Nível Local Multivariado

$$\underset{p\times 1}{y_t} = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma_\varepsilon) \qquad \text{(I-a)}$$

$$\mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim N(0, \Sigma_\eta) \qquad \text{(I-b)}$$

Seja a partição $\underset{p\times p}{\Sigma_\eta} = \begin{pmatrix} \Sigma_{11} \sim k\times k & \Sigma_{12} \sim k\times r \\ \underset{r\times k}{\Sigma_{12}'} & \Sigma_{22} \end{pmatrix}$ $\quad r = p-k$

Então (I-a,b) pode ser reparametrizado como:

$$y_t = \begin{pmatrix} \underset{k\times 1}{y_{1t}} \\ \underset{(r\times 1)}{y_{2t}} \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ \underset{r\times k}{\Pi} & I_r \end{pmatrix} \begin{pmatrix} \underset{k\times 1}{\mu_t^+} \\ \bar{\mu} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \qquad \text{(II-a)}$$

$$\underset{k\times 1}{\mu_{t+1}^+} = \mu_t^+ + \eta_t^+, \quad \eta_t^+ \sim N(0, \underset{k\times k}{\Sigma_\eta^+}) \qquad \text{(II-b)}$$

$$\underset{(r\times 1)}{\bar{\mu}_{t+1}} = \bar{\mu}_t + \bar{\eta}_t, \quad \bar{\eta}_t \sim N(0, \underset{r\times r}{\bar{\Sigma}_\eta})$$

Prova: De (I-b) $\mu_{t+1} = \mu_t + \eta_t \sim p\times 1$

$$\begin{matrix} (k\times 1) \rightarrow \\ (r\times 1) \rightarrow \end{matrix} \begin{pmatrix} \mu_{1,t+1} \\ \mu_{2,t+1} \end{pmatrix} = \begin{pmatrix} \mu_{1t} \\ \mu_{2t} \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix}$$

Seja a matriz $\underset{p\times p}{L} \hat{=} \begin{pmatrix} I_k & 0 \\ -\Pi & I_r \end{pmatrix}, \quad \underset{r\times k}{\Pi} = \underset{r\times k}{\Sigma_{21}} \underset{k\times k}{\Sigma_{11}^{-1}}$

$$L\mu_{t+1} = L\mu_t + L\eta_t \implies$$

$$\underbrace{\begin{bmatrix} I_k & 0 \\ -\Pi & I_n \end{bmatrix}\begin{bmatrix} \mu_{1,t+1} \\ \mu_{2,t+1} \end{bmatrix}}_{(L\,\mu_{t+1})} = \underbrace{\begin{bmatrix} I_k & 0 \\ -\Pi & I_n \end{bmatrix}\begin{bmatrix} \mu_{1t} \\ \mu_{2t} \end{bmatrix}}_{L\mu_t} + \underbrace{\begin{bmatrix} I_k & 0 \\ -\Pi & I_n \end{bmatrix}\begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}}_{L\eta_t}$$

$$\to I_k \mu_{1,t+1} = I_k \mu_{1,t} + I_k \eta_{1,t} \to \mu_{1,t+1} = \mu_{1,t} + \eta_{1,t}$$

$$\to -\Pi \mu_{1,t+1} + I_n \mu_{2,t+1} = -\Pi \mu_{1,t} + I_n \mu_{2t} - \Pi \eta_{1t} + I_n \eta_{2t}$$

$$\underbrace{\mu_{2,t+1} - \Pi \mu_{1,t+1}} = \underbrace{\mu_{2t} - \Pi \mu_{1t}} + \underbrace{\eta_{2t} - \Pi \eta_{1t}}$$

$$\overline{\mu}_{t+1} = \overline{\mu}_t + \overline{\eta}_t \quad , \text{ onde } \begin{cases} \overline{\mu}_t \hat{=} \mu_{2t} - \Pi \mu_{1t} \\ \overline{\eta}_t = \eta_{2t} - \Pi \eta_{1t} \end{cases}$$

E assim $L\,\mu_{t+1} = (\mu_{1,t+1}, \overline{\mu}_{t+1})'$

Por outro lado $\quad Z = L\eta_t$

$$E(Z) = 0$$

$$Var(Z) = E(ZZ') = L\,E(\eta_t \eta_t')\,L' = L\,\Sigma_n\,L'$$

$$L\Sigma_n L' = \begin{pmatrix} I_k & 0 \\ -\Pi & I_n \end{pmatrix}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} I_k & -\Pi' \\ 0 & I_n \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ -\Pi\Sigma_{11} + I_n\Sigma_{21} & -\Pi\Sigma_{12} + I_n\Sigma_{22} \end{pmatrix}\begin{pmatrix} I_k & -\Pi' \\ 0 & I_n \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{11} & \underbrace{-\Sigma_{11}\Pi' + \Sigma_{12}}_{0} \\ \underbrace{-\Pi\Sigma_{11} + \Sigma_{21}}_{0} & -\Pi\Sigma_{12} + \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix}$$

$$\Rightarrow -\Pi\Sigma_{11} + \Sigma_{21} = -\Sigma_{21}\underbrace{\Sigma_{11}^{-1}\Sigma_{11}}_{I} + \Sigma_{21} = -\Sigma_{21} + \Sigma_{21} = 0 \Rightarrow (-\Pi\Sigma_{11} + \Sigma_{21})' = 0$$

$$= -\Sigma_{11}\Pi' + \Sigma_{12} = 0$$

$$\Rightarrow -\Pi\Sigma_{12} + \Sigma_{22} = -\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} + \Sigma_{22}$$

E assim segue que

$$L\,\mu_{t+1} = \begin{pmatrix} \mu_{1,t+1} \\ \bar{\mu}_{t+1} \end{pmatrix} = \begin{pmatrix} \mu_{1,t+1} \\ \bar{\mu}_t \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \bar{\eta}_t \end{pmatrix},$$

$$\text{onde} \quad E\left[\begin{pmatrix} \eta_{1t} \\ \bar{\eta}_t \end{pmatrix} \begin{pmatrix} \eta_{1t} & \bar{\eta}_t \end{pmatrix}\right] = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \bar{\Sigma}_\eta \end{pmatrix}$$

$$\bar{\Sigma}_\eta = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

Definindo $\mu_{1t} = \mu_t^+$, a eq do estado fica

$$\begin{pmatrix} \mu_{t+1}^+ \\ \bar{\mu}_{t+1} \end{pmatrix} = \begin{pmatrix} \mu_t^+ \\ \bar{\mu}_t \end{pmatrix} + \underbrace{\begin{pmatrix} \eta_t^+ \\ \bar{\eta}_t \end{pmatrix}}_{\eta_t^*} \qquad E(\eta_t \eta_t^*) = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \bar{\Sigma}_\eta \end{pmatrix}$$

$$L\,\mu_t = \begin{pmatrix} \mu_t^+ \\ \bar{\mu}_t \end{pmatrix} \Rightarrow \mu_t = L^{-1}\begin{pmatrix} \mu_t^+ \\ \bar{\mu}_t \end{pmatrix}$$

$$L = \begin{pmatrix} I_k & 0 \\ -\Pi & I_n \end{pmatrix} \Rightarrow L^{-1} = \begin{pmatrix} I_k & 0 \\ \Pi & I_n \end{pmatrix}, \text{ e assim}$$

$$y_t = \underset{\sim}{\mu_t} + \underset{\sim}{\varepsilon_t} \Rightarrow y_t = L^{-1}\begin{pmatrix} \mu_t^+ \\ \bar{\mu}_t \end{pmatrix} + \underset{\sim}{\varepsilon_t}$$

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ \Pi & I_n \end{pmatrix}\begin{pmatrix} \mu_t^+ \\ \bar{\mu}_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

$\rightarrow$

$$
\left\{
\begin{array}{l}
\begin{matrix} y_{1t} \\ y_{2t} \end{matrix} = \begin{matrix} \mu_t^+ \\ \Pi \mu_t^+ + \bar{\mu}_t \end{matrix} + \begin{matrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{matrix} \\
\\
\mu_{t+1}^+ = \mu_t^+ + \eta_t^+ \\
\bar{\mu}_{t+1} = \bar{\mu}_t + \bar{\eta}_t
\end{array}
\right.
$$

tendências comuns $\Rightarrow$ posto $(\Sigma_\eta) = K < P$, e assim

$\hookrightarrow$ (I) apenas K tendencias são LI

(II) $(p-k) = r$ são LD

$$\bar{\Sigma}_\eta = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

e assim $\bar{\Sigma}_\eta = 0 \Rightarrow \bar{\mu}_{t+1} = \bar{\mu}_t = \bar{\mu}$

$$
\text{(III)} \quad \left\{
\begin{array}{l}
y_{1t} = \mu_t^+ + \varepsilon_{1t} \\
y_{2t} = \Pi \mu_t^+ + \bar{\mu} + \varepsilon_{2t} \\
\mu_{t+1}^+ = \mu_t^+ + \eta_t^+
\end{array}
\right.
$$

é o modelo de tendencias comuns

<u>Fatores de Carga</u> $\Rightarrow$ os fatores/componentes comuns podem ser, por construção, descorrelatados, e com variância unitária

Podemos re-escrever (III) como:

$$y_t = \Theta \mu_t^* + \mu_\theta + \varepsilon_t \quad , \varepsilon_t \sim N(0, \Sigma_\varepsilon)$$

$$\mu_{t+1}^* = \mu_t^* + \eta_t^* \quad , \eta_t^* \sim N(0, I_k)$$

$\mu_\theta \to (p \times 1)$

$\hookrightarrow$ K ~ zeros

$p-k \sim \bar{\mu}$

$\Theta \sim p \times k \Rightarrow$ matriz de cargas

$$\Theta = \Pi^+ (\Sigma_\eta^+)^{1/2} = (I, \Pi')'(\Sigma_\eta^+)^{1/2}$$

$$\begin{cases} y_t = \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} I_k \\ \Pi \end{pmatrix} \mu_t^+ + \begin{pmatrix} 0 \\ \bar{\mu} \end{pmatrix} + \varepsilon_t & \text{(III-a)} \\ \mu_{t+1}^+ = \mu_t^+ + \eta_t^+ & \text{(III-b)} \end{cases}$$

$$\Sigma_\eta^+ = \Sigma_\eta^{+1/2} \Sigma_\eta^{+1/2} \longrightarrow \begin{pmatrix} & * \\ 0 & \end{pmatrix}$$

Seja $\mu_t^* = \left(\Sigma_\eta^+\right)^{-1/2} \mu_t^+ \Rightarrow \mu_t^+ = \left(\Sigma_\eta^+\right)^{1/2} \mu_t^*$ Subs em III-a

$$\Rightarrow y_t = \begin{pmatrix} I_k \\ \Pi \end{pmatrix} \left(\Sigma_\eta^+\right)^{1/2} \mu_t^* + \begin{pmatrix} 0 \\ \bar{\mu} \end{pmatrix} + \varepsilon_t = \Pi^+ \left(\Sigma_\eta^+\right)^{1/2} \mu_t^* + \begin{pmatrix} 0 \\ \bar{\mu} \end{pmatrix} + \varepsilon_t$$

$$\Rightarrow y_t = \Theta \mu_t^* + \mu_\Theta + \varepsilon_t$$

$$\Rightarrow \mu_{t+1}^+ = \mu_t^+ + \eta_t^+$$

$$\left(\Sigma_\eta^+\right)^{-1/2} \mu_{t+1}^+ = \left(\Sigma_\eta^+\right)^{-1/2} \mu_t^+ + \Sigma_\eta^{+-1/2} \eta_t^+$$

$$\Rightarrow \mu_{t+1}^* = \mu_t^* + \eta_t^*, \quad \eta_t^* \sim N(0, I_k)$$

$$\eta_t^* = \Sigma^{-1/2} \eta_t^+$$

$$\mathrm{Var}(\eta_t^*) = E(\eta_t^* \eta_t^{*\prime}) = \Sigma_\eta^{+-1/2} \Sigma_\eta^+ \Sigma_\eta^{+-1/2}$$

$$= \Sigma_\eta^{+-1/2} \Sigma_\eta^{+1/2} \Sigma_\eta^{+1/2} \Sigma_\eta^{+-1/2} = I_k$$

$\Rightarrow$

$$\begin{cases} \overset{p\times 1}{y_t} = \overset{p\times k}{\Theta}\, \underset{(k\times 1)}{\mu_t^*} + \mu_\Theta + \varepsilon_t \\ \mu_{t+1}^* = \mu_t^* + \eta_t^*, \quad \eta_t^* \sim N(0, I_k) \end{cases}$$

Se $\left(\Sigma_\eta^+\right)^{1/2}$ é triang superior então Θ é tal que $\theta_{ij} = 0, j > i$
$i = 1, 2 .. k$

Ex $p = 3$
$k = 2$

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \\ \theta_{31} & \theta_{32} \end{pmatrix}_{3\times 2} \Rightarrow \begin{pmatrix} \theta_{11} & 0 \\ \theta_{21} & \theta_{22} \\ \theta_{31} & \theta_{32} \end{pmatrix}$$