3D Printing in Topology

Undergraduate Research

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Contents

1 Introduction

This report will provide the appropriate code to generate the desired knots below.

The variables to be used are

$(u, v) \in \mathbb{C}$ that define a point in $\mathbb{C}$,
$(x, y, z) \in \mathbb{R}$ that define a point in $\mathbb{R}^3$,
r, θ are two parameters, namely a radius and an angle.

Let's start by defining the inverse stereographic projection from $\mathbb{R}^3$ to $\mathbb{S}^3$, which is inside of $\mathbb{R}^4$ in which we identify as $\mathbb{C}^2$ We will need this idea during the computation. StandardDenominator will be a variable that is used throughout the calculations for nicer looking notation. gu will be the u component of the inverse stereographic projection of point $(x, y, z,)$. gv will be the v component of the inverse stereographic projection of point $(x, y, z,)$.

```
StandardDenominator = 1 + x^2 + y^2 + z^2;
gu[x_, y_, z_] := (2*x/StandardDenominator) + I*(2*y/StandardDenominator)
gv[x_, y_, z_] := (2*z/StandardDenominator) +
  I*((-1 + x^2 + y^2 + z^2)/StandardDenominator)
```

Next we will define a function pTinf that is a polynomial function from $\mathbb{C}^2$ to $\mathbb{C}$. This particular function's zero set is a trefoil going through infinity.

```
pTinf[u_, v_] := (u + I*v)^2 - (u - I*v)^3
```

Next we will define a function from $\mathbb{R}^3$ to $\mathbb{R}$ that returns a negative value if the polynomial evaluated at the stereographic inverse (x, y, z) is within radius r of the origin in $\mathbb{C}$, i.e. if (x, y, z) is within the tube of radius r around the trefoil knot. Note: remember this radius is not the radius in $\mathbb{R}^3$ but the radius in $\mathbb{C}$ the range of the defining polynomial pTinf.

```
inTinftube[x_, y_, z_, r_] :=
 Re[pTinf[gu[x, y, z], gv[x, y, z]]]^2 +
  Im[pTinf[gu[x, y, z], gv[x, y, z]]]^2 - r^2
```

From here we should have everything we need to plot the shapes in 3D. We can use a RegionPlot3D function to plot combinations of the knot, the rays, and boundary shapes that we will contain the knot it (in order to be able to print an object that does not stretch on to infinity...).

PlotPoints: The option PlotPoints will determine the "resolution" of the shape, i.e., how well the shape renders. For quick testing, set this value to something smaller, but not small enough to where holes start to appear in the shape.

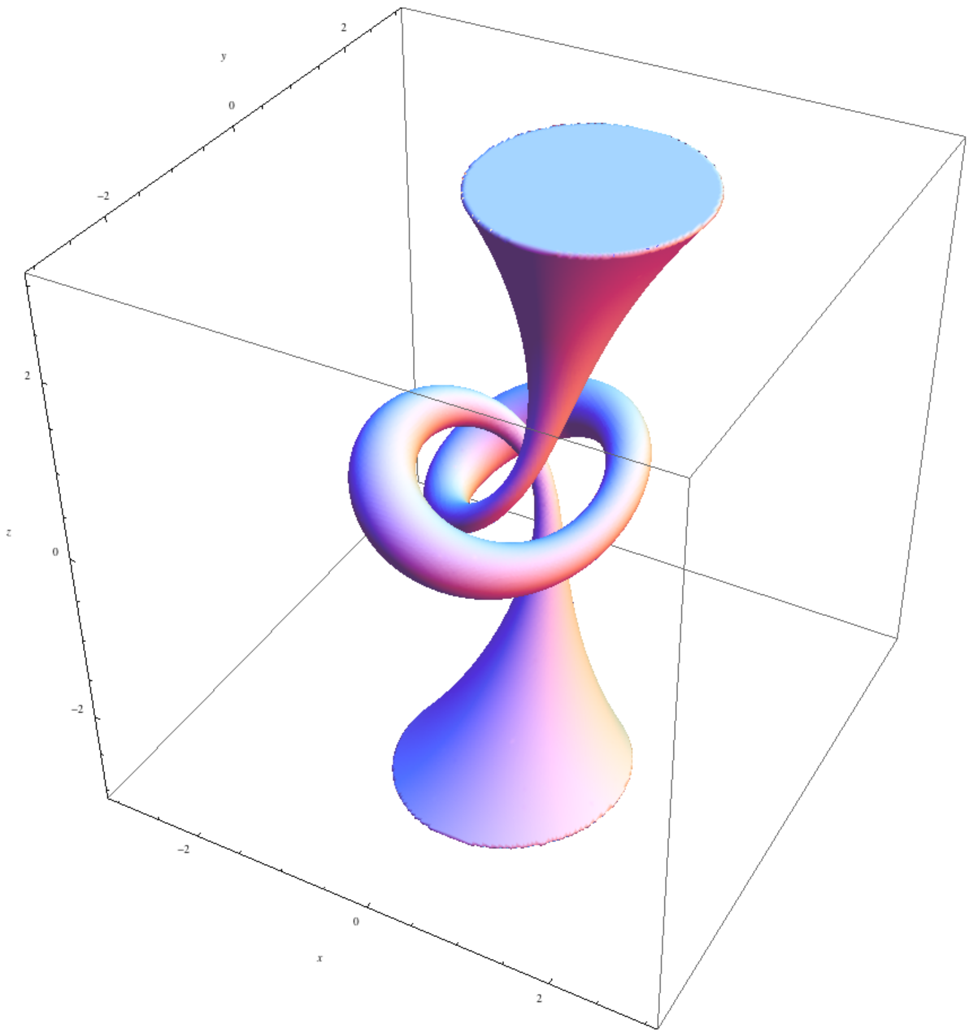
Computation Time: It's also worth noting that computation time will be much higher for a higher value of PlotPoints. To be exact, since this is a plot in $\mathbb{R}^3$, every time we double PlotPoints we will be multiplying our computation time by 8, since $2^3 = 8$.

```
onTinfpage[x_, y_, z_, θ_] :=
 -Sin[θ] * Re[pTinf[gu[x, y, z], gv[x, y, z]]] +
  Cos[θ] * Im[pTinf[gu[x, y, z], gv[x, y, z]]]
```

2 Trefoil Knot

So let's start by just plotting the knot by itself. To do this we will tell Mathematica to only plot the points that are inside inTinftube with radius $r = 0.8$, and we will use a boundary condition of a cylinder of radius 3 and height 6 centered at the origin. Notice that to use RegionPlot3D we have to define our surfaces and shapes by means of inequalities.

```
RegionPlot3D[(inTinftube[x, y, z, 1.0] < 0) && x^2 + y^2 < 9 && z > -3 && z < 3,
 {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, AxesLabel → Automatic, ImageSize → 850,
 PlotPoints → 200, Mesh → None]
```



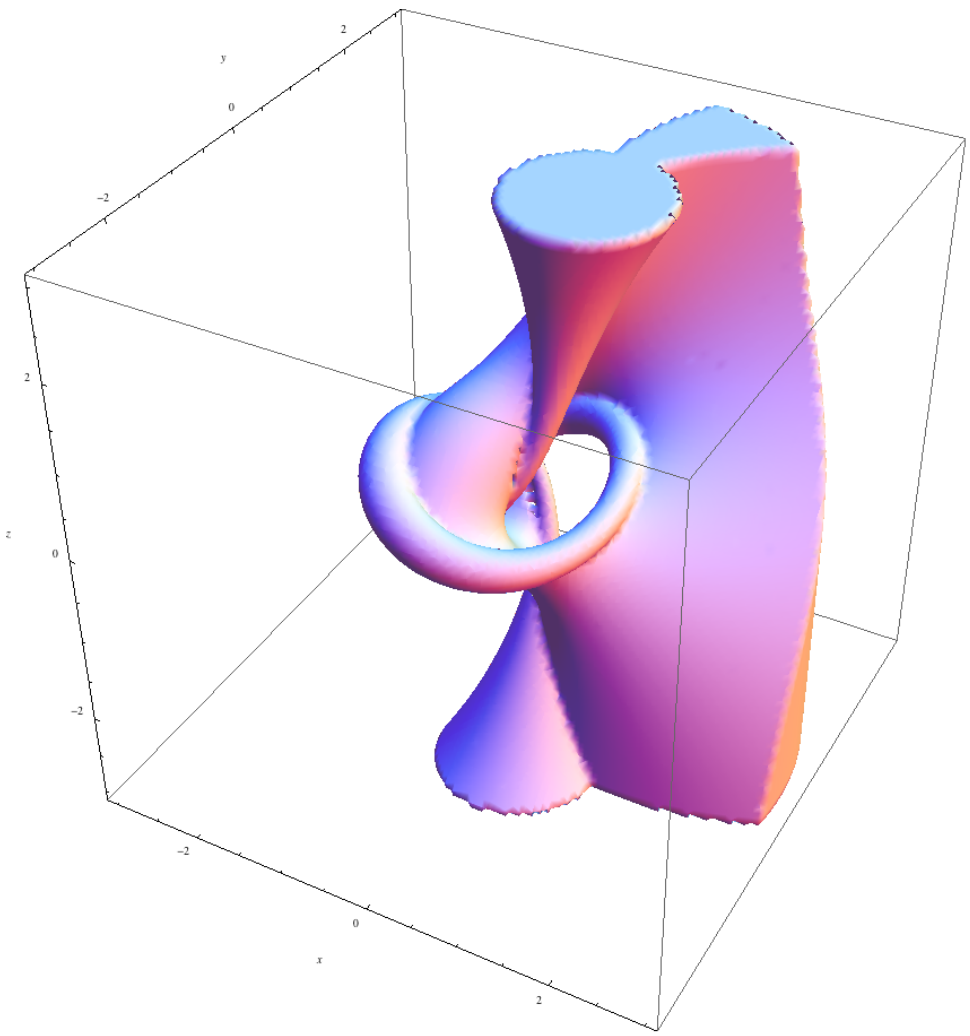
I uploaded this particular knot to **Thingiverse**, where one can download the .stl file to 3D print or manipulate.

3 Trefoil Knot Variations

3.1 Trefoil Knot and One Page

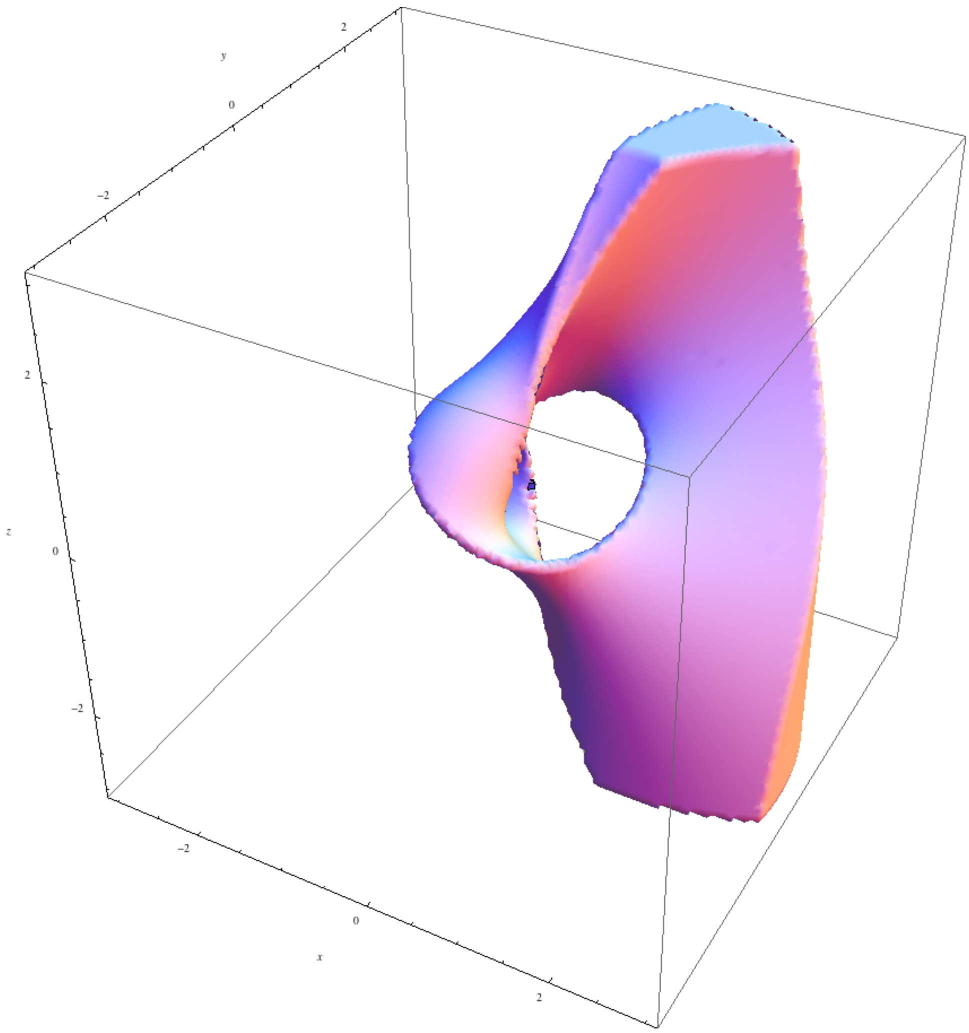
Now let's plot the knot and 1 page from $\frac{\pi}{6}$ to $\frac{\pi}{3}$. Note that even at a PlotPoints of 100, there are still some slight holes.

```
RegionPlot3D[
 ((onTinfpage[x, y, z, Pi / 6] < 0 && onTinfpage[x, y, z, Pi / 3] > 0) ||
    inTinftube[x, y, z, 0.8] < 0) && x ^ 2 + y ^ 2 < 9 && z > -3 && z < 3,
 {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, AxesLabel → Automatic, ImageSize → 850,
 PlotPoints → 100, Mesh → None]
```



We can also plot the page by itself.

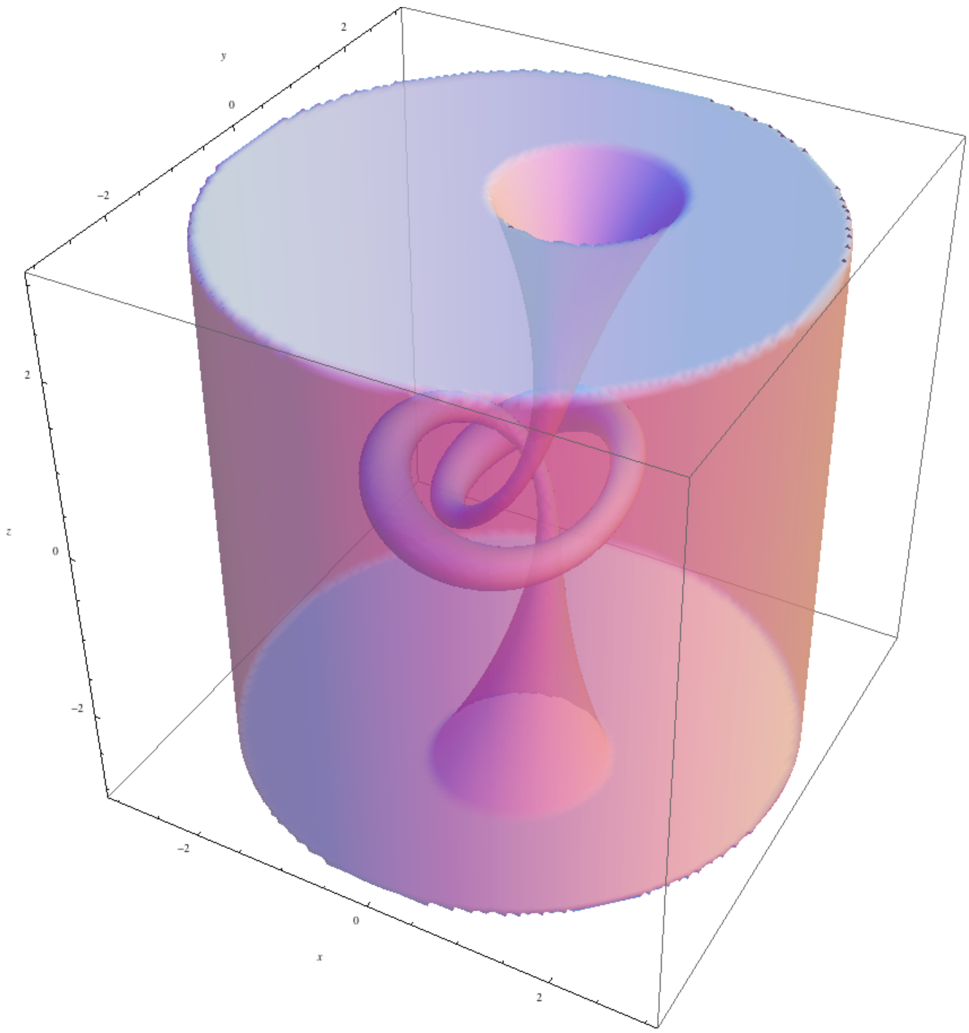
```
RegionPlot3D[
 ((onTinfpage[x, y, z, Pi / 6] < 0 && onTinfpage[x, y, z, Pi / 3] > 0) ||
    inTinftube[x, y, z, 0.8] < 0) && x ^ 2 + y ^ 2 < 9 && z > -3 && z < 3,
 {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, AxesLabel → Automatic, ImageSize → 850,
 PlotPoints → 100, Mesh → None]
```



3.2 The Knot Complement

Now let's plot the knot complement: the space outside the knot.

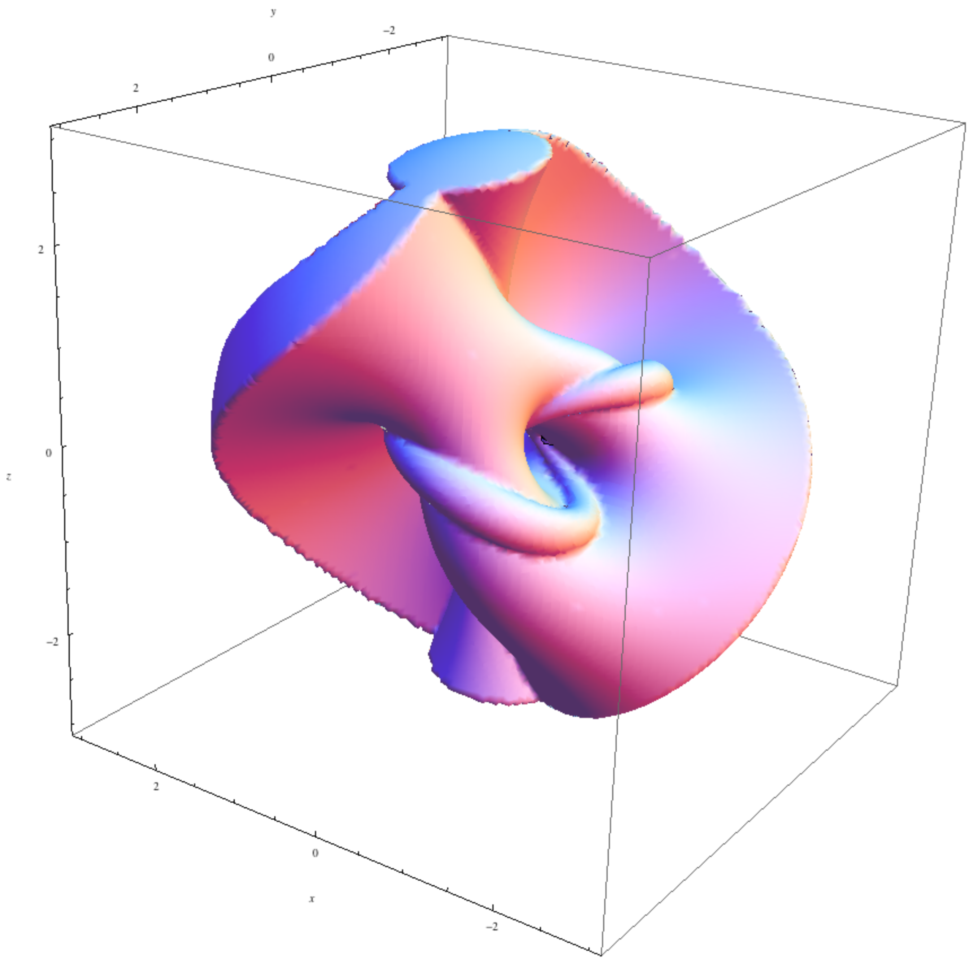
```
RegionPlot3D[(inTinftube[x, y, z, 0.8] > 0) && x^2 + y^2 < 9 && z > -3 && z < 3,
 {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, AxesLabel → Automatic, ImageSize → 850,
 PlotPoints → 100, Mesh → None, PlotStyle → Opacity[0.6]]
```



3.3 Trefoil Knot and Two Pages Bounded by a Circle

Let's now plot the standard trefoil knot with two pages, namely the sections from $\frac{\pi}{6}$ to $\frac{\pi}{3}$ and $\frac{7\pi}{6}$ to $\frac{4\pi}{3}$, all of which is bounded inside of a circle of radius 3 instead of our usual cylinder.

```
RegionPlot3D[(inTinftube[x, y, z, 0.8] > 0) && x^2 + y^2 < 9 && z > -3 && z < 3,
 {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, AxesLabel → Automatic, ImageSize → 850,
 PlotPoints → 100, Mesh → None, PlotStyle → Opacity[0.6]]
```



4 Current Tasks

I am currently working with the one page variation of the trefoil knot by making cuts to split the page up into pieces. Once the appropriate cuts have been made, I will print these pieces separately to attach to the standard trefoil knot via snaps or magnets.