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Comparing seasonal components for structural time series models

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Abstract

This paper discusses several encompassing representations for linear seasonal models in the structural framework. Their time and frequency domain properties are ascertained in a unifying framework, casting particular attention on the notion of 'smoothness' of the seasonal component. The shape of the forecast function is compared with that arising from a number of exponential smoothing algorithms. Finally, we investigate whether the specification of the seasonal model is likely to affect the out-of-sample predictive performance of the basic structural model. We conclude that the latter depends upon the features of the time series under investigation, and in particular on the degree of smoothness of the seasonal pattern.

Keywords: Exponential smoothing; Forecasting; Kalman filter; Seasonal adjustment; Smoothness

1. Introduction

It is a fact of life that seasonal fluctuations account for a major part of the variation of a wide spectrum of economic, social, and environmental phenomena. This explains the attention that they have received in the time series literature.

This paper focuses on seasonal models for a time series, y_t, $t = 1,2,\ldots,T$, observed with periodicity s, admitting the following representation:

$$y_t = \mu + \gamma_t + \epsilon_t \tag{1}$$

where μ is an intercept, γ_t denotes the seasonal effect at time t, and $\epsilon_t \sim \mathrm{NID}(0, \sigma^2_\epsilon)$. Several dynamic linear specifications for γ_t have been proposed in the structural approach, all complying with a definition of seasonality in terms of predictions, as the "part of the series which, when extrapolated, repeats itself over any one-year time period and averages out to zero over such a time period" (Harvey, 1989, p. 301). The feature shared by these models is that the seasonal sums, $U_t = S(L)\gamma_t$, where L is the lag operator and $S(L) = 1 + L + \cdots + L^{s-1}$ is the summation operator, has an invertible MA representation of order $s-2$ at most.

When μ is replaced by a stochastic trend, μ_t, model (1) is referred to (Harvey, 1989) as the *basic structural model* (BSM). In the sequel we will adopt the local level specification for this component:

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$$\begin{array}{lllr} \mu_t & = & \mu_{t-1} + \beta_{t-1} + \eta_t & \\ \beta_t & = & \beta_{t-1} + \zeta_t & \end{array} \qquad (2)$$

where η_t and ζ_t are mutually and serially independent disturbances with zero mean and variance σ^2_η and σ^2_ζ, respectively.

In this paper we review several equivalent representations for γ_t that encompass the linear models of stochastic seasonality proposed in the structural time series literature; particular emphasis will be placed on the specifications most commonly employed in empirical work such as the Harrison and Stevens' (1976, HS henceforth) model, Harvey's (1989) trigonometric seasonal model, Harvey and Todd's (1983, HT henceforth) dummy seasonal model, and a 'crude seasonal' specification.

We set off with the class of random walk seasonal models (Section 2), by which the effects associated with each season are allowed to vary over time according to a multivariate random walk with restrictions on the disturbance covariance matrix; the class of trigonometric seasonal models is dealt with in Section 3, where we argue that the trigonometric and the HS specifications are particular cases of the same underlying model and are obtained by imposing suitable restrictions on the covariance matrix of the innovations of the multivariate random walk. We also derive the restrictions under which the two are equivalent. Section 4 illustrates an encompassing representation for the quarterly case. In Section 5 we consider the West and Harrison (1997, WH henceforth) representation, which also encompasses the HT (1983) dummy seasonal model and the 'crude seasonal' specification.

The novel contribution of this paper is to establish a common ground for comparing the time and frequency domain properties of seasonal models. The key result is the stationary representation for $S(L)\gamma_t$ in terms of a moving average of order at most $s-2$ of linear combinations of a multivariate white noise process. Algorithms are then derived for the autocovariance function of the seasonal sums $S(L)\gamma_t$ (Section 5). The actual order of the MA representation for the seasonal sums and the role of rank deficiencies in the covariance matrix of the disturbances are dealt with.

In Section 6 we address the issue of characterising the smoothness of a seasonal model; on adoption of the Froeb and Koyak (1994) coefficient, it is possible to show that the HS model is smoother than the trigonometric one, which suffers from excess power at the Nyquist frequency. Now, it is usually thought that seasonal fluctuations should be slowly varying, as the underlying institutional factors and preferences change in the long run, but do not fluctuate wildly. By a formal argument we argue, however, that smoothness alone cannot provide a criterion upon which to judge the suitability of a particular model.

The last section is devoted to forecasting with the BSM under four different specifications of the model for seasonality. We discuss how the forecasts and their updating are related to alternative ad hoc and model-based forecasting procedures, and whether the out-of-sample predictive performance of the BSM is sensitive to the choice of the seasonal model.

2. The random walk representation of a seasonal model

In the time domain, a fixed (deterministic) seasonal pattern is modelled as: $\gamma_t = \boldsymbol{x}'_t\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is an $s\times 1$ vector containing the seasonal effects $\{\delta_j\}_{j=1}^{s}$, and $\boldsymbol{x}'_t = [D_{1t}, \ldots, D_{st}]$, with $D_{jt}=1$ in season j and 0 otherwise. The δ_j's measure the effect associated with the corresponding season and are restricted to sum up to zero, in order to ensure that model (1) is identifiable. Denoting $\boldsymbol{e}_{j,s} = [0, \ldots, 0, 1, 0, \ldots, 0]'$, a vector of length s with 1 in the jth position and 0 elsewhere, and

$\boldsymbol{i}_s = [1, 1, \ldots, 1]'$, an $s \times 1$ vector of ones, we have that $\boldsymbol{x}_t = \boldsymbol{e}_{j,s}$, $t = j \bmod s$; moreover, the zero-sum constraint is expressed as: $\boldsymbol{i}_s'\boldsymbol{\delta} = 0$.

A class of models of stochastic seasonality is derived by letting the coefficients $\boldsymbol{\delta}$ change over time according to a multivariate random walk:

$$\gamma_t = \boldsymbol{x}_t'\boldsymbol{\delta}_t,\ \ \boldsymbol{\delta}_t = \boldsymbol{\delta}_{t-1} + \boldsymbol{\omega}_t,\ \ \boldsymbol{\omega}_t \sim \mathrm{NID}(\boldsymbol{0}, \boldsymbol{\Omega}). \quad (3)$$

The stochastic counterpart of the zero sum constraint, $\boldsymbol{i}_s'\boldsymbol{\delta} = 0$, is enforced by

$$\boldsymbol{i}_s'\boldsymbol{\Omega} = \boldsymbol{0}_s'. \quad (4)$$

This implies $\boldsymbol{i}_s'\boldsymbol{\delta}_t = \boldsymbol{i}_s'\boldsymbol{\delta}_{t-1}$, which for $\boldsymbol{i}_s'\boldsymbol{\delta}_0 = 0$ implies in turn that $S(L)\gamma_t$ is a stationary zero mean process. Thus, the seasonal model arises as a systematic sample of a multivariate random walk, whose components measure the effect associated with a particular season. The constraint (4) induces a singularity in the covariance matrix $\boldsymbol{\Omega}$ that enforces one cointegration relation between the δ_{jt}'s.

By repeated substitution from (3):

$$\begin{aligned} S(L)\gamma_t &= \sum_{j=0}^{s-1} \boldsymbol{x}_{t-j}'\boldsymbol{\delta}_{t-j} \\ &= \sum_{j=0}^{s-1}\sum_{k=j}^{s-2} \boldsymbol{x}_{t-j}'\boldsymbol{\omega}_{t-j-k} + \left(\sum_{j=0}^{s-1} \boldsymbol{x}_{t-j}'\right)\boldsymbol{\delta}_{t-s+1} \\ &= \boldsymbol{x}_t'\boldsymbol{\omega}_t + (\boldsymbol{x}_t + \boldsymbol{x}_{t-1})'\boldsymbol{\omega}_{t-1} + \cdots \\ &\quad + (\boldsymbol{x}_t + \cdots + \boldsymbol{x}_{t-s+2})'\boldsymbol{\omega}_{t-s+2}. \end{aligned} \quad (5)$$

The last row of (5) states that $S(L)\gamma_t \sim MA(q)$, with $q \leq s-2$. As a matter of fact, there may be additional rank deficiencies implying for instance $(\boldsymbol{x}_t + \cdots + \boldsymbol{x}_{t-s+2})'\boldsymbol{\Omega} = \boldsymbol{0}'$. In general, $\mathrm{rank}(\boldsymbol{\Omega}) = q+1$ is necessary but not sufficient for $S(L)\gamma_t \sim MA(q)$; an illustration is provided in Section 5.

Model (3) is time-invariant if $\boldsymbol{\Omega}$ is a Toeplitz matrix; in such circumstance the autocovariance function of $U_t = S(L)\gamma_t$ does not depend on the season. Moreover, U_t is invertible if $\boldsymbol{\Omega}$ is positive semidefinite. The HS model is a special case of (3), which is obtained by choosing:

$$\boldsymbol{\Omega} = \sigma_\omega^2\left[\boldsymbol{I}_s - \frac{1}{s}\boldsymbol{i}_s\boldsymbol{i}_s'\right]. \quad (6)$$

This model is such that $U_t \sim MA(s-2)$, since the theoretical autocovariance of U_t at lag $s-2$ is nonzero and, precisely, defining $c_k = E(U_tU_{t-k})$, $c_{s-2} = \sigma_\omega^2 s^{-1}$. Representation (3) is amenable since the disturbances in $\boldsymbol{\omega}_t$ are season-specific. In Proietti (1998) it is argued that it can be generalised so as to account for seasonal heteroscedasticity in a simple fashion.

The constraint (4) can be made explicit expressing $\boldsymbol{\Omega} = \boldsymbol{\Theta}\tilde{\boldsymbol{\Omega}}\boldsymbol{\Theta}'$, for $\tilde{\boldsymbol{\Omega}}$ unconstrained and where

$$\boldsymbol{\Theta} = [\boldsymbol{I}_{s-1},\ -\boldsymbol{i}_{s-1}]'. \quad (7)$$

This allows us to reparameterise the seasonal model in terms of $s-1$ effects: indeed, setting $\tilde{\boldsymbol{\delta}}_t = (\boldsymbol{\Theta}'\boldsymbol{\Theta})^{-1}\boldsymbol{\Theta}'\boldsymbol{\delta}_t$, it is possible to write:

$$\gamma_t = \tilde{\boldsymbol{x}}_t'\tilde{\boldsymbol{\delta}}_t,\ \ \tilde{\boldsymbol{\delta}}_t = \tilde{\boldsymbol{\delta}}_{t-1} + \tilde{\boldsymbol{\omega}}_t,\ \ \tilde{\boldsymbol{\omega}}_t \sim \mathrm{NID}(\boldsymbol{0}, \tilde{\boldsymbol{\Omega}}) \quad (8)$$

where $\tilde{\boldsymbol{\delta}}_t$ is an $(s-1) \times 1$ vector containing the seasonal effects associated with the first $s-1$ seasons, and $\tilde{\boldsymbol{x}}_t' = \boldsymbol{x}_t'\boldsymbol{\Theta} = [\tilde{D}_{1t}, \ldots, \tilde{D}_{s-1,t}]$, with $\tilde{D}_{jt} = D_{jt} - D_{st}$, i.e. $\tilde{D}_{jt} = 1$, for $t = j \bmod s$, $\tilde{D}_{jt} = 0$, for $t \neq j \bmod s$, $\tilde{D}_{jt} = -1$, for $t = s \bmod s$. Hence, $\tilde{\boldsymbol{x}}_t = \boldsymbol{e}_{j,s-1}$, $t = j \neq s \bmod s$, and $\tilde{\boldsymbol{x}}_t = -\boldsymbol{i}_{s-1}$, $t = s \bmod s$. In particular, the HS model has $\tilde{\boldsymbol{\Omega}} = \sigma_\omega^2(\boldsymbol{\Theta}'\boldsymbol{\Theta})^{-1} = \sigma_\omega^2[\boldsymbol{I}_{s-1} - s^{-1}\boldsymbol{i}_{s-1}\boldsymbol{i}_{s-1}']$.

3. The trigonometric seasonal model

In the frequency domain, a fixed seasonal pattern is formulated in terms of $s-1$ effects associated with the amplitude of deterministic sine and cosine waves defined at the seasonal frequencies, $\lambda_j = 2\pi j/s$, $j = 1, 2, \ldots, [s/2]$, where $[s/2] = s/2$ for s even and $(s-1)/2$ if s is odd. If these effects are collected in the vector $\boldsymbol{\tau}$, we write: $\gamma_t = \boldsymbol{z}_t'\boldsymbol{\tau}$, with

$$\boldsymbol{z}_t' = [\cos\lambda_1 t, \sin\lambda_1 t, \dots, \cos\lambda_j t, \sin\lambda_j t, \dots, \cos\lambda_{[s/2]} t, \sin\lambda_{[s/2]} t].$$

When s is even, the last element of the vector is zero and is dropped, so that $\boldsymbol{z}_t$ always contains $s-1$ elements. Due to the periodic nature of the sine and cosine functions, there exists a linear mapping of the $\boldsymbol{z}_t$'s into the $\boldsymbol{x}_t$'s: $\boldsymbol{z}_t = \boldsymbol{H}'\boldsymbol{x}_t$, for the $(s-1)\times s$ matrix $\boldsymbol{H}' = [\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_s]$ (Canova & Hansen, 1995). By trigonometric identities $\sum_{j=0}^{s-1}\boldsymbol{z}_{t-j} = \boldsymbol{0}_{s-1}$.

The stochastic trigonometric seasonal model is obtained by letting $\boldsymbol{\tau}_t$ evolve as a multivariate RW:

$$\boldsymbol{\tau}_t = \boldsymbol{\tau}_{t-1} + \boldsymbol{\kappa}_t, \quad \boldsymbol{\kappa}_t \sim \mathrm{NID}(\boldsymbol{0}_{s-1}, \boldsymbol{K}). \tag{9}$$

The seasonal model proposed by Hannan, Terrell and Tuckwell (1970) arises for $\boldsymbol{K}$ diagonal, such that the diagonal elements vary with the frequency. It has been employed by Canova and Hansen (1995) and by Hylleberg and Pagan (1997) to explore issues relating to testing for unit roots at the seasonal frequencies. Harvey (1989) uses a different representation of this model, according to which the seasonal effect at time t results from the combination of $[s/2]$ stochastic cycles:

$$\gamma_t = \sum_{i=1}^{[s/2]} \gamma_{jt}, \quad \begin{bmatrix}\gamma_{jt}\\ \gamma_{jt}^{*}\end{bmatrix} = \begin{bmatrix}\cos\lambda_j & \sin\lambda_j\\ -\sin\lambda_j & \cos\lambda_j\end{bmatrix}\begin{bmatrix}\gamma_{j,t-1}\\ \gamma_{j,t-1}^{*}\end{bmatrix} + \begin{bmatrix}\omega_{j,t}\\ \omega_{jt}^{*}\end{bmatrix} \tag{10}$$

$j = 1, \dots, [s/2]$; for s even, the last component, defined at $\lambda_{s/2} = \pi$, reduces to

$$(1+L)\gamma_{s/2,t} = \omega_{s/2,t} \tag{11}$$

ω_{jt} and ω_{jt}^{*} are a set of serially and mutually independent sequences and if it is assumed that they have common variance σ_ω^2, this representation is equivalent to (9) with $\boldsymbol{K} = \sigma_\omega^2 \boldsymbol{I}_{s-1}$. The sequel model (9) with this specification for $\boldsymbol{K}$ will be referred to as the TS model.

Let us turn now to the relationship between (3) and the trigonometric seasonal model. In the light of the relationship $\boldsymbol{z}_t = \boldsymbol{H}'\boldsymbol{x}_t$, the trigonometric model of stochastic seasonality can be parameterised as (3), since $\gamma_t = \boldsymbol{z}_t'\boldsymbol{\tau}_t = \boldsymbol{x}_t'\boldsymbol{H}\boldsymbol{\tau}_t$, and (9) implies that $\boldsymbol{\delta}_t = \boldsymbol{H}\boldsymbol{\tau}_t$ is generated as a multivariate random walk with innovation covariance matrix:

$$\boldsymbol{\Omega} = \boldsymbol{H}\boldsymbol{K}\boldsymbol{H}'. \tag{12}$$

Note that by trigonometric identities $\boldsymbol{i}_s'\boldsymbol{H} = \boldsymbol{0}_{s-1}'$, so that (4) is automatically satisfied.

An interesting issue is what specification of $\boldsymbol{K}$ ensures that the trigonometric model is coincident with the HS model in (3) and (6): in view of (12) we must have

$$\sigma_\omega^2\left[\boldsymbol{I}_s - \frac{1}{s}\boldsymbol{i}_s\boldsymbol{i}_s'\right] = \boldsymbol{H}\boldsymbol{K}\boldsymbol{H}'.$$

Premultiplying and postmultiplying both sides respectively by $\boldsymbol{H}'$ and $\boldsymbol{H}$, and solving for $\boldsymbol{K}$, gives $\boldsymbol{K} = \sigma_\omega^2(\boldsymbol{H}'\boldsymbol{H})^{-1}$. For s odd, $\boldsymbol{K} = \tilde{\sigma}_\omega^2\boldsymbol{I}_{s-1}$, with $\tilde{\sigma}_\omega^2 = 2\sigma_\omega^2/(s-1)$, whereas for s even, $\boldsymbol{K} = \tilde{\sigma}_\omega^2\,\mathrm{diag}[1, 1, \dots, 1, 1/2]$, with $\tilde{\sigma}_\omega^2 = 2\sigma_\omega^2/s$. Hence, the last element of $\boldsymbol{\tau}_t$ has disturbance variance that is one-half of the other elements. In the Harvey's representation of the trigonometric seasonal model (10), this amounts to setting $\mathrm{Var}(\omega_{s/2,t}) = \sigma_\omega^2/2$ in (11).

4. Illustration: quarterly seasonality

In the quarterly case ($s = 4$) the assumption that $\boldsymbol{\Omega} = \{\Omega_{ij}\}$ is a Toeplitz matrix implies the following representation:

$$\boldsymbol{\Omega} = s_\omega^2\begin{bmatrix}1 & \sigma & -(1+2\sigma) & \sigma\\ \sigma & 1 & \sigma & -(1+2\sigma)\\ -(1+2\sigma) & \sigma & 1 & \sigma\\ \sigma & -(1+2\sigma) & \sigma & 1\end{bmatrix} \tag{13}$$

which depends on just two parameters, ς^2_ω and σ, the former acting as a scale parameter; the latter is constrained to lie in the interval $(-1, 0)$ in order to ensure that $\mathbf{\Omega}$ is positive semidefinite. The restrictions $\Omega_{31} = -\varsigma^2_\omega(1+2\sigma)$ and $\Omega_{41} = \varsigma^2_\omega\sigma$ arise as a consequence of the zero sum constraint imposed on the columns of $\mathbf{\Omega}$. In general, $\mathbf{\Omega}$ depends on $[s/2]$ parameters, e.g. on six parameters for monthly processes.

Note that (13) encompasses three commonly employed representations: the HS model, which arises for $\sigma = -1/3$, $\varsigma^2_\omega = 3\sigma^2_\omega/4$, the TS model, arising for $\sigma = -1/2$, and the trigonometric seasonal model with disturbance variances varying across the frequencies, $\boldsymbol{K} = \text{diag}(\sigma^2_{\omega_1}, \sigma^2_{\omega_1}, \sigma^2_{\omega_2})$, for which $\sigma = -\sigma^2_{\omega_2}/\varsigma^2_\omega$ and $\varsigma^2_\omega = \sigma^2_{\omega_1} + \sigma^2_{\omega_2}$.

5. The West–Harrison representation

Some seasonal models proposed in the literature cannot be represented as (3) with a time invariant $\mathbf{\Omega}$; nevertheless, they admit the so-called WH (1997) representation:

$$\gamma_t = \boldsymbol{e}'_{1,s}\mathbf{\Phi}_t, \;\; \mathbf{\Phi}_t = \boldsymbol{P}\mathbf{\Phi}_{t-1} + \boldsymbol{\omega}^\dagger_t,$$
$$\boldsymbol{\omega}^\dagger_t \sim \text{NID}(\mathbf{0}, \mathbf{\Omega}^\dagger) \tag{14}$$

where $\boldsymbol{i}'_s\mathbf{\Omega}^\dagger = \mathbf{0}'_s$, and $\boldsymbol{P}$ is the permutation matrix:

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{0}_{s-1} & \boldsymbol{I}_{s-1} \\ 1 & \mathbf{0}'_{s-1} \end{bmatrix}$$

characterised by the periodic property: $\boldsymbol{P}^{sn+j} = \boldsymbol{P}^j$, for integer n and $j = 1, \ldots, s$.

The connection with representation (3) is easily established writing $\boldsymbol{e}'_{j,s} = \boldsymbol{e}'_{1,s}\boldsymbol{P}^{t-1}$, and defining the process $\mathbf{\Phi}_t = \boldsymbol{P}^{t-1}\boldsymbol{\delta}_t$, so that $\boldsymbol{\omega}^\dagger_t = \boldsymbol{P}^{t-1}\boldsymbol{\omega}_t$, and $\mathbf{\Omega}^\dagger = \boldsymbol{P}^{t-1}\mathbf{\Omega}\boldsymbol{P}^{t-1\,\prime}$, provided $\mathbf{\Omega}^\dagger$ is time invariant. Obviously, as $\boldsymbol{i}'_s\boldsymbol{P}^{t-1} = \boldsymbol{i}'_s$, (4) implies $\boldsymbol{i}'_s\mathbf{\Omega}^\dagger = \mathbf{0}'_s$ and $\boldsymbol{i}'_s\mathbf{\Phi}_t = 0$. For the HS model $\mathbf{\Omega}^\dagger$ is given by (6), although $\omega^\dagger_{jt}$ is no longer interpreted as the disturbance pertaining to the jth season.

The time series properties of the parameterisation (14) can be ascertained from the following result:

$$\begin{aligned} S(L)\gamma_t &= \boldsymbol{e}'_{1,s}\sum_{j=0}^{s-1}\mathbf{\Phi}_{t-j} \\ &= \boldsymbol{e}'_{1,s}\left(\sum_{j=0}^{s-1}\sum_{k=0}^{j}\boldsymbol{P}^k\boldsymbol{\omega}^\dagger_{t-j} + \sum_{k=0}^{s-1}\boldsymbol{P}^k\mathbf{\Phi}_{t-s+1}\right) \\ &= \omega^\dagger_{1t} + (\omega^\dagger_{1,t-1} + \omega^\dagger_{2,t-1}) + \cdots \\ &\quad + (\omega^\dagger_{1,t-s+2} + \cdots + \omega^\dagger_{s-1,t-s+2}) \end{aligned} \tag{15}$$

which follows from $\boldsymbol{e}'_{1,s}\Sigma^{s-1}_{k=0}\boldsymbol{P}^k = \boldsymbol{i}'_s$. Hence, $S(L)\gamma_t \sim MA(q)$, $q \leq s-2$.

Setting $\mathbf{\Omega}^\dagger = \sigma^2_\omega\boldsymbol{r}^\dagger\boldsymbol{r}^{\dagger\prime}$, with $\boldsymbol{r}^\dagger = [-\boldsymbol{i}'_{s-1}, (s-1)]'$, the specification considered by WH (1997, p. 243) is obtained; with a terminology borrowed from Roberts (1982), see also Section 7, this will be referred to as the 'crude seasonal' specification (CS henceforth). Its interesting feature is that, although $\text{rank}(\mathbf{\Omega}^\dagger) = 1$, $U_t \sim MA(s-2)$. In fact, by the last row of (15), the theoretical autocovariance of U_t at lag $s-2$ is $\sigma^2_\omega(s-1)$. This illustrates that $\text{rank}(\mathbf{\Omega}^\dagger) = q+1$ is not sufficient to ensure $U_t \sim MA(q)$.

The constraint $\boldsymbol{i}'_s\mathbf{\Omega}^\dagger = \mathbf{0}'_s$ implies that the model (14) can be reparameterised in terms of a first order vector autoregression with dimension $s-1$. This amounts to writing $\mathbf{\Phi}_t = \boldsymbol{A}\mathbf{\Psi}_t$, for an $s \times (s-1)$ transformation matrix $\boldsymbol{A}$ with the property $\boldsymbol{i}'_s\boldsymbol{A} = \mathbf{0}'$. Hence,

$$\gamma_t = (\boldsymbol{e}'_{1,s}\boldsymbol{A})\mathbf{\Psi}_t, \;\; \mathbf{\Psi}_t = \boldsymbol{T}\mathbf{\Psi}_{t-1} + \boldsymbol{\omega}^*_t,$$
$$\boldsymbol{\omega}^*_t \sim \text{NID}(\mathbf{0}, \mathbf{\Omega}^*) \tag{16}$$

where $\boldsymbol{T} = \boldsymbol{A}^-\boldsymbol{P}\boldsymbol{A}$, $\boldsymbol{A}^-$ denoting the M-P inverse of $\boldsymbol{A}$, and $\mathbf{\Omega}^* = \boldsymbol{A}^-\mathbf{\Omega}^\dagger\boldsymbol{A}^{-\prime}$.

If we choose $\boldsymbol{A} = \mathbf{\Theta}$, as given in (7), then $\boldsymbol{e}'_{1,s}\mathbf{\Theta} = \boldsymbol{e}'_{1,s-1}$, so the measurement equation reduces to $\gamma_t = \boldsymbol{e}'_{1,s-1}\mathbf{\Psi}_t$. Moreover, the transition matrix takes the form:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{0}_{s-2} & \boldsymbol{I}_{s-2} \\ & -\boldsymbol{i}'_{s-1} \end{bmatrix}.$$

This is equivalent to dropping the last element, Φ_{st}, from (14) and replacing $\Phi_{s,t-1} = -\Sigma_{j=1}^{s-1}\Phi_{j,t-1}$ in the last but one recursion, $\Phi_{s-1,t} = \Phi_{s,t-1} + \omega^{\dagger}_{s-1,t}$; hence, the transition model for the first $s-2$ elements is unchanged whereas that for the element in position $(s-1)$ is amended. Finally, $\boldsymbol{\omega}^{*}_{t} = [\omega^{\dagger}_{1t}, \ldots, \omega^{\dagger}_{s-1,t}]'$.

The dummy seasonal (DS) model proposed by HT (1983) is formulated so as to require that the sum of the seasonal effects over a one year span is a zero mean white noise innovation:

$$\gamma_t = -\gamma_{t-1} - \cdots - \gamma_{t-s+1} + \omega_t \tag{17}$$

or equivalently $S(L)\gamma_t = \omega_t$, where $\omega_t \sim \mathrm{NID}(0, \sigma^2_{\omega})$. The DS model is actually a particular case of model (14) which has $\boldsymbol{\Omega}^{\dagger} = \sigma^2_{\omega}\boldsymbol{r}\boldsymbol{r}'$, with $\boldsymbol{r} = [0, \ldots, 0, -1, 1]'$. As a matter of fact, it can be written as (16) with $\boldsymbol{\Omega}^{*} = \sigma^2_{\omega}\boldsymbol{e}_{s-1,s-1}\boldsymbol{e}'_{s-1,s-1}$. The DS model can also be represented as in (3) with a time varying covariance matrix: $\boldsymbol{\Omega}_t = \sigma^2_{\omega}\boldsymbol{P}^{-(t-1)}\boldsymbol{r}\boldsymbol{r}'\boldsymbol{P}^{-(t-1)'}$. On the other hand, the HS model can be represented as (16) with $\boldsymbol{\Omega}^{*} = \sigma^2_{\omega}(\boldsymbol{I}_{s-1} - s^{-1}\boldsymbol{i}_{s-1}\boldsymbol{i}'_{s-1})$.

Table 1 recapitulates the structure of the matrices $\boldsymbol{\Omega}^{\dagger}$ and $\boldsymbol{\Omega}^{*}$ for the four models considered so far. The specification is completed by the first two moments of the initial distributions, which can be stated in terms of $\boldsymbol{\Psi}_0$, as $\boldsymbol{\Phi}_0 = \boldsymbol{\Theta}\boldsymbol{\Psi}_0$. If the process had started at time $-\kappa$, the mean and the covariance matrix of the initial state $\boldsymbol{\Psi}_0$ would be $\boldsymbol{0}$ and $\Sigma_{j=0}^{\kappa}\boldsymbol{T}^{j}\boldsymbol{\Omega}^{*}\boldsymbol{T}^{j'}$, respectively. For the HS and the TS models the latter is proportional to $\kappa\boldsymbol{\Omega}^{*}$, whereas for the DS and CS models it is a periodic function of κ, with period equal to s; however, when $-\kappa$ is a multiple of s, it is proportional to $\kappa(\boldsymbol{I}_{s-1} - s^{-1}\boldsymbol{i}_{s-1}\boldsymbol{i}'_{s-1})$ for the CS model and to a tridiagonal matrix with κ on the main diagonal and -0.5κ on the subdiagonals. Assuming that the process has started in the remote past amounts to letting κ go to infinity in the expressions above. An analytic algorithm (termed the exact initial Kalman filter) for inferences in the presence of such (often termed *diffuse*) initial conditions is presented in Koopman (1997).

6. The smoothness of a seasonal model

In order to ascertain the time domain properties of the various seasonal models, let us derive the autocovariance function of the seasonal sums, $U_t = S(L)\gamma_t$. Without loss of generality, let us assume $t = ns$, for a positive integer n, in (5), so that

$$\begin{aligned} U_t = \omega_{st} &+ (\omega_{s,t-1} + \omega_{s-1,t-1}) + \cdots \\ &+ (\omega_{s,t-s+2} + \omega_{s-1,t-s+2} + \cdots \\ &+ \omega_{2,t-s+2}) \end{aligned}$$

where ω_{jt} is the jth element of $\boldsymbol{\omega}_t$. Introducing the $s \times (s-1)$ matrix

Table 1
Disturbance covariance matrices for four standard seasonal models[a]

Model	$\boldsymbol{\Omega}^{\dagger}$	$\boldsymbol{\Omega}^{*}$
HS	$\sigma^2_{\omega}(\boldsymbol{I}_s - s^{-1}\boldsymbol{i}_s\boldsymbol{i}'_s)$	$\sigma^2_{\omega}(\boldsymbol{I}_{s-1} - s^{-1}\boldsymbol{i}_{s-1}\boldsymbol{i}'_{s-1})$
TS	$\sigma^2_{\omega}\boldsymbol{H}\boldsymbol{H}'$	$\sigma^2_{\omega}\boldsymbol{\Theta}^{-}\boldsymbol{H}\boldsymbol{H}'\boldsymbol{\Theta}^{-'}$
CS	$\sigma^2_{\omega}\boldsymbol{r}^{\dagger}\boldsymbol{r}^{\dagger'}$	$\sigma^2_{\omega}\boldsymbol{i}_{s-1}\boldsymbol{i}'_{s-1}$
DS	$\sigma^2_{\omega}\boldsymbol{r}\boldsymbol{r}'$	$\sigma^2_{\omega}\boldsymbol{e}_{s-1,s-1}\boldsymbol{e}'_{s-1,s-1}$

[a] Note: $\boldsymbol{i}_s = [1, 1, \ldots, 1]'$, $\boldsymbol{r}^{\dagger} = [-\boldsymbol{i}'_{s-1}, (s-1)]'$, $\boldsymbol{r} = [0, \ldots, 0, -1, 1]'$, $\boldsymbol{e}_{s-1,s-1} = [0, \ldots, 0, 1]'$, $\boldsymbol{\Theta}^{-} = (\boldsymbol{\Theta}'\boldsymbol{\Theta})^{-1}\boldsymbol{\Theta}'$, $\boldsymbol{\Theta} = [\boldsymbol{I}_{s-1}, -\boldsymbol{i}_{s-1}]'$, and the matrix $\boldsymbol{H}$ is given in Section 3.

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 1 & 1 \end{bmatrix}$$

$$U_t = [\mathrm{vec}(\boldsymbol{\Sigma}')]' \begin{bmatrix} \boldsymbol{\omega}_t \\ \boldsymbol{\omega}_{t-1} \\ \vdots \\ \boldsymbol{\omega}_{t-s+2} \end{bmatrix}.$$

The autocovariance at lag k, c_k, is computed as follows:

$$\begin{aligned} c_k &= E(U_t U_{t-k}) \\ &= [\mathrm{vec}(\boldsymbol{\Sigma}')]'(\boldsymbol{I}_{s-1} \otimes \boldsymbol{\Omega})\mathrm{vec}(\boldsymbol{\Sigma}_k') \end{aligned} \tag{18}$$

where $\boldsymbol{\Sigma}_k$ is obtained shifting the columns of $\boldsymbol{\Sigma}$ k positions to the right and filling with columns of zeros:

$$\boldsymbol{\Sigma}_k = \boldsymbol{\Sigma} \begin{bmatrix} & & \boldsymbol{I}_{s-k} \\ \boldsymbol{0}_{s-1,k} & \vdots & \\ & & \boldsymbol{0}_{s-k-1,s-k} \end{bmatrix}.$$

When the seasonal model is parameterised according to (14), the autocovariance function for U_t is (18) with $\boldsymbol{\Sigma} = [\boldsymbol{a}_1', \boldsymbol{a}_2', \ldots, \boldsymbol{a}_{s-1}']$, where $\boldsymbol{a}_j$ is an $s \times 1$ vector with j unit elements at the top and zeros at the bottom, and $\boldsymbol{\Omega}$ replaced by $\boldsymbol{\Omega}^{\dagger}$.

In the frequency domain, the various models can be compared in terms of the spectral density of the seasonal sums, U_t:

$$f(\lambda) = \frac{1}{2\pi}\left[c_0 + \sum_{k=1}^{s-2} c_k \cos \lambda k\right], \quad \lambda \in [0, \pi].$$

One possibility is to compare them with respect to the measure of smoothness introduced by Froeb and Koyak (1994): in particular, the coefficient of smoothness at cutoff frequency $\bar{\lambda}$ is the quantity:

$$\mathscr{S}(\bar{\lambda}) = \frac{1}{\bar{\lambda}}\int_0^{\bar{\lambda}} \ln f(\lambda)\mathrm{d}\lambda - \frac{1}{\pi - \bar{\lambda}}\int_{\bar{\lambda}}^{\pi} \ln f(\lambda)\mathrm{d}\lambda. \tag{19}$$

Thus, $\exp \mathscr{S}(\bar{\lambda})$ is approximately the ratio between the geometric mean of the spectral density at the low frequencies and the geometric mean at higher frequencies, and can be interpreted as the ratio between the average long-run variance to the average short-run variance. Note that when U_t is white noise, the smoothness coefficient is zero everywhere.

The left panel of Fig. 1 shows the natural logarithm of the standardised power spectra of U_t for the HS, TS and CS models when $s = 12$. The second has higher density at the frequency π, whereas CS displays local minima at the seasonal frequencies, λ_j, $j = 1, \ldots, 6$. The right panel shows the behaviour of (19) for $\bar{\lambda}$ ranging from $\pi/24$ to $\pi/2$: the message conveyed is that the HS model is smoother than the TS model; as a matter of fact, this results from the fact that the TS model is characterised by 'excess' power at π; furthermore, both are outperformed, in terms of smoothness, by CS. For the DS model, the smoothness coefficient is identically 0 for all $\bar{\lambda}$.

It is usually thought that seasonality should be a smoothly evolving component (see, e.g., Den Butter & Fase, 1991), as the institutional factors and preferences change slowly across time. Hence, there is a strong a priori for preferring smooth representations. This can be disputed, of course, as smoothness alone cannot be advocated to state the superiority of a seasonal model as is argued below; usually several criteria are employed jointly, which makes finding an optimal seasonal model a very complex task.

That smoothness itself does not provide a criterion for the selection of a particular seasonal model can be seen from the quarterly case, $s = 4$. In such a case, under the assumption that $\boldsymbol{\Omega}$ is a Toeplitz matrix, $\boldsymbol{\Omega}$ is given as in (13) and the smoothness coefficient depends upon the structural parameter σ alone. In Fig. 2 the behaviour of $\mathscr{S}(\pi/8)$ for σ varying in the range $(-1, 0)$, for which $\boldsymbol{\Omega}$ is positive semidefinite, is

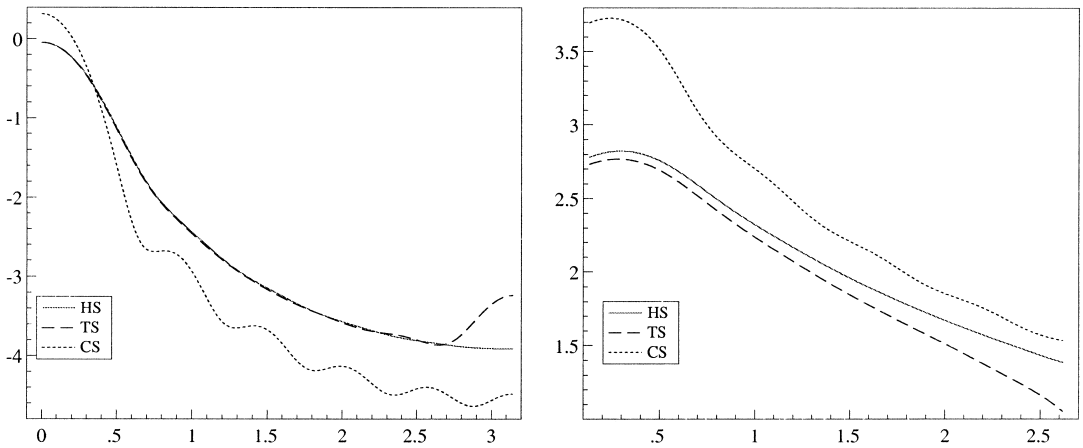

Fig. 1. Log-standardised spectra of U_t, $\ln[f(\lambda)/c_0]$, for three seasonal models (left panel) and Froeb and Koyak smoothness coefficient, $\mathcal{S}(\lambda)$ versus λ (right panel).

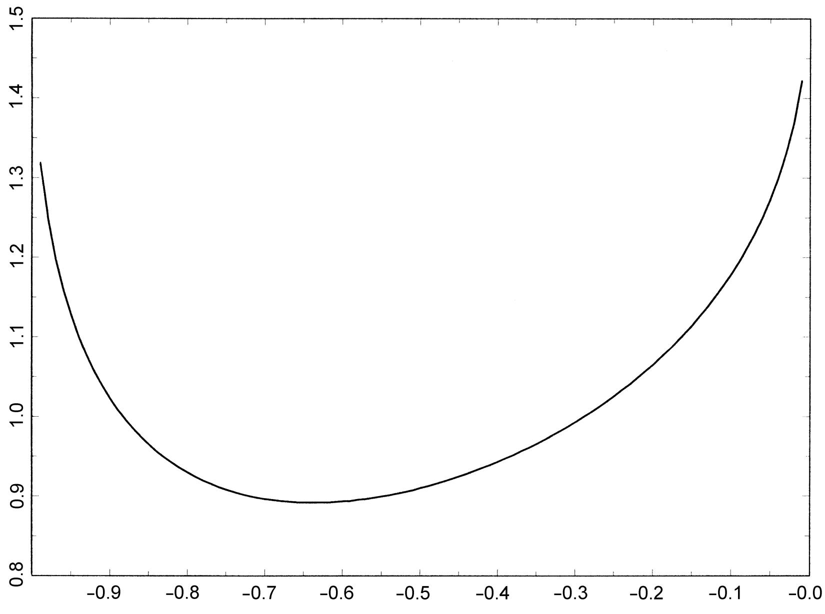

Fig. 2. $\mathcal{S}(\pi/8)$ as a function of the structural parameter σ.

presented. The shape of the function makes clear that the solution to the problem of maximising smoothness is trivial: as a matter of fact, there are two maxima, one for $\sigma = -1$, which corresponds to a non-invertible process at frequency $\pi/2$, and one for $\sigma = 0$, for which U_t is non-invertible at frequency π. We can also see that the TS model, which has $\sigma = -1/2$, is less smooth than the HS model ($\sigma = -1/3$).

7. Forecasting with the BSM

This section aims at discussing two fundamental issues arising with respect to the forecasts generated by the BMS with γ_t specified alternatively as HS, TS, DS, CS; in the first place we would like to establish how they relate to those produced by alternative ad hoc and model-based forecasting procedures; secondly, we would like to investigate whether the BSM predictive performance is sensitive to the choice of the seasonal model.

To address the first issue we shall refer to the WH representation (14). Usually, the hyperparameters, σ^2_ϵ, σ^2_η, σ^2_ζ, and σ^2_ω, will have to be estimated by maximising the diffuse likelihood function under the assumption that the disturbances, ϵ_t, η_t, ζ_t and $\boldsymbol{\omega}_t$, are mutually and serially uncorrelated Gaussian. Details are given in Koopman (1997); in the illustrations presented later in this section a computer programme implementing Koopman's exact initial Kalman filter has been written in the object oriented matrix programming language Ox 2.1 of Doornik (1998) and is available from the author. We note in passing that estimation of the DS and TS is available in Stamp 5.0 (Koopman, Harvey, Doornik & Shephard, 1995), whereas the package Ssfpack, linked to the Ox code (Koopman, Shephard & Doornik, 1999), features also HS, although relies upon an approximate solution to the initialisation of the filter.

The Kalman filter is the basic algorithm for inference and prediction; when it has reached a steady state (see, e.g., Harvey, 1989, pp. 118–125), the forecast function of the BSM takes on the following form:

$$\tilde{y}_{t+l|t} = \tilde{\mu}_t + l\tilde{\beta}_t + \tilde{\Phi}_{\{l \bmod s\},t}$$

where $\tilde{y}_{t+l|t} = E(y_{t+l}|Y_t)$, $Y_t = \{y_1, \dots, y_t\}$; the updating equations for its components are respectively:

$$\begin{array}{lllll} \tilde{\mu}_t & = & \tilde{\mu}_{t-1} & + & \tilde{\beta}_{t-1} + \alpha_\mu v_t \\ \tilde{\beta}_t & = & & & \tilde{\beta}_{t-1} + \alpha_\beta v_t \\ \tilde{\Phi}_{jt} & = & \tilde{\Phi}_{j+1,t-1} & + & \varrho_j v_t, \qquad j = 1,2, \dots, s-1 \\ \tilde{\Phi}_{st} & = & \tilde{\Phi}_{1,t-1} & + & \varrho_s v_t \end{array} \tag{20}$$

where $\tilde{\mu}_t = E(\mu_t|Y_t)$, $\tilde{\beta}_t = E(\beta_t|Y_t)$, $\tilde{\Phi}_{jt} = E(\Phi_{jt}|Y_t)$, $v_t = y_t - (\tilde{\mu}_{t-1} + \tilde{\beta}_{t-1} + \tilde{\Phi}_{1,t-1})$ is the one-step-ahead forecast error, and α_μ, α_β, ϱ_j, $j = 1, \dots, s$, are smoothing factors satisfying the following constraints:

$$0 \le \alpha_\mu, \alpha_\beta < 1, \quad \sum_{j=1}^{s} \varrho_j = 0, \quad \alpha_\mu + \varrho_1 \le 1.$$

The latter two constraints can be proved analytically; the first can be shown numerically, see Anderson and Moore (1979, p. 156), by exploring the behaviour of α_μ and α_β for all hyperparameter combinations. It should be noticed that, by virtue of the zero sum constraint on the ϱ_j's, the last updating equation can be dropped and the last but one rewritten as $\tilde{\Phi}_{s-1,t} = -\Sigma_{j=1}^{s-1}\tilde{\Phi}_{j,t-1} + \varrho_{s-1}v_t$, which amounts to using representation (16).

The shape of the forecast function is coincident with that of a number of exponential smoothing and model-based algorithms, namely:

1. Holt–Winters type forecasting procedure — Crude seasonal model (Roberts, 1982): for three given smoothing constants, $0 \le \alpha_\mu, \alpha_\beta, \rho < 1$,
$$\varrho_1 = \rho, \quad \varrho_j = -\rho/(s-1), \;\; j = 2, \dots, s.$$
2. Corrected Additive Seasonal Holt–Winters (Newbold, 1988, p. 115): for three given smoothing constants, $0 \le A,B,C < 1$,

$$\alpha_\mu = A + C(1 - A)/s, \quad \alpha_\beta = AB,$$

$$\varrho_1 = \frac{s-1}{s} C(1 - A),$$

$$\varrho_j = -C(1 - A)/s, \quad j = 2, \ldots, s.$$

The pattern of the ϱ_j's is the same as for the crude seasonal model (constant across j for $j > 1$), but the updating coefficients are functionally related to the three smoothing constants.

3. Holt–Winters type forecasting procedure — EWR seasonal model (Roberts, 1982): for three given smoothing constants, $0 \le \alpha_\mu, \alpha_\beta, \rho < 1$,

$$\varrho_1 = 1 - \rho^{s-1}, \quad \varrho_j = -\rho^{j-2}(1 - \rho),$$
$$j = 2, \ldots, s.$$

The seasonal model is based on the notion of *Exponentially Weighted Regression* (EWR) and it is advocated on the grounds that the information content of the previous months on the current seasonal effect dies away geometrically. The notion of EWR can be further employed to restrict $\alpha_\mu = 1 - \alpha^2$, and $\alpha_\beta = (1 - \alpha)^2$, where $\alpha \in (0, 1)$.

4. Airline model: $\Delta\Delta_s y_t = (1 + \theta L)(1 + \Theta L^s)\xi_t$, $-1 \le \theta$, $\Theta < 0$, $\xi_t \sim \mathrm{NID}(0, \sigma^2)$. The updating coefficients are given in Box, Pierce and Newbold (1987, p. 281) and can be rewritten as follows:

$$\alpha_\mu = (1 + \theta)\left[1 - (1 + \Theta)\frac{s+1}{2s}\right] + \frac{1}{s}(1 + \Theta), \quad \alpha_\beta = s^{-1}(1 + \theta)(1 + \Theta)$$

$$\varrho_1 = \frac{s-1}{s}(1 + \Theta) - \frac{s-1}{2s}(1 + \theta)(1 + \Theta),$$

$$\varrho_j = \frac{s + 1 - 2(j-1)}{2s}(1 + \theta)(1 + \Theta) - \frac{1}{s}(1 + \Theta).$$

The noticeable feature is that the ϱ_j's decrease linearly for $j > 1$.

In the structural framework, the pattern of coefficients associated with v_t in the updating formulae depends on the variance parameters σ^2_ϵ, σ^2_η, σ^2_ζ, σ^2_ω; their analytic derivation in terms of these quantities involves the solution of an algebraic Riccati equation for the one-step-ahead state covariance matrix, which will usually prove quite complicated. Nevertheless, we can conclude that the structural approach is more flexible, in that the pattern of the updating coefficient is allowed to adapt to the nature of the series under investigation. Imposing patterns to the ϱ_j's, as in Roberts' crude and ERW schemes, may yield forecasts with higher mean square error. Furthermore, the representation of the components, and in particular their orthogonality, prevents the insurgence of anomalous forecasting schemes, such as those arising for the Airline model when $\theta < 1 - 2s/[(s-1)(1 + \Theta)]$, in which circumstance ϱ_1 is greater than one, and it would be difficult to consider (20) as an error learning procedure, since the innovations v_t would enter the updating of the seasonal effect amplified by a factor greater than 1.

7.1. The airline series

The series of natural logarithms of monthly passenger totals in international air travel (Box, Jenkins & Reinsel, 1994), available from January 1949 to December 1960, is considered here to illustrate the pattern of the smoothing factors in (20) for the BMS. The estimated parameter values of the Airline model are $\hat{\theta} = -0.402$, $\hat{\Theta} = -0.557$, $\hat{\sigma}^2 = 0.00135$, and the pattern of coefficients ϱ_j is given in Table 1 of Box et al. (1987). In our notation, $\varrho_1 = 0.257$, $\varrho_j = 0.097 - 0.020(j - 1)$; moreover, $\alpha_\mu = 0.50$ and $\alpha_\beta = 0.02$.

The corresponding estimates for four specifications of the BSM, differing only for the

Table 2
Parameter estimates (multiplied by 10^5) and prediction error variance

Model	$\hat{\sigma}^2_\eta$	$\hat{\sigma}^2_\zeta$	$\hat{\sigma}^2_\omega$	$\hat{\sigma}^2_\epsilon$	pev
HS	2902	0	219	2482	0.00138
TS	2983	0	36	2344	0.00139
DS	6995	0	641	1295	0.00152
CS	2865	0	18	2595	0.00138

seasonal model, are reported in Table 2, along with the prediction error variance (pev), which represents the variance of the one-step-ahead prediction errors, v_t, in the steady state and is directly comparable to $\hat{\sigma}^2$.

The steady state coefficients ϱ_j are plotted in Fig. 3. For the HS and the CS models their pattern is virtually identical; TS displays a similar pattern, but with a more jagged appearance; finally, the DS specification is characterised by a lower value for ϱ_1 and by $\varrho_j \approx 0$ for $j = 3, \ldots, 10$. Moreover, for HS, TS and CS $\alpha_\mu = 0.460$ and $\alpha_\beta = 0.001$, whereas for DS $\alpha_\mu = 0.679$ and $\alpha_\beta = 0.001$.

Hence, the forecasts produced by DS are likely to differ from TS, HS and CS, which are in turn almost indistinguishable. On the other hand, they will differ from those of the Airline model, which updates respectively by a greater and a lesser amount the slope and the seasonals. Fig. 4 compares the forecasts for lead times up to 24 months, generated by the Airline and the BMS models with HS and DS specifications (estimated using the observations up to Dec. 1959). For this particular illustration the HS model minimises the sum of the squares of the forecast errors.

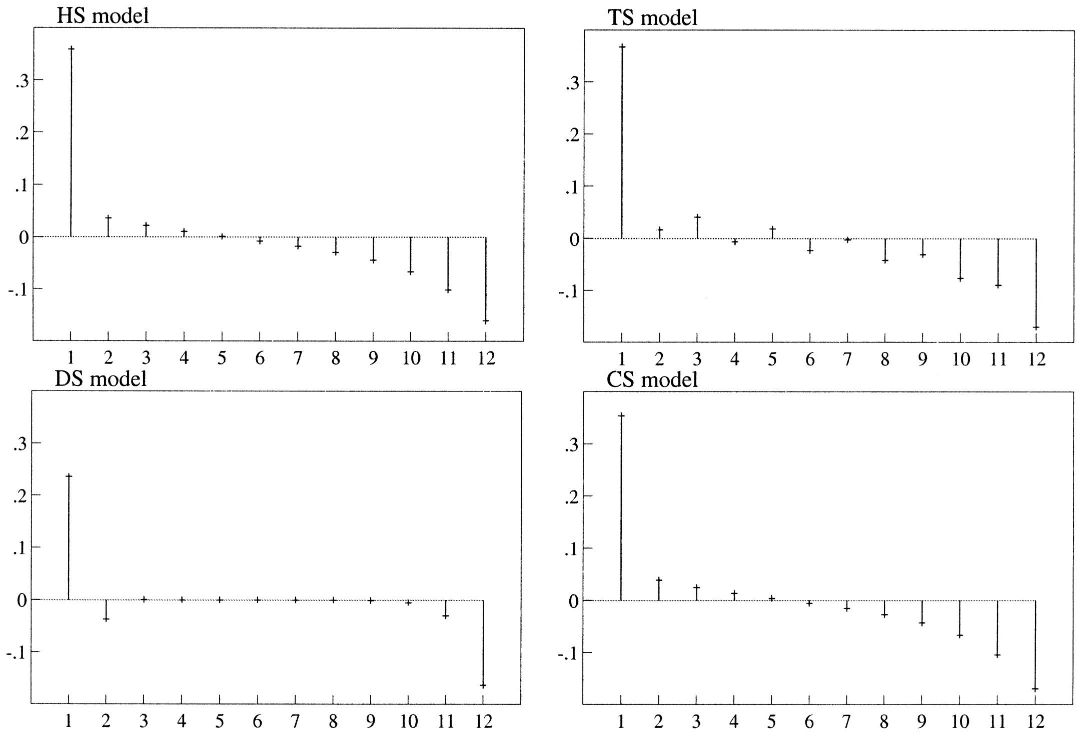

Fig. 3. Airline series, pattern of the updating coefficients ϱ_j, $j = 1, \ldots, 12$, for four BSM specifications.

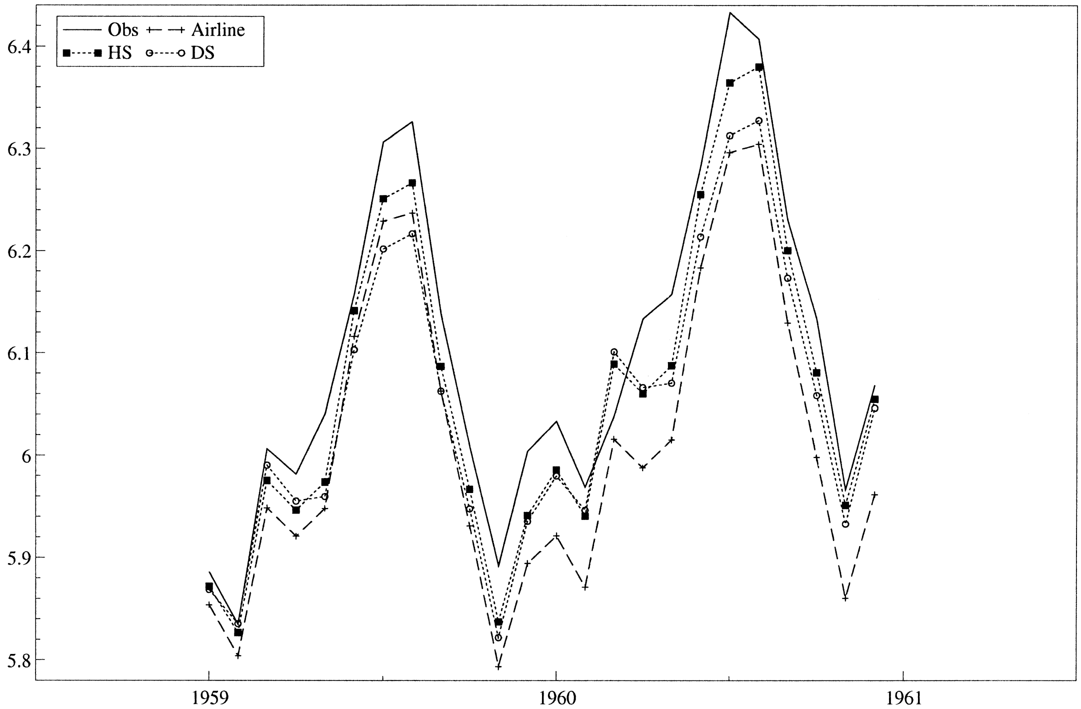

Fig. 4. Airline series, plot of the observed values and forecasts for 1,2, . . . ,24 months ahead, from the origin December 1959.

7.2. US Industrial production

In order to address the second issue, namely whether the choice of the seasonal model affects the BMS predictive performance, we consider a set of monthly US industrial production series for two-digit manufacturing industries, along with the *Total*, *Durable* and Nondurable Manufacturing series, for a number of 23 series altogether. This set, which has been studied by Miron (1996), is made available electronically by the Federal Reserve Board at the URL `www.bog.frb.fed.us` and covers the period 1947.1–1996.12.

Comparison of out-of-sample multistep mean square forecast error (MSFE) has been based on the following rolling forecast exercise: starting from 1989.12, the four alternative specifications of the BSM are estimated using the observations through a given forecast origin and l-steps-ahead forecasts, $l = 1, \dots, 12$, are computed. The procedure is repeated shifting the forecast origin by one month until the end of the sample period is reached; this yields a total of 84 one-step-ahead forecast errors and 72 twelve-steps-ahead forecast errors for the four specifications of the BSM.

The results can be summarised as follows: for the vast majority of series the predictive performance of HS, TS and CS at all forecast horizons is virtually indistinguishable and represents a significant improvement over DS. A typical case is the *Nondurable Manufacturing* series plotted in Fig. 5 (logarithms) along with the estimated spectral density of the seasonal and regular differences. Especially at shorter forecast horizons, the MSFE reduction from using HS rather than DS seems to be quite high,

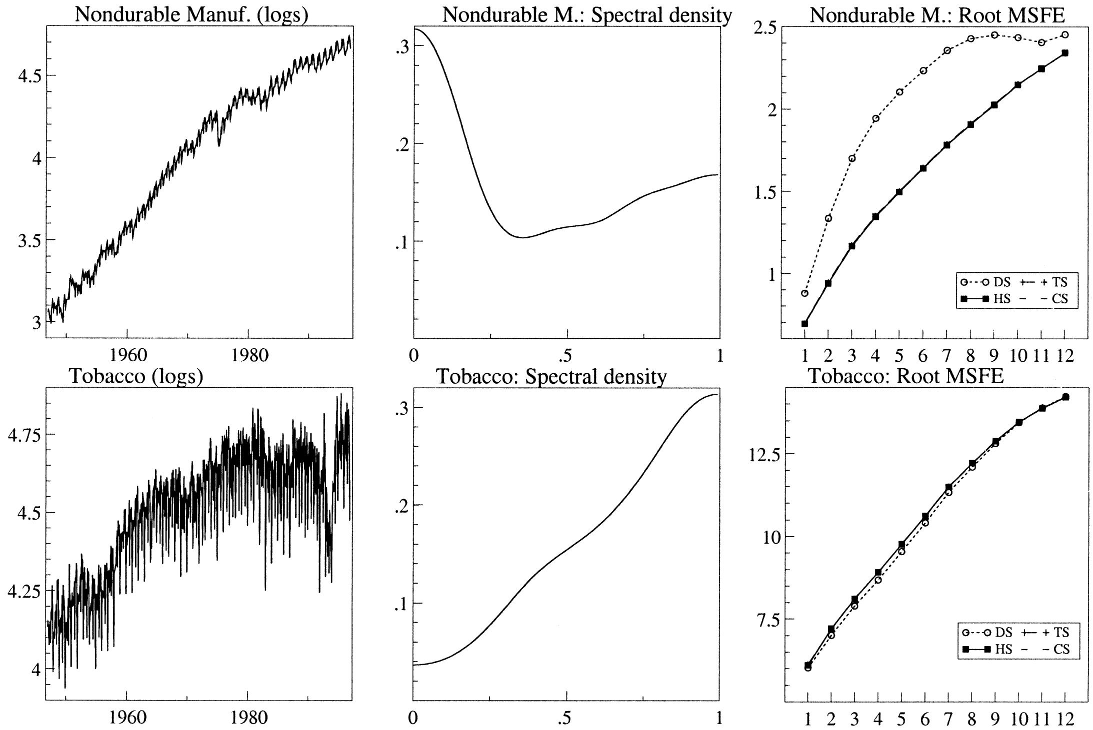

Fig. 5. US Index of industrial production for Nondurable manufacturing and Tobacco, Jan. 1947–Dec. 1996: series in logs, estimated spectral density of $\Delta\Delta_{12} \ln y_t$, and root mean square forecast errors at forecast horizons 1, . . . ,12.

as is evident from the third panel. Notice that the root MSFE is practically identical for HS, TS and CS.

These results, however, should not be taken as evidence that the DS specification should not be entertained. As a matter of fact, in the same data set it is possible to present a case, the index of industrial production for *Tobacco*, in which the forecasting performance of the DS specification is slightly superior (see the last panel of Fig. 5). The comparison of the spectral density of the $\Delta\Delta_{12} \ln y_t$ with that of the *Nondurable* series reveals that the smoothness properties of the two series are rather different; for the latter the contribution of the lower frequencies is substantially greater than in the former case.

In conclusion, the performance of the four specifications depends upon the features of the time series under investigation, and in particular the relative smoothness of the seasonal pattern; for a wide class of macroeconomic time series, for which seasonality changes slowly, the DS displays the worst out-of-sample performance and is not to be recommended; nevertheless, none of the models considered in this comparison can be advocated as the optimal seasonal model: in fact, every specification captures a different kind of seasonality and is not, strictly speaking, nested in other models.

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