

Bisection method (Bolzano)

Suppose you have to locate the root of the equation $f(x)=0$ in an interval say (x_0, x_1) , let $f(x_0)$ and $f(x_1)$ are of opposite signs such that $f(x_0)f(x_1) < 0$

Then the graph of the function crossed the x-axis between x_0 and x_1 which exists the existence of at least one root in the interval (x_0, x_1) .

The desired root is approximately defined by the mid point $x_2 = \frac{x_0 + x_1}{2}$ if $f(x_2) = 0$ then x_2 is the root of the equation otherwise the root lies either between x_0 and x_2 or x_1 and x_2

Example:

Carry out the five iterations for the function $f(x) = 2x \cos(2x) - (x+1)^2$

Note: All the calculations should be done in the radians.

Solution:

$$f(x) = 2x \cos(2x) - (x+1)^2$$

$$f(-1) = 2(-1) \cos(-2) - (-1+1)^2 = -2(-0.4161) = +0.8322 > 0$$

$$f(0) = 2(0) \cos(0) - (0+1)^2 = -1 = -1 < 0$$

so the root lies between 0 and -1 as $f(0)f(-1) < 0$

$$x_2 = \frac{0-1}{2} = -0.5$$

$$f(-0.5) = 2(-0.5) \cos(-1) - (-0.5+1)^2 = -0.5403 - 0.25 = -0.7903 < 0$$

so root lies between -1 and -0.5 as $f(-1)f(-0.5) < 0$

$$x_3 = \frac{-0.5-1}{2} = -0.75$$

$$f(-0.75) = 2(-0.75) \cos(-1.5) - (-0.75+1)^2 = -0.106 - 0.0625 = -0.1686 < 0$$

so root lies between -1 and -0.75 as $f(-1)f(-0.75) < 0$

$$x_4 = \frac{-0.75-1}{2} = -0.875$$

$$f(-0.875) = 2(-0.875) \cos(-1.75) - (-0.875+1)^2 = 0.3119 - 0.015625 = 0.296275 > 0$$

so root lies between -0.875 and -0.75 as $f(-0.75)f(-0.875) < 0$

$$x_5 = \frac{-0.75-0.875}{2} = -0.8125$$

$$f(-0.8125) = 2(-0.8125) \cos(-1.625) - (-0.8125+1)^2 = 0.0880 - 0.0351 = 0.052970 > 0$$

so root lies between -0.8125 and -0.75 as $f(-0.75)f(-0.8125) < 0$

$$x_5 = \frac{-0.75-0.8125}{2} = -0.78125$$

Regula-Falsi method (Method of false position)

Here we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ have opposite signs. Intermediate value property suggests that the graph of the $y=f(x)$ crosses the x -axis between these two points and therefore, a root lies between these two points.

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Example

Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$,

- (i) if the root lies between 2 and 4
- (ii) if the root lies between 2 and 3.

Comment on the results.

Solution

Let

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29.$$

Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4.

Taking $x_1 = 2$, $x_2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{4 - 2}{29 - (-9)} (29) = 4 - \frac{2(29)}{38} = 4 - 1.5263 = 2.4736$$

Now

$$f(x_3) = 2.4736^3 - 9(2.4736) + 1 = 15.13520 - 22.2624 + 1 = -6.12644.$$

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3 .

The second approximation to the root is given as

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.4736 - \frac{2.4736 - 4}{-6.12644 - 29} (-6.12644) \\ &= 2.4736 - \frac{-1.5264}{-35.12644} (-6.12644) = 2.4736 - (0.04345)(-6.12644) \\ &= 2.4736 + 0.26619 = 2.73989 \end{aligned}$$

Therefore

$$f(x_4) = 2.73989^3 - 9(2.73989) + 1 = 20.5683 - 24.65901 + 1 = -3.090707.$$

Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.73989 - \frac{2.73989 - 4}{-3.090707 - 29} (-3.090707) = 2.86125$$

$$= 2.73989 - \frac{-1.26011}{-32.090707} (-3.090707) = 2.73989 + 0.039267(3.090707) = 2.73989 + 0.121363 = 2.86125$$

Now

$$f(x_5) = 2.86125^3 - 9(2.86125) + 1 = 23.42434 - 25.75125 + 1 = -1.326868.$$

(ii)

Here

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1.$$

Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3.

Taking $x_1 = 2$, $x_2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3 - 2}{1 + 9} (1)$$

$$= 3 - \frac{1}{10} = 2.9$$

$$f(x_3) = 2.9^3 - 9(2.9) + 1 = 24.389 - 26.1 + 1 = -0.711$$

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3 .

The second approximation to the root is given as

$$x_4 = 2.9 - \frac{2.9 - 3}{-0.711 - 1} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711)$$

$$= 2.9 - (0.05844)(-0.711) = 2.9 + 0.04156 = 2.94156$$

$$f(x_4) = -0.0207$$

$$f(x_4) = 2.94156^3 - 9(2.94156) + 1 = 25.45265 - 26.47404 + 1 = -0.0207$$

Now, we observe that $f(x_2)$ and $f(x_4)$ are of opposite signs; the third approximation is obtained from

$$x_5 = 2.94156 - \frac{2.94156 - 3}{-0.0207 - 1} (-0.0207) = 2.94156 - \frac{-0.05844}{-1.0207} (-0.0207)$$

$$= 2.94156 - (-0.05725)(-0.0207) = 2.94275$$

$$f(x_5) = 2.94275^3 - 9(2.94275) + 1 = 25.48356 - 26.48475 + 1 = -0.0011896$$

We observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_5 , when the interval considered is $(2, 3)$, is closer to the root.

Example:

Use method of false position to solve $e^{-x} + 2^{-x} + 2 \cos x - 6 = 0 \quad 1 \leq x \leq 2$

Solution:

$$f(x) = e^x + 2^{-x} + 2 \cos x - 6$$

$$x_0 = 1, \quad x_1 = 2$$

now

$$x_{n+1} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$f(1) = e^1 + 2^{-1} + 2 \cos 1 - 6 = 2.7182 + 0.5 + 2(0.5403) - 6 = -1.7011$$

$$f(2) = e^2 + 2^{-2} + 2 \cos 2 - 6 = 7.3886 + 0.25 + 2(-0.4161) - 6 = 0.8068$$

now for $n = 1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 2 - \frac{2 - 1}{0.8068 + 1.7011} (0.8068)$$

$$x_2 = 2 - \frac{1}{2.5079} (0.8068) = 1.6783$$

$$f(1.6783) = e^{1.6783} + 2^{-1.6783} + 2 \cos(1.6783) - 6 = -0.5457$$

now for $n = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 1.6783 - \frac{1.6783 - 2}{(-0.5457) - 0.8068} (-0.5457)$$

$$x_3 = 1.6783 - \frac{(-0.3217)}{(-1.3525)} (-0.5457) = 1.6783 + 0.12979 = 1.8081$$

$$f(1.8081) = e^{1.6783} + 2^{-1.8081} + 2 \cos(1.8081) - 6 = -0.8575$$

now for $n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 1.8081 - \frac{1.8081 - 1.6783}{(-0.08575) + 0.5457} (-0.08575)$$

$$x_3 = 1.8081 - \frac{0.1298}{0.45995} (-0.08575) = 1.6783 + 0.12979 = 1.8323$$

$$f(1.8323) = e^{1.8323} + 2^{-1.8323} + 2 \cos(1.8323) - 6 = 0.1199$$

now for $n = 4$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 1.8323 - \frac{1.8323 - 1.8081}{0.01199 + 0.08575} (0.01199)$$

$$x_5 = 1.8323 - \frac{0.0242}{0.09774} (0.01199) = 1.8323 - 0.00296 = 1.8293$$

$$f(1.8293) = e^{1.8293} + 2^{-1.8293} + 2 \cos(1.8293) - 6 = -0.000343$$

now for $n = 5$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) = 1.8293 - \frac{1.8293 - 1.8323}{-0.000343 - 0.01199} (-0.000343)$$

$$x_6 = 1.8293 - \frac{(-0.003)}{-0.012333} (-0.000343) = 1.8293$$

Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of $f(x)=0$ the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

setting $f(x_{n+1}) = 0$ gives,

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

thus on simplification, we get ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0, 1, 2, \dots$$

Example

Find the first three iteration of the equation $f(x) = x - 0.8 - 0.2 \sin x$ in the interval $[0, \pi/2]$.

Solution

$$f(0) = 0 - 0.8 - 0.2 \sin(0) = 0 - 0.8 - 0.2(0) = -0.8$$

$$\begin{aligned} f(1.57) &= 1.57 - 0.8 - 0.2 \sin(1.75) \\ &= 1.57 - 0.8 - 0.2(0.99999) \\ &= 1.57 - 0.8 - 0.199998 = 0.570002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0) = 1 - 0.2 \cos(0) = 1 - 0.2 = 0.8$$

here $|f(0)|$ is greater then $x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-0.8}{0.8} = 1$$

now

$$\begin{aligned} f(1) &= 1 - 0.8 - 0.2 \sin(1) \\ &= 1 - 0.8 - 0.1683 \\ &= 0.0317 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(1) = 1 - 0.2 \cos(1) = 1 - 0.1081 = 0.8919$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{0.0317}{0.8919} = 1 - 0.0355 = 0.9645$$

$$\begin{aligned} f(0.9645) &= 0.9645 - 0.8 - 0.2 \sin(0.9645) \\ &= 0.9645 - 0.8 - 0.1645 \\ &= 0.0002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0.9645) = 1 - 0.2 \cos(0.9645) = 1 - 0.11396 = 0.88604$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.9645 - \frac{0.0002}{0.88604} = 0.9645 - 0.00022 = 0.9643$$

Example

Perform three iteration of the equation $\ln(x-1) + \cos(x-1) = 0$ when $1.2 \leq x \leq 2$. Use Newton Raphson method to calculate the root.

Solution

Here

$$\ln(x-1) + \cos(x-1) = 0 \text{ when } 1.2 \leq x \leq 2$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$\begin{aligned} f(1.2) &= \ln(1.2-1) + \cos(1.2-1) \\ &= -1.6094 + 0.9801 = -0.6293 \end{aligned}$$

$$\begin{aligned} f(2) &= \ln(2-1) + \cos(2-1) \\ &= 0 + 0.5403 = 0.5403 \end{aligned}$$

$$\text{now } f(x) = \ln(x-1) + \cos(x-1)$$

$$f'(x) = \frac{1}{x-1} - \sin(x-1)$$

$$\begin{aligned} f'(1.2) &= \frac{1}{1.2-1} - \sin(1.2-1) \\ &= 5 - 0.1986 = 4.8014 \end{aligned}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{-0.6293}{4.8014} = 1.2 + 0.1311 = 1.3311$$

$$\begin{aligned} f(1.3311) &= \ln(1.3311-1) + \cos(1.3311-1) \\ &= -1.1053 + 0.9457 = -0.1596 \end{aligned}$$

$$\begin{aligned} f'(1.3311) &= \frac{1}{1.3311-1} - \sin(1.3311-1) \\ &= 3.0202 - 0.3251 = 2.6951 \end{aligned}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3311 - \frac{-0.1596}{2.6951} = 1.3311 + 0.0592 = 1.3903$$

$$\begin{aligned} f(1.3903) &= \ln(1.3903-1) + \cos(1.3903-1) \\ &= -0.9408 + 0.9248 = -0.016 \end{aligned}$$

$$\begin{aligned} f'(1.3903) &= \frac{1}{1.3903-1} - \sin(1.3903-1) \\ &= 2.5621 - 0.3805 = 2.1816 \end{aligned}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.3903 - \frac{-0.016}{2.1816} = 1.3903 + 0.0073 = 1.3976$$

Secant Method

The secant method is modified form of Newton-Raphson method. If in Newton-Raphson method; we replace the derivative $f'(x_n)$ by the difference ratio, i.e.,

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Where x_n and x_{n-1} are two approximations of the root we get

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_n f(x_n) - x_n f(x_{n-1}) - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Provided $f(x_n) \neq f(x_{n-1})$

Example

Do three iterations of secant method to find the root of

$$f(x) = x^3 - 3x + 1 = 0,$$

Taking

$$x_0 = 1, x_1 = 0.5$$

$$n = 1,$$

$$f(x_0) = f(1) = 1^3 - 3(1) + 1 = -1$$

$$f(x_1) = f(0.5) = 0.5^3 - 3(0.5) + 1 = 0.125 - 1.5 + 1 = -0.375$$

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{(1)(-0.375) - (0.5)(-1)}{-0.375 - (-1)} = 0.2 \end{aligned}$$

$$n = 2,$$

$$f(x_2) = f(0.2) = 0.2^3 - 3(0.2) + 1 = 0.008 - 0.6 + 1 = 0.408$$

$$\begin{aligned} x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ &= \frac{(0.5)(0.408) - 0.2(-0.375)}{0.408 - (-0.375)} = 0.3563 \end{aligned}$$

$$n = 3,$$

$$f(x_3) = f(0.3563) = 0.3563^3 - 3(0.3563) + 1 = 0.04523 - 1.0689 + 1 = -0.02367$$

$$\begin{aligned} x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ &= \frac{(0.2)f(0.3563) - 0.3563f(0.2)}{f(0.3563) - f(0.2)} \end{aligned}$$

$$x_5 = 0.3473, f(x_5) = -0.0000096$$

$$\text{and } |x_5 - x_4| = 0.0004.$$