TREND ESTIMATION AND THE HODRICK-PRESCOTT FILTER.

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The article analyses the relationship between unobserved component trend-cycle models and the Hodrick-Prescott filter. Consideration is given to the consequences of using an inappropriate smoothing constant and the effect of changing the observation interval.

Key words and phrases: Signal-noise ratios, smoothing constant, stochastic cycles, stochastic trends, unobserved components models.

1. Stochastic trends

The fundamental time series models for trend analysis is one made up of a stochastic trend component, μ_t , and a random irregular term, ε_t . Other components, such as seasonals and cycles, may be added if required. Akaike (1980) was one of the first to suggest the use of such unobserved components models. His proposal, excluding the seasonal component, was

(1.1)
$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim NID(0, \sigma_{\varepsilon}^2), \quad t = 1, \dots, T$$

with

(1.2)
$$\mu_{t} = \mu_{t-1} + \beta_{t-1}, \\ \beta_{t} = \beta_{t-1} + \zeta_{t}, \quad \zeta_{t} \sim NID(0, \sigma_{\zeta}^{2}).$$

The irregular and slope disturbances, ε_t and ζ_t , respectively, are mutually independent and the notation $NID(0, \sigma^2)$ denotes normally and independently distributed with mean zero and variance σ^2 . The signal-noise ratio, $q = \sigma_{\zeta}^2/\sigma_{\varepsilon}^2$, plays the key role in determining how observations should be weighted for prediction and signal extraction. The higher is q, the more past observations are discounted in forecasting the future. Similarly a higher q means that the closest observations receive a bigger weight when signal extraction is carried out. The trend is known as an *integrated random walk* trend. When estimated it tends to be relatively smooth and indeed fitting the continuous time version of the model is known to be equivalent to fitting a cubic spline; see Wecker and Ansley (1983).

The statistical treatment of such unobserved component models is based on the state space form (SSF), as described in Harvey (1989) and Kitagawa and Gersch (1996). Once a model has been put in SSF, the Kalman filter yields

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estimators of the components based on current and past observations. Signal extraction, which refers to estimation of components based on all the information in the sample, is based on smoothing recursions which run backwards from the last observation. Predictions are made by extending the Kalman filter forward. The unknown variance parameters are estimated by constructing a likelihood function from the one-step ahead prediction errors, or innovations, produced by the Kalman filter.

The filter proposed by Hodrick and Prescott (1997)—usually referred to as the HP filter—is equivalent to the smoothed trend obtained from the model in (1.1) and (1.2). For quarterly data Hodrick and Prescott (1997) proposed a value of q = 1/1600, where 1600 is referred to as the smoothing constant; we will denote this filter as HP(1600). The HP filter is widely used for detrending macroeconomic time series, such as Gross Domestic Product (GDP), so as to yield a cycle. Harvey and Jaeger (1993) observed that, for US GDP, the HP filter gives a very similar trend to the one produced by fitting an unobserved components model in which the irregular component in (1.1) is replaced by a stochastic cycle. In this paper we investigate why this is the case by examining the weights associated with different smoothing filters and showing how filters may be matched up by comparing their gain functions. We then apply these techniques to US Investment and argue that a more appropriate value for the smoothing constant in an approximating HP filter is 1/32000 rather than 1/1600. This can have important practical implications, particularly at the end of the sample.

2. Frequency domain analysis

The effect of any linear filter, $w(L) = \sum w_j L^j$, where w_j , $j = 0, \pm 1, \pm 2, ...$ are fixed weights and L is the lag operator, can be obtained from the frequency response function, which is found by replacing L by $\exp(-i\lambda)$, where λ denotes frequency in radians. The gain is the modulus of the frequency response function. Assuming the original series to be stationary, the gain shows how the amplitude at each frequency is affected. Squaring the gain gives the factor by which the spectrum of the original series must be multiplied to give the spectrum of the filtered series. When the filter is symmetric, the frequency response function is real and if it is nowhere negative it is the same as the gain. A comparison of gains can be used to give an indication of the closeness of two filters.

Writing $w(e^{-i\lambda})$ somewhat more compactly as $w(\lambda)$, the gain for extracting the stochastic trend in (1.1) can be obtained from the Wiener-Kolmogorov (WK) filter² $w(L) = q/(q + |1 - L|^4)$, and expressed as

$$(2.1) w(\lambda) = \frac{1}{1 + q^{-1}(2 - 2\cos\lambda)^2} = \frac{1}{1 + q^{-1}2^4\sin^4(\lambda/2)}, 0 \le \lambda \le \pi.$$

If $\lambda_{0.5}$ is the frequency for which the gain equals one-half, the corresponding

¹ The original working paper appeared in 1980.

² We follow Whittle (1983, p. 12) in adopting the convention that $|\theta(L)|^2 = \theta(L)\theta(L^{-1})$.

signal-noise ratio is

(2.2)
$$q(\lambda_{0.5}) = \left[2\sin(\lambda_{0.5}/2)\right]^4, \quad 0 < \lambda_{0.5} < \pi.$$

A frequency of $\lambda_{0.5} = 0.1583$ corresponds to a period of 39.70 quarters or 9.93 years; see Gomez (2001). Hence, $q(\lambda_{0.5}) = q(0.1583) = 1/1600$. The gain for the HP detrending filter is $1 - w(\lambda)$. It is sometimes argued that the rationale for the HP(1600) filter is that it tends to cut out frequencies corresponding to periods above 9.93 years. However, as we will see later, it can be misleading to apply this argument to nonstationary time series.

3. Models with cycles

We now consider what happens when a stochastic cycle is added to the model. There are several questions of interest. Firstly, how does the presence of a cycle affect the weights and the behaviour of the extracted trend? Secondly, is the ratio of the slope variance to the total variance of the stationary component the best guide to the signal-noise ratio in an approximating HP filter and if not how might the signal-noise ratio in an approximating HP filter be found? Thirdly, what are the consequences of an inappropriate choice of signal-noise ratio (smoothing constant). Fourthly, what are the consequences of changing the observation interval?

The trend-cycle model is

$$y_t = \mu_t + \psi_t + \varepsilon_t, \quad t = 1, \dots, T.$$

An autoregressive model is often adopted for the cycle, ψ_t . Another possibility, used in Harvey and Jaeger (1993) and Koopman *et al.* (2006), is the stochastic cycle

(3.1)
$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos \lambda_c & \sin \lambda_c \\ -\sin \lambda_c & \cos \lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}, \quad t = 1, \dots, T,$$

where λ_c is frequency in radians and κ_t and κ_t^* are two mutually independent Gaussian white noise disturbances with zero means and common variance σ_{κ}^2 . Given the initial conditions that the vector $(\psi_0, \psi_0^*)'$ has zero mean and covariance matrix $\sigma_{\psi}^2 \mathbf{I}$, it can be shown that for $0 \leq \rho < 1$, the process ψ_t is stationary and indeterministic with zero mean, variance $\sigma_{\psi}^2 = \sigma_{\kappa}^2/(1-\rho^2)$ and τ -th order autocorrelation $\rho^{\tau} \cos \lambda_c \tau$. The spectrum of ψ_t displays a peak, centered around λ_c , which becomes sharper as ρ moves closer to one. The reduced form is an ARMA(2,1) process in which the autoregressive part has complex roots. The complex root restriction allows a clearer separation into trend and cycle, particularly when combined with the smooth trend restriction.

4. Weights and gains

For the integrated random walk, the ratio of the slope variance to the total variance of the stationary component is $q_c = \sigma_{\zeta}^2/(\sigma_{\psi}^2 + \sigma_{\varepsilon}^2)$. To simplify matters,

let $\sigma_{\varepsilon}^2 = 0$, so that $q_c = \sigma_{\zeta}^2/\sigma_{\psi}^2$. The WK filter for extracting the trend is

$$w_c(L) = \frac{q_c}{q_c + (1 - \rho^2)\{c(L)c(L^{-1}) + s(L)s(L^{-1})\}(1 - L^2)(1 - L^{-2})},$$

where

$$c(L) = \frac{1 - \rho \cos \lambda_c L}{1 - 2\rho \cos \lambda_c L + \rho^2 L^2} \quad \text{and} \quad s(L) = \frac{\rho \sin \lambda_c L}{1 - 2\rho \cos \lambda_c L + \rho^2 L^2}.$$

This becomes

$$w_c(L) = \frac{q_c |1 - 2\rho\cos\lambda_c L + \rho^2 L^2|^2}{q_c |1 - 2\rho\cos\lambda_c L + \rho^2 L^2|^2 + (1 - \rho^2)\{1 + \rho^2 - \rho\cos\lambda_c L - \rho\cos\lambda_c L^{-1}\}|1 - L^2|^2}.$$

Setting L=1 confirms that the weights sum to one.

Figure 1 shows the weights, calculated by the algorithm of Koopman and Harvey (2003), for extracting the trend using the HP(1600) filter, while Fig. 2 shows the weights from the trend-cycle model when there is no irregular and the cycle parameters are $\rho=0.9$ and $\lambda_c=.314$, the latter corresponding to a period of 20 quarters. The signal-noise ratios are the same, that is $q_c=q=1/1600$. The salient feature of these figures is the wider bandwidth³ induced by the inclusion of the cycle. Letting $w_c(L)=\sum_j w_{c,j}L^j$ and $w(L)=\sum_j w_jL^j$ denote the HP filter, the sum of the absolute differences in the weights, $\sum_j |w_{c,j}L^j-w_jL^j|$, is 0.342.

The gain associated with $w_c(L)$ is

$$w_c(\lambda) = \left[1 + \frac{1}{q_c} \frac{(2 - 2\cos\lambda)^2 (1 + \rho^2 - 2\rho\cos\lambda_c\cos\lambda)(1 - \rho^2)}{1 + \rho^4 + 4\rho^2\cos^2\lambda_c - 4(\rho + \rho^3)\cos\lambda_c\cos\lambda + 2\rho^2\cos2\lambda} \right]^{-1}$$

as opposed to (2.1). The two gains are shown in Fig. 3 for the same parameter settings as were used to calculate the weights shown in Figs. 1 and 2. Including the cycle cuts out more high frequency movement. A measure of the difference in gains is

$$dg = \int_0^{\pi} |w_c(e^{-i\lambda}) - w(e^{-i\lambda})| d\lambda.$$

Evaluating this using Simpson's rule gave 0.0454.

The value of q_c that makes $w_c(\lambda) = 0.5$ is

$$q_c = [2\sin(\lambda/2)]^4 \frac{(1+\rho^2 - 2\rho\cos\lambda_c\cos\lambda)(1-\rho^2)}{1+\rho^4 + 4\rho^2\cos^2\lambda_c - 4(\rho+\rho^3)\cos\lambda_c\cos\lambda + 2\rho^2\cos2\lambda}.$$

If we set $\lambda = 0.1583$, then $q_c = 0.002157$ rather than 0.000625 and the inverse is 464 rather than 1600. A plot of the trend-cycle gain with q_c in $w_c(\lambda)$ equal to 0.002157 is shown by the heavier dot-dash line in Fig. 3. As can be seen, this gain is slightly sharper than the HP gain in that it cuts out the higher frequencies

³ A reasonable measure of bandwidth in the figures would be the range of the positive values of the weights. This is approximately 40 for HP(1600)—the same as the period obtained by setting the gain equal to 0.5.

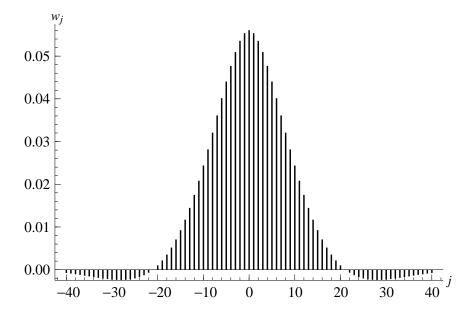


Figure 1. Weights for HP(1600) filter.

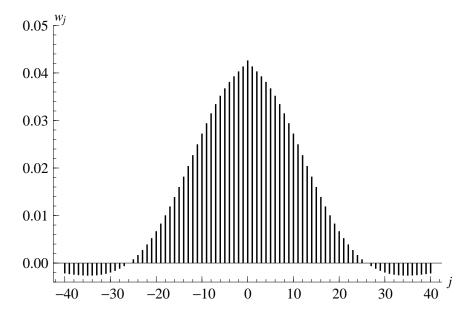


Figure 2. Weights for trend plus cycle with $q_c = 1/1600$.

more rapidly after $\lambda=0.1583$. The dg criterion is clearly smaller at 0.0177. The criterion of setting gains to 0.5 at the same frequency does not necessarily minimise dg but it comes close.

Calculating the weights for $w_c(L)$ with $1/q_c$ set to 464 shows them to be much

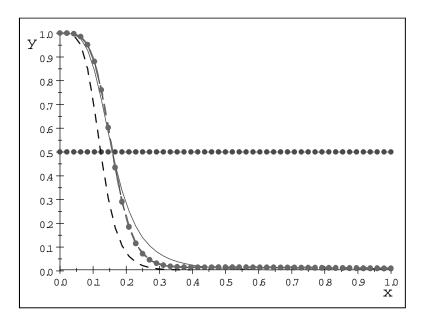


Figure 3. Gains for HP(1600) filter (continuous line), trend-cycle model (1600) before adjustment (dash) and trend-cycle model (464) after adjustment (dot-dash).

closer to those for HP(1600) with the sum of absolute differences now equal to 0.111. For quarterly US GDP from 1954:1 to 1989:4, Harvey and Jaeger (1993) estimated the period, $2\pi/\lambda_c$, to be 22.2 with $\rho = .92$ and found the inverse of the signal-noise ratio to be 508. This is not far from 464 and so it is not surprising that Harvey and Jaeger (1993) observed that the model based trend was similar to the HP(1600) trend.

The trend-cycle filter is sensitive to the choice of period, $P = 2\pi/\lambda_c$. If the period is increased, the gain shifts to the left, cutting out more of the lower frequencies. Decreasing the period shifts it to the right; when $q_c = 1/1600$, a period of about three years (P = 12) gives a gain close to that of HP(1600). An expression for the relationship between the HP smoothing constant, H = 1/q, and the inverse of the signal-noise ratio in the trend-cycle model, $Q = 1/q_c$, can be obtained by equating the gains at a value of 0.5.

5. Consequences of an inappropriate choice of smoothing constant

A smoothing constant of 1600 for quarterly data has been taken as a standard by many researchers in macroeconomics. In the frequency domain this value translates into a cut-off at a period of about ten years. The flaw in the case for determining the smoothing constant on the basis of a fixed cut-off point at this or any other predetermined value are brought home by considering what happens with co-integrated series exhibiting balanced growth. For example, Investment and GDP are usually assumed to have a common trend, but it is an established stylized fact that the variance of the cycle in investment is greater than that

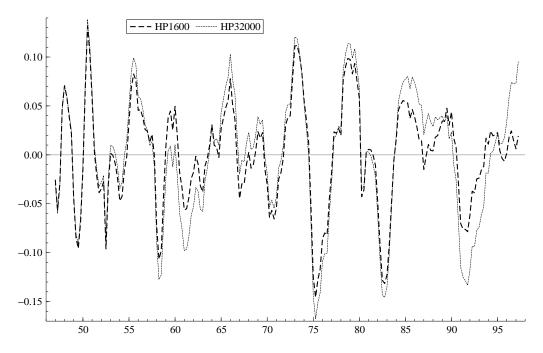


Figure 4. Detrended US investment using HP filters.

in GDP. Thus the signal-noise ratios in the individual series must be different. A factor of around 20 to 30 for the ratio of the investment to the GDP cycle emerges, irrespective of whether individual series are estimated or whether a bivariate model is estimated with a common trend; see, for example, Harvey and Trimbur (2003).

The gain of the detrending filter from a trend plus cycle model with the signal-noise ratio appropriate for GDP divided by 30 is now approximated quite well by HP(32000). The cut-off frequency⁴ in this case corresponds to a period of 21 years. The effect is shown in Fig. 4 which plots quarterly US investment, from 1947Q1 to 1997Q2, detrended by HP(32000) and HP(1600). As can be seen, using HP(1600) gives a smaller standard deviation—about 80% of that of HP(32000). The tendency for too small a smoothing constant to diminish the standard deviation can be confirmed by plotting the spectrum of a detrended trend plus cycle model. This theoretical analysis also points to other effects such as a slightly smaller period as indicated by the peak in the spectrum. From the practical point of view, an even more serious consequence of the smoothing constant being too small is that the large gap at the end of the series does not show up as it is absorbed within the trend.

 $^{4 \}lambda_{0.5} = 2 \sin^{-1}(q^{1/4}/2) = 2 \sin^{-1}((1/32000)^{1/4}/2)$. The 32000 was obtained by matching the gain of the HP filter to that of the trend-cycle filter.

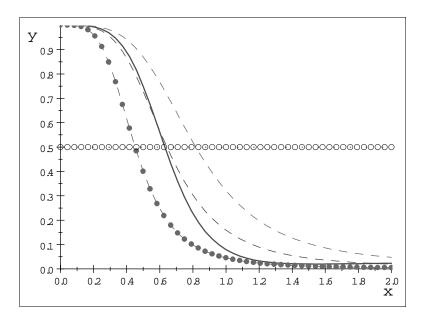


Figure 5. Gains for HP(25, 6.25, 2.5) filters (dashed lines, left to right) and stock trend-cycle model.

6. Changing the observation interval

For a stock variable, the analysis of the implications of a change in observation interval is relatively straightforward since the variance of the stationary component remains the same. At first sight this seems to imply that q_c is multiplied by δ , the observation interval. However, as δ increases, the serial correlation becomes weaker and as δ goes to infinity it becomes white noise. To be more specific, the specification of a stochastic cycle at an interval of δ is such that ρ becomes ρ^{τ} and λ_c becomes $\delta\lambda_c$; see Harvey (1989, p. 312). When $\rho = 0.9$ and $\lambda_c = 0.314$, moving from a quarterly to an annual interval results in the first order autocorrelation falling from 0.86 to 0.20. Figure 5 shows the gain calculated⁵ by dividing the smoothing constant by 4^3 and adjusting ρ and λ_c as above. For HP(6.25) the gain of 0.5 corresponds to a period just below ten years. The gain for the trend-cycle filter is quite close to that of HP(6.25)—at least for low frequencies—but it is well to the right of HP(25) which is an approximation that some authors have suggested for a stock.

For a flow variable, the analysis is more complicated because we first have to find out the implication for aggregating a cycle; see Harvey (1989, p. 313–7) and Chambers and McGarry (2002, p. 397–8). Precise results for a flow have not been worked out, but a rough approximation suggests that HP(2.5) may have a gain that is about the same as that of an aggregated trend-cycle model

⁵ Calculating the gain from $w_c(\lambda)$ takes no account of the fact that making observations less frequently induces a correlation between the level and slope disturbances; see Harvey (1989, p. 312). However, this turns out to make very little difference since when the gain is plotted it is virtually indistinguisable from the gain calculated assuming that the level and slope are mutually uncorrelated.

at $\lambda = 0.5$.

7. Conclusion

For quarterly US GDP the detrended series extracted by a trend-cycle model is similar to that obtained from the HP filter with a smoothing constant of 1600 and our analysis shows that the weights and gains are close. For an annual stock variable, a comparison of gains shows that an HP filter with a smoothing constant of around 6.25 might be expected to produce a similar result to a trend-cycle model when a smoothing constant of 1600 is a good choice for quarterly data. However, as the example of investment illustrates, fixing the smoothing constant on the basis of what works for US GDP is problematic.

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