Estimação de Sazon. estoc. por HS

1) Supor $\mu_t = \mu_{t-1} + \eta_t$, $\eta_t \sim N(0, \sigma^2_\eta)$

$$HS \Rightarrow \gamma_t = \underset{\sim}{x}'_t \, \underset{\sim}{\delta}_t \qquad \underset{\sim}{\delta}_t = (\delta_{1t}\ \delta_{2t} \dots \delta_{st})'$$

$$\hookrightarrow (x_{1t}, x_{2t}, \dots, x_{st})' \quad x_{jt} = \begin{cases} 1, & \text{se } t = s \\ 0, & \text{c.c.} \end{cases}$$

$$\delta_{tj} = \delta_{t-1,j} + \omega_{jt}$$

$$\omega_{jt} \sim N(0, \sigma^2_\omega)$$

$$\Downarrow$$

$$\sum_{j=1}^{s} \omega_{jt} = 0$$

Para simplificar supor $s = 4$

$$y_t = \mu_t + \gamma_t + \varepsilon_t = \mu_t + \underset{\sim}{x}'_t \underset{\sim}{\delta}_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$$

$$y_t = [1 \quad \underset{\sim}{x}'_t] \begin{bmatrix} \mu_t \\ \underset{\sim}{\delta}_t \end{bmatrix} + \varepsilon_t$$

$$\begin{pmatrix} \mu_t \\ \delta_{1t} \\ \delta_{2t} \\ \delta_{3t} \\ \delta_{4t} \end{pmatrix} = \underset{5\times5}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix}} \underset{5\times1}{\begin{pmatrix} \mu_{t-1} \\ \delta_{1,t-1} \\ \delta_{2,t-1} \\ \delta_{3,t-1} \\ \delta_{4,t-1} \end{pmatrix}} + \underset{5\times4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}} \underset{4\times1}{\begin{pmatrix} \eta_t \\ \omega_{1t} \\ \omega_{2t} \\ \omega_{3t} \end{pmatrix}}$$

2) Outra forma $\Rightarrow$ colocando restrição em $\underset{\sim}{Q}$

Como vimos se $\underset{\sim}{Q}_{Sazt} \hat{=} \sigma^2_\omega \left(I_4 - \frac{1}{4} i_4 i_4' \right) \Rightarrow \sum_{j=1}^{s} \delta_{jt} = 0$

$$\underset{\sim}{Q}_{Sazt} = \sigma^2_\omega \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right] = \begin{pmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix} \sigma^2_\omega = \underset{\sim}{Q}_{Sazt}$$

$$y_t = [1 \quad \underset{\sim}{x}'_t] \begin{bmatrix} \mu_t \\ \underset{\sim}{\delta}_t \end{bmatrix} + \varepsilon_t$$

$$\begin{pmatrix} \mu_t \\ \underset{\sim}{\delta}_t \end{pmatrix} = \underset{\sim}{I}_5 \begin{pmatrix} \mu_{t-1} \\ \underset{\sim}{\delta}_{t-1} \end{pmatrix} + \underset{\sim}{I}_5 \underbrace{\begin{pmatrix} \eta_t \\ \underset{\sim}{\omega}_t \end{pmatrix}}_{\underset{\sim}{\eta}_t}$$

$\underset{\sim}{\omega}_t \to 4\times1$

$$E(\underset{\sim}{\eta}_t \underset{\sim}{\eta}_t') = \begin{pmatrix} \sigma^2_\eta & 0 \\ 0 & \underset{\sim}{Q}_{Sazt} \end{pmatrix}$$

$\Longrightarrow$

Em ambos os casos os estimadores filtrados, atualizados e suavizados de $\underset{\sim}{\gamma}_t$ também seguirão a restrição teórica $\sum_{j=1}^{s} \gamma_{jt} = 0$, $\forall t$.