Semester Project: Application of Stochastic calculus to Mathematical Finance

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Contents

1 Abstract

The aim of this project is to study chapter 12 of *Stochastic Differential Equations: An Introduction with Applications* by B. Øksendal [2] and provide a summary that can be studied as independently as possible from the other chapters with additional details for some arguments and more illustrations. In particular, we will provide complete explanations and some examples for the study of Complete Market, see Section 6, illustrate the results of Corollary 7.6 with the generalised Black Scholes model in Theorem 7.7 and show an example of price calculation with the American call in Section 7.3. In the following, we assume that the reader is familiar with the basic notions of stochastic calculus (Brownian motion, martingale, Ito integral, Ito formula...)

2 Acknowledgement

Before stating some useful preliminary results and definition, I would like to first thank Dr.Cheuk Yin Lee for having supervised this project and answered all my questions on the following material.

3 Notation and preliminary results

We denote by $B(t) \in \mathbb{R}^m$ the standard m-dimensional Brownian motion and $\mathcal{F}_t^{(m)}$ the natural filtration associated to $B(t)$.

Definition 3.1. *$\mathcal{V}^{m\times n}(S,T)$ denotes the set of $m \times n$ matrices $v = [v_{ij}(t,\omega)]$ where each entry $v_{ij}(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ satisfies:*

(i) *$(t,\omega) \to v_{ij}(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.*

(ii) *$v_{ij}(t,\omega)$ is $\mathcal{F}_t$-adapted*

(iii) $\mathbb{E}[\int_S^T v_{ij}(t,\omega)^2\, dt] < \infty$

If m=1, we simply write $\mathcal{V}^n(S,T)$ instead of $\mathcal{V}^{1\times n}(S,T)$

Definition 3.2 (Uniformly elliptic Itô diffusion)**.** *A process $Y(t) \in \mathbb{R}^k$ is an uniformly elliptic Itô diffusion if it has the form*

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t), \quad Y(0) = y \tag{1}$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times l}$ are given Lipschitz continuous functions and there exists a constant $c > 0$ such that

$$x^\top \sigma(y)\sigma^\top(y)x \geq c|x|^2 \tag{2}$$

for all $x \in \mathbb{R}^k$, $y \in \mathbb{R}^k$.

Definition 3.3. *Let X_t be an Itô diffusion[1] satisfying*

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \qquad t \geq s \tag{3}$$

for some Lipschitz continuous functions $b(x) \in \mathbb{R}^n$ and $\sigma(x) \in \mathbb{R}^{n\times m}$. Then we know that for any initial condition $X(s) = x$, there exists a unique solution of (3) that we will denote by $X_t = X_t^{s,x}; t \leq s$. If $s = 0$ we write X_t^x and we define

$$\mathbb{E}^x[f(X_t)] \coloneqq \mathbb{E}[f(X_t^x)] \tag{4}$$

for any bounded Borel function f and time $t \geq 0$.

[1] More information on Itô diffusion can be found in chapter 7 and 8 of Oksendal [2]

Theorem 3.4 (Girsanov theorem with Novikov condition). *Suppose a process $u(t,\omega) \in \mathcal{V}^m(0,T)$ satisfies the Novikov condition:*

$$\mathbb{E}\left[\exp\Big(\frac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\right] < \infty.$$

Define the measure Q on $\mathcal{F}_T^{(m)}$ by

$$dQ(\omega) = \exp\Big(-\int_0^T u(t,\omega)\,dB(t) - \frac{1}{2}\int_0^T u^2(t,\omega)\,dt\Big)dP(\omega). \tag{5}$$

Then

$$\tilde{B}(t) \coloneqq \int_0^t u(s,\omega)\,ds + B(t) \tag{6}$$

is an $\mathcal{F}_t^{(m)}$-Brownian motion w.r.t Q. It means that by adding the drift coefficient $u(t,\omega)$, the standard Bronian motion under P becomes a Bromnian motion under Q.

Theorem 3.5 (Representation theorem). *Let F be an $\mathcal{F}_T^m$-adapted random process with $\mathbb{E}[F^2] < \infty$. Then there exists a unique process $\phi(t,\omega) \in \mathcal{V}^m(0,T)$ such that:*

$$F(\omega) = \mathbb{E}[F] + \int_0^T \phi(t,\omega)dB(t). \tag{7}$$

Theorem 3.6 (Martingale representation theorem). *Let $F(t)_{t\in[0,T]}$ be a martingale with respect to the Brownian filtration $\mathcal{F}_t^m$ with $\mathbb{E}[F(T)^2] < \infty$. Then there exists a unique process $\phi(t,\omega) \in \mathcal{V}^m(0,T)$ such that:*

$$F(t,\omega) = \mathbb{E}[F(0)] + \int_0^t \phi(t,\omega)dB(t) \quad \text{for all } 0 \le t \le T. \tag{8}$$

Lemma 3.7. *The linear span of random variables of the type*

$$\exp\Big\{\int_0^T h(t)dB_t - \frac{1}{2}\int_0^T h^2(t)dt\Big\}; \quad h \in L^2[0,T] \text{ (deterministic)}$$

is dense in $L^2(\mathcal{F}_T, P)$.

I expect the reader to be familiar with most of those results except maybe for the last one for which a proof can be found in Oksendal [2] at p.50.

4 Introduction

In this Section we define the key objects we will study.

Definition 4.1. ***a)*** *A market is an $\mathcal{F}_t^{(m)}$-adapted $(n+1)$-dimensional Itô process $X(t) = (X_0(t), X_1(t), ..., X_n(t))$; $0 \le t \le T$ which satisfies the following SDE:*

$$dX_0(t) = \rho(t,\omega)X_0(t)dt; \;\; X_0(0) = 1 \tag{9}$$

where $\rho(t,\omega)$ is a bounded process and

$$dX_i(t) = \mu_i(t,\omega)dt + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)dB_j(t) \tag{10}$$

where μ_i denotes the components of the $n \times 1$ drift vector μ and σ_{ij} are the elements of the $n \times m$ diffusion matrix σ.
In other words:

$$d\hat{X}(t) = \mu(t,\omega)dt + \sigma(t,\omega)dB(t); \tag{11}$$

where we define $\hat{X} = (X_1(t), X_2(t), ..., X_n(t))$ the risky assets part of the market by opposition to the risk free asset X_0.

b) *The market $X(t)_{t\in[0,T]}$ is called normalized (or discounted) if $X_0(t) \equiv 1$.*

c) *A portfolio in the market $X(t)_{t\in[0,T]}$ is an $n+1$-dimensional (t,ω)-measurable and $\mathcal{F}_t^{(m)}$-adapted stochastic process*

$$\theta(t,\omega) = (\theta_0(t,\omega), \theta_1(t,\omega), ...\theta_n(t,\omega)); \;\; 0 \le t \le T. \tag{12}$$

d) *The value at time t of a portfolio $\theta(t)$ is defined by*

$$V(t,\omega) = V^{\theta}(t,\omega) = \theta(t) \cdot X(t) = \sum_{i=0}^{n} \theta_i(t)X_i(t) \tag{13}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^{n+1}$

e) *The portfolio $\theta(t)$ is called self-financing if the following regularity condition holds*

$$\int_0^T \left\{ |\theta_0(s)\rho(s)X_0(s) + \sum_{i=1}^{n} \theta_i(s)\mu_i(s)| + \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \theta_i(s)\sigma_{ij}(s) \right]^2 \right\} ds < \infty \; a.s. \tag{14}$$

and

$$dV(t) = \theta(t) \cdot dX(t) \tag{15}$$

i.e.

$$V(t) = V(0) + \int_0^t \theta(s) \cdot dX(s) \; for \; t \in [0,T]. \tag{16}$$

Remark. **a)** We can view $\theta_i(t)$ as the portion of our investment at time t into the asset $X_i(t)$ and $V(t)$ as the total value of our portfolio.

b) The risk free asset $X_0(t)$ has no diffusion part in its SDE, is explicitly given by

$$X_0(t) = \exp\bigg(\int_0^t \rho(s,\omega)ds\bigg), \tag{17}$$

and we can define:

$$\xi(t) := X_0^{-1}(t) = \exp\bigg(-\int_0^t \rho(s,\omega)ds\bigg) > 0 \text{ for all } t \in [0,T]; \tag{18}$$

Since $\xi(t)$ is positive we can view it as a different unit of the price: given any market $\{X(t)\}_{t\in[0,T]}$ we define its normalized or discounted version:

$$\overline{X}(t) := \xi(t)X(t) \tag{19}$$

which satisfies:

$$\overline{X}_0(t) \equiv 1.$$

c) To make sense of the self financing condition (15), it can be helpful to view its implication in discrete time, we would have:

$$V(t_k) - V(t_{k-1}) = \theta(t_{k-1}) \cdot \big(X(t_k) - X(t_{k-1})\big); \tag{20}$$

i.e. the difference of the value of our Portfolio between time t_{k-1} and t_k depends exclusively of the change of the price of the market. No money is injected or withdrawn from the portfolio.

d) One can show that if $\theta_1(t), ..., \theta_n(t)$ are chosen, for any initial value $V^\theta(0)$ of our portfolio there exists a unique $\theta_0(t)$ such that the portfolio $\theta_0(t), \theta_1(t), ..., \theta_n(t)$ is self financing and has initial value $V^\theta(0)$. It is given by

$$\theta_0(t) = V^\theta(0) + \xi(t)A(t) + \int_0^t \rho(s)A(s)\xi(s)ds. \tag{21}$$

where $A(t) := \sum_{i=1}^n \Big(\int_0^t \theta_i(s)dX_i(s) - \theta_i(t)X_i(t)\Big)$. The proof can be found in [2] at p.272.

5 Arbitrage Portfolio

Definition 5.1. *A portfolio $\theta(t)$ is said to be admissible if it is self financing, the regularity condition (14) holds and $V^\theta(t)$ is (t,ω) a.s. lower bounded, i.e. there exists $K = K(\theta) < \infty$ such that*

$$V^\theta(t,\omega) \geq -K \textit{ for a.a } (t,\omega) \in [0,T] \times \Omega. \tag{22}$$

Definition 5.2. *An admissible portfolio is called an arbitrage (in the market $\{X(t)\}_{t\in[0,T]}$) if the corresponding value process V^θ satisfies $V^\theta(0) = 0$ and*

$$V^\theta(T) \geq 0 \textit{ a.s and } P[V^\theta(T) > 0] > 0. \tag{23}$$

In other words, there is an arbitrage opportunity in the market $\{X(t)\}_{t\in[0,T]}$ if there exists some portfolio $\theta(t)$ for which the initial cost of the correspoding investment is null and that produces a positive payoff at time T with positive probability.

Lemma 5.3. *Suppose there exists a measure Q on $\mathcal{F}_T^{(m)}$ such that $P\sim Q$ and such that the normalized price process $\{\overline{X}(t)\}_{t\in[0,T]}$ is a local martingale with respect to Q. Then the market has no arbitrage.*

Remark. Actually we can prove very similarly the converse, that is the equivalence of existence of arbitrage for the normalized price process and the normalized one. It is often more useful to word with the latter as it makes computations easier while keeping the important properties.

Definition 5.4 (Equivalent Martingale Measure (EMM))**.** *A measure $Q\sim P$ such that the normalized price process $\{\overline{X}(t)\}_{t\in[0,T]}$ is a (local) martingale w.r.t Q is called an equivalent (local) martingale measure.*

In other terms according to Lemma 5.3, if there exists an EMM for the normalized price process, we can rule out arbitrage in our market. We will now state the main result of this section that characterize arbitrage in a market.

Theorem 5.5. *:*

a) Suppose there exists a process $u(t,\omega) \in \mathcal{V}^m(0,T)$ such that, with $\hat{X}(t,\omega) = (X_1(t,\omega),\ldots,X_n(t,\omega))$,

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)\hat{X}(t,\omega) \quad \textit{for a.a } (t,\omega) \tag{24}$$

and such that

$$E\left[\exp\left(\frac{1}{2}\int_0^T u^2(t,\omega)dt\right)\right] < \infty.$$

Then the market has no arbitrage.

b) (Karatzas [1], Th.0.2.4) Conversely if the market $\{X(t)\}_{t\in[0,T]}$ has no arbitrage, then there exists an $\mathcal{F}_T^{(m)}$-adapted, (t,ω)-measurable process $u(t,\omega)$ such that

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)\hat{X}(t,\omega) \quad \textit{for a.a } (t,\omega)$$

6 Complete Market

If the market $\{X(t)\}_{t\in[0,T]}$ allows arbitrage it doesn't make sense to study completeness, so in this section we will assume that there exists a process $u \in \mathcal{V}^m(0,T)$ such that:

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)\hat{X}(t,\omega) \tag{25}$$

We then set the measure Q to be the one generated by u as in the Girsanov theorem statement (3.4).

Definition 6.1. ***a)*** *A (European) contingent T-claim is a lower bounded $\mathcal{F}_T^{(m)}$-measurable random variable $F(\omega) \in L^2(Q)$.*

b) *We say that the claim $F(\omega)$ is attainable (in the market $\{X(t)\}_{t\in[0,T]}$) if there exists an admissible portfolio $\theta(t)$ and a real number z such that*

$$F(\omega) = V_z^\theta := z + \int_0^T \theta(t) dX(t) \ \ a.s. \tag{26}$$

and such that

$$\overline{V}_z^\theta = z + \int_0^t \xi(s)\theta(s)\cdot\sigma(s)d\tilde{B}(s); \ 0 \le t \le T \text{ is a } Q\text{-martingale.} \tag{27}$$

If such a $\theta(t)$ exists, we call it a replicating or hedging portfolio for F. Note that passing to the expectation, we get $z = E_Q[F\xi(T)]$.

c) *The market $\{X(t)\}_{t\in[0,T]}$ is called complete if every T-claim is attainable.*

In other words, the market is complete if for every claim $F(\omega)$ we can find an initial investment z and a replicating portfolio $\theta(t)$ which generates a value V_z^θ at time T which equals F a.s.

Theorem 6.2. *The market $\{X(t)\}_{t\in[0,T]}$ is complete if and only if $\sigma(t,\omega)$ has a left inverse $\Lambda(t,\omega)$ for a.a (t,ω), i.e. there exists an $\mathcal{F}_t^{(m)}$-adapted matrix valued process $\Lambda(t,\omega) \in \mathbb{R}^{m\times n}$ such that*

$$\Lambda(t,\omega)\sigma(t,\omega) = I_m.$$

We first make the following observation:

Lemma 6.3 (Admissible portfolio and discounted market)**.** *Let $\theta(t)$ be a portfolio. Then $\theta(t)$ is admissible for the market $\{X(t)\}_{t\in[0,T]}$ if and only if it is also an admissible portfolio for the discounted market $\overline{X}(t)_{t\in[0,T]}$, i.e.*

$$\begin{aligned} V^\theta(t) &= V^\theta(0) + \int_0^t \theta(s)\, dX(s) \\ \Leftrightarrow \overline{V}^\theta(t) &= \overline{V}^\theta(0) + \int_0^t \theta(s)\, d\overline{X}(s) \end{aligned}$$

Proof. The discounted value process is:

$$\overline{V}^\theta(t) := \xi(t)V^\theta(t) = \theta(t)\cdot\overline{X}(t). \tag{28}$$

As we assumed $\rho(t)$ to be bounded $V^\theta(t)$ is lower bounded if and only if so is $\overline{V}^\theta(t)$. Suppose that $\theta(t)$ is self financing for the market $\{X(t)\}_{t\in[0,T]}$, we then have:

$$
\begin{aligned}
d\overline{V}^{\theta}(t) &= \xi(t)dV^{\theta}(t) + V^{\theta}(t)d\xi(t) \\
&= \xi(t)\theta(t)\cdot dX(t) - \rho(t)\xi(t)V^{\theta}(t)dt \\
&= \theta(t)\xi(t)[dX(t) - \rho(t)X(t)dt] \\
&= \theta(t)d\overline{X}(t)
\end{aligned}
$$

i.e. $\theta(t)$ is also self financing for the discounted market.

Conversely, suppose that $\theta(t)$ is self financing for the discounted market $\overline{X}(t)_{t\in[0,T]}$, we then have:

$$
\begin{aligned}
dV^{\theta}(t) &= d\big(X_0(t)\overline{V}^{\theta}(t)\big) && (29) \\
&= X_0(t)d\overline{V}^{\theta}(t) + \overline{V}^{\theta}(t)dX_0(t) && (30) \\
&= X_0(t)\theta(t)\cdot d\overline{X}(t) + \rho(t)X_0(t)\overline{V}^{\theta}(t)dt && (31) \\
&= \theta(t)X_0(t)[d\overline{X}(t) + \rho(t)\overline{X}(t)dt] && (32) \\
&= \theta(t)dX(t) && (33)
\end{aligned}
$$

□

Proof of Theorem 6.2. (i) First assume that $\sigma(t,\omega)$ has a left inverse $\Lambda(t,\omega)$. We first note that for an admissible portfolio $\theta(t)$ and an initial investment z value function with initial condition $V^{\theta}(0) = z$ is:

$$V^{\theta}(t) = z + \int_0^t \theta(s)dX(s);$$

and thus by Lemma (6.3) the discounted value function becomes:

$$
\begin{aligned}
\overline{V}^{\theta}(t) &= z + \int_0^t \theta(S) d\overline{X}(s) \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s)\, d\overline{X}_i(s) && (d\overline{X}_0(s) := 0) \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s) \Big(-\rho(s)\xi(s)X_i(s)ds + \xi(s)dX(s)\Big) \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s)\, \xi(s)\Big(-\rho(s)X_i(s)ds + \big(\mu_i(s)ds + \sigma_i(s)dB(s)\big)\Big) \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s)\, \xi(s)\Big(\big(\mu_i - \rho(s)X_i(s)\big)ds + \sigma_i(s)dB(s)\Big) \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s)\, \xi(s)\Big(\sigma_i u_i(s)ds + \sigma_i(s)dB(s)\Big) && \text{by (25)} \\
&= z + \int_0^t \sum_{i=1}^{n} \theta_i(s)\, \xi(s)\sigma_i d\tilde{B}(s) && (34)
\end{aligned}
$$

Let F be a T-claim. We thus want to prove that there exists an replicating portfolio $\theta(t)$ and an initial investment z such that

$$
\begin{aligned}
V^{\theta}(T) &= F(\omega) \text{ a.s} \\
\Leftrightarrow \overline{V}^{\theta}(T) &= \xi(T)F(\omega) \text{ a.s}
\end{aligned}
$$

with $V^{\theta}(0) = z$ and

$$\xi(T)F(\omega) = \overline{V}^{\theta}(T) = z + \int_0^T \xi(t) \sum_{i=1}^{n} \theta_i(t)\sigma_i(t) d\tilde{B}(t) \tag{35}$$

is a Q-Martingale.

But by using Lemma (3.5), with measure Q and the Q-Browmnian motion $\tilde{B}(t)$, we know that we also have a unique representation

$$\xi(T)F(\omega) = \mathbb{E}_Q[\xi(T)F] + \int_0^T \phi(t,\omega)\, d\tilde{B}(t)$$

For some $\phi(t,\omega) = (\phi_1(t,\omega), \phi_2(t,\omega), ..., \phi_m(t,\omega)) \in \mathbb{R}^m$.
By comparing this representation to (34), we see that we can set $z = \mathbb{E}_Q[\xi(T)F]$, and choose $\hat{\theta}(t) = (\theta_1(t), \theta_2(t), ..., \theta_n(t))$ such that

$$\xi(t) \sum_{i=1}^{n} \theta_i(t)\sigma_{ij}(t) = \phi_j(t); \qquad 1 \le j \le m$$

i.e. such that

$$\xi(t)\hat{\theta}(t)\sigma(t) = \phi(t)$$

But we assumed that σ has a left inverse Λ, so we can easily find the solution

$$\tilde{\theta}(t,\omega) = X_0(t)\phi(t,\omega)\Lambda(t,\omega).$$

With $\hat{\theta}(t)$ chosen we know that there exists a unique $\theta_0(t)$ given by (21) such that the portfolio $\theta(t)$ is self-financing and has initial value $V^\theta(0) = z$.

Moreover, we have defined $\hat{\theta}(t)$ such that $\overline{V}^\theta(t) = z + \int_0^t \phi(s)d\tilde{B}(s)$ which is a Q-martingale as an integral with respect to a Brownian motion. This means that

$$\xi(t)V^\theta(t) = \mathbb{E}_Q[\xi(T)V^\theta(T)|\tilde{\mathcal{F}}^{(m)}] = \mathbb{E}_Q[\xi(T)F|\tilde{\mathcal{F}}^{(m)}],$$

where $\tilde{\mathcal{F}}^{(m)}$ is the natural filtration for $\tilde{B}(t)$ and since F and $\xi(t)$ are lower bounded, we finally have that so is $V^\theta(t)$. Hence $\theta(t)$ is admissible and the market $\{X(t)\}_{t\in[0,T]}$ is complete.

(ii) Conversely, assume now that the market $\{X(t)\}_{t\in[0,T]}$ is complete. Choose $\phi(t,\omega) \in \mathcal{V}^m(0,T)$ and define $F(\omega) := X_0(T)\int_0^T \phi(t,\omega)d\tilde{B}(t)$. Then, by completeness there exists an admissible portfolio $\theta = (\theta_0, \hat{\theta})$ such that $\overline{V}^\theta(t) = \int_0^t \xi(s)\hat{\theta}(s)\sigma(s)\,d\tilde{B}(s)$ is a Q- martingale and

$$\int_0^T \phi(t,\omega)d\tilde{B}(t) = \xi(T)F(\omega) = \int_0^T \xi(t)\hat{\theta}(t)\sigma(t)\,d\tilde{B}(t).$$

But then

$$\int_0^t \phi\,d\tilde{B} = \mathbb{E}_Q[\xi(T)F|\tilde{\mathcal{F}}^{(m)}] = \int_0^t \xi\hat{\theta}\sigma\,d\tilde{B} \qquad \text{a.s. for all } t\in[0,T].$$

Note that $\mathbb{E}_Q[\xi(T)F|\tilde{\mathcal{F}}_t^{(m)}]$ is a martingale with respect to $\tilde{\mathcal{F}}_t^{(m)}{}_{t\in[0,T]}$ and thus by uniqueness of the representation given by the martingale representation theorem (See theorem (3.6)) we have $\phi(t,\omega) = \xi(t,\omega)\hat{\theta}(t,\omega)\sigma(t,\omega)$ for a.a. (t,ω). This implies that $\phi(t,\omega)$ belongs to the linear span of the rows $\{\sigma_i(t,\omega)\}_{i=1}^n$ of $\sigma(t,\omega)$. Since this apply in particular to all process $\phi = (\phi_1, \phi_2, ..., \phi_m)$ with

$$\phi_i = \exp\Big\{\int_0^T h_i(t)dB_t - \frac{1}{2}\int_0^T h_i^2(t)dt\Big\}$$

for some deterministic functions $(h_i(t))_{i=1}^m$ in $L^2[0,T]$, by applying Lemma 3.7 on every component of ϕ, it follows that the linear span of $\{\sigma_i\}_{i=1}^n$ is the whole of $\mathbb{R}^m$ for a.a. (t,ω). So rank $\sigma(t,\omega) = m$ and there exists $\Lambda(t,\omega)\in\mathbb{R}^{m\times n}$ such that

$$\Lambda(t,\omega)\sigma(t,\omega) = I_m.$$

□

Corollary 6.4. ***a)*** *If $n = m$ then the market is complete if and only if $\sigma(t,\omega)$ is invertible for a.a. (t,ω).*

b) *If the market is complete then*

$$\operatorname{rank}\sigma(t,\omega) = m \text{ for a.a. } (t,\omega).$$

In particular, $n \geq m$.
Moreover, the process $u(t,\omega)$ satisfying (25) is unique and is given by:

$$u(t,\omega) = \Lambda(t,\omega)[\mu(t,\omega) - \rho(t,\omega)\tilde{X}(t,\omega)].$$

Example 6.1. We now give both a complete normalised market and a non complete normalised market example ($\rho = 0$). Those are taken from the Exercise section of Oksendal[2].

a) $(n = m = 2)$
$dX_1(t) = 3dt + dB_1(t) + dB_2(t)$,
$dX_2(t) = -dt + dB_1(t) - dB_2(t)$

In other terms, $\mu = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. σ is clearly invertible with inverse $\Lambda = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. The no arbitrage condition (25) is satisfied as by choosing $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we have $\sigma u = \mu$. Moreover the existence of the left inverse of σ, Λ, ensures completeness by Theorem 6.2.

b) $(n = 2, m = 3)$
$dX_1(t) = dt + dB_1(t) + dB_2(t) - dB_3(t)$,
$dX_2(t) = 5dt - dB_1(t) + dB_2(t) + dB_3(t)$,

In this example, $\mu = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$. The market is arbitrage free as by choosing $u = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, we have $\sigma u = \mu$. Nevertheless, it isn't complete as we can actually choose any u of the form $\begin{pmatrix} a \\ 3 \\ a+2 \end{pmatrix}$, $a \in \mathbb{R}$ which contradicts Corollary 6.4. We thus know that the market is not complete, but let's dive further and actually find a non attainable T-claim. Choose $F(\omega) = g(\tilde{B}_1(T))$ where $g : \mathbb{R} \to \mathbb{R}$ is bounded. Suppose by absurd that there exists an initial investment z and an admissible portfolio $(\theta_0(t), \hat{\theta}(t)) = (\theta_0(t), \theta_1(t), \theta_2(t))$ such that

$$
\begin{aligned}
g(\tilde{B}_1(T)) &= V^\theta(T) \\
&= \overline{V}^\theta(T) \\
&= z + \int_0^T \hat{\theta}(t)\sigma(t)d\tilde{B}(t) \\
&= z + \int_0^T (\theta_1(t) - \theta_2(t))d\tilde{B}_1(t) + (\theta_1(t) + \theta_2(t))d\tilde{B}_2(t) + (\theta_2(t) - \theta_1(t))d\tilde{B}_3(t)
\end{aligned}
$$

and is a Q-Martingale. But by the martingale representation theorem (3.6) applied on the 3-dimensional Brownian motion $\tilde{B}(t) = (\tilde{B}_1(t), \tilde{B}_2(t), \tilde{B}_3(t))$ we also know that there exists a unique process $\phi(t,\omega) = (phi_1(t,\omega), \phi_2(t,\omega), \phi_3(t,\omega))$ in $\mathcal{V}^m(0,T)$ such that

$$
g(\tilde{B}_1(T)) = \mathbb{E}_Q[g(\tilde{B}_1(T))] + \int_0^T \phi_1(t)d\tilde{B}_1(t) + \phi_2(t)d\tilde{B}_2(t) + \phi_3(t)d\tilde{B}_3(t)
$$

By uniqueness we must then have a.a.

$$
\begin{aligned}
\phi_1(t) &= \theta_1(t) - \theta_2(t) \\
\phi_2(t) &= \theta_1(t) + \theta_2(t) \\
\phi_3(t) &= \theta_2(t) - \theta_1(t)
\end{aligned}
$$

But also by the martingale representation theorem (3.6) applied on the 1-dimensional Brownian motion $\tilde{B}_1(t)$ we must have $\phi_2 = \phi_3 = 0$. It then follows that $\phi_1 = -\phi_3 = 0$ and

$$
g(\tilde{B}_1(T)) = \mathbb{E}_Q[g(\tilde{B}_1(T))]
$$

which leads to a contradiction by choosing a non constant function g. The reasoning might seem a bit convoluted at first glance, but we actually just illustrated that the system $\hat{\theta}(t)\sigma = \phi$ was underdetermined as we had more Brownian motion $\tilde{B}_i$ to deal with ($m = 3$) then θ_i in $\hat{\theta}$ ($n = 2$) to play with.

7 Option Pricing

7.1 European option

Let $F(\omega)$ be a T-claim. A *European option* on the claim F is the guarantee to be paid the amount $F(\omega)$ at time $t = T > 0$. The question is then how to price this option.

Lemma 7.1 (Buyer's price of the European contingent claim)**.** *Let $F(\omega)$ be a T-claim and $p(F)$ be maximal price the buyer is willing to pay to get the* *guarantee to be paid the amount $F(\omega)$ at time $t = T > 0$. Then*

$$
\begin{aligned}
p(F) = \sup\{y;\ &\text{There exists an admissible portfolio } \varphi \\
&\text{such that } V_y^{\varphi}(T,\omega) := -y + \int_0^T \varphi(s)dX(s) \geq -F(\omega)\ a.s\ \}
\end{aligned}
$$

Lemma 7.2 (Seller's price of the European contingent claim)**.** *Let $F(\omega)$ be a T-claim and $q(F)$ be the minimal price the Seller is willing to accept to offer the guarantee to be paid the amount $F(\omega)$ at time $t = T > 0$. Then*

$$
\begin{aligned}
q(F) = \inf\{z;\ &\text{There exists an admissible portfolio } \psi \\
&\text{such that } V_y^{\psi}(T,\omega) := z + \int_0^T \psi(s)dX(s) \geq F(\omega)\ a.s\ \}
\end{aligned}
$$

In other terms, we suppose rational behavior of the Buyer/Seller. For instance if I am the buyer, I will accept a price y only if I can use $-y$ as an initial investment to hedge to time T a value $V_y^{\theta}(T,\omega)$ which gives me a non negative payoff a the end

$$V_y^{\theta}(T,\omega) + F(\omega) \geq 0 \ \ a.s.$$

for a total initial investment $y - y = 0$.

Definition 7.3. *If $p(F) = q(F)$ we call this common value* the price (at t=0) of the (European) T-contingent claim $F(\omega)$.

Two important examples of European contingent claim are

a) **the European call**, where

$$F(\omega) = (X_i(T,\omega) - K)^+$$

for some $i \in \{1, 2, ..., n\}$ and some $K > 0$. This option gives the owner the right (but not the obligation) to buy one unit of the security i of the market at a predetermined price K (the exercise price) at time T. Assuming rational behavior, the buyer will exercise this option if and only if $X_i(T,\omega) \geq K$.

b) Similarly, **the European put** option gives the owner the right (but not the obligation) to sell one unit of the security i of the market at a predetermined price K at time T. The payoff function for this option is:

$$F(\omega) = (K - X_i(T,\omega))^+$$

Theorem 7.4. ***a)*** *Suppose there exists a process $u \in \mathcal{V}^m(0,T)$ that satisfies (25) and let Q be the measure generated by u as in the statement of Theorem (3.4). Then*

$$\text{ess } \inf F(\omega) \leq p(F) \leq \mathbb{E}_Q[\xi(T)F] \leq q(F) \leq \infty.$$

b) *Suppose, in addition to the condition in 1), that the market $\{X(t)\}_{t\in[0,T]}$ is complete. Then the price of the (European) T-claim F is*

$$p(F) = \mathbb{E}_Q[\xi(T)F] = q(F).$$

7.2 How to hedge an attainable claim?

We have seen in Section 6 that if $V_z^\theta(t)$ is the value process of an admissible portfolio $\theta(t)$ for the market $\{X(t)\}_{t\in[0,T]}$, then $\overline{V}_z^\theta := \xi(t)V_z^\theta(t)$ is the value process of $\theta(t)$ for the normalized market $\{\overline{X}(t)\}_{t\in[0,T]}$ (Lemma 6.3). Hence we have

$$\xi(t)V_z^\theta(t) = z + \int_0^t \theta(s)\, d\overline{X}(s)$$

If furthermore there exists a process $u \in \mathcal{V}^m(0,T)$ that satisfies (25) and we let Q be the measure generated by u and $\tilde{B}$ the Brownian motion under Q as in the statement of Theorem (3.4). Then

$$\xi(t)V_z^\theta(t) = z + \int_0^t \xi(s)\hat{\theta}(s)\sigma(s)\, d\tilde{B}(s)$$

Therefore the portfolio $\theta(t)$ needed to hedge a given T-claim F solves

$$\xi(t,\omega)\hat{\theta}(t,\omega)\sigma(t,\omega) = \phi(t,\omega),$$

i.e.

$$\hat{\theta(t)} = X_0(t)\phi(t)\Lambda(t), \tag{36}$$

where $\Lambda(t,\omega)$ is the left inverse of $\sigma(t,\omega)$ and $\phi(t,\omega)$ is a the unique process in $\mathcal{V}^m(0,T)$ given by the representation theorem (Theorem 3.5) such that

$$\xi(T)F(\omega) = z + \int_0^T \phi(t,\omega)d\tilde{B}(t).$$

The problem is that this process $\phi(t,\omega)$ is given by an existence theorem and a priori we have no idea of what it exactly is. Nevertheless, we can actually characterize $\phi(t,\omega)$ for some special T-claims.

Theorem 7.5. *Let $Y(t) \in \mathbb{R}^k$ be an uniformly elliptic Itô and assume that $h : \mathbb{R}^k \to \mathbb{R}$ is a given function such that*

$$\left\{\frac{\partial}{\partial y_i} E_Q^y[h(Y(T-t))]\right\}_{i=1}^k \quad \textit{exists} \tag{37}$$

and

$$E_Q^y\left[\int_0^T \phi^2(t,\omega)dt\right] < \infty, \tag{38}$$

where

$$\phi(t,\omega) = \sum_{i=1}^k \frac{\partial}{\partial y_i} E_Q^y[h(Y(T-t))]_{y=Y(t)}\sigma_i(Y(t)). \tag{39}$$

Then we have the Itô representation formula

$$h(Y(T)) = E_Q[h(Y(T))] + \int_0^T \phi(t,\omega)d\tilde{B}(t). \tag{40}$$

In order to use this result to find $\phi(t,\omega)$ in, we consider the case where the equation for the market is *Markovian*, i.e.

$$\begin{aligned} dX_0(t) &= \rho(X(t))X_0(t)dt; \quad X_0(0) = 1, \\ dX_i(t) &= \mu_i(t)(X(t))dt + \sigma_i(X(t))dB(t); \quad 1 \leq i \leq n. \end{aligned}$$

We can rewrite this equation in term of $\tilde{B}$ using (6) and (25).

$$dX_0(t) = \rho(X(t))X_0(t)dt; \quad X_0(0) = 1, \tag{41}$$

$$dX_i(t) = \rho(X(t))X_i(t)dt + \sigma_i(X(t))d\hat{B}(t); \quad 1 \leq i \leq n. \tag{42}$$

Therefore by writing $h(X(t)) = \frac{h_0(X(t))}{X_0(t)} = \xi(t)h_0(X(t))$ we get the following result

Corollary 7.6. *Let $X(t) = (X_0(t), \dots, X_n(t))$ be given by (41)-(42) and assume that $h_o : \mathbb{R}^{n+1} \to \mathbb{R}$ is a given function such that*

$$\left\{ \frac{\partial}{\partial x_i} E_Q^x[\xi(T-t)h_0(X(T-t))] \right\}_{i=1}^n \quad \textit{exists} \tag{43}$$

and

$$E_Q^x \left[\int_0^T \phi^2(t,\omega)dt \right] < \infty, \tag{44}$$

where

$$\phi(t,\omega) = \sum_{i=1}^n \frac{\partial}{\partial x_i} E_Q^x[\xi(T-t)h_0(Y(T-t))]_{x=X(t)}\sigma_i(X(t)). \tag{45}$$

Then we have the Itô representation formula

$$\xi(T)h_0(Y(T)) = E_Q[\xi(T)h_0(X(T))] + \int_0^T \phi(t,\omega)d\tilde{B}(t). \tag{46}$$

Using this theorem or corollary can be a bit tricky because of the nature of the expression we give for $\phi(t,\omega) = \sum_{i=1}^n \frac{\partial}{\partial x_i} E_Q^x[\xi(T-t)h_0(Y(T-t))]_{x=X(t)}\sigma_i(X(t))$. To make more sense of it we will show an application of Corollary 7.6 in the proof of Theorem 7.7.

The generalized Black-Scholes Model

We now address the special case where the market has just two securities $X_0(t), X_1(t)$ which are Itô processes of the form

$$\begin{aligned} dX_0(t) &= \rho(t,\omega)X_0(t); \\ dX_1(t) &= \alpha(t,\omega)X_1(t)dt + \beta(t,\omega)X_1(t)dB(t), \end{aligned} \tag{47}$$

where $B(t)$ is 1-dimensional and $\alpha(t,\omega),\beta(t,\omega)$ are 1-dimensional processes in $\bigcap\limits_{T>0}\mathcal{V}^m(0,T)$. In this case (47) has the explicit solution (use Itô 's lemma on $\ln(X_1(t))$)

$$X_1(t) = X_1(0)\exp\Big(\int_0^t \beta(s,\omega)dB(s) + \int_0^t (\alpha(s,\omega) - \frac{1}{2}\beta^2(s,\omega))ds\Big). \tag{48}$$

The equation for the process u satisfying (25) becomes:

$$X_1(t)\beta(t,\omega)u(t,\omega) = X_1(t)\alpha(t,\omega) - X_1(t,\omega)\rho(t,\omega)$$

which has the solution

$$u(t,\omega) = \beta^{-1}(t,\omega)[\alpha(t,\omega) - \rho(t,\omega)] \text{ if } \beta(t,\omega) \neq 0 \tag{49}$$

and the regularity condition for u becomes:

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \frac{\alpha(s,\omega) - \rho(s,\omega)}{\beta^2(s,\omega)}ds\Big)\Big] < \infty$$

We thus have an equivalent martingale measure Q and the market has no arbitrage. Moreover, in this case $\sigma = \beta(t,\omega)X_1(t) \neq 0$ a.a. and thus has an inverse which means that the market is also complete.
If we add some conditions on ρ, β and F we get the following:

Theorem 7.7 (**The generalised Black-Scholes formula**). *Suppose $X(t) = (X_0(t), X_1(t))$ is given by*

$$dX_0(t) = \rho(t)X_0(t); \qquad X_0(0) = 1 \tag{50}$$

$$dX_1(t) = \alpha(t,\omega)X_1(t)dt + \beta(t)X_1(t)dB(t); \qquad X_1(0) = x_1 > 0 \tag{51}$$

where $\rho(t),\beta(t)$ are deterministic and

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \frac{(\alpha(t,\omega) - \rho(t))^2}{\beta^2(t)}\Big)\Big] < \infty$$

- ***a)*** *Then the market $\{X(t)\}_{t\in[0,T]}$ is arbitrage free and complete and the price at time $t = 0$ of the European T-claim $F(\omega) = f(X_1(T,\omega))$, where $\mathbb{E}_Q[f(X_1(T,\omega)] < \infty$, is*

$$p = \frac{\xi(t)}{\delta\sqrt{2\pi}}\int_{\mathbb{R}} f\Big(x_1\exp\Big[y + \int_0^T(\rho(s) - \frac{1}{2}\beta^2(s))ds\Big]\Big)\exp\big(-\frac{y^2}{2\delta^2}\big)dy \tag{52}$$

where $\xi(T) = \exp(-\int_0^T \rho(s)\,ds)$ and $\delta^2 = \int_0^T \beta^2(s)ds$.

- ***b)*** *If $\rho,\alpha,\beta \neq 0$ are constants and $f \in C^1(\mathbb{R})$, then the self-financing portfolio $\theta(t) = (\theta_0(t),\theta_1(t))$ is given by*

$$\theta_1(t,\omega) = \frac{1}{\sqrt{2\pi(T-t)}}\int_{\mathbb{R}} f'(X_1(t,\omega)\exp\{\beta x + (\rho - \frac{1}{2}\beta^2)(T-t)\}) \cdot \exp\Big(\beta x - \frac{x^2}{2(T-t)} - \frac{1}{2}\beta^2(T-t)\Big)dx \tag{53}$$

and θ_0 is given by (21).

Proof. **a)** We know from Theorem 7.4 and by (48) that the price $p = p(F) = q(F)$ is, with $x_1 = X_1(0)$,

$$p = \xi(T)\mathbb{E}_Q\left[f\left(x_1 \exp\Big(\int_0^T \beta(s)d\tilde{B}(s) + \int_0^T (\rho(s) - \frac{1}{2}\beta^2(s))\,ds\Big)\right)\right]$$

But we know by Itô isometry that the random variable $Y := \int_0^T \beta(s)d\tilde{B}(s)$ is normally distributed with mean 0 and variance $\delta^2 := \int_0^T \beta^2(t)\,dt$ and therefore we get

$$p = \frac{\xi(t)}{\delta\sqrt{2\pi}}\int_{\mathbb{R}} f\Big(x_1 \exp\Big[y + \int_0^T (\rho(s) - \frac{1}{2}\beta^2(s))ds\Big]\Big)\exp\big(-\frac{y^2}{2\delta^2}\big)dy$$

b) The portfolio we seek is by (36)

$$\theta_1(t,\omega) = X_0(t)(\beta X_1(t,\omega))^{-1}\phi(t,\omega)$$

where we have to be a bit careful to compute $\phi(t,\omega)$ using Corollary 7.6 with $h(y) = f(y)$ as we demonstrate

$$\begin{aligned}
\phi(t,\omega) &= \frac{\partial}{\partial x_1}\mathbb{E}_Q^{(x_0,x_1)}\Big[\xi(T-t)f(X_1(T-t))\Big]_{x=(X_0(t),X_1(t))}\beta X_1(t,\omega)\\
&= \frac{\partial}{\partial x_1}\Big(\frac{1}{x_0}e^{-\rho(T-t)}\mathbb{E}_Q^{(x_0,x_1)}\Big[f(X_1(T-t))\Big]_{x=(X_0(t),X_1(t))}\Big)\beta X_1(t,\omega)\\
&= \frac{\partial}{\partial x_1}\Big(\frac{1}{e^{\rho t}}e^{-\rho(T-t)}\mathbb{E}_Q^{x_1}\Big[f(X_1(T-t))\Big]_{x_1=X_1(t)}\Big)\beta X_1(t,\omega) \qquad (\text{ since } X_0(t) = e^{\rho t})\\
&= \frac{\partial}{\partial x_1}\mathbb{E}_Q^{x_1}\Big[e^{-\rho T}f(X_1(T-t))\Big]_{x_1=X_1(t)}\beta X_1(t,\omega)
\end{aligned}$$

and

$$X_1(t) = x_1\exp\{\beta\tilde{B}(t) + (\rho - \frac{1}{2}\beta^2)t\}$$

Hence

$$\begin{aligned}
\theta_1(t,\omega) &= e^{\rho t}(\beta X_1(t,\omega))^{-1}\frac{\partial}{\partial x_1}\mathbb{E}_Q^{x_1}\Big[e^{-\rho T}f(X_1(T-t))\Big]_{x_1=X_1(t)}\beta X_1(t,\omega)\\
&= e^{\rho(t-T)}\frac{\partial}{\partial x_1}\mathbb{E}\Big[f(x_1\exp\{\beta B(T-t) + (\rho - \frac{1}{2}\beta^2)(T-t)\})\Big]_{x_1=X_1(t)}\\
&= e^{\rho(t-T)}\frac{\partial}{\partial x_1}\mathbb{E}\Big[f'(x_1\exp\{\beta B(T-t) + (\rho - \frac{1}{2}\beta^2)(T-t)\})\\
&\qquad\cdot\exp\{\beta B(T-t) + (\rho - \frac{1}{2}\beta^2(T-t)\}\Big]_{x_1=X_1(t)}\\
&= \frac{1}{\sqrt{2\pi(T-t)}}\int_{\mathbb{R}} f'(X_1(t,\omega)\exp\{\beta x + (\rho - \frac{1}{2}\beta^2)(T-t)\})\\
&\qquad\cdot\exp\Big(\beta x - \frac{x^2}{2(T-t)} - \frac{1}{2}\beta^2(T-t)\Big)dx
\end{aligned}$$

□

Corollary 7.8. *(The classical Black-Scholes formula)*

a) *Suppose $X(t) = (X_0(t), X_1(t))$ is the classical Black-Scholes market*

$$dX_0(t) = \rho X_0(t); \qquad X_0(0) = 1 \tag{54}$$
$$dX_1(t) = \alpha\ X_1(t)dt + \beta X_1(t)dB(t); \qquad X_1(0) = x_1 > 0 \tag{55}$$

where $\rho, \alpha,$ and β are constant. Then the price of the European call option with payoff

$$F(\omega) = (X_1(T,\omega) - K)^+ \tag{56}$$

where $K > 0$ is a constant, is

$$p = x_1 \Phi(\eta + \frac{1}{2}\beta\sqrt{T}) - Ke^{-\rho T}\Phi(\eta - \frac{1}{2}\beta\sqrt{T}) \tag{57}$$

where

$$\eta = \beta^{-1}T^{-\frac{1}{2}}\Big(\ln\frac{x_1}{K} + \rho T\Big)$$

and

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}}x^2\,dx; \qquad y \in \mathbb{R} \tag{58}$$

is the standard normal distribution function.

b) *The replicating portfolio $\theta(t) = (\theta_0(t), \theta_1(t))$ for this claim F is given by*

$$\theta_1(t,\omega) = \Phi\Big(\beta^{-1}(T-t)^{-\frac{1}{2}}\Big(\ln\frac{X_1(t)}{K} + \rho(T-t) + \frac{1}{2}\beta^2(T-t)\Big)\Big) \tag{59}$$

with $\theta_0(t,\omega)$ determined by (21) and $V^\theta(0) = p$.

7.3 American option

The difference between European and American option is that in the latter case the buyer of the option is free to choose any exercise time τ before or at the given expiration time T (and the guaranteed payoff may depend on both τ and ω.) τ can depends on ω, but it has to be in a way that the decision to exercise before or at time t only depends on past information. The idea is that we can't use information from future time $t' > t$ to make our exercise decision at time t. Mathematically, it simply means that

$$\{\omega; \tau(\omega) \geq t\} \in \mathcal{F}_t^{(m)},$$

i.e. τ is a $\mathcal{F}_t^{(m)}$-stopping time.

Definition 7.9. *An american contingent T-claim is an $\mathcal{F}_t^{(m)}$-adapted, (t,ω)-measurable and a.s lower bounded continuous stochastic process $F(t) = F(t,\omega)$: $t \in [0,T], \omega \in \Omega$. An American option on such a claim $F(t,\omega)$ gives the owner the right (but not the obligation) to choose any stopping time $\tau(t,\omega) \geq 0$ as exercise time for the option, resulting in a payment $F(\tau(\omega),\omega)$ to the owner.*

Similarly as for the European options, one can show: (see Okesendal p.300 [2])

Lemma 7.10 (Buyer's price of the American option)**.** *Let $F(t) = F(t,\omega)$ be an American contingent claim and $p_A(F)$ be the maximal price a buyer is willing to pay to get the possibility to choose any stopping time $(t,\omega) \geq 0$ as exercise time for the option. Then*

$$p_A(F) = \sup \Big\{ y;\ \text{There exists an admissible portfolio } \varphi$$
$$\text{such that } V^{\varphi}_{-y}(\tau(\omega),\omega) := -y + \int_0^{\tau(\omega)} \varphi(s)\, dX(s) \geq -F(\tau(\omega),\omega) \text{ a.s } \Big\}$$

Lemma 7.11 (Seller's price of the American option)**.** *Let $F(t) = F(t,\omega)$ be an American contingent claim and $q_A(F)$ be the minimal price a seller is willing to receive to offer the possibility to choose any stopping time $(t,\omega) \geq 0$ as exercise time for the option. Then*

$$q_A(F) = \inf \Big\{ z;\ \text{There exists an admissible portfolio } \psi$$
$$\text{such that } V^{\psi}_{z}(t,\omega) := z + \int_0^{t} \psi(s)\, dX(s) \geq F(t,\omega) \text{ a.s } \Big\}$$

Theorem 7.12 (Pricing formula for American options)**.** ***a)*** *Suppose there exists a process $u \in \mathcal{V}^m(0,T)$ that satisfies (25) and let Q be the measure generated by u as in the statement of Theorem (3.4). Let $F(t) = F(t,\omega); t \in [0,T]$ be an American contingent T-claim such that*

$$\sup_{\tau \leq T} \mathbb{E}_Q[\xi(\tau)F(\tau)] < \infty$$

Then

$$p_A(F) \leq \sup_{\tau \leq T} \mathbb{E}_Q[\xi(\tau)F(\tau)] \leq q_A(F) \leq \infty$$

b) *Suppose in addition to the conditions in a) that the market $\{X(t)\}_{t\in[0,T]}$ is complete. Then*

$$p_A(F) = \sup_{\tau \leq T} \mathbb{E}_Q[\xi(\tau)F(\tau)] = q_A(F)$$

We conclude by giving an example with the calculation of the price of the American call. It corresponds to Exercise **12.14.** p.312 in Oksendal [2]

Example 7.1 (The American call)**.** Suppose $X(t) = (X_0(t), X_1(t))$ is the classical Black-Scholes market

$$dX_0(t) = \rho X_0(t); \qquad X_0(0) = 1 \tag{60}$$
$$dX_1(t) = \alpha\ X_1(t)dt + \beta X_1(t)dB(t); \qquad X_1(0) = x_1 > 0 \tag{61}$$

where ρ, α, and β are constant. If the American T-claim is given by

$$F(t,\omega) = (X_1(t,\omega) - K)^+, \qquad 0 \leq t \leq T,$$

then the corresponding option is called *the American call*.
According to Theorem 7.12 the price of an American call is given by

$$p_A(F) = \sup_{\tau \le T} \mathbb{E}_Q[e^{-\rho\tau}(X_1(\tau) - K)^+].$$

We will now prove that more precisely

$$p_A(F) = e^{-\rho T}\mathbb{E}_Q\big[\big(X_1(T) - K\big)^+\big],$$

i.e. that in this case it is always optimal to exercise the American call at the terminal time T, if at all.

Proof. Define

$$Y(t) \coloneqq e^{-\rho t}\big(X_1(t) - K\big).$$

We claim that $Y(t)$ is a Q-submartingale. First notice that the market is complete and that the process u that satisfies 25 is given by

$$\begin{aligned}\beta X_1 u &= \alpha X_1 - \rho X_1 \\ \Rightarrow u &= \frac{\alpha - \rho}{\beta}.\end{aligned}$$

We can thus define the corresponding equivalent martingale measure Q and the Brownian motion under Q, $\tilde{B}(t) \coloneqq B(t) + \int_0^t \frac{\alpha-\rho}{\beta}\, ds$. We thus get

$$\begin{aligned}X_1(t) &= x_1 \exp\Big\{(\alpha - \frac{\beta^2}{2})t + \beta B(t)\Big\} \\ &= x_1 \exp\Big\{(\rho - \frac{\beta^2}{2})t + \beta\tilde{B}(t)\Big\}\end{aligned}$$

Thus for any $t > s$

$$\begin{aligned}\mathbb{E}_Q\big[Y(t)|\mathcal{F}_s\big] &= \mathbb{E}_Q\big[\big(x_1 \exp\{-\frac{\beta^2}{2}t + \beta\tilde{B}(t)\} - Ke^{-\rho t}\big)|\mathcal{F}_s\big] \\ &= x_1 \exp\{-\frac{\beta^2}{2}s + \beta\tilde{B}(s)\} - Ke^{-\rho s} + Ke^{-\rho s} - Ke^{-\rho t} \\ &\ge Y(s)\end{aligned}$$

where we used in the second equality that $\big\{x_1 \exp\{-\frac{\beta^2}{2}t + \beta\tilde{B}(t)\}\big\}_{t\in[0,T]}$ is a martingale under Q (it satisfies Novikov condition). Thus $Y(t)$ is indeed a Q sub martingale and we can use Jensen inequality to show that $Z(t) \coloneqq e^{-\rho t}\big(X_1(t) - K\big)^+$ is also a Q-submartingale as

$$\begin{aligned}\mathbb{E}_Q[Z(t)|\mathcal{F}_s] \ge \mathbb{E}_Q[Y(t)|\mathcal{F}_s]^+ &\ge 0 \\ &\ge \mathbb{E}_Q[Y(t)|\mathcal{F}_s] \ge Y(S)\end{aligned}$$

Finally we can use Doob's optional sampling theorem on $Z(t)$ to conclude that

$$\mathbb{E}_Q[Z(t)] \ge \mathbb{E}_Q[Z_{T\wedge\tau}] = \mathbb{E}_Q[Z_\tau] \quad \text{for all stopping time } \tau \le T$$

and

$$p_A(F) = \sup_{\tau \le T} \mathbb{E}_Q[e^{-\rho\tau}(X_1(\tau) - K)^+] = e^{-\rho T}\mathbb{E}_Q\big[\big(X_1(T) - K\big)^+\big]$$

$\square$

References

[1] Ioannis Karatzas, Steven E Shreve, I Karatzas, and Steven E Shreve. *Methods of mathematical finance*, volume 39. Springer, 1998.

[2] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.