvature of a manifold. Reflecting on the covariant nature of the equations above it is easy to see that if the curvature vanishes for a particular coordinate system, it will so in every other. The curvature tensor is also commonly represented with a lowered first index, $R_{\sigma\gamma\mu\nu}$.

One of the main properties of the curvature tensor is called the Bianchi Identity. We can find it by performing cyclic permutations of the covariant derivative of (1.31):

$$\nabla_\lambda R^\sigma{}_{\gamma\mu\nu} + \nabla_\sigma R^\gamma{}_{\lambda\mu\nu} + \nabla_\gamma R^\lambda{}_{\sigma\mu\nu} = 0. \tag{1.32}$$

There are two important contractions of (1.31). The first is called the Ricci tensor. We find it by contracting the first and third indexes of the curvature tensor;

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}. \tag{1.33}$$

The second is called the Ricci scalar. It is the contraction of the indexes in the Ricci Tensor:

$$R = R^\mu{}_\mu. \tag{1.34}$$

To create a symmetric $(0, 2)$ tensor from the Ricci Tensor and the Ricci Scalar, (1.32) can be contracted twice; resulting in

$$\nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla^\mu R g_{\mu\nu} = 0. \tag{1.35}$$

Applying the linearity of the covariant derivative we define the Einstein Tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.36}$$

which sustain the following property:

$$\nabla^\mu G_{\mu\nu} = 0. \tag{1.37}$$

1.2 General Relativity

Now that we paved the way with the necessary mathematical introduction, we can proceed to generalize the gravitational theory. First and foremost, let's go about how one might take a law of physics usable in flat space to curved spacetime. A sort of straightforward way is called the minimal-coupling principle. The minimal-coupling principle is simply an algorithm for rewriting a given law in a covariant manner. Such transformation can be made by transforming variables into tensors, and the partial derivative into covariant derivatives.

As previously hinted, gravity experienced by a free-falling particle should be merely an expression of the selected coordinate system; gravity is no different than a geometrical force. Hence, it seems reasonable to replace the gravitational field with a geometrical entity, *e.g.*, the metric tensor, or the curvature tensor.

Moreover, another important entity in gravitation is mass. Since we know that mass is a specific representation of energy, we should substitute it with a tensor that encompasses all its possible forms. We want a $(0, 2)$ symmetric tensor that contains information about the distribution of energy in a system. Additionally, this tensor should extend the continuity equation to our curved spacetime.

The described tensor is known; it is the energy-momentum tensor $T_{\mu\nu}$. It can be thought of as a matrix of all combinations of the flux of μ momentum across surfaces of constant x^ν. $T_{\mu\nu}$ is constructed in a way that

$$\nabla^\mu T_{\mu\nu} = 0, \tag{1.38}$$

i.e., the total energy is conserved. (1.38) is the covariant form of the continuity equation.

The gravitational field is described by the Poisson Equation:

$$\nabla^2 \Phi = 4\pi G\rho. \tag{1.39}$$

By following the minimal-coupling principle we can propose a proportionality between the second-order derivative of a $(0, 2)$ symmetric geometry-related tensor and the energy-momentum tensor.

Unfortunately, since the (1.16) is metric-compatible, direct derivations of the metric are not a great candidate. However, the Ricci Tensor does embody this twice-differentiated metric in itself. Sadly, $R_{\mu\nu}$ is not consistent with (1.38).

The required tensor was already derivated in the last section by twice contracting (1.32). The Einstein Tensor respects all requirements for the equivalence between energy and space-time curvature. Finally, we may propose the following field equation:

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.40}$$

We can find κ by solving the equation for a massive body at rest in a weak gravitational field. The energy-momentum tensor will be just the energy density across time, *i.e.*, the classical energy density $\rho = T_{00}$. The metric will be the Minkowski flat space plus small perturbations,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.41}$$

known as the weak-field limit.

A few algebra yields

$$\nabla^2 h_{00} = -\kappa\rho, \tag{1.42}$$

where the similarity with (1.39) is evident. Moreover, using the weak-field limit for non-relativistic particles, the geodesic equation shows us

$$h_{00} = -2\Phi. \tag{1.43}$$

With the last two equations, we can see that

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.44}$$

is the generalization of (1.39). (1.44) is Einstein's Field Equation. It brings to light the subtle reality that energy is the curvature of spacetime.

1.3 Cosmology

In 1917, Einstein applied his theory to the cosmos by proposing the existence of a cosmological constant, Λ, to balance the universe and keep it static. This idea was based on the prevailing belief at the time that the universe was static and unchanging.

However, in the 1920s, Alexander Friedmann disregarded the need for a closed universe, instead, he theorized the possibility of a dynamical cosmos with a variable radius. He used (1.44) to show that the universe could be either expanding or contracting, which challenged Einstein's idea of a static universe.

To find such solutions, we begin with the cosmological principle. The cosmological principle states that the universe is spatially homogeneous and isotropic. These two characteristics give rise to rotational and translational symmetry. Constraining our possible metrics to those with such symmetries we are forced to select from a set of maximally symmetric spaces.

The set of maximally symmetric spaces contains those spaces that have the same number of Killing vector fields as the Euclidian Space, but with arbitrary constant curvature. This arbitrarity allows three groups: zero, positive, and negative curvature. The only difference within these groups is the scaling, therefore, we will only consider $(-1, 01)$ as possible values.

Since the curvature is constant in such spaces the curvature tensor, (1.31), is a Lorentz-invariant in a locally flat coordinate system $x^{\hat{\mu}}$. Therefore, we can build $R_{\hat{\sigma}\hat{\gamma}\hat{\mu}\hat{\nu}}$ from another known invariant tensor, *e.g.*, the metric. By trying to match the symmetries of (1.31), we arrive at the following proportionality:

$$R_{\hat{\sigma}\hat{\gamma}\hat{\mu}\hat{\nu}} = A\left(g_{\hat{\sigma}\hat{\mu}}g_{\hat{\gamma}\hat{\nu}} - g_{\hat{\sigma}\hat{\nu}}g_{\hat{\gamma}\hat{\mu}}\right). \tag{1.45}$$

The same is true for every other coordinate system. Moreover, we can contract booth sides twice yielding the general form of the Riemann Curvature Tensor for a maximally symmetric manifold;

$$R_{\sigma\gamma\mu\nu} = \frac{R}{n(n-1)} \left(g_{\sigma\mu} g_{\gamma\nu} - g_{\sigma\nu} g_{\gamma\mu} \right). \tag{1.46}$$

We are interested in $3D$ spaces, which will compose the leaves of constant time. By restricting our curvature to those of three dimensions, we define the Gaussian curvature as

$$K = \frac{R}{6}. \tag{1.47}$$

Let's begin by assuming a general spatially-dynamic metric

$$ds^2 = -dt^2 + a(t) d\sigma^2, \tag{1.48}$$

where $d\sigma^2$ is the metric of a spherically-symmetric 3-space given by

$$d\sigma^2 = e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \tag{1.49}$$

By computing the curvature of this general metric and using the curvature tensor for maximally symmetric spaces we can fix

$$e^{2\beta(r)} = \frac{1}{1 - Kr^2}. \tag{1.50}$$

Rewriting everything and parametrizing the radius, $r = \sqrt{\alpha} r$, to restrain $K = \alpha k$ we arrive at the Friedmann-Lemaître-Robertson-Walker Metric (FLRW Metric)

$$ds^2 = -dt^2 + a(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \tag{1.51}$$

We have $a(t)$ as the scale factor and $k = -1, 0, +1$ as the curvature.

Prior to solving Einstein's field equation, it is necessary to describe the energy within the universe. The fundamental assumption is that the cosmos is permeated by a perfect fluid composed of various entities. The following energy-momentum tensor can describe such fluid:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{1.52}$$

The structure of (1.52) can be understood by the isotropy of the perfect fluid. Isotropy results in the diagonal form of this tensor. Moreover, it also explains the equal valued pressure in all three directions.

With cosmo's energy content and its related metric, we can solve (1.44) for the variable radius $a(t)$. Although the process is straightforward, it is long and mechanical. First, we find all Christoffel symbols for the FLRW metric. Secondly, with all possible symbols, we compute the Riemann tensor; we will contract to find the Ricci tensor and contract again for the Ricci scalar. Now, we are capable of analyzing the time and spatial portions of the Field Equation separately. By doing so, we arrive at two equations:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a(t)^2}, \tag{1.53}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho(t) + 3p\right). \tag{1.54}$$

These are respectively the First and Second Friedmann Equations. (1.53) is derived from the time portion of the Field Equation, and the (1.54) is derived from a substitution of (1.53) into the spatial portion of the Field Equation.

Not long after Friedmann's remarks on Relativistic Cosmology, the Belgian priest Georges Lemaître arrived at similar equations and theorized about the cosmos at $t = 0$. In 1927 he proposed that the universe began as a "primordial atom", which exploded creating all energy components of the universe. Moreover, his calculations showed a linear relationship between the velocity of stellar bodies and their distances from an observer. In 1929, Hubble published his measurement of the relation between the redshift of galaxies and their distances, solidifying the idea of a cosmological expansion. The Hubble-Lemaître Law states

$$v = H_0 d, \tag{1.55}$$

where H_0 is the current Hubble constant, v is the relative velocity, and d is the distance. By replacing the velocity of a stellar body with the velocity of the expansion and the distance by the radius we can also write the Hubble parameter as

$$H = \frac{\dot{a}}{a}. \tag{1.56}$$

The Hubble parameter, H, represents the rate at which the universe is expanding, and the current rate, H_0, is one of the parameters that can be determined using observational data.

It is useful to define a critical energy density required for the universe to have zero curvature, or $k = 0$. By constraining the universe in this way, we can solve the first Friedmann equation (1.53) for ρ_{crit};

$$\rho_{crit} = \frac{3H^2}{8\pi G}. \tag{1.57}$$

A general density parameter in terms of ρ_{crit} is given by

$$\Omega = \frac{\rho}{\rho_{crit}}. \tag{1.58}$$

Before we proceed with the analysis of these parameters, it is necessary to understand the distribution of energy density in the universe and how each component evolves with the expansion of the universe. For any perfect fluid (1.52) is conserved, *i.e.*, $\nabla_\mu T^\mu{}_\nu = 0$. The time component of this derivation yields

$$\nabla_\mu T^\mu{}_0 = \partial_0 T^0{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^\alpha_{\mu 0} T^\mu{}_0 = 0. \tag{1.59}$$

Computing $T^\mu{}_\nu$, and replacing the symbols obtained from the FLRW metric we find the fluid equation;

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{1.60}$$

Each component of the cosmo has a unique equation of state

$$p = w\rho. \tag{1.61}$$

Replacing (1.61) in (1.60), and solving for ρ in terms of a we find

$$\rho = \rho_0 a^{-3(1+w)}. \tag{1.62}$$

The visible components of the universe are matter and radiation. Since the equation of state for each is known, their evolutions are respectively

$$\rho_m = \rho_{m0} a^{-3}, \tag{1.63}$$

$$\rho_r = \rho_{r0} a^{-4}. \tag{1.64}$$

Additionally, another type of energy may arise as a possible interpretation of the cosmological constant Λ. The most general form of the Field Equation has Λ as an integration constant. One can think of this constant as a part of the energy-momentum tensor that emerges from the vacuum. The central principle regarding the vacuum energy density is how it should evolve with the variation of the scale factor. As the size of the cosmos increases or decreases, the amount of vacuum energy changes accordingly. Therefore, its density should remain constant across the evolution of the universe. For $w = -1$, we can see in (1.62) a constant vacuum energy density;

$$\rho_\Lambda = \rho_{\Lambda 0}. \tag{1.65}$$

Moreover, one vital corollary of $w = -1$ is

$$p = -\rho_\Lambda. \tag{1.66}$$

Finally, the energy density for the whole universe can be written as

$$\rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda(t) = \rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_\Lambda. \tag{1.67}$$

Dividing by the critical density, we have the density parameter

$$\Omega = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda, \tag{1.68}$$

We have omitted the subscript indicating that these parameters represent their current values for simplicity. Just like H_0, the set of parameters $(\Omega_m, \Omega_r, \Omega_\Lambda)$ can be nummerically fixed with observational data.

We are, at last, close to an equation that couples all of our cosmological parameters of interest. We can define a final parameter by rewriting the first Friedmann Equation in terms of the Hubble and density parameters. Dividing (1.53) by H_0^2, identifying (1.57), and defining

$$\Omega_c = -\frac{k}{H_0^2} \tag{1.69}$$

we can solve the result for H and find

$$H^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_c}{a^2} + \Omega_\Lambda\right). \tag{1.70}$$

The equation (1.70) lacks an observable variable. However, using the Geodesic Equation (1.25), one can find a relation between the scale factor $a(t)$ and the visible wavelength. To find this equality, we first define the four-momentum of a particle

$$P^\mu = \frac{dx^\mu}{d\lambda} = (E, p_r, p_\theta, p_\phi). \tag{1.71}$$

From the relativistic energy-momentum relation a massless particle, such as a photon, will have a four-momentum with zero magnitudes. Furthermore, a photon travels radially across the universe, therefore $p_\theta = p_\phi = 0$. Computing $g_{\mu\nu}P^\mu P^\nu$ we find that

$$-E^2 + \frac{a^2}{1-kr^2}p_r^2 = 0. \tag{1.72}$$

Replacing (1.71) into (1.25) and using the chain rule, the time component of the result is

$$E\frac{dE}{dt} = -\Gamma^0_{rr}p_r^2 = -a\dot{a}\frac{1}{1-kr^2}p_r^2. \tag{1.73}$$

Replacing p_r^2 from the energy-momentum relation into the equation above we arrive at a solvable relation between energy and the scale factor.

$$\frac{1}{E}\frac{dE}{dt} = -\frac{1}{a}\frac{da}{dt}. \tag{1.74}$$

Taking $a_0 = 1$, (1.74) yields

$$E = \frac{E_0}{a}. \tag{1.75}$$

A commonly used observable is the redshift, defined as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}. \tag{1.76}$$

By rewriting (1.75) in terms of λ_{obs} and λ_{em}, an equation of $a(z)$ emerges;

$$a = \frac{1}{1+z}. \tag{1.77}$$

This relation is fundamental. We shall use it to know the scale factor at the moment of the emission of the light we are observing. In close, we can replace the scale factor in (1.70) by (1.77), yielding the Parameter Equation as a function of the redshift;

$$H^2 = H_0^2 \left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_c(1+z)^2 + \Omega_\Lambda\right). \tag{1.78}$$

The redshift can be used to estimate the distance of a light-emitting object. We can first create a grid over the cosmos and use it to map fixed distances between objects. This way, even though the grid is expanding, the measurement within the grid remains constant. We call this the comoving distance,

$$\chi(z) = \int_0^z \frac{dz}{H(z)}. \tag{1.79}$$

A value of redshift will always be at a fixed comoving distance $\chi(z)$.

We can also estimate the distance using the flux of light measured by an observer. This is the luminosity distance, d_L, and it can be calculated in terms of (1.79) with

$$d_L = (1+z)\frac{H_0^{-1}}{\sqrt{|\Omega_k|}} S_k\left[H_0\sqrt{|\Omega_k|}\chi(z)\right]. \tag{1.80}$$

$S_k(\chi)$ is one of three possible functions depending on the curvature;

$$S_k(\chi) = \begin{cases} sin(\chi), & k = 1 \\ \chi, & k = 0 \\ sinh\chi, & k = -1 \end{cases}. \tag{1.81}$$

Finally, we can use the magnitude of an object to estimate its distance. The distance modulus, μ, is the difference between the apparent magnitude and the absolute magnitude. It is related to (1.80) by

$$\mu = 5\log_1 0\left(\frac{d_L}{10}\right). \tag{1.82}$$

2 Statistical Methods

The primary focus of data analysis is a set of data, observations, $\{x_i\}$, and an underlying relationship between these measurements. These assumptions are largely based on the belief that there is a deeper reality beyond the measurements themselves. Our goal is to extract meaningful information from these data. For us, this relationship is represented by the probability distribution P. The probability of a measurement x given a hypothesis θ is given by $P(x|\theta)$.

2.1 Something about estimators

2.1.1 Max Likelihood Estimators

2.1.2 Something about likelihood

2.2 Bayes

2.3 Something about parameters

3 Supernovaes as Standard Candles

3.1 Supernovaes Gaussian Distribution

4 Results

5 Conclusion

Os resultado batero tudo.

5.1 Future Researches

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Referências