1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.
2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the Bisection method on the following intervals to find p_3.
   a. $[-2, 1.5]$ b. $[-1.25, 2.5]$
3. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.
   a. $[0, 1]$ b. $[1, 3.2]$ c. $[3.2, 4]$
4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.
   a. $[-2, -1]$ b. $[0, 2]$ c. $[2, 3]$ d. $[-1, 0]$
5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.
   a. $x - 2^{-x} = 0$ for $0 \le x \le 1$
   b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$
   c. $2x\cos(2x) - (x+1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$
   d. $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$
6. Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.
   a. $3x - e^x = 0$ for $1 \le x \le 2$
   b. $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$
   c. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and $2 \le x \le 4$
   d. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 0.5$ and $0.5 \le x \le 1$

1. $p_3 = 0.625$

3. The Bisection method gives:

a. $p_7 = 0.5859$ **b.** $p_8 = 3.002$ **c.** $p_7 = 3.419$

5. The Bisection method gives:

a. $p_{17} = 0.641182$ **b.** $p_{17} = 0.257530$

c. For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.

d. For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

   a. $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$
   b. $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$
   c. $g_3(x) = \left(\dfrac{x+3}{x^2+2}\right)^{1/2}$
   d. $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

2. a. Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.
   b. Which function do you think gives the best approximation to the solution?

3. The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

   a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$
   b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$
   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$
   d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

   a. $p_n = p_{n-1}\left(1 + \dfrac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$
   b. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$
   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$
   d. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

5. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on [1, 2]. Use $p_0 = 1$.

6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on [1, 2]. Use $p_0 = 1$.

7. Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

8. Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

9. Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

10. Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

11. For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

    a. $x = \dfrac{2 - e^x + x^2}{3}$
    b. $x = \dfrac{5}{x^2} + 2$
    c. $x = (e^x/3)^{1/2}$
    d. $x = 5^{-x}$
    e. $x = 6^{-x}$
    f. $x = 0.5(\sin x + \cos x)$

1. For the value of x under consideration we have
   a. $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$
   b. $x = \left(\frac{x + 3 - x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$
   c. $x = \left(\frac{x+3}{x^2+2}\right)^{1/2} \Leftrightarrow x^2(x^2+2) = x + 3 \Leftrightarrow f(x) = 0$
   d. $x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$
3. The order in descending speed of convergence is (b), (d), (a). The sequence in (c) does not converge.
5. With $g(x) = (3x^2 + 3)^{1/4}$ and $p_0 = 1$, $p_6 = 1.94332$ is accurate to within 0.01.
7. Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \le g(x) =\le g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \le \frac{1}{4}$, for $0 \le x \le 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that
   $$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$
   For the bound to be less than 0.1, we need $n \ge 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.
9. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.
11. **a.** With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$. **b.** With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.
    c. With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$. **d.** With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.
    e. With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$. **f.** With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?

3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.
- **a.** Use the Secant method.
- **b.** Use the method of False Position.
- **c.** Which of **a.** or **b.** is closer to $\sqrt{6}$?

4. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.
- **a.** Use the Secant method.
- **b.** Use the method of False Position.

5. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
- **a.** $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$
- **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$
- **c.** $x - \cos x = 0, \quad [0, \pi/2]$
- **d.** $x - 0.8 - 0.2\sin x = 0, \quad [0, \pi/2]$

6. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
- **a.** $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $1 \le x \le 2$
- **b.** $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \le x \le 2$
- **c.** $2x\cos 2x - (x-2)^2 = 0$ for $2 \le x \le 3$ and $3 \le x \le 4$
- **d.** $(x-2)^2 - \ln x = 0$ for $1 \le x \le 2$ and $e \le x \le 4$
- **e.** $e^x - 3x^2 = 0$ for $0 \le x \le 1$ and $3 \le x \le 5$
- **f.** $\sin x - e^{-x} = 0$ for $0 \le x \le 1$ $3 \le x \le 4$ and $6 \le x \le 7$

7. Repeat Exercise 5 using the Secant method.

8. Repeat Exercise 6 using the Secant method.

9. Repeat Exercise 5 using the method of False Position.

10. Repeat Exercise 6 using the method of False Position.

11. Use all three methods in this Section to find solutions to within 10^{-5} for the following problems.
- **a.** $3xe^x = 0$ for $1 \le x \le 2$
- **b.** $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$

12. Use all three methods in this Section to find solutions to within 10^{-7} for the following problems.

a. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and for $2 \le x \le 4$

b. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 1/2$ and for $1/2 \le x \le 1$

13. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to (1, 0). [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]

14. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = 1/x$ that is closest to (2, 1).

15. The following describes Newton's method graphically: Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \ne 0$ on $[a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$, and let $p_0 \in [a, b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \ge 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

16. Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

17. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

a. Method of False Position

b. Secant method

c. Newton's method

Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

18. The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi)\arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

a. Bisection method

b. Method of False Position

c. Secant method

19. The iteration equation for the Secant method can be written in the simpler form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.

20. The equation $x^2 - 10\cos x = 0$ has two solutions, ± 1.3793646. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

a. $p_0 = -100$ **b.** $p_0 = -50$ **c.** $p_0 = -25$

d. $p_0 = 25$ **e.** $p_0 = 50$ **f.** $p_0 = 100$

1. $p_2 = 2.60714$

3. **a.** 2.45454 **b.** 2.44444 **c.** Part (b) is better.

5. **a.** For $p_0 = 2$, we have $p_5 = 2.69065$. **b.** For $p_0 = -3$, we have $p_3 = -2.87939$.
c. For $p_0 = 0$, we have $p_4 = 0.73909$. **d.** For $p_0 = 0$, we have $p_3 = 0.96434$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:
a. $p_{11} = 2.69065$ **b.** $p_7 = -2.87939$ **c.** $p_6 = 0.73909$ **d.** $p_5 = 0.96433$

9. Using the endpoints of the intervals as p_0 and p_1, we have:
a. $p_{16} = 2.69060$ **b.** $p_6 = -2.87938$ **c.** $p_7 = 0.73908$ **d.** $p_6 = 0.96433$

11. **a.** Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.

b. Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

13. For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

15. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

17. **a.** For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.
b. For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.
c. For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

19. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if p_{n-1} and p_{n-2} are nearly equal.