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Elementary Sieve Methods. BH 227. 3/11, 4pm.
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Preliminaries. Throughout this material, n denotes a positive integer, p denotes a prime, x denotes a positive real number, and $\mathcal{A}$ denotes the set of positive integers less than or equal to x.

Definitions. We let $\mu(n)$ denote the Möbius function evaluated at n, defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

For a fixed prime p, we distinguish some number of congruence classes mod p and let $\mathcal{A}_p$ denote the set of $n \leq x$ in one of these classes. For a squarefree integer d, we define $\mathcal{A}_d = \cap_{p|d} \mathcal{A}_p$. For a positive real number z, we define $\mathcal{P} = \{p \text{ prime} : p < z\}$ and $P(z) = \prod_{p \in \mathcal{P}} p$. We also define $S(\mathcal{A}, \mathcal{P}, z) = \#(\mathcal{A} \setminus \cup_{p|P(z)} \mathcal{A}_p)$.

For a function $f : \mathbb{R} \to \mathbb{R}$ we write $f(x) = O(g(x))$ if $g : \mathbb{R} \to \mathbb{R}^+$ and there is a positive constant A such that $|f(x)| \leq A g(x)$ for all $x \in \mathbb{R}$.

Theorem 1. (*The fundamental property of the Möbius function*)

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2. (*The Möbius inversion formula*) Let f, g be real valued functions defined on $\mathbb{N}$.

$$f(n) = \sum_{d|n} g(d) \qquad \text{if and only if} \qquad g(n) = \sum_{d|n} \mu(d) f(n/d)$$

Theorem 3. (*Partial Summation*) Let $c_1, c_2, \ldots$ be a sequence of real numbers and set $S(x) := \sum_{n \leq x} c_n$. If $f : [1, \infty) \to \mathbb{R}$ has continuous derivative in $(1, \infty)$, then for positive integer x we have

$$\sum_{n \leq x} c_n f(n) = S(x) f(x) - \int_1^x S(t) f'(t) dt$$

Theorem 4. (*Selberg's Sieve*) Let $f : \mathbb{N} \to \mathbb{R}$ be a multiplicative function and write $\#\mathcal{A}_d = \frac{x}{f(d)} + R_d$. For any sequence $\{\lambda_d\}$ of real numbers with $\lambda_1 = 1$,

$$S(\mathcal{A}, \mathcal{P}, z) \leq x \sum_{\substack{d_1, d_2 | P(z) \\ d_1, d_2 \leq z}} \frac{\lambda_{d_1} \lambda_{d_2}}{f([d_1, d_2])} + O\Big(\sum_{\substack{d_1, d_2 | P(z) \\ d_1, d_2 \leq z}} |\lambda_{d_1}| |\lambda_{d_2}| |R_{[d_1, d_2]}| \Big)$$

Theorem 5. (*Selberg's Sieve*) With $\#\mathcal{A}_d$ as in Theorem 4,

$$S(\mathcal{A}, \mathcal{P}, z) \leq \frac{x}{V(z)} + O\Big(\sum_{\substack{d_1, d_2 | P(z) \\ d_1, d_2 \leq z}} |R_{[d_1, d_2]}| \Big)$$

where $V(z) := \sum_{\substack{d|P(z) \\ d \leq z}} \frac{\mu^2(d)}{f_1(d)}$ and f_1 is the Möbius inverse to f.

Lemma 6. With $V(z)$ as in Theorem 5, $V(z) \geq \sum_{d \leq z} \frac{1}{\tilde{f}(d)}$ where $\tilde{f}$ is f extended completely multiplicatively.

Theorem 7. (*Selberg's Sieve for Primes*) Let $\pi(x)$ denote the number of primes less than x.

$$\pi(x) = O\Big(\frac{x}{\log x} \Big)$$

Lemma 8. Let $\tau(n)$ denote the number of positive divisors of an integer n. Then $\sum_{n \leq x} \tau(n) = x \log x + O(x)$.

Theorem 9. (*Selberg's Sieve for Twin Primes*) Let $\pi_2(x)$ denote the number of twin primes less than x.

$$\pi_2(x) = O\Big(\frac{x}{\log^2(x)} \Big)$$

Theorem 10. (*Brun's Theorem*) The sum $\sum_{\substack{p \\ p+2 \text{ prime}}} \frac{1}{p}$ converges.