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Fourier's Theorem

Fourier, not being noble, could not enter the artillery, although he was a second Newton.
—François Jean Dominique Arago

Trigonometry has come a long way since its inception more than two thousand years ago. But three developments, more than all others, stand out as having fundamentally changed the subject: Ptolemy's table of chords, which transformed trigonometry into a practical, computational science; De Moivre's theorem and Euler's formula $e^{ix} = \cos x + i \sin x$, which merged trigonometry with algebra and analysis; and Fourier's theorem, to which we devote this last chapter.

Jean Baptiste Joseph Fourier was born in Auxerre in north-central France on March 21, 1768. By the age of nine both his father and mother had died. Through the influence of some friends of the family, Fourier was admitted to a military school run by the Benedictine order, where he showed an early talent in mathematics. France has had a long tradition of producing great scientists who also served their country in the military: René Descartes (1596–1650), the soldier-turned-philosopher who invented analytic geometry; Gaspard Monge (1746–1818), who developed descriptive geometry and who in 1792 became minister of the marines; Jean Victor Poncelet (1788–1867), who wrote his great work on projective geometry while a prisoner of war in the aftermath of Napoleon's retreat from Moscow in 1812; and the two Carnots, the geometrician Lazar Nicolas Marguerite Carnot (1753–1823), who became one of France's great military leaders, and his son, the physicist Nicolas Léonard Sadi Carnot (1796–1832), who began his career as a military engineer and went on to lay the foundations of thermodynamics. Young Fourier wished to follow the tradition and become an artillery officer; but being of the wrong social class (his father was a tailor), he was only able to get a mathematics lectureship in the military school. This, however, did not deter him from getting involved in public life: he actively supported the French Revolution in 1789 and later was arrested for defending the victims of the terror, barely escaping the guillotine. In the end Fourier was rewarded for his activities and in 1795 was offered a professorship at the prestigious École Polytechnique in Paris, where Lagrange and Monge were also teaching.

In 1798 Emperor Napoleon Bonaparte launched his great military campaign in Egypt. A man of broad interests in the arts and sciences, Napoleon asked several prominent scholars to join him, among them Monge and Fourier. Fourier was appointed governor of southern Egypt and in that capacity organized the workshops of the French occupation forces. Following the French defeat at the hands of the British in 1801, he returned home and became prefect of the district of Grenoble. Among his administrative duties was the supervision of road construction and drainage projects, all of which he executed with great ability. And if that was not enough to keep him busy, he was also appointed secretary of the Institut d'Égypts, and in 1809 completed a major work on ancient Egypt, *Préface historique.*

One often marvels at the enormous range of activities of many eighteenth- and nineteenth-century scholars. At the very time Fourier was exercising his administrative duties, he was deeply engaged in his mathematical research. He worked in two seemingly unrelated fields: the theory of equations, and mathematical physics. When only sixteen, he found a new proof of Descartes' rule of signs about the number of positive and negative roots of a polynomial. His became the standard proof found in modern algebra texts. He began working on a book entitled *Analyse des équations déterminées*, in which he anticipated linear programming. However, Fourier died before completing this work (it was edited for publication in 1831 by his friend Louis Marie Henri Navier). He also pioneered dimensional analysis—the study of relations among physical quantities based on their dimensions.

But it is in mathematical physics that Fourier left his greatest mark. He was particularly interested in the manner in which heat flows from a region of high temperature to one of lower temperature. Newton had already studied this question and found that the rate of cooling (drop in temperature) of an object is proportional to the difference between its temperature and that of its surroundings. Newton's law of cooling, however, governs only the *temporal* rate of change of temperature, not its *spatial* rate of change, or gradient. This latter quantity depends on many factors: the heat conductivity of the object, its geometric shape, and the initial temperature distribution on its boundary. To deal with this problem one must use the analytic tools of the continuum, in particular partial differential equations (see p. 53). Fourier showed that to solve such an equation one must express the initial temperature distribution as a sum of infinitely many sine and cosine terms—a *trigonometric* or *Fourier series*. Fourier began work on this subject as early as 1807 and later expanded it in his major work, *Theorié analytique de la chaleur* (Analytic theory of heat, 1822), which became a model for some of the great nineteenth-century treatises on mathematical physics.

Fourier died tragically in Paris on May 16, 1830, after falling from a flight of stairs. Few portraits of him survive. A bust created in 1831 was destroyed in World War II. A second bust, erected in his hometown in 1849, was melted down by the German occupiers, who used the metal for armament; but the mayor of Auxerre got word of the impending disaster and managed to rescue two bas-reliefs of the bust, and fortunately these survived. In 1844 the archeologist Jacques Joseph Champollion-Figeac (brother of the Egyptologist Champollion mentioned in the Prologue) wrote Fourier's biography, entitled *Fourier, Napoleon et les cent jours*.[1]

In his work Fourier was guided as much by his sound grasp of physical principles as by purely mathematical considerations. His motto was: "Profound study of nature is the most fertile source of mathematical discoveries." This brought him biting criticism from such purists as Lagrange, Poisson, and Biot, who attacked his "lack of rigor"; one suspects, however, that political motivations and personal rivalry played a role as well. Ironically, Fourier's work in mathematical physics would later lead to one of the purest of all mathematical creations—Cantor's set theory.

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The basic idea behind Fourier's theorem is simple. We know that the functions $\cos x$ and $\sin x$ each have period 2π, the functions $\cos 2x$ and $\sin 2x$ have period $2\pi/2 = \pi$, and in general the functions $\cos nx$ and $\sin nx$ have period $2\pi/n$. But if we form any *linear combination* of these functions—that is, multiply each by a constant and add the results—the resulting function still has period 2π (fig. 92). This leads us to the following observation:

Let $f(x)$ be any "reasonably behaved" periodic function with period 2π; that is, $f(x + 2\pi) = f(x)$ for all x in its domain.[2] We

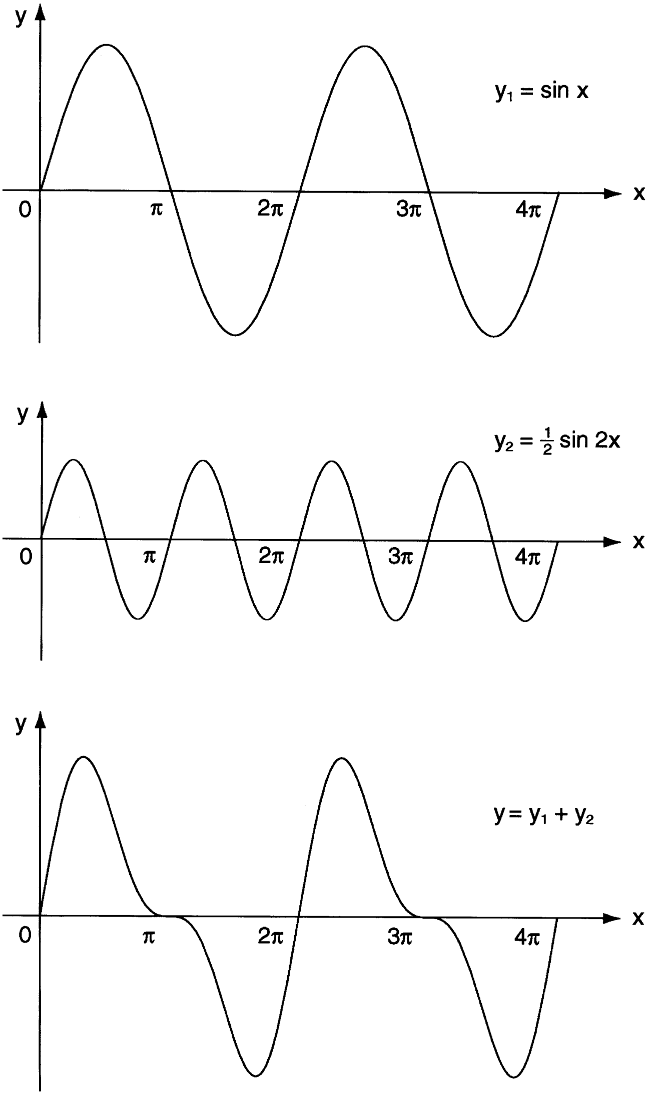

FIG. 92. Graphs of $\sin x$, $(\sin 2x)/2$, and their sum.

form the *finite* sum

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx \\ &\quad + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx \\ &= \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx), \end{aligned} \tag{1}$$

where the coefficients a_m and b_m are real numbers (the reason for dividing a_0 by 2 will become clear later); the subscript n under the $S(x)$ indicates that the sum depends on the number of sine and cosine terms present. Since $S_n(x)$ is the sum of terms of the form $\cos mx$ and $\sin mx$ for $m = 1, 2, 3, \ldots,$ it is a periodic function of x with period 2π; the nature of this function, of course, depends on the coefficients a_m and b_m (as well as on n). We now ask: is it possible to determine these coefficients so that the sum (1), for large n, will approximate the *given* function $f(x)$ in the interval $-\pi < x < \pi$? In other words, can we determine the a_m's and b_m's so that

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) \tag{2}$$

for every point in the interval $-\pi < x < \pi$? Of course, we require that the approximation should improve as n increases, and that for $n \to \infty$ it should become an equality; that is, $\lim_{n\to\infty} S_n(x) = f(x)$. If this indeed is possible, we say that the series (2) *converges* to $f(x)$ and write

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx). \tag{3}$$

In what follows we shall assume that the series (2) does indeed converge to $f(x)$ in the interval $-\pi < x < \pi$,[3] and we will show how to determine the coefficients.[4] Our starting point is the three integration formulas

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \neq 0 \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \text{for all } m \text{ and } n,$$

known as the *orthogonality relations* for the sine and cosine (these formulas can be proved by using the product-to-sum formulas for each integrand and then integrating each term separately; note that when both m and n are zero, the integrand in the second formula is 1, so that we get $\int_{-\pi}^{\pi} dx = 2\pi$).

To find the coefficients a_m for $m = 1, 2, 3, \ldots,$ we multiply equation (3) by $\cos mx$ and integrate it term-by-term over the interval $-\pi < x < \pi$.[5] In view of the orthogonality relations, all terms on the right side of the equation will be zero except the term $(a_m \cos mx) \cdot \cos mx = a_m \cos^2 mx = a_m(1 + \cos 2mx)/2$, whose integral from $-\pi$ to π is πa_m. We thus get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad m = 1, 2, 3, \cdots. \tag{4}$$

To find a_0 we repeat the process; but since we now have $m = 0$, multiplying equation (3) by $\cos 0x = 1$ leaves it unchanged, so we simply integrate it from $-\pi$ to π; again all terms will be zero except the term $(a_0/2) \int_{-\pi}^{\pi} dx = (a_0/2)(2\pi) = \pi a_0$. We thus get

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx. \tag{5}$$

Note that equation (5) is actually a special case of equation (4) for $m = 0$; this is why we chose the constant term in equation (3) as $a_0/2$. Had we chosen it to be a_0, the right side of equation (5) would have to be divided by 2.

Finally, to get b_m we multiply equation (3) by $\sin mx$ and again integrate from $-\pi$ to π; the result is

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \quad m = 0, 1, 2, \cdots. \tag{6}$$

Equations (4) through (6) are known as Euler's formulas (yes, two more formulas named after Euler!), and they allow us to find each coefficient of the Fourier series. Of course, depending on the nature of $f(x)$, the actual integration may or may not be executable in terms of the elementary functions; in the latter case we must resort to numerical integration.

Let us now apply this procedure to some simple functions. Consider the function $f(x) = x$, regarded as a periodic function over the interval $-\pi < x < \pi$; its graph has the saw-tooth shape shown in figure 93. Because this is an *odd* function (that is, $f(-x) = -f(x)$), the integrand in the first of Euler's equations is odd; and since the limits of integration are symmetric

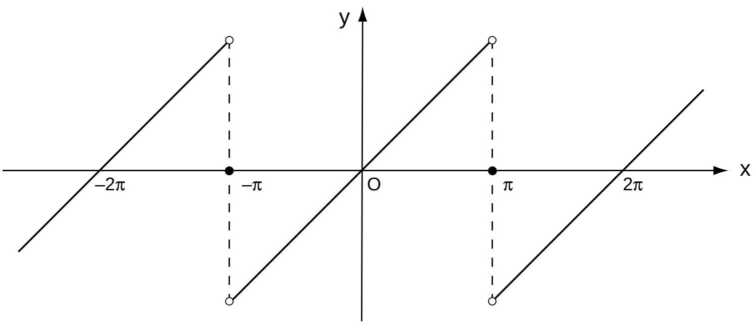

FIG. 93. Graph of the periodic function $f(x) = x$, $-\pi < x < \pi$.

with respect to the origin, the resulting integral will be zero for all $m = 0, 1, 2, \ldots$. Thus all the a_m's are zero, and our series will consist of sine terms only. For the b_m's we have

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin mx\, dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin mx\, dx.$$

Integration by parts leads to

$$b_m = \frac{2(-1)^{m+1}}{m}.$$

We thus have

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - + \cdots\right). \tag{7}$$

Figure 94 shows the first four partial sums of this series; we clearly see how the sine waves pile up near $x = \pm\pi$, but it is not so obvious that the series actually converges to the saw-tooth graph of figure 93 for *every* point of the interval, including the points of discontinuity at $x = \pm n\pi$. Indeed, in Fourier's time the fact that an infinite sum of smooth sine waves may converge to a function whose graph is anything but smooth was met with a great deal of disbelief.[6] But so were Zeno's paradoxes two thousand years earlier! When it comes to infinite processes, we can always expect some surprises around the corner.

Since equation (7) holds for any value of x, let us put in it some specific values. For $x = \pi/2$ we get

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right).$$

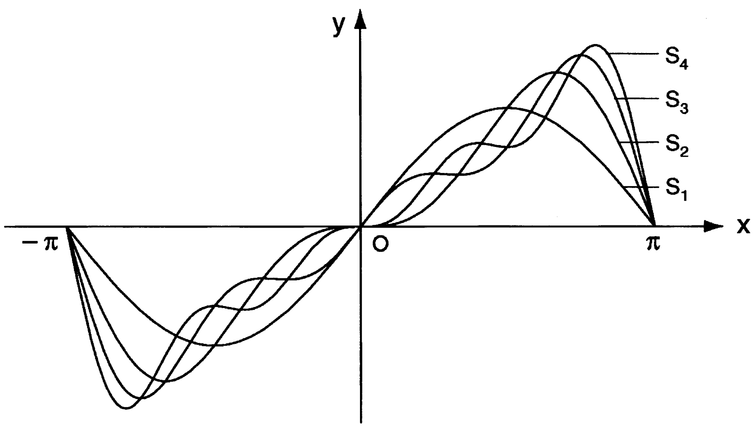

FIG. 94. First four partial sums of the Fourier expansion of $f(x) = x$, $-\pi < x < \pi$.

Dividing by 2 gives us the Gregory-Leibniz series (p. 159). For $x = \pi/4$ we get, after some labor,

$$\frac{\pi\sqrt{2}}{4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + + - - \cdots,$$

a little-known formula that connects the reciprocals of the odd integers with π and $\sqrt{2}$ (note that the right side of this series has the same terms as the Gregory-Leibniz series, but their signs alternate every two terms).

For the *even* function $f(x) = x^2$ (again regarded as a periodic function over the interval $-\pi < x < \pi$) we obtain, after twice integrating by parts, a Fourier series of cosine terms only:

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - + \cdots\right). \tag{8}$$

Substituting $x = \pi$ and simplifying results in

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

This is the famous formula that Euler discovered in 1734 in an entirely different and nonrigorous manner (see chapter 12). Many other series can be obtained in a similar way, as shown in figure 95.

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We have formulated Fourier's theorem for functions whose period is 2π, but it can easily be adjusted to functions with an arbitrary period P by the substitution $x' = (2\pi/P)x$. It then becomes more convenient to formulate the theorem in terms of the *angular frequency* ω (omega), defined as $\omega = 2\pi/P$. Fourier's theorem then says that any periodic function can be written as the sum of infinitely many sine and cosine terms whose angular frequencies are $\omega, 2\omega, 3\omega$, and so on. The lowest of these frequencies (i.e., ω) is the *fundamental frequency*, and its higher multiples are known as *harmonics*.

The word "harmonic," of course, comes to us from music, so let us digress for a moment into the world of sound. A *musical sound*—a tone—is produced by the regular, periodic vibrations of a material body such as a violin string or the air column of a flute. These regular vibrations produce in the ear a sense of pitch that can be written as a note on the musical staff. By contrast, nonmusical sounds—noises—are the result of irregular, random vibrations, and they generally lack a sense of pitch. Music, then, is the realm of periodic vibrations.[7]

23.7 $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$$\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

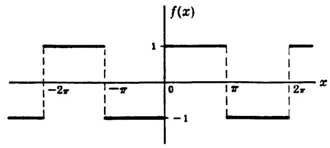

Fig. 23-1

23.8 $f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

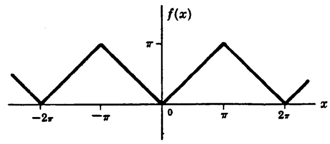

Fig. 23-2

23.9 $f(x) = x, \; -\pi < x < \pi$

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

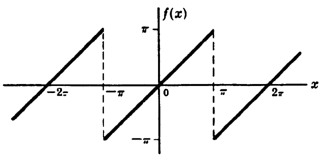

Fig. 23-3

23.10 $f(x) = x, \; 0 < x < 2\pi$

$$\pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

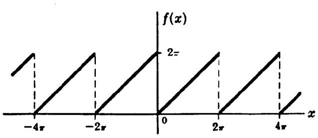

Fig. 23-4

23.11 $f(x) = |\sin x|, \; -\pi < x < \pi$

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \cdots\right)$$

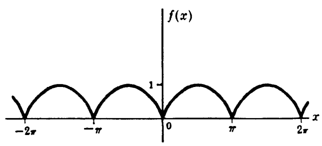

Fig. 23-5

Fig. 95. Fourier expansion of some elementary functions.

The pitch of a musical sound is determined by the frequency of its vibrations: the higher the frequency, the higher the pitch. For example, the note C ("middle C" on the staff) corresponds to a frequency of 264 hertz, or cycles per second; the note A above C, to 440 hertz, and the note C′ one octave above C, to 528 hertz.[8] Musical *intervals* correspond to frequency *ratios*: an octave corresponds to the ratio 2 : 1, a fifth to 3 : 2, a fourth to 4 : 3, and so on (the names "octave," "fifth," and "fourth" derive from the positions of these intervals in the musical scale).

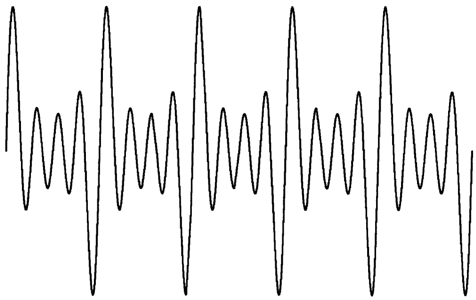

Fig. 96. Sound wave of a musical tone.

The simplest musical tone is a *pure tone*; it is produced by a sine wave, or—to use a term from physics—by *simple harmonic motion*.[9] A pure tone can be generated by an electronic synthesizer, but all natural musical instruments produce tones whose wave profiles, while periodic, are rather complicated (fig. 96). Nevertheless, these tones can always be broken down into their simple sine components—their *partial tones*—according to Fourier's theorem. Musical tones, then, are *compound tones*, whose constituent sine waves are the harmonics of the fundamental (lowest) frequency.[10]

The harmonics of a musical sound are not a mere mathematical abstraction: a trained ear can actually hear them. In fact, it is these harmonics that give a tone its characteristic "color"—its musical texture. The brilliant sound of a trumpet is due to its rich harmonic content; the sound of a flute is poor in harmonics, hence its mellow color (fig. 97). Each instrument has its characteristic *acoustic spectrum*—its signature of harmonic components. Amazingly, the human ear can split a compound tone into its component pure tones and hear each one of them separately, like a prism that splits white light into its rainbow colors

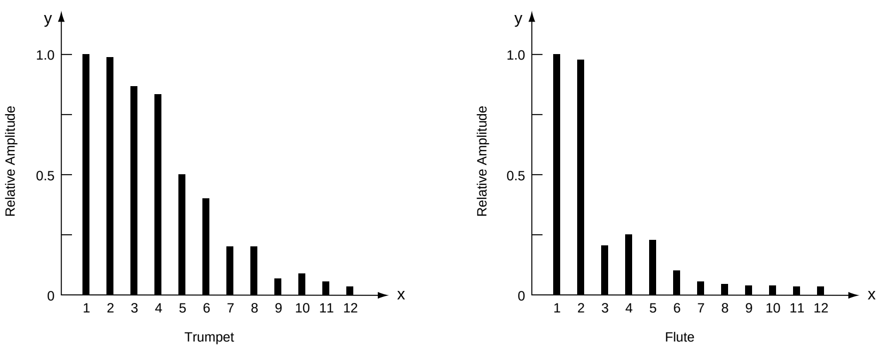

Fig. 97. Acoustic spectrum of a trumpet (left) and a flute (right).

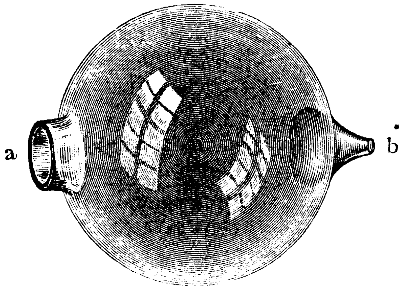

FIG. 98. Resonator.

(whence the name "spectrum"). The ear, in effect, is a Fourier analyzer.[11]

In the nineteenth century these ideas were new: scientists as well as musicians found it difficult to believe that a musical tone is actually the algebraic sum of all its harmonic components. The great German physicist and physiologist Herman Ludwig Ferdinand von Helmholtz (1821-1894) demonstrated the existence of partial tones by using *resonators*—small glass spheres of various sizes, each capable of enhancing one particular frequency in a compound tone (fig. 98). A series of these resonators formed a primitive Fourier analyzer analogous to the human ear. Helmholtz also did the reverse: by combining different simple tones of various frequencies and amplitudes, he was able to imitate the sound of actual musical instruments, anticipating the modern electronic synthesizer.

When we write the set of harmonics 1, 2, 3, . . . in musical notation, we get the sequence of notes shown in figure 99. This sequence is known as the *harmonic series*, and it plays a crucial

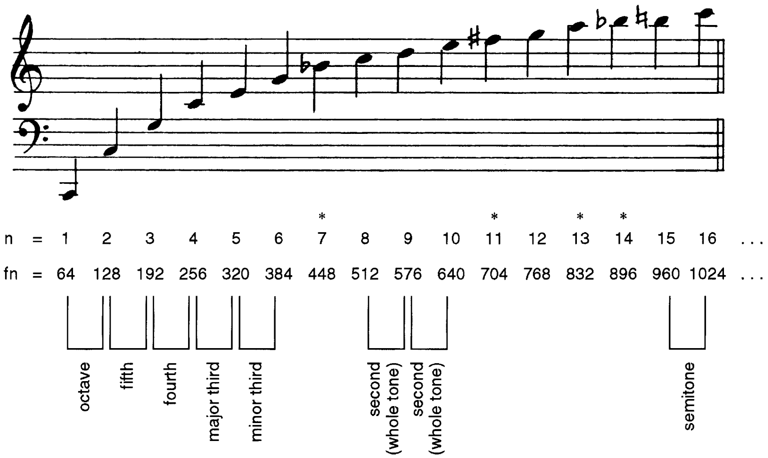

FIG. 99. The harmonic series.

role in musical theory: it is from this series that the fundamental musical intervals are derived.[12] That this series should have the same name as the mathematical series $1 + 1/2 + 1/3 + \cdots$ is no coincidence: the terms of the latter are precisely the periods of the harmonics in the former. Moreover, each term of the series $1 + 1/2 + 1/3 + \cdots$ is the *harmonic mean*[13] of the two terms immediately preceding and following it. These are just two examples of the numerous occurrences of the word "harmonic" in mathematics, reflecting the intimate ties that connect these two great creations of the human mind.

The importance of Fourier's theorem, of course, is not limited to music: it is at the heart of all periodic phenomena. Fourier himself extended the theorem to *nonperiodic* functions, regarding them as limiting cases of periodic functions whose period approaches infinity. The Fourier series is then replaced by an *integral* that represents a continuous distribution of sine waves over all frequencies. This idea proved of enormous importance to the development of quantum mechanics early in our century. The mathematics of Fourier's integral is more complicated than that of the series, but at its core are the same two functions that form the backbone of all trigonometry: the sine and cosine.[14]

NOTES AND SOURCES

1. There is no biography of Fourier in English. A brief sketch of his life can be found in Eric Temple Bell, *Men of Mathematics* (Harmondsworth, U.K.: Penguin Books, 1965), vol. 1, chap. 12. The biographical sketch of Fourier in this chapter is based in part on the article on Fourier by Jerome R. Ravetz and I. Grattan-Guiness in the *DSB*.

2. By "reasonably behaved" we mean that $f(x)$ is *sectionally smooth* on $-\pi < x < \pi$, i.e., that it is continuous and differentiable there except possibly at a finite number of finite jump discontinuities. At a jump discontinuity, we define $f(x)$ as $[f(x^-) + f(x^+)]/2$, that is, the mean between the values of $f(x)$ just to the left and right of the point in question. For a complete discussion, see Richard Courant, *Differential and Integral Calculus* (London: Blackie & Son, 1956), vol. 1, chap. 9.

3. Convergence is assured under the conditions stipulated in note 2.

4. The situation is somewhat analogous to the expansion of a function $f(x)$ in a power series $\sum_{i=0}^{n} a_i x^i$: we must determine the coefficients so that the sum will approximate the function at each point in the interval of convergence.

5. Term-by-term integration is permissible under the conditions mentioned in note 2.

6. For an interesting historical episode relating to this issue, see Paul J. Nahin, *The Science of Radio* (Woodbury, N.Y.: American Institute of Physics, 1995), pp. 85–86.

7. However, in our own era this traditional distinction has all but disappeared: witness the never-ending debate between classical music connoisseurs and rock fans as to what constitutes "real" music.

8. These frequencies are in accordance with the international standard known as *concert pitch*, in which A = 440 hertz. *Scientific pitch* is based on C = 256 hertz and has the advantage that all octaves of C correspond to powers of two; in this pitch A = 426.7 hertz.

9. The term "pure tone" refers both to a sine and a cosine vibration. This is because the human ear is not sensitive to the relative phase of a tone; that is, $\sin \omega t$ and $\sin(\omega t + \varepsilon)$ sound the same to the ear.

10. Strictly speaking there is a distinction between *overtones* in general—any set of higher frequencies present in a tone—and *harmonics*, those overtones whose frequencies are integral multiples of the fundamental frequency. Most musical instruments produce harmonic overtones, but some—notably drums and percussion—have nonharmonic components that cause their pitch to be less well defined.

11. By contrast, the eye does not have this capability: when blue and yellow light are superimposed, the result appears as green.

12. However, the presence of two different ratios, 9 : 8 and 10 : 9, for a whole tone causes difficulties when a melody is transposed (translated) from one scale to another. For this reason all modern instruments are tuned according to the *equal-tempered scale*, in which the octave consists of twelve equal semitones, each with the frequency ratio $(\sqrt[12]{2}) : 1$. The numerical value of this ratio is 1.059, slightly less than the *just intonation* semitone 16 : 15 = 1.066. See my article, "What is there so Mathematical about Music?" *Mathematics Teacher*, September 1979, pp. 415–422.

13. The harmonic mean H of two positive numbers a and b is defind as $H = 2ab/(a + b)$. From this it follows that $1/H = (1/a + 1/b)/2$, i.e., the reciprocal of the harmonic mean is the arithmetic mean of the reciprocals of a and b. As an example, the harmonic mean of 1/2 and 1/4 is 1/3.

14. Fourier series have also been generalized to nontrigonometric functions, with appropriate orthogonality relations analogous to those for the sine and cosine. For details, see any text on advanced applied mathematics.

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