Resource-Bounded Type Theory: Compositional Cost Analysis via Graded Modalities

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Abstract

We present a compositional framework for certifying resource bounds in typed programs. Terms are typed with synthesized bounds b drawn from an abstract resource lattice $(L, \preceq, \oplus, \bot)$, enabling uniform treatment of time, memory, gas, and domain-specific costs. We introduce a graded feasibility modality $\Box_r$ with counit and monotonicity laws. Our main result is a syntactic cost soundness theorem: if a closed term has synthesized bound b under budget r, its operational cost is bounded by b. We provide a cost-indexed semantic model in the presheaf topos $\mathbf{Set}^L$ where types are functors from bounds to value-cost pairs and the cost function is a natural transformation to the internal lattice. We prove constructive completeness via reification. A case study demonstrates end-to-end reasoning for binary search. We discuss integration with Lean 4 via structural cost proxies and CI regression testing.

1 Purpose: Why Resource-Bounded Type Systems?

Formal methods typically certify *what holds*, not *what it costs*. In many settings—safety-critical controllers, on-device ML, blockchain contracts, and time-bounded autonomous agents—a proof without a resource budget is insufficient: a correct component that exceeds its resource envelope is unusable. We propose a type system where resource boundedness is first-class: typing judgments synthesize compositional cost bounds alongside correctness proofs.

Goal (compositional resource bounds). For a typed term t (program/proof) and a resource budget $r \in L$, we synthesize a bound b via typing rules such that:

$$\Gamma \vdash_{r;\,\mathsf{b}} t : A$$

The synthesized bound b is checked against the budget ($\mathsf{b} \preceq r$) and over-approximates operational cost. We prove this connection syntactically (Theorem 2.9) and validate it semantically via a categorical model in $\mathbf{Set}^L$ (§3).

1.1 Motivating Examples

Time-critical systems: Automotive braking must compute correct results within 10 ms.

Memory-constrained devices: Mobile inference must fit within 4 GB.

Blockchain contracts: Smart contracts must not exceed gas limits.

Energy budgets: Embedded systems have strict power envelopes.

Composite constraints: Real-time systems may require bounds on (time × memory × depth) as a single structured resource.

1.2 Our Approach

We present a type system with:

1. An *abstract resource lattice* $(L, \preceq, \oplus, \bot)$ as the foundational order structure.
2. A *compositional bound-synthesis discipline* where typing rules compose costs structurally.
3. A *feasibility modality* $\Box_r$ that certifies resource compliance.
4. A *syntactic cost soundness theorem* proved by induction on typing derivations.
5. A *categorical semantic model* in $\mathbf{Set}^L$ with natural transformations.
6. A *constructive completeness theorem* via reification.

1.3 Key Contribution: Abstraction

By parameterizing over an arbitrary lattice, we avoid premature commitment to specific dimensions and their interaction semantics (e.g., how do time and memory compose?). A single formal development accommodates all concrete instantiations—time, memory, gas, multi-dimensional tuples, or custom domains—without repetition.

1.4 Contributions

1. A compositional calculus for resource-bounded terms over abstract lattices (§2.2).
2. Type soundness theorems: preservation, progress, and **cost soundness** for the simply-typed fragment (§2.5).
3. A categorical semantic model in $\mathbf{Set}^L$ where types are presheaves and cost is a natural transformation (§3).
4. **Constructive completeness** via reification (§3.8).
5. A graded modality $\Box_r$ for feasibility certification (§2.6).
6. A case study (binary search) demonstrating compositional reasoning (§7).
7. Instantiation examples: time-only, multi-dimensional, gas budgets (§3.9).
8. Engineering integration patterns for Lean 4 (§5).

1.5 Scope of the Formalization

The formal development in this paper covers the *simply-typed lambda calculus* (STLC) extended with resource bounds. Our calculus includes:

- **Base types:** Booleans, natural numbers
- **Type constructors:** Products $(A \times B)$, functions $(A \to B)$
- **Graded modality:** $\Box_r A$ for resource certification

We prove type soundness, cost soundness (Theorem 2.9), semantic soundness (Theorem 3.12), and constructive completeness (Theorem 3.14) for this simply-typed fragment.

Extensions beyond simple types. Section 4 sketches extensions to dependent types, resource-indexed universes, and size-dependent recursion, but these are *not formalized* in the current paper. Examples like binary search (§7) use Lean's native termination checking with bounds proven as separate theorems, illustrating the pattern of resource-bounded reasoning but not providing full synthesis of size-indexed bounds within the type system.

Structurally-decreasing recursion. We include a rule for structurally-decreasing recursion (depth-bounded, independent of input size). Size-dependent cost analysis (e.g., $O(n \log n)$ for sorting) requires dependent types and is future work.

1.6 Design Principles

P1. Abstraction over concreteness. Resources are elements of an abstract lattice; no hardcoded dimensions.

P2. Budgets are first-class. Resource bounds appear in types and judgments.

P3. Compositional rules. Bound synthesis mirrors term structure, independent of lattice instantiation.

P4. Monotonicity. If $r_1 \preceq r_2$, then $\Box_{r_1} A \to \Box_{r_2} A$ (weakening).

P5. Categorical semantics. Types are presheaves in $\mathbf{Set}^L$ with natural cost extraction.

Notation. We write L for an abstract resource lattice; $r, \rho, \sigma \in L$ are resource elements; $\preceq$ is the lattice order; $\oplus$ is the join (least upper bound or monoidal operation); $\bot$ is the bottom element (zero resource); $\Box_r A$ denotes feasibility at budget r.

2 A Minimal Calculus for Resource-Bounded Types

2.1 Abstract Resource Lattices

Definition 2.1 (Resource lattice)**.** *A* resource lattice *is a structure* $L = (L, \preceq, \oplus, \bot)$ *where:*

- $(L, \preceq)$ *is a partially ordered set (poset).*
- $\oplus : L \times L \to L$ *is a binary operation (join or monoidal sum).*
- $\bot \in L$ *is the least element (zero resource).*
- $\oplus$ *is associative, commutative, and monotone in both arguments.*
- $\bot$ *is the identity:* $r \oplus \bot = r$ *for all* $r \in L$*.*
- *(Optional)* L *is distributive, forms a lattice with* $\sqcap$*, or satisfies other algebraic properties.*

Examples.

1. **Single-dimension (time):** $L = \mathbb{N}$ with $\preceq = \leq$, $\oplus = +$, $\bot = 0$.
2. **Multi-dimensional:** $L = \mathbb{N}^3$ (time, memory, depth) with pointwise order and componentwise operations:
$$(\mathsf{t}, \mathsf{m}, \mathsf{d}) \oplus (\mathsf{t}', \mathsf{m}', \mathsf{d}') = (\mathsf{t} + \mathsf{t}', \mathsf{m} + \mathsf{m}', \max(\mathsf{d}, \mathsf{d}')).$$
3. **Power sets:** $L = 2^D$ (set of resources, D a domain) with $\preceq = \subseteq$, $\oplus = \cup$, $\bot = \emptyset$.

4. **Blockchain gas:** $L = \mathbb{N}$ with $\preceq = \leq$, $\oplus = +$, $\bot = 0$ (identical to time, but interpreted as gas units).

5. **Custom lattices:** Any structure satisfying the axioms; e.g., lattices of energy levels, bandwidth capacities, or domain-specific constraints.

2.2 Core Typing Rules

Judgments carry a resource context and a synthesized bound:

$$\Gamma \vdash_{r;\,\mathsf{b}} t : A.$$

Here $r \in L$ is the available resource budget, and $\mathsf{b} \in L$ is the synthesized bound (to be checked as $\mathsf{b} \preceq r$). The core typing rules are shown in Fig. 1.

$$\Gamma, x : A \vdash_{r;\,\bot} x : A \quad \text{(Var)}$$

$$[0.5em]\frac{\Gamma, x : A \vdash_{r;\,b} t : B}{\Gamma \vdash_{r;\,b} \lambda x.t : A \to B} \quad \text{(Lam)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b_f} f : A \to B \qquad \Gamma \vdash_{r;\,b_a} a : A}{\Gamma \vdash_{r;\,b_f \oplus b_a \oplus \delta_{\text{app}}} f\ a : B} \quad \text{(App)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b_a} a : A \qquad \Gamma \vdash_{r;\,b_b} b : B}{\Gamma \vdash_{r;\,b_a \oplus b_b} (a, b) : A \times B} \quad \text{(Pair)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b_c} c : \mathsf{Bool} \qquad \Gamma \vdash_{r;\,b_t} t : A \qquad \Gamma \vdash_{r;\,b_f} f : A}{\Gamma \vdash_{r;\,b_c \oplus (b_t \sqcup b_f) \oplus \delta_{\text{if}}} \textbf{if } c \textbf{ then } t \textbf{ else } f : A} \quad \text{(If)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b} t : A \quad b \preceq r}{\Gamma \vdash_{r;\,b} \text{box}_r(t) : \Box_r A} \quad \text{(Box)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b} t : \Box_{r'} A}{\Gamma \vdash_{r;\,b \oplus \delta_{\text{unbox}}} \text{unbox}(t) : A} \quad \text{(Unbox)}$$

$$[0.5em]\frac{\Gamma \vdash_{r;\,b} t : \Box_{r_1} A \quad r_1 \preceq r_2}{\Gamma \vdash_{r;\,b} t : \Box_{r_2} A} \quad \text{(Monotone)}$$

Figure 1: Core typing rules for STLC with resource bounds over abstract lattice L. The cost of application, conditional, and unboxing are $\delta_{\text{app}}, \delta_{\text{if}}, \delta_{\text{unbox}} \in L$ (lattice-dependent constants). The rule (Box) requires the synthesized bound b to be within the budget r. The (Lam) rule uses *latent cost abstraction*: the function carries its body cost b.

Notes on composition. In concrete instantiations (e.g., $L = \mathbb{N}$), $\sqcup$ becomes max and $\oplus$ becomes $+$. The cost of application is typically $\delta_{\text{app}} = 1$ step; the cost of conditionals is typically $\delta_{\text{if}} = 1$ per branch decision; unboxing costs $\delta_{\text{unbox}} = 1$.

Latent cost abstraction. The (Lam) rule does not add overhead for constructing the closure. Instead, the function "carries" its body cost b, which is paid when the function is applied. This design simplifies the cost soundness proof (Theorem 2.9).

Recursion (Structurally-Decreasing Case). For structurally-decreasing recursion (e.g., tree traversal), we bound the depth:

$$\frac{\Gamma, f{:}A \to B, x{:}A \vdash_{r;\, b} t : B}{\Gamma \vdash_{r;\, \mathrm{depth}(r) \oplus b} \mathsf{fix}\ f.\lambda x.t : A \to B} \quad \text{(Rec)}$$

Here $\mathrm{depth}(r) \in L$ extracts or approximates the recursion depth from r (lattice-dependent). This rule applies when the cost b is provably independent of input size.

Remark 2.2 (Recursion: Current Approach)**.** *The rule (Rec) handles only structurally-decreasing recursion where the cost b does not depend on input size. For size-dependent costs (e.g., sorting, searching):*

- *In Lean: Use well-founded recursion with* `termination_by`
- *Bound lemma: Proved separately (e.g.,* `binarySearch_cost_bound`*)*
- *Composition: Bound lemmas are used in subsequent typing derivations*

Example:

```
theorem binarySearch_bound (arr : Array ) (target : ) :
  cost (binarySearch arr target)  log arr.size + 5 := by
  -- manual proof
```

A dependent typing rule that internalizes size-indexed bounds is future work (§8). For now, separate lemmas suffice for examples (§7).

2.3 Reading the Typing Judgment: Plain English

The typing judgment $\Gamma \vdash_{r;\, \mathsf{b}} t : A$ may appear cryptic at first glance. We provide an intuitive reading for practitioners and students.

Component breakdown.

- Γ **(context):** A list of assumptions about variables. Think of it as "what we already know" or the environment where variables are declared. Example: $x : \mathsf{Nat}, y : \mathsf{Bool}$ means "we know x is a natural number and y is a boolean."
- $\vdash$ **(turnstile):** The judgment operator. Read as "proves that" or "types as." In programming contexts: "the type checker accepts that..."
- r **(budget):** The maximum resources we are allowed to spend. Example: if $r = 100$, we have 100 time steps (or memory units, or gas) available. Think of this as your bank account balance before shopping.
- b **(bound):** The synthesized cost of running the term—how much the term actually costs. Example: if $\mathsf{b} = 37$, the term costs 37 resources to execute. Think of this as the price tag on the item you're buying. We require $\mathsf{b} \preceq r$ (the cost must fit within the budget).
- t **(term):** The program, expression, or proof we're analyzing.
- A **(type):** The type of the term—what kind of value it produces. Example: $\mathsf{Nat} \to \mathsf{Nat}$ is a function from numbers to numbers.

Full reading. The judgment $\Gamma \vdash_{r;\mathsf{b}} t : A$ reads:

> "Under context Γ, with budget r, the term t has type A and synthesized cost bound b."

More colloquially: "Given the assumptions in Γ, with a resource budget of r, the term t has type A and costs b resources to execute (where $\mathsf{b} \leq r$)."

Why both r and b? One is a *limit* (budget r), the other is a *measurement* (bound b). The budget sets a hard limit on resources (like going shopping with \$100 in your wallet). The bound tells us the actual cost (the items you buy cost \$73). The check $\mathsf{b} \preceq r$ verifies the program fits within the budget (\$73 $\leq$ \$100).

The three-level picture. Understanding resource bounds requires seeing three levels:

1. **Budget (r):** What you allocate or claim (upper limit)
2. **Synthesized bound (b):** What the typing rules compute (structural analysis)
3. **Actual cost (k):** What operationally happens when you run the term (real consumption)

The typing judgment establishes $k \preceq \mathsf{b} \preceq r$, where k is proven by Theorem 2.9.

2.4 Operational Semantics

We define a big-step call-by-value operational semantics with explicit cost accounting.

Values. The set of values Val is defined by:

$$v \in \mathsf{Val} ::= \lambda x{:}A.\, t \mid \langle v_1, v_2 \rangle \mid \mathsf{tt} \mid \mathsf{ff} \mid \mathrm{box}_r(v)$$

Big-step evaluation with cost. We write $t \Downarrow v \triangleright k$ for "t evaluates to value v with cost $k \in L$." The judgment is defined inductively by the rules in Fig. 2.

Lemma 2.3 (Determinism of evaluation)**.** *If $t \Downarrow v_1 \triangleright k_1$ and $t \Downarrow v_2 \triangleright k_2$, then $v_1 = v_2$ and $k_1 = k_2$.*

Proof. By induction on the derivation of $t \Downarrow v_1 \triangleright k_1$. □

2.5 Metatheory: Type and Cost Soundness

We establish soundness properties for the simply-typed fragment.

2.5.1 Substitution Lemmas

Lemma 2.4 (Typing substitution)**.** *If $\Gamma, x{:}A \vdash_{r;b} t : B$ and $\Gamma \vdash_{r;b_v} v : A$ where $v \in \mathsf{Val}$, then $\Gamma \vdash_{r;b} t[x := v] : B$.*

Proof. By induction on the typing derivation of $\Gamma, x{:}A \vdash_{r;b} t : B$. The base case (Var) either replaces x with v (taking v's type A and bound $b_v \preceq b$ by weakening) or leaves another variable unchanged. The inductive cases follow from the IH and the fact that substitution commutes with term constructors. □

Lemma 2.5 (Cost substitution)**.** *If $\Gamma, x{:}A \vdash_{r;b} t : B$ and $\Gamma \vdash_{r;b_v} v : A$ where $v \in \mathsf{Val}$, then there exist $w \in \mathsf{Val}$ and $k \in L$ such that $t[x := v] \Downarrow w \triangleright k$ with $k \preceq b$.*

$$\frac{}{v \Downarrow v \triangleright \bot} \quad (\mathsf{Val})$$

$$[0.5em]\frac{t \Downarrow v \triangleright k_1 \qquad u \Downarrow w \triangleright k_2}{\langle t, u\rangle \Downarrow \langle v, w\rangle \triangleright k_1 \oplus k_2} \quad (\text{Pair})$$

$$[0.5em]\frac{t \Downarrow \langle v, w\rangle \triangleright k}{\pi_1 t \Downarrow v \triangleright k \oplus \delta_\pi} \quad (\text{Fst}) \qquad \frac{t \Downarrow \langle v, w\rangle \triangleright k}{\pi_2 t \Downarrow w \triangleright k \oplus \delta_\pi} \quad (\text{Snd})$$

$$[0.5em]\frac{c \Downarrow \mathsf{tt} \triangleright k_c \qquad t \Downarrow v \triangleright k_t}{\textbf{if } c \textbf{ then } t \textbf{ else } u \Downarrow v \triangleright k_c \oplus k_t \oplus \delta_{\text{if}}} \quad (\text{IfT})$$

$$[0.5em]\frac{c \Downarrow \mathsf{ff} \triangleright k_c \qquad u \Downarrow v \triangleright k_u}{\textbf{if } c \textbf{ then } t \textbf{ else } u \Downarrow v \triangleright k_c \oplus k_u \oplus \delta_{\text{if}}} \quad (\text{IfF})$$

$$[0.5em]\frac{f \Downarrow \lambda x.t \triangleright k_f \qquad a \Downarrow v \triangleright k_a \qquad t[x := v] \Downarrow w \triangleright k_b}{f\ a \Downarrow w \triangleright k_f \oplus k_a \oplus \delta_{\text{app}} \oplus k_b} \quad (\text{App})$$

$$[0.5em]\frac{t \Downarrow v \triangleright k}{\text{box}_r(t) \Downarrow \text{box}_r(v) \triangleright k} \quad (\text{Box}) \qquad \frac{t \Downarrow \text{box}_{r'}(v) \triangleright k}{\text{unbox}(t) \Downarrow v \triangleright k \oplus \delta_{\text{unbox}}} \quad (\text{Unbox})$$

Figure 2: Big-step evaluation with cost. Constants $\delta_\pi, \delta_{\text{if}}, \delta_{\text{app}}, \delta_{\text{unbox}} \in L$ represent per-operation costs.

Proof. By induction on the typing derivation of $\Gamma, x{:}A \vdash_{r;\,b} t : B$.

- **Var:** If $t = x$, then $t[x := v] = v \in \mathsf{Val}$, so $v \Downarrow v \triangleright \bot$ and $\bot \preceq b$.
- **Lam:** If $t = \lambda y.s$, then $t[x := v] = \lambda y.s[x := v]$ is already a value, evaluating with cost $\bot \preceq b$.
- **App, Pair, If, etc.:** Apply IH to sub-terms and use the evaluation rules; costs compose according to the typing bounds.

□

Lemma 2.6 (Budget weakening)**.** *If $\emptyset \vdash_{r_1;\,b} t : A$ and $r_1 \preceq r_2$, then $\emptyset \vdash_{r_2;\,b} t : A$.*

Proof. By induction on the typing derivation. All rules preserve the bound b while allowing the budget to increase. The only place the budget appears in the rules is:

- (Box): Requires $b \preceq r$. If $b \preceq r_1 \preceq r_2$, then $b \preceq r_2$ by transitivity.
- All other rules: Budget r appears uniformly in premises and conclusion.

Therefore the derivation can be lifted to the larger budget r_2. □

2.5.2 Type Soundness

Theorem 2.7 (Preservation)**.** *If $\Gamma \vdash_{r;\,\mathsf{b}} t : A$ and $t \Downarrow v \triangleright k$, then $\Gamma \vdash_{r;\,\mathsf{b}'} v : A$ for some $\mathsf{b}' \preceq \mathsf{b}$.*

Proof. By induction on the evaluation derivation $t \Downarrow v \triangleright k$, using Lemma 2.4 in the (App) case. □

Theorem 2.8 (Progress)**.** *If $\emptyset \vdash_{r;\,\mathsf{b}} t : A$, then either $t \in \mathsf{Val}$ or there exist v and k such that $t \Downarrow v \triangleright k$.*

Proof. By induction on the typing derivation. Canonical forms ensure that closed, well-typed non-values can step. □

2.5.3 Cost Soundness

Theorem 2.9 (Cost soundness)**.** *If $\emptyset \vdash_{r;\,\mathsf{b}} t : A$, then there exist $v \in \mathsf{Val}$ and $k \in L$ such that $t \Downarrow v \triangleright k$ with $k \preceq \mathsf{b} \preceq r$.*

Proof. By induction on the typing derivation of $\emptyset \vdash_{r;\,\mathsf{b}} t : A$. We show that the operational cost k is bounded by the synthesized bound b.

Base cases:

- **Var:** Contradicts $\emptyset$ (no variables in empty context).
- **Lam:** $t = \lambda x.s$ is already a value. By rule (Val), $t \Downarrow t \triangleright \bot$, and $\bot \preceq b$ holds trivially.

Inductive cases:

- **App:** $t = f\,a$ with typing derivation:

$$\frac{\emptyset \vdash_{r;\,b_f} f : A \to B \qquad \emptyset \vdash_{r;\,b_a} a : A}{\emptyset \vdash_{r;\,b_f \oplus b_a \oplus \delta_{\mathrm{app}}} f\,a : B}$$

  By IH on f, there exist $\lambda x.s \in \mathsf{Val}$ and $k_f \in L$ with $f \Downarrow \lambda x.s \triangleright k_f$ and $k_f \preceq b_f$.

  By IH on a, there exist $v \in \mathsf{Val}$ and $k_a \in L$ with $a \Downarrow v \triangleright k_a$ and $k_a \preceq b_a$.

  By the typing derivation of $\lambda x.s$, we have $x{:}A \vdash_{r;\,b_f} s : B$ (using latent cost abstraction: the body bound equals the function bound).

  By Lemma 2.5, $s[x := v] \Downarrow w \triangleright k_b$ for some $w \in \mathsf{Val}$ and $k_b \preceq b_f$.

  By rule (App), $f\,a \Downarrow w \triangleright k_f \oplus k_a \oplus \delta_{\mathrm{app}} \oplus k_b$.

  Since $k_f \preceq b_f$, $k_a \preceq b_a$, and $k_b \preceq b_f$, and $\oplus$ is monotone:

$$k_f \oplus k_a \oplus \delta_{\mathrm{app}} \oplus k_b \preceq b_f \oplus b_a \oplus \delta_{\mathrm{app}} = \mathsf{b}$$

- Other cases (Pair, Projection, If, Box, Unbox, Monotone) follow similarly by applying the IH and using monotonicity of $\oplus$.

In all cases, we have established $t \Downarrow v \triangleright k$ with $k \preceq \mathsf{b}$. For terms typed under (Box), we additionally have $\mathsf{b} \preceq r$ from the side condition, yielding $k \preceq \mathsf{b} \preceq r$. □

Corollary 2.10 (Operational cost bounded by budget)**.** *If $\emptyset \vdash_{r;\,\mathsf{b}} \mathrm{box}_r(t) : \Box_r A$, then t evaluates with cost $k \preceq r$.*

Proof. By Theorem 2.9, $\mathrm{box}_r(t) \Downarrow \mathrm{box}_r(v) \triangleright k$ with $k \preceq \mathsf{b} \preceq r$ (using the side condition of (Box)). □

Remark 2.11 (Syntactic proof)**.** *The proof of Theorem 2.9 proceeds entirely by induction on typing derivations. No semantic model is required for this result. The theorem is syntactically verifiable and can be mechanized in a proof assistant by straightforward structural recursion on typing derivations.*

2.6 Feasibility Modality as Graded Interior

We treat $\Box_r$ as a *graded modality* indexed by $r \in L$. The modality $\Box_r A$ represents "a computation of type A certified to consume at most r resources."

Core laws.

$$\Box_r A \to A \quad (\textit{counit } \varepsilon) \qquad \frac{r_1 \preceq r_2}{\Box_{r_1} A \to \Box_{r_2} A} \quad (\textit{monotonicity})$$

The counit law says: if you have a certified computation, you can execute it (via unbox). Monotonicity says: if a computation is certified for budget r_1, it also fits within any larger budget $r_2 \succeq r_1$.

Cost-aware boxing (introduction). Promotion is conditional:

$$\text{If } \Gamma \vdash_{r;\, b} t : A \text{ with } b \preceq r, \text{ then } \mathrm{box}_r(t) : \Box_r A.$$

You can only obtain $\Box_r A$ by constructing A within budget r; there is no unconditional rule $A \to \Box_r A$.

Remark 2.12 (Why not comultiplication?)**.** *Graded comonads in category theory typically include a comultiplication law:*

$$\Box_{r_1 \oplus r_2} A \to \Box_{r_1} \Box_{r_2} A$$

This would represent staged computation*: factoring a computation into an outer part (budget r_1) that produces a suspended inner part (budget r_2). Since our examples involve monolithic recursive algorithms rather than staged evaluation, we omit comultiplication as a core law. Extensions that require staging (e.g., lazy evaluation, thunks) can add it as needed.*

3 Categorical Semantic Model in $\mathbf{Set}^L$

We provide a semantic interpretation in the presheaf topos $\mathbf{Set}^L$, where types are functors from resource bounds to sets of value-cost pairs. The resource lattice L is internalized as a presheaf, and the cost function is a natural transformation. This validates that the typing rules compose correctly with respect to operational costs.

3.1 The Presheaf Topos $\mathbf{Set}^L$

Definition 3.1 (The topos of presheaves on L)**.** *View the resource lattice $(L, \preceq)$ as a category:*

- *Objects: elements $r \in L$*
- *Morphisms: $r_1 \to r_2$ exists iff $r_1 \preceq r_2$*
- *Composition: transitivity of $\preceq$*

The presheaf topos $\mathbf{Set}^L$ has:

- *Objects: functors $F : L \to \mathbf{Set}$ (covariant presheaves)*
- *Morphisms: natural transformations $\alpha : F \Rightarrow G$*

For a presheaf $F : L \to \mathbf{Set}$:

- *Each resource bound $r \in L$ gives a set $F(r)$*

- *Each $r_1 \preceq r_2$ gives a transition map $F(r_1 \preceq r_2) : F(r_1) \to F(r_2)$*
- *Functoriality: $F(\mathrm{id}_r) = \mathrm{id}_{F(r)}$ and $F(r_2 \preceq r_3) \circ F(r_1 \preceq r_2) = F(r_1 \preceq r_3)$*

Remark 3.2 (Covariant vs. contravariant)**.** *We use covariant presheaves (functors $L \to \mathbf{Set}$, not $L^{op} \to \mathbf{Set}$) because resource bounds are cumulative: if a computation fits within budget r_1, it also fits within any larger budget $r_2 \succeq r_1$. This monotonicity is captured by the transition maps $F(r_1) \to F(r_2)$.*

3.2 The Internal Lattice $\mathbb{L}$

Definition 3.3 (The downset presheaf)**.** *Define the presheaf $\mathbb{L} : L \to \mathbf{Set}$ by:*

$$\mathbb{L}(r) = {\downarrow} r = \{a \in L \mid a \preceq r\}$$

For $r_1 \preceq r_2$, the transition map $\mathbb{L}(r_1 \preceq r_2) : \mathbb{L}(r_1) \to \mathbb{L}(r_2)$ is the inclusion:

$$\iota_{r_1, r_2}(a) = a \quad \text{for } a \in \mathbb{L}(r_1)$$

This is well-defined because $a \preceq r_1 \preceq r_2$ implies $a \preceq r_2$ by transitivity, so $a \in \mathbb{L}(r_2)$.

Lemma 3.4 (Functoriality of $\mathbb{L}$)**.** *$\mathbb{L}$ is a functor from L to $\mathbf{Set}$.*

Proof. Identity and composition hold because the maps are inclusions. □

Remark 3.5 (Internal lattice structure)**.** *The presheaf $\mathbb{L}$ internalizes the resource lattice L in the topos $\mathbf{Set}^L$. Elements of $\mathbb{L}(r)$ are costs that fit within budget r. The transition maps reflect monotonicity: expanding the budget enlarges the set of affordable costs.*

3.3 Internal Monoidal Structure

Definition 3.6 (Internal operations on $\mathbb{L}$)**.** *The lattice operations $\oplus$ and $\bot$ on L induce natural transformations:*

Internal join: $\widetilde{\oplus} : \mathbb{L} \times \mathbb{L} \Rightarrow \mathbb{L}$ *with components:*

$$\widetilde{\oplus}_r : \mathbb{L}(r) \times \mathbb{L}(r) \to \mathbb{L}(r), \quad (a, b) \mapsto a \oplus b$$

This is well-defined because $a, b \preceq r$ and monotonicity of $\oplus$ imply $a \oplus b \preceq r$.

Internal unit: $\widetilde{\bot} : \mathbf{1} \Rightarrow \mathbb{L}$ *where $\mathbf{1}$ is the terminal presheaf ($\mathbf{1}(r) = \{*\}$ for all r), with components:*

$$\widetilde{\bot}_r : \{*\} \to \mathbb{L}(r), \quad * \mapsto \bot$$

This is well-defined because $\bot \preceq r$ for all r (bottom element).

Lemma 3.7 (Naturality of internal operations)**.** *$\widetilde{\oplus}$ and $\widetilde{\bot}$ are natural transformations.*

Proof. Straightforward check of commutativity of the naturality squares; both are based on inclusions and the algebraic laws of $\oplus$ and $\bot$. □

3.4 Type Interpretation as Presheaves

Definition 3.8 (Cost-indexed type interpretation)**.** *For each type A, we define a presheaf $[\![A]\!] : L \to \mathbf{Set}$ where*

$$[\![A]\!](r) = \{(v, k) \in \mathsf{Val} \times L \mid v \in \mathsf{Val}_A,\ k \preceq r,\ \exists r' \succeq r.\, \emptyset \vdash_{r';\, k} v : A,\ v \Downarrow v \triangleright k\}.$$

For $r_1 \preceq r_2$, the transition map is inclusion:

$$[\![A]\!](r_1 \preceq r_2) : (v, k) \mapsto (v, k).$$

Remark 3.9. *Elements of $[\![A]\!](r)$ are value–cost pairs where the cost fits within budget r and is justified by a typing derivation and evaluation.*

3.5 The Cost Natural Transformation

Definition 3.10 (Cost as a natural transformation)**.** *The cost function is a natural transformation*

$$\mathsf{cost} : [\![A]\!] \Rightarrow \mathbb{L}$$

with components

$$\mathsf{cost}_r : [\![A]\!](r) \to \mathbb{L}(r), \quad (v, k) \mapsto k.$$

Theorem 3.11 (Naturality of cost)**.** cost *is a natural transformation in* $\mathbf{Set}^L$*.*

Proof. For $r_1 \preceq r_2$, both paths in the naturality square send (v, k) to k, by inclusion on both $[\![A]\!]$ and $\mathbb{L}$. □

3.6 Type Constructors in the Topos

We only sketch the main constructions; details follow standard presheaf semantics.

Base types.

$$[\![\mathsf{Bool}]\!](r) = \{(\mathsf{tt}, \bot), (\mathsf{ff}, \bot)\}, \qquad [\![\mathsf{Nat}]\!](r) = \{(n, \bot) \mid n \in \mathbb{N}\}.$$

Products.

$$[\![A \times B]\!](r) = \{(\langle v_1, v_2\rangle, k_1 \oplus k_2) \mid (v_1, k_1) \in [\![A]\!](r_1),\ (v_2, k_2) \in [\![B]\!](r_2),\ k_1 \oplus k_2 \preceq r\}.$$

Functions. Informally,

$[\![A \to B]\!](r) = \{(\lambda x.t, k_{\text{body}}) \mid$ applying the function to any $(v, k_a) \in [\![A]\!]$ yields $(w, k_b) \in [\![B]\!]$ with $k_a \oplus k_b \oplus \delta_{\text{app}}$

Box. For $s \in L$,

$$[\![\Box_s A]\!](r) = \{(\text{box}_s(v), k) \mid (v, k) \in [\![A]\!](r),\ k \preceq s\}.$$

3.7 Semantic Soundness

Theorem 3.12 (Semantic soundness)**.** *If $\emptyset \vdash_{r;\, b} t : A$ and $t \Downarrow v \triangleright k$, then*

1. *$k \preceq b$ (by Theorem 2.9);*
2. *$(v, k) \in [\![A]\!](b)$;*
3. *$\mathsf{cost}_b(v, k) = k$.*

Proof. (1) is Theorem 2.9. For (2), we use the definition of $[\![A]\!]$ plus the typing derivation and evaluation; for (3) we use Definition 3.10. □

3.8 Constructive Completeness via Reification

Definition 3.13 (Reification function)**.** *Reification* reify_A *is defined by structural recursion on* A*:*

$$\begin{aligned}
\mathsf{reify}_{\mathsf{Bool}}((v, 0)) &= v, \\
\mathsf{reify}_{A\times B}((\langle v_1, v_2\rangle, k_1 \oplus k_2)) &= \langle \mathsf{reify}_A(v_1, k_1), \mathsf{reify}_B(v_2, k_2)\rangle, \\
\mathsf{reify}_{A\to B}((\lambda x.t, k)) &= \lambda x.t, \\
\mathsf{reify}_{\Box_s A}((\mathrm{box}_s(v), k)) &= \mathrm{box}_s(\mathsf{reify}_A(v, k)).
\end{aligned}$$

Theorem 3.14 (Constructive completeness)**.** *For all types* A*, bounds* $b \in L$*, and* $(v, k) \in [\![A]\!](b)$*:*

1. *There exists* $r \succeq k$ *with* $\emptyset \vdash_{r;\,k} \mathsf{reify}_A(v, k) : A$*;*
2. $\mathsf{reify}_A(v, k) \Downarrow v \triangleright k$*;*
3. $k \preceq b$*.*

Moreover the derivation and evaluation proof are obtained by structural recursion on A*.*

Proof. By induction on the structure of A, using the presheaf definitions and the typing rules. The crucial step is that each semantic constructor mirrors a syntactic constructor. □

Corollary 3.15 (Full adequacy)**.** *The presheaf model in* $\mathbf{Set}^L$ *is sound and complete for the calculus:*

- *Soundness: Theorem 3.12;*
- *Completeness: Theorem 3.14.*

3.9 Concrete Instantiations

3.9.1 Single-Dimension: Time Only

$L = \mathbb{N}$, $\preceq = \leq$, $\oplus = +$, $\bot = 0$. Then Theorem 2.9 reads $k \leq b \leq r$.

3.9.2 Multi-Dimensional: (Time, Memory, Depth)

$L = \mathbb{N}^3$ with pointwise order and

$$(\mathsf{t}, \mathsf{m}, \mathsf{d}) \oplus (\mathsf{t}', \mathsf{m}', \mathsf{d}') = (\mathsf{t} + \mathsf{t}', \mathsf{m} + \mathsf{m}', \max(\mathsf{d}, \mathsf{d}')).$$

3.9.3 Gas (Blockchain)

Same structure as time ($L = \mathbb{N}$), interpreted as gas units; only the constants $\delta\_$ change.

4 Dependent Layer (Sketch)

We sketch extensions; no proofs in this paper.

4.1 Resource-Indexed Universes

We postulate universes $\mathcal{U}_r$ for each $r \in L$, with cumulativity $\mathcal{U}_{r_1} \subseteq \mathcal{U}_{r_2}$ if $r_1 \preceq r_2$, and a complexity measure $\mathsf{complexity}$ controlling membership.

4.2 Dependent Recursion

Size-dependent bounds (e.g. $O(n \log n)$) require dependent products and suprema $\bigsqcup_x b(x)$ in the lattice, which we leave to future work. In the current paper, such bounds appear as separate lemmas rather than synthesized types.

4.3 Identity Types and Equality

Extending the resource model to identity types and higher structure (paths, univalence) is left open; transport and equivalences would need explicit cost accounting.

5 Engineering in Lean 4

We consider two roles for Lean 4:

5.1 Deep Embedding: Mechanizing the Theory

Here Lean is used to formalize the calculus, operational semantics, and theorems.

```
-- Syntax (sketch)
inductive Term where
  | var : Nat  Term
  | lam : Term  Term
  | app : Term  Term  Term
  | box : Resource  Term  Term
  -- ...

-- Typing judgments
inductive HasType :
    Context  Resource  Resource  Term  Type  Prop where
  | var : ...
  | lam : ...
  | app : ...
  | box : ...
  -- ...

-- Big-step evaluation with cost
inductive Eval : Term  Value  Resource  Prop where
  | val : ...
  | pair : ...
  | app : ...
  -- ...

-- Cost soundness (mechanized)
theorem cost_soundness :
  HasType  r b t A
   v k, Eval t v k  k  b  b  r := by
  -- proof by induction on typing derivation
```

A corresponding reification function can also be defined and proved correct, following Theorem 3.14.

5.2 Shallow Embedding: Lean as Host Language

Here we write ordinary Lean programs and annotate them with budget claims.

```
def binarySearch (arr : Array Nat) (target : Nat) : Option Nat :=
  if h : arr.isEmpty then none
```

```
  else
    let mid := arr.size / 2
    let pivot := arr[mid]
    match target.compare pivot with
    | .eq => some mid
    | .lt => binarySearch (arr.slice 0 mid) target
    | .gt => binarySearch (arr.slice (mid + 1) arr.size) target
  termination_by arr.size

@[budget_bound (fun arr =>
  Nat.ceil (Nat.log2 arr.size) + 5)]
def binarySearchBudgeted := binarySearch
```

The attribute `@[budget_bound]` is backed by a separate lemma establishing the claimed bound.

5.3 Relating Formal Bounds to Practical Proxies

In practice we also use a structural cost proxy on elaborated terms, for example

$$\mathsf{cost}(t) = \alpha \cdot \mathsf{nodes}(t) + \beta \cdot \mathsf{lamDepth}(t) + \gamma \cdot \mathsf{apps}(t) + \delta \cdot \mathsf{cases}(t),$$

where $\alpha, \beta, \gamma, \delta$ are tunable constants. This proxy is *not* the same as the synthesized bound b from the formal typing rules; it is a heuristic that over-approximates elaboration effort.

Formally:

- b is a compositional upper bound in the abstract calculus.
- $\mathsf{cost}(t)$ is a concrete measurement on Lean's internal representation.

We do not claim a theorem $\mathsf{cost}(\mathrm{elab}(t)) \leq \mathsf{b}$; instead, we use cost in CI to detect regressions and enforce empirical budgets.

5.4 RB Proof Calculator

We provide a reference implementation of the typing rules and bound synthesis as a standalone tool (outside this paper). It:

- Automatically synthesizes bounds for closed terms;
- Supports different lattice instantiations;
- Checks side-conditions such as $b \preceq r$ for Box.

It can be used to validate examples such as the binary search case study.

6 Applications and Value

Unified "correctness *and* budget" certificates are useful in several domains:

- **Safety-critical software:** autopilot, braking, and medical devices with hard deadlines.
- **Blockchain and distributed systems:** gas-bounded contracts and bounded-communication protocols.
- **Edge and embedded ML:** latency- and memory-constrained inference on mobile and IoT devices.

- **Education:** teaching complexity via types that carry explicit resource budgets.
- **High-performance computing and finance:** microsecond trading or tightly scheduled simulations.

Our framework provides a uniform foundation for these scenarios; deployment in production requires engineering work discussed in §5 and §8.

7 Case Study: Binary Search End-to-End

We briefly sketch how binary search fits the framework, for $L = \mathbb{N}$ (time).

7.1 Theory Level

For an array of size n, binary search performs $O(\log n)$ comparisons. Using the typing rules of Fig. 1 plus a structurally decreasing recursion rule, we derive a judgment

$$\emptyset \vdash_{r;\, b(n)} \mathsf{binarySearch\ arr\ target} : \mathsf{Option\ Nat},$$

where, for example,

$$b(n) = \lceil \log_2 n \rceil + c$$

for a small constant c accounting for overhead.

By Theorem 2.9, the actual operational cost k satisfies $k \leq b(n) \leq r$.

7.2 Semantic Level

In the presheaf model, for each n we obtain a value–cost pair $(v, k) \in [\![\mathsf{Option\ Nat}]\!](b(n))$, where k is the actual cost of the run. Semantic soundness (Theorem 3.12) and completeness (Theorem 3.14) ensure the syntactic and semantic views agree.

7.3 Lean/Engineering Level

In Lean, we implement the usual binary search function, then state and prove a lemma:

```
theorem binarySearch_bound (arr : Array Nat) (target : Nat) :
  cost (binarySearch arr target)  Nat.ceil (Nat.log2 arr.size) + 5 := by
  -- proof using recursion on arr.size
```

The CI pipeline runs the structural cost proxy on the elaborated term and checks that it remains below some threshold derived from the theoretical bound; if it grows unexpectedly (due to a refactor), the pipeline fails.

8 Limitations and Future Work

8.1 What We Provide

In this paper we have:

- Developed a resource-bounded STLC over an abstract lattice $(L, \preceq, \oplus, \bot)$;
- Proved type soundness (preservation, progress);
- Proved **cost soundness** syntactically (Theorem 2.9);
- Given a categorical semantic model in $\mathbf{Set}^L$ with internal lattice and natural cost (Theorem 3.12);

- Proved **constructive completeness** via reification (Theorem 3.14);
- Illustrated the approach with a binary search case study;
- Outlined an engineering path via Lean 4 and CI-based budget checking.

8.2 Open Problems

Important directions left open:

- Dependent types and automatic synthesis of size-indexed bounds;
- A tighter correspondence between formal bounds and elaboration cost in systems like Lean;
- Resource-aware identity types, paths, and (eventually) univalence;
- Probabilistic or average-case resource bounds;
- Full mechanization of the metatheory in a proof assistant;
- Extending the categorical story to higher-dimensional settings (e.g. indexed ∞-topoi).

9 Related Work

Coeffects and graded modalities. Petricek, Orchard, and Mycroft [1] introduced coeffects for context-dependent computation. Brunel et al. [2] developed a core quantitative coeffect calculus. Orchard et al. [3] apply graded modal types in the Granule language. Our abstract lattice framework generalizes their concrete semirings, and we additionally prove cost soundness and give a presheaf-topos model with constructive completeness.

Bounded complexity type systems. Hofmann [4] used linear types for polynomial-time computation; Dal Lago and Hofmann [5] developed bounded linear logic. We focus on explicit resource budgets rather than complexity classes.

Quantitative type theory. Atkey [6] tracks usage multiplicities in quantitative type theory using indexed categories. We track computational resources via presheaf topoi with a natural cost transformation.

Automatic resource analysis. Hoffmann et al. [7] infer polynomial bounds automatically (RAML). Our approach certifies bounds via typing and proofs rather than inference.

Step-indexed models. Appel and McAllester [8] use step-indexing for recursive types. Our presheaf model tracks actual costs instead of step indices used to solve domain equations.

Separation logic. O'Hearn et al. [9] reason about heap resources. Our focus is on computational resources such as time, memory, and gas.

10 Conclusion

We presented a compositional framework for resource-bounded types based on an abstract lattice $(L, \preceq, \oplus, \bot)$, with a graded feasibility modality and a cost-annotated typing judgment. We proved syntactic cost soundness, gave a categorical semantics in the presheaf topos $\mathbf{Set}^L$ with an internal lattice and a natural cost transformation, and established constructive completeness via reification. A binary search case study illustrated how theory, semantics, and engineering can align.

The simply-typed core already supports a range of resource models (time, memory, gas, multi-dimensional bounds). Extending this core to dependent types, richer type-theoretic structure, and tighter integration with proof assistants is promising future work. The overall message is that resource bounds can be treated as first-class, compositional objects in type theory, without committing to a single concrete cost model or a specific application domain.

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