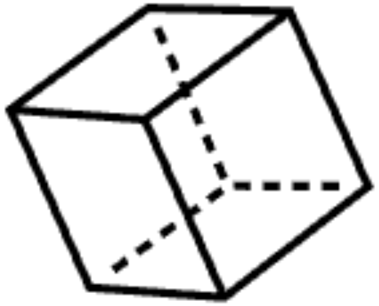
ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY

First Week

Least Square Method



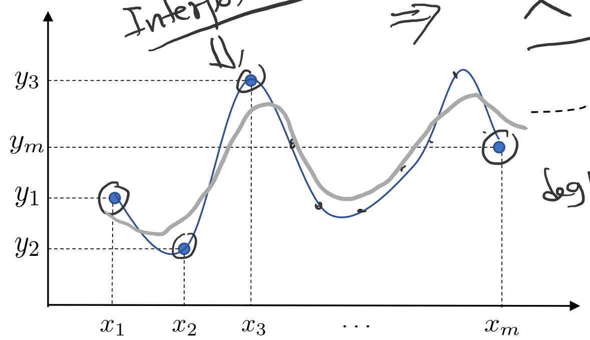
1. Least Squares Approximation

1.1. Polynomial approximation

$$\begin{bmatrix} x_1^0 & x_1^1 & \cdots & x_1^{m-1} \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Data: $\{(x_i, y_i) | i = 1, \cdots, \underline{m}\}$

Interpolation $\Rightarrow$ $\underline{X \cdot A = Y}$



deg $\underline{m-1}$

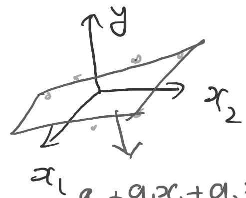
$$P_n = \sum_{j=0}^{n} \underset{1}{\underline{a_j}} x^j \text{ for } n \leq m-1$$

unknown,

$$\Big(\underset{\substack{\| \\ m-1}}{\underline{P_n(x_i) = y_i}} \quad i = 1, 2, \cdots, m \Big)$$

Question: How to $\underline{\text{minimize}}$ $\underset{\substack{\| \\ \|E\|_2^2}}{E} = \sum_{i=1}^{m}(y_i - P_n(x_i))^2$? $\rightarrow$ "$\underline{\text{Normal Equation}}$"

1.1. Polynomial approximation



unknown a_j

$a_0 + a_1x_1 + a_2x_2 = f(x_1, x_2)$

Question: How to minimize $E = \sum_{i=1}^{m} (y_i - P_n(x_i))^2$? → "Normal Equation"

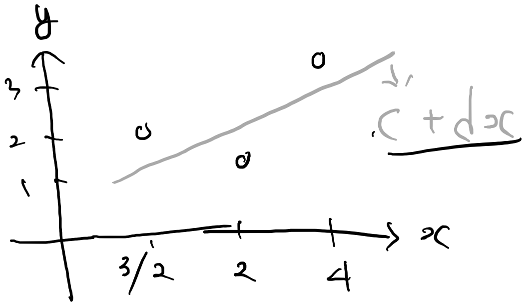
Matrix equation

$\frac{\partial E}{\partial a_j} = 0$

$$
\underbrace{\begin{bmatrix}
\sum_{i=1}^m x_i^0 & \sum_{i=1}^m x_i^1 & \cdots & \sum_{i=1}^m x_i^n \\
\sum_{i=1}^m x_i^1 & \sum_{i=1}^m x_i^2 & \cdots & \sum_{i=1}^m x_i^{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^m x_i^n & \sum_{i=1}^m x_i^{n+1} & \cdots & \sum_{i=1}^m x_i^{2n}
\end{bmatrix}}_{X}
\underbrace{\begin{bmatrix}
a_0 \\ a_1 \\ \vdots \\ a_n
\end{bmatrix}}_{A}
=
\underbrace{\begin{bmatrix}
\sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i x_i \\ \vdots \\ \sum_{i=1}^m y_i x_i^n
\end{bmatrix}}_{Y}
$$

$(n+1) \times (n+1)$ $\quad$ $n+1$

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3/2 \\ 1 & 4 \end{pmatrix}$ $X = \begin{pmatrix} c \\ d \end{pmatrix}$ $Y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$



$\{(2, 1), (3/2, 2), (4, 3)\}$

$AX = Y$

$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \Rightarrow c + 2d = 1$

L_2-Norm : $\|Y - AX\|_2^2 = (1 - (c+2d))^2 + (2 - (c + \frac{3}{2}d))^2 + (3 - (c+4d))^2$

E

$\Rightarrow \frac{\partial E}{\partial c} = 0, \quad \frac{\partial E}{\partial d} = 0$

$AX = Y$: <u>No solution</u>.

(A : rectangular)

Sol :

$$A^t A \hat{X} = A^t Y \Rightarrow \hat{X} = \underbrace{(A^t A)^{-1} (A^t Y)}_{\Downarrow}$$

$$\hat{X} \overset{?}{\Leftrightarrow} X$$

Find a minimum of $\| AX - Y \|_2^2 \Rightarrow \hat{X}$ is a sol.

where $\hat{X} = (A^t A)^{-1} (A^t Y)$.

Let $f(X) = \| AX - Y \|_2^2 = \| Y - AX \|_2^2$

$$= (Y - AX)^t \, (Y - AX)$$

$\underline{Y^t - X^t A^t}$ $\quad = \left(\underline{Y^t Y} - \underline{X^t A^t Y} - \underline{Y^t A X} + \underline{X^t A^t A X} \right)$

$\Rightarrow$ Minimum of $f(X) \Leftrightarrow \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\nabla f = - \underline{A^t Y} - \underline{A^t Y} + 2 A^t A X = 0$$

$$\Rightarrow \quad A^t A X = A^t Y \Rightarrow \underline{X} = \underline{(A^t A)^{-1} (A^t Y)}$$

1.2. Exponential approximation

Question: How to minimize $E = \sum_{i=1}^{m} (y_i - P(x))^2$ for $P(x) = be^{ax}$?

$y = be^{ax} \implies \ln y = \ln b + ax$

Let $\ln y = \hat{y}, \ \ln b = a_0, \ a = a_1$ then,

$$\begin{bmatrix} \sum_{i=1}^{m} x_i^0 & \sum_{i=1}^{m} x_i^1 \\ \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} \hat{y}_i \\ \sum_{i=1}^{m} \hat{y}_i x_i \end{bmatrix}$$

1.2. Exponential approximation

Example 1.2.1

Consider the collection of data in the first three columns

i	x_i	y_i	$\ln y_i$	x_i^2	$x_i \ln y_i$
1	1.00	5.10	1.629	1.0000	1.629
2	1.25	5.79	1.756	1.5625	2.195
3	1.50	6.53	1.876	2.2500	2.814
4	1.75	7.45	2.008	3.0625	3.514
5	2.00	8.46	2.135	4.0000	4.270
	7.50		9.404	11.875	14.422