Solving Economics and Finance Problems with MatLab

Semester Project: Estimating the Heston model

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User guide - How to run the program

The main program is called "Pricing.M". It can be used to price options assuming the stock price and its variance follows a process according to Heston (1993).
The general procedure can be controlled through the "GENERAL SET-UP" section. The "Settings"-field is the control form of the program. Then option- and Heston-parameters have to be specified as desired or as estimated. These parameters will be used for all consecutive sub-sections.
Two pricing methods are used in the program, the COS-FFT method and a standard Monte-Carlo-Simulation pricing framework. Additionally an estimation of the Heston-parameters can be conducted on the basis of the differential evolution algorithm written by Storn. Furthermore the program allows optionally to benchmark the Heston-Model against the classical Black-Scholes Model.

(b) COS-FFT Pricing: The COS-FFT parameters can be controlled through the INPUT section. The OUTPUT runs the pricing function with the help of two functions ($UK(\cdot)$ and $cfHes(\cdot)$).
(c) Estimation: The "Settings"-field allows to turn the estimation program on and off. If it is on, it calls the program "Rundeopt" and the differential evolution algorithm starts the least-squares minimization. Five Heston parameters will then be estimated - ρ λ, η, $\bar{u}$ and u_0.
(d) MC-Simulation Pricing: Equivalent to part (b) the MC-Simulation parameters can be adjusted in the INPUT section. The OUTPUT runs the simulation which follows straightforward a two-dimensional discretized stochastic-process.
(*) Black-Scholes-Benchmark: It prices the option according to the standard BS formula. It makes use of the function $BlackScholes(\cdot)$.
Results The last section controls the output of the results.

More details about the program and the functions are described in the program itself.

Description of the economic problem

The aim of the Heston Model is to price options based on the Heston stochastic volatility model (Heston, 1993). Although the Black-Scholes formula is often quite successful in explaining stock option prices, it does have known biases. The Black-Scholes model makes the strong assumption that stock re-

turns are normally distributed with known mean and variance. Therefore the stock has only one volatility and the implied volatility is neither depended on time to maturity nor on strike prices.
The Heston models allows for stochastic volatility, meaning volatility changes randomly over time. In comparison to other stochastic volatility models, Heston (1993) has developed a tractable closed-form solution for European option pricing with stochastic volatility that also permits correlation between asset returns and volatility (Nandi, P.2, 1996). If volatility is uncorrelated with the stock price, the probability for very low and high terminal stock prices increases, what would explain the volatility smile (Hull, P. 409ff., 2012). If volatility is negatively correlated with the stock price the probability for very low stock prices is increased and the probability for very high stock prices decreased, what leads to decreasing implied volatility and a volatility skew (Hull, P.606,P.727, 2012). The spot asset at time t follows the following diffusion process (Heston, P.328, 1993):

$$dS(t) = \mu S dt + \sqrt{v(t)} S dz_1(t)$$

where $z_1(t)$ is a Wiener process. The volatility follows an Ornstein-Uhlenbeck process, which can be written as the familiar square root process:

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dz_2(t)$$

where $z_2(t)$has correlation ρ with $z_1(t)$.

In a second step the Heston model is estimated by non-linear least-squares. The instantaneous volatility is itself an estimated parameter, as according Kearns (1992) using the time series of Black-Scholes implied volatilities to estimate the parameters and determine the level of volatility in a stochastic volatility model is inconsistent.

Mathematical Methods

In the program three kinds of mathematical methods are used, COS-FFT, Differential Evolution and Monte Carlo.

COS-FFT

For the Heston model a closed-form solution in form of a characteristic equation, the Fourier transform, is available. From the characteristic function the p.d.f. can be obtained by integration. The basic idea of the COS-FFT

method is to write a cosine expansion for the density function (Fang, F. & Oosterlee, C.W., 2008). Simple transformations make the COS-FFT able to calculate the p.d.f. of a characteristic function on a general interval. The density of the probabilistic function of the Heston model is then used to calculate option prices. The convergence rate of COS-FFT is exponential, making it faster and more efficient than the state-of-the-art FFT. In the Heston model N=50 already yields excellent results.
The characteristic function used in the program coincides with the version used by Fang, F. and Oosterlee, C. W.(P. 834, 2008).

$$\varphi_{hes}(\omega;u_0) = \exp\left(i\omega\mu\Delta t + \frac{u_0}{\eta^2}\left(\frac{1-\exp(-D\Delta t)}{1-G\exp(-D\Delta t)}\right)(\lambda - i\rho\eta\omega - D)\right)\cdot$$
$$\exp\left(\frac{\lambda\bar{v}}{\eta^2}\left(\Delta t(\lambda - i\rho\eta\omega - D) - 2\ln\left(\frac{1-G\exp(-D\Delta t)}{1-G}\right)\right)\right),$$

with

$$D = \sqrt{(\lambda - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2)} \quad \text{and} \quad G = \frac{\lambda - i\rho\eta\omega - D}{\lambda - i\rho\eta\omega + D}$$

The COS-FFT corresponding is:

$$v(x,t_0,u_o) \approx K\exp(-r\Delta t)Re\left\{\sum_{k=0}^{N-1,}\varphi_{hes}\left(\frac{k\pi}{b-a};u_0\right)\exp(ik\pi\frac{x-a}{b-a})U_k\right\}$$

The payoff function for European put options according to Fang, F. & Oosterlee, C.W. (2008) is adjusted, as the proposed one is obviously false and not leads to consistent results with the Monte Carlo Simulation:

$$U_k{}^{put} = \frac{2}{b-a}\left(\chi_k(0,a) - \psi_k(0,a)\right)$$

According Fang, F. and Oosterlee, C. W.(P. 842, 2008) the size of the truncation range (a-b) can be approximated well by the quantity $stdv\ (\bar{u} + \bar{u}\eta)^{0.5}$.

Differential Evolution

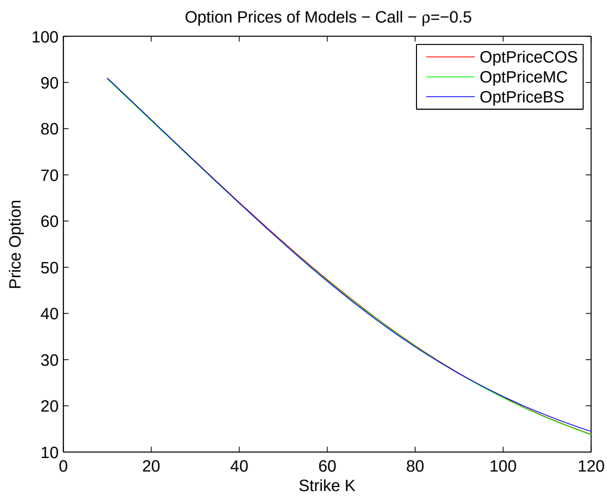
Differential Evoultion is a very simple population based, stochastic function minimizer which is very powerful at the same time (Price, K. & Storn, R., 2012). Differential Evolution is the most powerful tool for global optimization A whole population of trial vectors is tacked in a mutation and a selection phase until a defined threshold or the maximum number of steps is reached. Differential evolution is not very precise, but there is the possibility to find a global minimum in non-convex functions. Differential evolution is able to work with all possible functions in n dimensions.

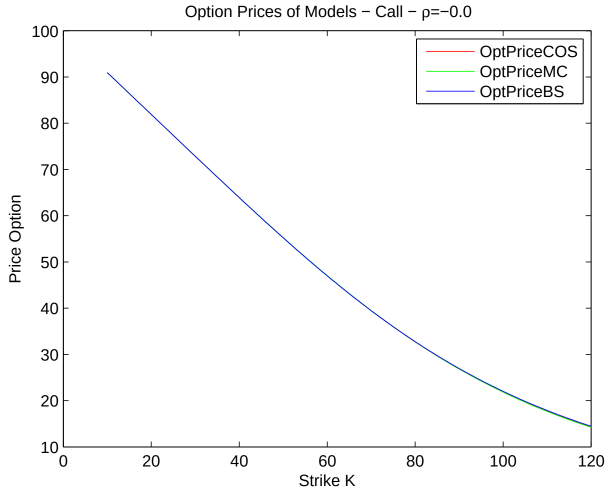
Monte-Carlo

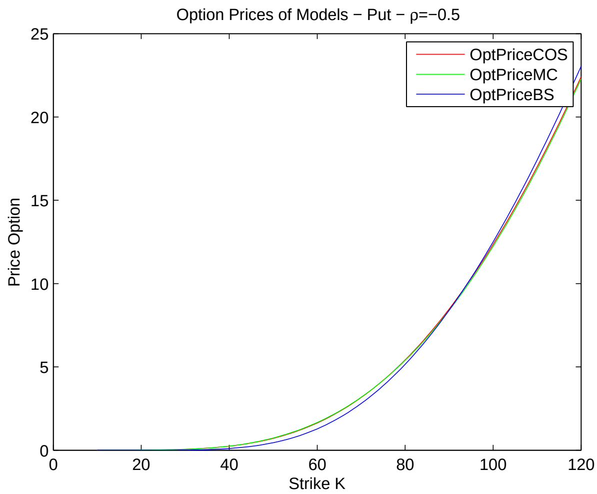
The Monte Carlo method is a numerical method of solving mathematical problems by random sampling. In the program the diffusion process of the Heston model is simulated with a Monte Carlo simulation. The simulated stock prices at terminal time are then used to calculate the option prices. As two processes have to be simulated simultaneously and a high number of simulations and of time steps are needed to get precise results, this approach is very inefficient and imprecise.
Due to the fact that the random numbers are generated jointly in a Mxn matrix in our program the Monte Carlo simulation is very fast. But this needs a lot of memory, what results in a maximum number of simulations of about 50000 simulations and 250 time steps. .

Interpretation

As we can see in the figures the COS-FFT and the MC-Simulation deliver consistent and reasonable results. Compared to the Black-Scholes case, we observe striking differences for the put option with a negative ρ. The prices are in line with the observed volatility skew, as the Heston model contains more probability mass in the left tail of the implied distribution (through the negative correlation (ρ) between the asset price process and the variance process). The volatility smile is not directly observable from the graphs, although our data of the estimated prices show higher prices for out-of-the-money call options compared to the BS model.

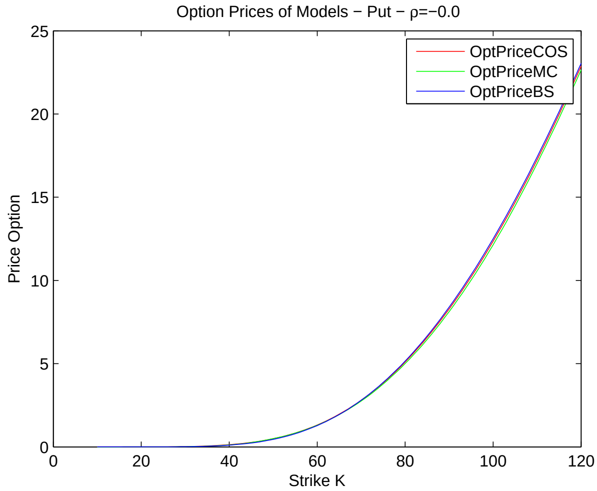






As you can see table 1 and 2 the estimation delivers nonsensical estimates for the parameters and also the function value seems certainly too high.

To have a target precision of 0.1 USD for a predicted option, the function value has to be smaller than 4.9. As Nandi (1996) or Bakshi, Cao, and Chen (1997) use similar data (intra day data on S&P 500 index options), our data can not be a reason for our unreasonable estimates.
Although, the mean absolute pricing error in the SV model is always lower than in the Black-Scholes model, still substantial mispricing are observed for deep out-of-the-money options (Nandi, P. 1, 1996). This can not be an explanation of our results, as there are no deep out-of-the-money options in our sample. Another explanations by Nandi (P.1, 1996), is that the degree of mispricing is related to bid-ask-spreads (a form of transaction costs) on



Parameter Heston	$\bar{u}$	λ	η	ρ	U_0
Parameter DE	best(1)	best(2)	best(3)	best(4)	best(5)
Estimate	0.0165384	1.95984e-014	0.209407	-0.797463	0.0552042

Table 1: Iteration: 2375, Best: 4271.545651, F_weight: 0.750000, F_CR: 0.800000, I_NP: 50

options. Unfortunately we are lacking the relevant information of our sample to draw any inference. The main difference between our estimation and the one of Nandi is that he corrects his objective function for autocorrelation in the residuals, leading to a more complicated objective function (Nandi, P.11, 1996). To correct for autocorrelation he uses of course the Option prices and the fitted Option prices the day before too. But his estimates seem to be reasonable (Nandi, P.28, 1996).

Parameter Heston	$\bar{u}$	λ	η	ρ	U_0
Parameter DE	best(1)	best(2)	best(3)	best(4)	best(5)
Estimate	0.051685	2.77064	0.305998	-1	0.0530704

Table 2: Iteration: 1093, Best: 4217.420883, F_weight: 0.700000, F_CR: 0.500000, I_NP: 100

List of Reference

Bakshi, G., Cao, C. and Chen, Z. (1997). Empirical Performance of Alternative Option Pricing Models, *Journal of Finance*, 52, 2003- 2049

Fang, F. and Oosterlee, C. W.(2008). A Novel Pricing Method for European Options Based on Fourier-COSINE Series Expansions. *SIAM J. SCI. COMPUT.* , Vol. 31, No. 2, pp. 826-848

Heston, S. L.(1993). A Closed-Form Solution for Options with Stochastic Volatility with Application to Bond and Currency Options. *The Review of Financial Studies* Vol.6, No. 2, 327-343.

Hull, J. C.(2012). *Options, Futures, And Other Derivatives* Pearson Education Limited: Essex.

Kearns, P. (1992). Pricing Interest Rate Derivative Securities When Volatility is Stochastic. Working Paper, University of Rochester

Nandi, S. (1996). Pricing and Hedging Index Options under Stochastic Volatility: An Empirical Examination. Working Paper, Federal Reserve of Atlanta

Price, K. and Storn, R. (2012). *http://www.icsi.berkeley.edu/ storn/code.html*