Some Notes on Sheaves

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The word 'sheaf' calls to mind a bundle of some sort, and this image is well-suited to the mathematical object bearing this name. Sheaves are used in mathematics to associate data to a topological space, compatible with the topology, in the sense that the data respects the inclusion of open sets. To make this precise, we first examine the notion of presheaf.

Definition: Given a topological space X and a category $\mathcal{C}$, a *presheaf* is a contravariant functor $\mathcal{F} : \mathfrak{X} \to \mathcal{C}$, where $\mathfrak{X}$ denotes the space X, viewed as a poset under subset inclusion. Explicitly,

(I). For $U \subseteq X$ an open set, $\mathcal{F}(U)$ is an object of $\mathcal{C}$

(II). If $V \subseteq U$ is an inclusion of open sets, there is a morphism $p_{U,V} = \mathcal{F}(V \subseteq U) : \mathcal{F}(U) \to \mathcal{F}(V)$.

Since the empty set $\varnothing$ is an initial object in $\mathfrak{X}$, we see that $\mathcal{F}(\varnothing)$ is terminal in $\mathcal{F}(\mathfrak{X})$. For example, if the category is $\mathfrak{Ab}$, we typically have $\mathcal{F}(\varnothing) = 0$. Similarly, $\mathcal{F}(X)$ is initial, since X is terminal in $\mathfrak{X}$.

For a presheaf to 'retain' the structure of the topological space, we should check its behaviour with intersections and unions. Firstly, it is clear from the definition that finite intersections and infinite unions of open sets U correspond to objects of $\mathcal{F}(\mathfrak{X})$. If $\{V_i\}_{i=1,\ldots,n}$ is a finite family of open sets, by (II) we have arrows $p_{V_j,\, \bigcap_{i=1}^n V_i} : \mathcal{F}(V_j) \to \mathcal{F}(\bigcap_{i=1}^n V_i)$ for $j = 1, \ldots, n$, so that the intersection inclusion is reversed in $\mathcal{F}(\mathfrak{X})$. Similarly, for an arbitrary family $\{U_i\}_{i\in I}$ we have arrows $p_{\bigcup_{i\in I} U_i, U_k}$ for all $k \in I$.

Though the definition of presheaf is relatively simple, its usefulness is not immediately clear. To motivate the development of sheaf theory, it can be helpful to consider a natural context in which sheaves arise. In differential geometry, one associates to a smooth manifold $\mathfrak{M}$ the tangent space of vectors at each point of the manifold. One looks at the germ of functions on open sets about points to answer questions about the local structure of the manifold. In some circumstances, information on the global structure may also be obtained from this study.
The ring of smooth functions $C^\infty(\mathfrak{M})$ on $\mathfrak{M}$ maps canonically via restriction onto $C^\infty(U)$ for an open set $U \subseteq \mathfrak{M}$, which makes C^∞ a presheaf on the manifold, viewed as a topological space. The germ at point P is obtained by identifying all functions which agree on some open neighbourhood of P. This makes the germ the direct limit of the spaces $C^\infty(U)$, with maps the restriction maps. We will later see that the direct limit formed in this way for a general sheaf similarly encodes the interesting local information for a sheaf. This object is so useful that it has its own name in sheaf theory, the *stalk* associated to the sheaf C^∞ at point P.