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Analysis 2 course

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Introduction générale

Bienvenue dans ce cours d'Analyse II, conçu pour les étudiants en sciences informatiques, option programmation et entrepreneuriat.

Les concepts mathématiques avancés que nous allons aborder ne sont pas de simples abstractions : ils constituent des outils concrets pour modéliser, analyser et optimiser des problèmes que vous rencontrerez dans les domaines du développement logiciel, de la gestion des données, de l'intelligence artificielle ou encore dans la création d'entreprises technologiques.

Dans un monde numérique en constante évolution, la capacité à formuler rigoureusement un problème, à comprendre son comportement à travers des fonctions multivariables, des intégrales multiples ou des équations différentielles, vous place en position de concevoir des solutions innovantes, robustes et performantes.

Ce cours vise ainsi à vous transmettre non seulement des compétences analytiques solides, mais aussi à vous aider à en saisir l'impact concret dans votre futur métier : que ce soit pour créer un algorithme de prédiction, optimiser un réseau, ou estimer des valeurs dans un modèle économique.

En mathématiques comme en entrepreneuriat, ce que vous construisez dépend des fondations que vous posez.

Objectifs pédagogiques du cours

1. Comprendre et manipuler les fonctions de plusieurs variables, et les appliquer à la modélisation de phénomènes numériques.
2. Maîtriser le calcul d'intégrales simples et multiples, avec des applications en analyse de données et inférence.
3. Résoudre des équations différentielles d'ordre 1 et 2 dans un contexte informatique.
4. Développer une intuition rigoureuse autour de la continuité, de la différentiabilité, et des extrema.
5. Démontrer l'utilité de l'analyse dans des projets entrepreneuriaux, en lien avec la gestion de croissance, l'optimisation, et les prévisions.

Pourquoi ce cours est essentiel pour vous ?

1. Parce qu'un bon développeur ne code pas seulement : il modélise et comprend la structure des problèmes qu'il résout.

2. Parce que toute optimisation (d'un algorithme, d'un coût, d'un temps de calcul) repose sur des fondements analytiques.
3. Parce que les produits numériques s'appuient souvent sur des données continues : prédire, régulariser, lisser... cela se fait avec des intégrales et des dérivées.
4. Parce qu'un entrepreneur tech doit comprendre les bases mathématiques des outils qu'il vend ou intègre (moteurs de recherche, modèles prédictifs, systèmes dynamiques...).
5. Parce que les recruteurs et investisseurs valorisent des profils qui combinent créativité, rigueur et culture scientifique solide.

Une histoire pour commencer

Un jour, un jeune développeur travaillait sur une application de recommandation musicale. Les tests échouaient, les données étaient bruitées, les résultats instables. Il s'apprêtait à abandonner, quand son mentor lui dit :

« Tu ne peux pas apprendre à l'algorithme ce que tu ne comprends pas toi-même. Reviens aux fondations. »

Le jeune homme reprit ses notes, redériva les équations, comprit le modèle, et corrigea l'erreur. L'application devint un succès.

« En mathématiques comme en entreprise, les raccourcis coûtent plus cher que la rigueur. »

<u>**Essai littéraire**</u>

Aujourd'hui 10 Juin 2025, dans la ville de Koudougou,
battue par le vent et la poussière dès 7h.
Ce matin-là, le ciel de la ville était lourd.
Le vent chaud, chargé de poussière, étouffait les rues.
Il faisait déjà 36°C avant même 7h30min.
Le genre de matin où l'on préférerait rester à l'ombre,
au frais, loin des salles de classe chauffées par le béton et pleine de vent poussiéreux.
Le prof de maths savait que peu d'étudiants viendraient.
Mais il partit quand même dès 6h30min,
en se disant que « s'il y a ne serait-ce qu'un seul étudiant,
il méritera tout son cours d'Analyse II».

Et pourtant ?
Quand je suis arrivé pour commencer, vous étiez là.
Pas un, pas dix, mais une centaine.
Assis, silencieux, qui avaient bravé le même vent,
la même fatigue, les mêmes doutes, poussière, prêts à apprendre.

« Aujourd'hui, nous sommes peu.
Mais ce que vous venez de faire,
vous lever, vous déplacer, venir malgré tout,
c'est exactement ce qu'on appelle la discipline.
Et c'est elle, bien plus que le talent ou la chance, qui bâtit les grandes réussites. »

Alors j'ai compris quelque chose de simple mais puissant :
« Le succès appartient à ceux qui avancent même quand personne ne les regarde. »
Et,
Ce n'est pas l'environnement qui forge le destin ou le succès d'un étudiant,
mais plutôt, sa discipline et sa volonté.

Merci à vous, chers étudiants de CS/27,
pour votre engagement, votre présence, et votre courage silencieux.
Votre discipline et votre détermination sont une véritable source d'inspiration.

Puisse ce cours d'Analyse II contribuer, à sa manière,

à vous doter des outils nécessaires pour bâtir l'avenir que vous imaginez.

Avec respect et admiration,

Dr. OUEDRAOGO

General Introduction

Welcome to the course *Analysis II*, designed specifically for students in Computer Science with a focus on programming and entrepreneurship.

The advanced mathematical concepts we will explore are far from mere theoretical constructs. They are practical tools that allow us to model, analyze, and optimize problems frequently encountered in software development, data management, artificial intelligence, and the creation of technology-driven enterprises.

In an ever-evolving digital world, the ability to rigorously formulate a problem and to understand its behavior through multivariable functions, multiple integrals, or differential equations positions you to design innovative, robust, and high-performing solutions.

This course thus aims not only to equip you with solid analytical competencies but also to help you appreciate their tangible impact on your future profession whether it involves building predictive algorithms optimizing networks, or estimating key parameters in an economic model.

In both mathematics and entrepreneurship, what you build depends on the foundations you lay.

Learning Objectives

1. Understand and manipulate functions of several variables, and apply them to the modeling of digital phenomena.
2. Master single and multiple integral calculus, with applications in data analysis and inference.
3. Solve first and second-order differential equations in computational contexts.
4. Develop rigorous intuition regarding continuity, differentiability, and extrema.
5. Demonstrate the relevance of analysis in entrepreneurial projects related to growth management, optimization, and forecasting.

Why Is This Course Essential for You ?

1. Because a skilled developer does more than just write code, they model and understand the structure of the problems they solve.
2. Because all optimization of an algorithm, a cost, or computation time rests on analytical foundations.

3. Because digital products often rely on continuous data : prediction, regularization, smoothing, all of which depend on derivatives and integrals.
4. Because a tech entrepreneur must grasp the mathematical underpinnings of the tools they integrate or commercialize (search engines, predictive models, dynamic systems, etc.).
5. Because recruiters and investors value profiles that combine creativity, analytical rigor, and a strong scientific culture.

A Story to Begin With

Once, a young developer was working on a music recommendation app. The tests kept failing, the data was noisy, the outputs unstable. Ready to give up, he was stopped by his mentor, who said :

> *"You cannot teach the algorithm what you do not understand yourself. Go back to the foundations."*

The young man went back to his notes, re-derived the equations, understood the model, and fixed the bug. The application became a success.

"In mathematics, as in business, shortcuts cost more than rigor."

Literary Essay

Today, June 10, 2025, in the city of Koudougou,
beaten by wind and dust as early as 7 a.m.
That morning, the sky over the city was heavy.
The hot wind, thick with dust, choked the streets.
It was already 36°C before even 7 :30 a.m.
The kind of morning where you'd rather stay in the shade,
somewhere cool, far from concrete-heated classrooms filled with dusty wind.
The math teacher knew few students would come.

But he left anyway at 6 :30 a.m.,
telling himself, "If there is even one student,
he will deserve the full Analysis II lecture."

And yet ?
When I arrived to begin, you were there.
Not one, not ten, but a hundred.

Seated, silent, having braved the same wind,
the same fatigue, the same doubts and dust, ready to learn.

"Today, we are few.
But what you have just done,
getting up, moving, coming despite it all,
that is exactly what we call discipline.
And it is discipline far more than talent or luck that builds great achievements."

Then I understood something simple yet powerful :
"Success belongs to those who keep going even when no one is watching."
And,
It is not the environment that shapes a student's destiny or success,
but rather, their discipline and determination.

Thank you, dear students of CS/27,

for your commitment, your presence, and your silent courage.

Your discipline and determination are a true source of inspiration.

May this Analysis II course, in its own way,

equip you with the tools you need to build the future you imagine.

With respect and admiration,

Dr. OUEDRAOGO

Chapitre 1

Numerical functions of several variables

1.1 Generality

1.1.1 Notions and definitions

Definition 1.1.1 *1. Let E be a part of $\mathbb{R}^n$. The function f defined by :*

$$f : E \longrightarrow \mathbb{R}^n$$
$$x = (x_1, x_2, \cdots, x_n) \longmapsto y = (y_1, y_2, \cdots, y_n)$$

2. Let E be a part of $\mathbb{R}^n$ and $f : E \longrightarrow \mathbb{R}$ a function. We call graph of f, the surface $\{(x, y) \in \mathbb{R}^{n+1}\ y = f(x)\}$, with $x \in \mathbb{R}^n$

Exemple 1.1.2 *Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $X \longrightarrow x^2 + y^2$. The graph of this function f is a paraboloid of revolution*

Definition 1.1.3 *We call the set of definition of the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the set defined by*

$$\mathcal{D}_f = \{x \in \mathbb{R}^n : f(x) \quad exists\}$$

Exemple 1.1.4 *Determine and represent the definition set of the functions below :*

$$f(x, y) = \frac{2xy}{\sqrt{4 - x^2 - y^2}}, \qquad h(x, y) = ln(x^2 + y^2 - 4x - 2y + 2) + \sqrt{x + 2}$$

1.2 Limit and continuity

Definition 1.2.1 *1. Let E be a vector space on $\mathbb{R}^n$, the norm on E is the application $N : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\forall x, y \in \mathbb{R}^n, N(x) = ||x|| \in \mathbb{R}^+$ verifying :*

- $N(x) = 0 \Leftrightarrow x = 0_E$
- $N(\lambda x) = |\lambda| N(x)$
- $N(x + y) \leq N(x) + N(y)$.

We have for example the norms $N_1(x_1, \cdots, x_n) = \sqrt{x_1^2 + \cdots + x_n^2}$,
$N_2(x_1, \cdots, x_n) = |x_1| + \cdots + |x_n|$ *and* $N_\infty(x_1, \cdots, x_n) = sup(|x_1|, \cdots, |x_n|)$

2. Let $x, y \in \mathbb{R}^n$. *The distance associated with* x *and* y *to the* $||.||$ *norm noted* $d(x, y)$ *can be defined by :*

$$d(x, y) = ||x - y||$$

and verifying $x, y \in E$

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $\forall x, y, z \in E, d(x, z) \leq d(x, y) + d(y, z)$

In the following, we will assume for the most part that $d(x, y) = N_1(x, y)$.

Definition 1.2.2 *Let* $f : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a function with several real variables.* f *admits in* $x_0 \in \mathbb{R}^n$ *the limit* l *if :*

$$\forall \varepsilon > 0, \exists \eta_0 > 0; \quad d(x, x_0) \leq \eta_0 \Rightarrow |f(x) - l| \leq \varepsilon$$

Exemple 1.2.3 *1. Let* $f(x, y) = \frac{xy}{x^2+y^2}$ *We have* $f(x, 0) = 0$ *and* $\lim_{x\to 0} f(x, 0) = 0$. *Then* $f(0, y) = 0$ *and* $\lim_{y\to 0} f(0, y) = 0$. *Finally* $f(x, x) = \frac{1}{2}$ *and* $\lim_{x\to 0} f(x, x) = \frac{1}{2} \neq 0$.
Therefore f *does not have a limit at* $(0, 0)$. *Determine the limit at* $(0, 0)$ *of* h *given by* $h(x, y) = \frac{xy^2}{x+y}$.

Proposition 1.2.4 *let* f *be a function,* $a, b \in \mathbb{R}$. *If there exist* $l \in \mathbb{R}$ *and* $S : r \longmapsto S(r)$ *such that* $|f(a + rcos(\theta); b + rsin(\theta) - l)| \leq S(r)$ *and* $\lim_{r\to 0} S(r) = 0$ *then* $\lim_{(x,y)\to(a,b)} f(x, y) = l$.

Exemple 1.2.5 *Using the previous proposition, given eventually the limit of* f *in* $(0, 0)$*, with* $f = \frac{2x^3}{x^2+y^2}$

Definition 1.2.6 *1. A function defined on* $E \subset and\mathbb{R}^n$ *is continuous at a point* x_0 *if*

$$lim_{x\to x_0} f(x) = f(x_0)$$

2. the function f *is continuous on* E *if it is continuous in any point of* E

1.3 Partial function and derivative

Definition 1.3.1 *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a multivariate function. We call $i-th$ partial at point $a = a_1, a_2, \cdots, a_n$, the function f_i define on $E_1 = \{x \in \mathbb{R}\ (a_1, a_2, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_n)\}$ by*

$$f(x) = f(a_1, a_2, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_n)$$

Exemple 1.3.2 *Let $f(x, y, z) = xz - 3y^2 + 2z$ and $a = (1, -1, 2)$. The partial functions of f are :*

$$f_1(x) = f(x, -1, 2); \qquad f_2(x) = f(1, x, 2); \qquad f_3(x) = f(1, -1, x).$$

Definition 1.3.3 *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a multivariate function. If the $i-th$ partial function of f in a is derivable in a_1, then its derivative (with respect to x_i) is called the $i - th$ partial derivative of f in a and is denoted $\frac{\partial f}{\partial x_i}(a)$. We have :*

$$f'_{x_i}(a) = \frac{\partial f}{\partial x_i}(a) = \lim_{x \to a_i} \frac{f(a_1, a_2, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_n) - f(a)}{x_i - a_i}$$

If $f'_{x_i}(a)$ exists $\forall i = 1; \cdots; n$ then f is derivable in a

Exemple 1.3.4 *To give*

1.3.1 Jacobian

Definition 1.3.5 *Let $B = (e_j)_{1\leq j\leq n}$ be a base for E, and $B' = (e'_i)_{1\leq i\leq p}$ a base for F, $f : E \longrightarrow F$.*
Then the Jacobian matrix of f is defined by :

$$\mathcal{J}_f(x_0) = \left(\frac{\partial f_i(x_0)}{\partial x_j} \right)_{1\leq i\leq p; 1\leq j\leq n}$$

We have :

$$\mathcal{J}_a(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_p} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

In the case $dimF = dimE = n$, then the determinant of

$$\mathcal{J}_f(x_0)$$

is called the Jacobian of f in x_0. We note

$$det(\mathcal{J}_f(x_0)) = \frac{d(f_1, f_2, \cdots, f_n)}{d(x_1, x_2, \cdots, x_n)}(x_0)$$

Exemple 1.3.6 *Let*

$$\begin{aligned}
f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
(r,\theta) &\longmapsto (x,y) = (rcos\theta, rsin\theta) \\
&and \\
h : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
(r,\theta,\varphi) &\longmapsto (x,y,z) = (rsin\theta cos\varphi, rsin\theta sin\varphi, rcos\theta)
\end{aligned}$$

Determine the Jacobians of f and h in (r,θ) and in (r,θ,φ)

1.3.2 Partial derivatives of order greater than 1

Let $f : E \supset U \longrightarrow F$. If f admits first partial derivative functions on U and these are derivable, then we say that f admits partial derivatives of order 2 on U

Notation

Partial derivative functions of order 2 of f in a :

$$\mathcal{D}_{i,j}f(a) = \frac{\partial}{\partial x_j}(\frac{\partial f}{\partial x_i})(a) = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad ; \qquad i,j = \overline{1, \cdots, n}$$

• Si $i = j$, $\mathcal{D}_{i,i}f(a) = \frac{\partial^2 f}{\partial x_i^2}(a)$, these derivatives are called homogeneous derivatives.
• If $i \neq j$, $\mathcal{D}_{i,j}f(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$, they are called mixed derivatives.
By iteration, we can define the partial derivatives of order $k \in \mathbb{N}^*$

$$\mathcal{D}_{1,2,3,\cdots,k}f(a) = \frac{\partial^k f}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \cdots \partial x_{j_n}}(a)$$

Definition 1.3.7 *f is said to be of class c^k $(k \in \mathbb{N})$ on U an open of E, when $\forall j_1, j_2, \cdots, j_k \in \{1, 2, 3, \cdots, n\}$, f admits a partial derivative of order k continuous. This derivative is written $\frac{\partial^k f}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \cdots \partial x_{j_n}}(a)$*

Definition 1.3.8 *If f is a function derivable at a point $a \in \mathbb{R}^n$ then we define the partial* *derivatives of order 2 of f by derivation of the first partial derivatives. They are written :*

$$f''_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem 1.3.9 ***Schwarz's theorem***

Let f be a function with continuous first and second partial derivatives in the neighbourhood of (x_0, y_0). Then

$$f''_{xy} = f''_{yx}$$

i.e

$$\frac{\partial^2 f}{\partial x \partial y}(a) = \frac{\partial^2 f}{\partial y \partial x}(a)$$

Exemple 1.3.10 *Let*

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (\frac{2x^2 y}{x+y}, sin(\frac{1}{x^2})) \end{aligned}$$

Determine the first and second partial derivatives of f

Remarque 1.3.11 *The Schwarz theorem generalizes to higher order partial derivatives*

1.4 Notion of differential

Definition 1.4.1 *Let f be a function defined on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$.*

1. *We say that f is differentiable in $(a_1; \cdots; a_n)$, if there exist $h = (h_1; \cdots; h_n)$, $b = (b_1; \cdots; b_n) \subset \mathbb{R}^n$ such that :*

$$f(a_1 + h_1; \cdots; a_n + h_n) - f(a_1; \cdots; a_n) = h_1 b_1 + \cdots + h_n b_n + \sqrt{h_1^2 + \cdots + h_n^2}\varphi(h_1; \cdots; h_n)$$

with $\lim_{h \to 0} \varphi(h_1; \cdots; h_n) = 0$ or

$$\lim_{(h_1;\cdots;h_n)\to(0;\cdots;0)} \frac{f(a_1 + h_1; \cdots; a_n + h_n) - f(a_1; \cdots; a_n)}{\sqrt{h_1^2 + \cdots + h_n^2}} = l \in \mathbb{R}$$

In addition, it is shown that $b_i = f'_{x_i}(a)$.

2. *We call the differential of f in a, the linear application df defined by :*

$$df : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (h_1; \cdots; h_n) \longmapsto f'_{x_1} h_1 + \cdots + f'_{x_n} h_n$$

In particular if we denote the linear application dx_i by $dx_i = h_i$, then

$$df = f'_{x_1}dx_1 + \cdots + f'_{x_n}dx_n$$

Proposition 1.4.2 ***Properties of the differential***

1. Continuity : *A function f differentiable at a point is continuous at that point.*

2. Linearity : *Let $f, g : E \longrightarrow \mathbb{R}$ be two functions defined on a part E of $\mathbb{R}^n$. If f and g are differentiable in $A \in E$, then $d(f+g)(A) = df(A) + dg(A)$ and $d(\lambda f)(A) = \lambda df(A)$.*

3. Composition : *Let $f, g : E \longrightarrow \mathbb{R}$ be two functions defined on a part E of $\mathbb{R}^n$ and differentiable in $A \in E$, and $h : E \longrightarrow \mathbb{R}$ differentiable in $g(A)$, then hircg is differentiable in A and the differential $d(h \circ g)(A) = dh(g(A)) \times dg(A)$.*

Theorem 1.4.3 *If f is a differentiable function, then it is continuous and has partial derivatives. The converse is true and the first derivatives are continuous.*

Property 1.4.4 *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $X_0 \in E$. If $\forall i = 1; \cdots ; n, X \longmapsto \frac{\partial f_j}{\partial x_i}(X)$ exists in the neighbourhood of X_0 and is continuous in X_0, then f is differentiable in X_0*

Exemple 1.4.5 *Let f be defined on $\mathbb{R}^2$ by $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 \quad otherwise \end{cases}$*

Is it differentiable in $(0,0)$?

Definition 1.4.6 ***1.*** *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^1. We call the gradient of f in $(x_0, y_0) \in E$, the unique vector $\nabla f(x_0, y_0) \in \mathbb{R}^2$ such that :*

$$\nabla f(x_0, y_0) = (\frac{\partial f}{\partial x}(x_0, y_0); \frac{\partial f}{\partial y}(x_0, y_0))$$

and $\forall h = (h_1; h_2)$, we have $df((x_0, y_0))h = (\nabla f((x_0, y_0); h)$.

2. *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^2. We call the Laplacian of f in $(x_0, y_0) \in E$, the real $\Delta f(x_0, y_0) = \nabla^2 f(x_0, y_0) = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$*

We also establish the following theorem on the derivative of compound functions :

Property 1.4.7 *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ a function of class C^1, υ, ω two real functions of classes C^1 on $I \subset \mathbb{R}$, then if $\forall t \in I$, $F(t) = f(\upsilon(t), \omega(t)) \in C^1$ then*

$$F'(t) = \frac{\partial f}{\partial x}(\upsilon(t), \omega(t))\upsilon'(t) + \frac{\partial f}{\partial y}(\upsilon(t), \omega(t))\omega'(t)$$

Exemple 1.4.8 *Let $\upsilon(t) = cos(t)$ and $\omega(t) = e^t$. Let f be a function defined on $\mathbb{R}^2$ by $f(x, y) = arcsin(x) + ln(y^2)$.*
Determine $F'(t)$ if $F(t) = f(\upsilon(t), \omega(t))$.

We deduce the following theorem :

Property 1.4.9 *Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^1, $h, k \in \mathbb{R}$. Then $\forall(x, y) \in E$ such that $(x + h, y + k) \in E$, we have*

$$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x}(x+\theta h, y+\theta k) + h\frac{\partial f}{\partial y}(x+\theta h, y+\theta k), \qquad \theta \in]0,1[\qquad \mathit{2}$$

1.5 Extrema

We study the behaviour of a function of several variables with real values. Such a function can have extremal values : minima (smallest values) or maxima (largest values) on the whole domain of definition or on a part. They are called extrema

Definition 1.5.1 ***1.*** *Let $f : E \longrightarrow \mathbb{R}$, be a function defined on a part of $E \subset \mathbb{R}^n$. We say that f has a global maximum (respectively a global minimum) at the point $x \in E$ if we have : $f(x) \leq f(a)$ (respectively $f(x) \geq f(a)$). The maximum (respectively the minimum) is called strict if $f(x) < f(a)$ (respectively $f(x) > f(a)$)* ***2.*** *We say that f admits a local maximum (respectively a minimum) in $a \in E$, if we can find a number $r > 0$ such that $x \in E$ and $||x - a|| < r$ leads to $f(x) \leq f(a)$ (respectively $f(x) > f(a)$)*

Global extrema are also called absolute extrema. In the remainder of this chapter, we will limit ourselves to the study of local extrema

Definition 1.5.2 *Let f be a function of class C^1 of $E \subset \mathbb{R}^n$. We say that $a \in E$ is a critical point of f if*

$$\nabla f(a) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}(a) = \cdots = \frac{\partial f}{\partial x_n}(a) = 0$$

We say that a is a cancellation point of its gradient.

Property 1.5.3 *Let $f : E \longrightarrow \mathbb{R}$, be a function and E be an open of $\mathbb{R}^n$. If f admits a local extremum in a then $\nabla f(a) = 0$*

Remarque 1.5.4 *The reciprocal of the previous theorem is false.*

1.5.1 Extremum on $\mathbb{R}^n$

Definition 1.5.5 ***Revisions***

1. *The array of n rows and n columns is called a square matrix of order n, denoted by $A = (a_{i,j})_{1\leq i,j\leq n}$.*

2. *We call the transpose of a matrix A, noted t_A, the matrix having as rows the columns of A and as columns the rows of A.*

3. *A matrix is said to be symmetric if ${}^tA = A$.*

4. *We call the characteristic polynomial $P_\lambda(A)$ defined by $P_\lambda(A) = det(A - \lambda I_n)$ where $\lambda \in \mathbb{R}$ and I is the identity matrix of $\mathcal{M}_n(\mathbb{R})$.*

5. *The solutions of $P_\lambda(A) = 0$ are called eigenvalues of a matrix A*

Definition 1.5.6 *Let f be a function of class C^2 on an open $E \subset \mathbb{R}^n$ and $a \in E$. We call the Hesian matrix of f in a (or the Hesian of f in a) and we note $\mathcal{H}_a(f)$, the matrix noted :*

$$\mathcal{H}_a(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} & \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

The quadratic form of f in a is thus defined by :

$$q_a(h) = \sum_{i=i}^{n} \sum_{j=i}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Let $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $a \in E$. When $n = 1$, to know if a critical point a is a local maximum or a local minimum, we study the second derivative (when it exists) :

- $f^{"}(a) > 0$, then $f(a)$ is a local minimum.
- $f^{"}(a) < 0$, then $f(a)$ is a local maximum.

If $f^{"}(a) = 0$ is a local maximum, then $f(a) = 0$ is a local maximum and additional higher derivative calculations are required. This point can be an inflection point, a maximum or a minimum.

In the case of several variables, instead of $f^{"}$, we study the Hessian.

Property 1.5.7 *Let f be a function of class C^1 on an open E of $\mathbb{R}^n$. Let $a \in E$ be a critical point, $q_a(h)$ the quadratic form and $\mathcal{H}_a(f)$ the Hessian matrix of f in a. If*

i. $\forall h \neq 0$, $q_a(h) \geq 0$ *(respectively $q_a(h) \leq 0$)*

ii. *If $\mathcal{H}_a(f)$ has a positive and a negative eigenvalue,*

then f ***has no local extremum in*** *a.*

We will limit ourselves to the study of extrema for functions of two variables :

1.5.2 Extremum for $n = 2$

Definition 1.5.8 *Let a be a critical point of a function f. We define the coefficients r, s and t by :*

$$r = \frac{\partial^2 f}{\partial x^2}(a); \qquad s = \frac{\partial^2 f}{\partial x \partial y}(a) = \frac{\partial^2 f}{\partial y \partial x}(a); \qquad t = \frac{\partial^2 f}{\partial y^2}(a).$$

Thus we establish :

Theorem 1.5.9 *Let a be a critical point of f :*
If $rt - s^2 > 0$ and $r > 0$ then f admits a ***local minimum*** *at a.*
If $rt - s^2 > 0$ and $r < 0$ then f admits a ***local maximum*** *at a .*
If $rt - s^2 < 0$ then f does not admit a ***extreme*** *in a. We then say that a is a saddle point or a neck point.*
If $rt - s^2 = 0$ then we can say nothing.

Remarque 1.5.10 *In the case $rt - s^2 = 0$, it is necessary to return to the usual definition of the extremum.*
If s is a col point, it is necessary to demonstrate that f takes positive and negative values in a neighbourhood of s.

Exemple 1.5.11 ***Exercise*** *Determine the possible extrema of $f(x, y)$ and $h(x, y)$ where*

$$f(x, y) = 4xy + y^2 + x^2 + 2y; \qquad h(x, y) = 4yx + \frac{1}{3}y^3 - 2x^2 - 5y.$$

1.6 Taylor's formula

Let f be a function of class c^{k+1} on an open U. Let $a \in U$. If for $h \in \mathbb{R}^n$, $n \geq 1, a + h \in U$, then the Taylor limited expansion (L.D.) of f in the neighbourhood of a to order k is given by :

$$f(a+h) = f(a) + \left(h_1 \tfrac{\partial f}{\partial x_1}(a) + h_2 \tfrac{\partial f}{\partial x_2}(a) + \cdots + h_n \tfrac{\partial f}{\partial x_n}(a) \right)$$
$$+ \frac{1}{2!}\left(h_1 \tfrac{\partial f}{\partial x_1}(a) + h_2 \tfrac{\partial f}{\partial x_2}(a) + \cdots + h_n \tfrac{\partial f}{\partial x_n}(a) \right) + \cdots + \frac{1}{k!}\left(h_1 \tfrac{\partial f}{\partial x_1}(a) + h_2 \tfrac{\partial f}{\partial x_2}(a) + \cdots + h_n \tfrac{\partial f}{\partial x_n}(a) \right)$$
$$+ ||k||^{k+1}\varepsilon(h)$$

where $\lim_{h \to 0} \varepsilon(h) = 0$ and the quantity $\left(h_1 \tfrac{\partial f}{\partial x_1}(a) + h_2 \tfrac{\partial f}{\partial x_2}(a) + \cdots + h_n \tfrac{\partial f}{\partial x_n}(a) \right)$ is calculated by Newton's formula

1.6.1 Special cases

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^1. The D. L of order 2 of f in a is written

$$f(a+h) = f(a) + \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a) + \frac{1}{2}[\sum_{j=1}^{n} h_j^2 \frac{\partial^2 f}{\partial x_j^2}(a) + 2 \sum_{1 \leq i < j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a)] + \Theta(||h||^2)$$

Exemple 1.6.1 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, (x, y, z) \longmapsto tan(\frac{xy}{z})$.
Determine the L.D. of order 2 of f in the neighbourhood of $a = (0, \frac{\pi}{4}, 1)$.

1.7 Implicit function

1.7.1 Implicit function theorem

Let $f : \mathbb{R}^n \supset U \longrightarrow \mathbb{R}$ be of class c^1, $a \in U$ such that $f(a) = 0$ and $\frac{\partial f}{\partial x_n} \neq 0$, then there exists :

i. An open V of $\mathbb{R}^{n+1}$ containing $(a_1, a_2, \cdots, a_{n-1})$

ii. I is an open of $\mathbb{R}$ containing a_n

iii. $\varphi : V \longrightarrow I$ of class C^1 such that $x_n = \varphi(x_1, x_2, \cdots, x_{n-1})$ It is said that φ implicitly defines the equation $f(x_1, x_2, \cdots, x_n) = 0$ in the neighbourhood of a.

In addition

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \cdots, x_{n-1}) = \frac{-\frac{\partial f}{\partial x_i}(x_1, x_2, \cdots, x_n)}{\frac{\partial f}{\partial x_n}(x_1, x_2, \cdots, x_n)}$$

Exemple 1.7.1 *Let the equation $arctan(xy) + 1 = e^{x+y}$ (1).*

1. Show that (1) implicitly defines a function f in the neighbourhood of 0.

2. Determine the D.L of order 3 of f in the neighbourhood of 0.

Chapitre 2

Multiple Integrals

2.1 Introduction to Integration

Integrals play a fundamental role in mathematical analysis and engineering. They allow us to quantify continuous quantities such as areas, volumes, arc lengths, and the work done by a force. In electromechanics, they are particularly useful in signal analysis, the study of electrical circuits, and modeling dynamic systems. This chapter aims to introduce and deepen understanding of Riemann integration, emphasizing its properties and applications.

2.2 Some Reminders

2.2.1 Definitions

Definition 2.2.1 *Let $f : I \to \mathbb{R}$ be a function defined on an interval I. A **primitive** (or antiderivative) of f is any differentiable function F on I such that :*

$$F'(x) = f(x), \quad \forall x \in I.$$

Definition 2.2.2 *The **indefinite integral** of f is the set of all its primitives, denoted by :*

$$\int f(x)dx = F(x) + C,$$

*where C is an arbitrary constant called the **constant of integration**.*

2.2.2 Properties

- **Additivity** : $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

– **Linearity** :

$$\int (\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx.$$

Chasles' Relation : For any $c \in [a, b]$, we have :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

- **Monotonicity / Positivity**

  Let f and g be integrable functions on $[a, b]$ with $a \leq b$. If $f \geq g$ on $[a, b]$, then :

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$$

  In particular, if $f \geq 0$ on $[a, b]$, then :

$$\int_a^b f(x)\, dx \geq 0.$$

- **Triangle Inequality**

  Let f be integrable on $[a, b]$. Then :

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

  More generally, for any order of a and b :

$$\left| \int_a^b f(x)\, dx \right| \leq |b - a| \sup_{x \in [a,b]} |f(x)|.$$

- **Strict Positivity**

  Assume f is **continuous and positive**.

  – If there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$, then :

$$\int_a^b f(x)\, dx > 0.$$

  – If $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Integration by Parts

If u and v are differentiable functions, then :

$$\int u\, dv = uv - \int v\, du.$$

Example : Compute $\int xe^x dx$.

Substitution

If $x = g(t)$, then :

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt.$$

Example : Compute $\int_1^4 \frac{dx}{\sqrt{x}}$ by setting $x = t^2$.

Cauchy-Schwarz Inequality : For two integrable functions $f(x)$ and $g(x)$:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right).$$

2.3 Some common functions

2.3.1 Exponential function

Let $f(z) = e^z$, $z \in \mathbb{C}$. Putting $z = x + iy$, we have $f(z) = P(x, y) + iQ(x, y)$ where $P(x, y) = e^x cosy$ and $Q(x, y) = e^x siny$.

a. Taylo's derivative :

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

b. Some properties :

- $\forall z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1} \times e^{z_2}$.
- $\forall z \in \mathbb{C}$, $e^{z+2i\pi} = e^z$ we say that e^z is periodic of period $2i\pi$.
- $\forall z \in \mathbb{C}$, $|e^z| = e^{Re(z)}$
- $\forall z \in \mathbb{C}$, $arge^z = Im(z) + 2k\pi$, $k \in \mathbb{Z}$.

2.3.2 Trigonometric functions

a. $\forall z \in \mathbb{C}$,

- $cosz = \frac{e^{iz}+e^{-iz}}{2}$.
- $sinz = \frac{e^{iz}-e^{-iz}}{2i}$.
- $tanz = \frac{sinz}{cosz} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$.

b. Some properties :

$$\bullet cosz = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\bullet sinz = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
$$\bullet cos^2 z + sin^2 z = 1$$
$$\bullet cos(z + 2\pi) = cosz$$
$$\bullet sin(z + 2\pi) = sinz$$

2.3.3 Inverse Trigonometric Functions

- **arcsin(x)** - Inverse of sine

  $y = \arcsin(x) \Leftrightarrow \sin(y) = x$

  Domain : $x \in [-1, 1]$, Range : $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- **arccos(x)** - Inverse of cosine

  $y = \arccos(x) \Leftrightarrow \cos(y) = x$

  Domain : $x \in [-1, 1]$, Range : $y \in [0, \pi]$
- **arctan(x)** - Inverse of tangent

  $y = \arctan(x) \Leftrightarrow \tan(y) = x$

  Domain : $x \in \mathbb{R}$, Range : $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

2.3.4 Hyperbolic Functions

- **sinh(x)** - Hyperbolic sine

  $\sinh(x) = \frac{e^x - e^{-x}}{2}$

  Domain : $\mathbb{R}$, Range : $\mathbb{R}$
- **cosh(x)** - Hyperbolic cosine

  $\cosh(x) = \frac{e^x + e^{-x}}{2}$

  Domain : $\mathbb{R}$, Range : $[1, +\infty[$
- **tanh(x)** - Hyperbolic tangent

  $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

  Domain : $\mathbb{R}$, Range :$(-1, 1)$

b. Property :

$$chz = cos(iz), \quad shz = -isin(iz)$$

2.3.5 Inverse Hyperbolic Functions

– **arcsinh(x)** - Inverse hyperbolic sine

arsinh(x) $= \ln(x + \sqrt{x^2 + 1})$

Domain : $\mathbb{R}$, Range : $\mathbb{R}$

– **arccosh(x)** - Inverse hyperbolic cosine

arcosh(x) $= \ln(x + \sqrt{x^2 - 1})$

Domain : $[1, +\infty)$, Range : $[0, +\infty[$

– **arctanh(x)** - Inverse hyperbolic tangent

artanh(x) $= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

Domain : $(-1, 1)$, Range :$\mathbb{R}$

2.3.6 Remarks

– Inverse functions are useful in solving integrals and appear in many areas of physics, signal processing, and geometry.

– Hyperbolic functions are commonly used in differential equations and describe the shape of hanging cables (catenary curves).

2.3.7 Logarithm function

a. Let $\forall z\mathbb{C}$, write $z = |z|e^{i\theta}$ where $\theta = argz$.

So $lnz = ln|z|e^{i\theta} = ln|z| + lne^{i\theta} = ln|z| + i\theta$.

Then $lnz = ln|z| + i\theta$.

Thus $Relnz = ln|z|$ and $Imlnz = \theta = argz$.

b. Property :

• $ln(z_1z_2) = lnz_1 + lnz_2$.

• $ln(\frac{z_1}{z_2}) = lnz_1 - lnz_2$.

• $ln(z^n) = nlnz$.

2.3.8 Common Antiderivatives

Function $f(x)$	**Interval**	**Primitive** $F(x)$
e^{ax}, $a \in \mathbb{C}^*$	$\mathbb{R}$	$\frac{e^{ax}}{a} + C$
$\cosh(ax)$, $a \in \mathbb{R}^*$	$\mathbb{R}$	$\frac{\sinh(ax)}{a} + C$
$\sinh(ax)$, $a \in \mathbb{R}^*$	$\mathbb{R}$	$\frac{\cosh(ax)}{a} + C$
$\cos(ax)$, $a \in \mathbb{R}^*$	$\mathbb{R}$	$\frac{\sin(ax)}{a} + C$
$\sin(ax)$, $a \in \mathbb{R}^*$	$\mathbb{R}$	$-\frac{\cos(ax)}{a} + C$
x^a, $a \neq -1$	$]0, +\infty[$	$\frac{x^{a+1}}{a+1} + C$
$\frac{1}{x^2+a^2}$, $a > 0$	$\mathbb{R}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{a^2-x^2}}$, $a > 0$	$]-a, a[$	$\arcsin\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{a^2+x^2}}$, $a > 0$	$\mathbb{R}$	$\operatorname{arsinh}\left(\frac{x}{a}\right) = \ln(x + \sqrt{x^2+a^2}) + C$
$\frac{1}{\sqrt{x^2-a^2}}$, $a > 0$	$]a, +\infty[$	$\operatorname{arcosh}\left(\frac{x}{a}\right) = \ln(x + \sqrt{x^2-a^2}) + C$
$\frac{1}{a^2-x^2}$, $a > 0$	$]-a, a[$	$\frac{1}{2a} \ln\left\lvert\frac{x+a}{x-a}\right\rvert + C$
$\tan(x)$	$]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi[$	$-\ln\lvert\cos(x)\rvert + C$
$\frac{1}{\sin(x)}$	$]k\pi, (k+1)\pi[$	$-\ln\lvert\tan(\frac{x}{2})\rvert + C$
$\frac{1}{\cos(x)}$	$]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi[$	$-\ln\lvert\tan(\frac{\pi}{4} + \frac{x}{2})\rvert + C$
$\ln(x)$	$]0, +\infty[$	$x\ln(x) - x + C$
$\arctan(x)$	$\mathbb{R}$	$x\arctan(x) - \ln(\sqrt{1+x^2}) + C$
$\arcsin(x)$	$]-1, 1[$	$x\arcsin(x) - \sqrt{1-x^2} + C$
$\arccos(x)$	$]-1, 1[$	$x\arccos(x) + \sqrt{1-x^2} + C$
$\operatorname{arsinh}(x)$	$\mathbb{R}$	$x\operatorname{arsinh}(x) - \sqrt{1+x^2} + C$
$\operatorname{arcosh}(x)$	$[1, +\infty[$	$x\operatorname{arcosh}(x) - \sqrt{x^2-1} + C$
$\operatorname{artanh}(x)$	$]-1, 1[$	$x\operatorname{artanh}(x) + \ln(\sqrt{1-x^2}) + C$

2.4 Exercises

Exercise 1 : Compute

1. $\int_0^2 (3x^2 + 2x)\,dx$
2. $\int_0^1 e^x\,dx$
3. $\int_0^\pi \sin x\,dx$
4. $\int_0^1 xe^x\,dx$
5. $\int_1^4 \frac{dx}{\sqrt{x}}$, using the substitution $x = t^2$

Exercise 2 : Compute

1. $\int x^2 \ln(x)\,dx$

2. $\int \frac{1}{(1+u^2)^2}\, du$. Let $u = \tan(x)$.
3. $\int \frac{x}{\sqrt{1+x}}\, dx$. Let $u = \sqrt{1+x}$.
4. $\int x\sqrt{1+x^2}\, dx$. Let $u = 1+x^2$.
5. $\int \arcsin(x)\, dx$.
6. $\int xe^{\sqrt{x}}\, dx$. Let $t = \sqrt{x}$.

Let

$$
\begin{aligned}
f : \mathbb{R}^n &\longrightarrow \mathbb{R} \\
x &\longmapsto f(x)
\end{aligned}
$$

where $x = (x_1, x_2, \cdots, x_n)$ Let $D \subset \mathbb{R}^n$. The area of f which projects onto D is called the integral of order n on D. It is noted

$$\int_D f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

2.5 Double Integral

2.5.1 Pavement case

Definition 2.5.1 *A domain D of $\mathbb{R}^2$ is said to be paved if it is written in the form :*

$$D = \{(x, y) \in \mathbb{R}^2 \ a \leq x \leq b, c \leq y \leq d\}$$

or

$$D = [a, b] \times [c.d]$$

where a, b, c and d are real constants.

Exemple 2.5.2 *To be constructed in class*

Definition 2.5.3 *The quaternion defined by*

$$I = \int_a^b \int_c^d f(x, y) dx dy = \int_a^b (\int_c^d f(x, y) dy) dx = \int_c^d (\int_a^b f(x, y) dx) dy$$

2.5.2 Fubini's Theorem

Exemple 2.5.4 *Let us calculate*

$$I = \int_D xy dx dy$$

where $D = [4, 8; 1, 2] = [4; 8] \times [1; 2]$

Remarque 2.5.5 *Let*

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x)$$

f *is said to be homogeneous of degree* $k \in \mathbb{N}$ *if* $\forall \alpha \in \mathbb{R}, f(\alpha x) = \alpha^k f(x)$

2.5.3 Case of any bounded domain

Definition 2.5.6 *A bounded domain* D *of* $\mathbb{R}^2$ *is defined as follows :*

$$D = \{(x, y) \in \mathbb{R}^2 \ a \leq x \leq b; \quad \varphi_1(x) \leq b \leq \varphi_2(x)\}$$

or

$$D = \{(x, y) \in \mathbb{R}^2 \ \psi_1 \leq x \leq \psi_2(x) \quad \text{and} \quad c \leq y \leq d\}$$

where a, b, c *and* d *are constants ;* $\varphi_1, \varphi_2, \psi_1$ *and* ψ_2 *are continuous functions*

Exemple 2.5.7 *To be constructed in class of examples of representation*

Exemple 2.5.8 *Compute*

$$I = \int\int_D e^{-y^2} dxdy \qquad \text{where} \qquad S = AOB \quad \text{with} \quad O(0,0); A(0,1); B(1,1)$$

$$J = \int\int_D x^2 y dxdy$$

where D *is limited by the lines* $y = -x^2$ *and* $x = y^2$

2.6 Change of variable

Let $\int\int_D f(x, y)dxdy$ be calculated. A change of variable is often motivated by the difficult access to a primitive of f or by a domain D where it is difficult to show the limits of x and y. We then define the change of variable as follows :

$$\varphi : \Delta \longrightarrow D$$
$$(u, v) \longmapsto (x, y) = (\varphi_1(u, v), \varphi_2(u, v))$$

where φ_1 and φ_2 are continuous functions. i.e.

$$x = \varphi_1(u, v) \quad \textit{and} \quad y = \varphi_2(u, v)$$

where φ_1 and φ_2 are continuous functions.

In this case

$$I = \int\int_{\Delta} f(\varphi_1(u, v), \varphi_2(u, v))|\mathcal{J}|dudv$$

where $\mathcal{J}$ denotes the Jacobian of φ in (x, y) with respect to (u, v). We have $\mathcal{J} = \frac{d\varphi(x,y)}{d(u,v)}$

2.6.1 Applications

Exemple 2.6.1 *Calculate*

$$\int\int_{D} (x^2 + y^2)dxdy$$

where D is bounded by the lines $y = x$ and $y = x\sqrt{3}$ and the arc of circle $x^2 + y^2 = 8$ of the 1st dial.

Hint : make a change of variable. (polar coordinates)

Exemple 2.6.2 *Calculate*

$$I_1 = \int\int_{D} (x^2 + y^2)dxdy$$

where

$$D = \{(x, y) \in \mathbb{R}^2 \quad | \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

Consider the change of variable : $x = aucos\theta$, $y = busin\theta$ with $u \geq 0$ and $\theta \in [0, 2\pi]$.

Exemple 2.6.3 *Calculate*

$$I_2 = \int\int_{\Delta} (x + y)dxdy$$

where

$$\Delta = \{(x, y) \in \mathbb{R}^2 \sqrt{x} + \sqrt{y} \geq 1; \sqrt{1 - x} + \sqrt{1 - y} \geq 1\}$$

Consider the change of variable.

2.7 Triple integrals

2.7.1 Examples

Exemple 2.7.1 *Calculate*

$$I_1 = \int\int\int_{D_1} dxdydz$$

where

$$D_1 = \{(x, y, z) \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq 9\}$$

Exemple 2.7.2 *Calculate*

$$I_2 = \int\int\int_{D_2} 3z^2 - x^2 - y^2 dxdydz$$

where

$$D_2 = \{(x, y, z) \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq R^2\}$$

Exemple 2.7.3 *Calculate*

$$I_3 = \int\int\int_{D_3} 3z^2 - x^2 - y^2 dxdydz$$

where

$$D_3 = \{(x, y, z) \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq 2Rz\}$$

Chapitre 3

First-order linear differential equations

3.1 General Presentation

In this part of the course, we will discuss differential equations.

What is a differential equation ?

Here is an example : $y' - 2xy = 5e^{x^2}$, where x denotes a variable, y a function, and y' its derivative.

This means we are looking for a **function** $y(x)$ defined on an interval I and satisfying :

$$y'(x) - 2xy(x) = 5e^{x^2}.$$

We can understand the terminology through this example. We have written above a mathematical expression containing an equality ; thus, it is an **equation**.

Moreover, this equation involves a derivative. The branch of mathematics that studies derivatives of functions is called **differential calculus**, and equations involving derivatives are called **differential equations**.

In general, a differential equation is an equation of the type $f(x, y, y') = 0$, where f is a given formula and y denotes the unknown function we are seeking.

To express this form in the previous example, we rewrite it as :

$$y' - 2xy - 5e^{x^2} = 0.$$

The **order** of a differential equation corresponds to the highest order of derivative appearing in the equation. Here, we have differentiated the function y **once**, resulting in a formula of the type $f(x, y, y') = 0$; thus, we say that the differential equation is of **first order**.

The equation : $y'' + 3xy + \cos(y) + x^5 = 0$ is a **second-order** differential equation.

Differential equations are equations linking a function and its derivative. This type of equation naturally arises in physics. For example, if $y(x)$ denotes the position of an object at time x, then $y'(x)$ represents its velocity, and $y''(x)$ its acceleration. A differential equation modeling such a phenomenon implies that the object's velocity depends on its position.

3.1.1 Differential Equations and Integration

You have already solved differential equations.

Exercise 1. *Solve the following differential equation :* $y' = x$.

Solution : $y' = x$ means $y'(x) = x$, which leads to

$$\int y'(x)\, dx = \int x\, dx,$$

$$y(x) = \frac{x^2}{2} + c, \quad \text{where } c \in \mathbb{R}.$$

This exercise demonstrates two things :

- Solving a differential equation of the type $y' = f(x)$ corresponds to finding the **antiderivative** of $f(x)$.
- A **differential equation** has infinitely many solutions because it includes a constant c. This is referred to as the **general solution** of the differential equation.

Exercise 2. *Find the function y satisfying*

$$y' = x, \quad \text{with } y(1) = 7.$$

3.1.2 Solutions of a Differential Equation

A differential equation, like any equation, does not necessarily have solutions !
This situation arises when **none** can be found, meaning the solution cannot be expressed in standard forms.
Example : $(y')^2 + 1 = 0 \Rightarrow (y')^2 = -1$.
This differential equation has no solutions. Indeed, throughout this course, it is implied that we consider only real-valued functions. Therefore, in all cases, $(y')^2$ is non-negative, whereas the equation requires it to be equal to -1, which is impossible. There also exist differential equations for which we cannot provide explicit formulas to express the solutions. For example : $y' - y^2 + x = 0$, another example : $y' = e^{x^2}$. It is possible to show that these *differential equations*

have solutions, but these cannot be expressed using standard elementary formulas involving polynomials, cosine, sine, exponentials, and logarithms. So how do

How Were These Drawings Obtained ?

The following idea is attributed to Euler (1707-1783).

At each point in the plane, we can compute and represent a direction vector of the tangent to an integral curve. We have seen that at $x = -1$ and $y = 3$, we have $y' = 10$.

Therefore, a direction vector of the tangent to the integral curve at this point is $(1, 10)$. We can thus draw the tangent to the integral curve passing through the point $(-1, 3)$.

Next, the idea is to say that locally, the curve resembles its tangent. So, we draw a "small segment" of the tangent in place of the curve. This yields the adjacent figure, where we have represented the vectors of the vector field to assist the illustration.

Then, we extend this drawing by repeating this process from the "end" of the tangent obtained on the curve. This results in the broken line shown in Figure 1.5.

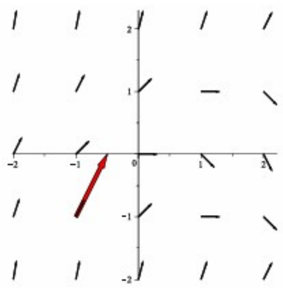

FIGURE 1.4 – Représentation d'une portion de tangente en un point.

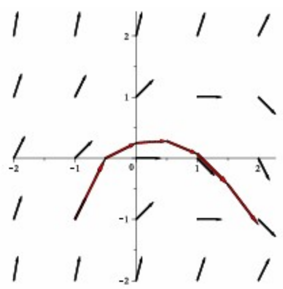

FIGURE 1.5 – Représentation d'une solution via la méthode d'Euler.

Thus, we have a broken line that resembles a solution. By taking smaller tangent segments, we obtain a smoother curve.

3.2 Methods for Solving First-Order Linear Differential Equations

In the following, we will present a method to obtain a formula giving the solutions of differential equations of a certain type.
We will consider *linear* first-order differential equations. These are equations of the type :

$$y' + \alpha(x)y = \gamma(x),$$

where α and γ are given functions.
The solution method consists of three steps, which we will illustrate using the example : $y'+2xy = 3x$. From this example, we will deduce the general method.

3.2.1 Homogeneous Equation

The homogeneous equation associated with $y' + \alpha(x)y = \gamma(x)$ is :

$$y' + \alpha(x)y = 0.$$

In our example, this gives : $y' + 2xy = 0$.

The first step is to solve the associated homogeneous equation. Here's how we proceed with the example $y' + 2xy = 0$.

First, we note that the constant function equal to 0 is a solution to this equation.
Indeed, if $y(x) = 0$, then $y'(x) = 0$, and we have :

$$0 + 2x \times 0 = 0,$$

so the equation is indeed *satisfied*. One way to view this is to say that 0 is a solution.

Now, consider a function y that is a solution to the differential equation and such that y is not the constant zero function.
Since two integral curves do not intersect, it follows that y will have a constant sign, meaning that y is either strictly **positive** or strictly **negative**.
Moreover, y satisfies the equation, so we have :

$$y'(x) + 2xy(x) = 0 \Rightarrow y'(x) = -2xy(x).$$

$$\Rightarrow \frac{y'(x)}{y(x)} = -2x, \quad \text{this division is possible since } y(x) \neq 0.$$

$$\Rightarrow \int \frac{y'(x)}{y(x)}dx = \int -2xdx = -x^2 + c_1, \quad c_1 \in \mathbb{R}.$$

$$\Rightarrow \ln(|y(x)|) = -x^2 + c_1 \Rightarrow |y(x)| = e^{-x^2+c_1} = e^{c_1}e^{-x^2}.$$

$$\Rightarrow y(x) = ce^{-x^2}, \quad \text{where } c \in \mathbb{R}.$$

We note that when $c = 0$, we obtain the constant function equal to 0, which had already been identified as a solution.

By applying this approach to the general case of the homogeneous equation $y' + \alpha(x)y = 0$, we obtain :

Theorem 3.2.1 *The solution, denoted $y_0(x)$, of the homogeneous equation $y' + \alpha(x)y = 0$ is of the form :*

$$y_0(x) = ce^{-A(x)}, \quad \textit{where } c \in \mathbb{R},$$

$$\textit{and } A(x) = \int \alpha(x)dx.$$

We denote the solution as y_0 to emphasize that the right-hand side of the differential equation under study is zero.
In our example, we have shown that $y_0(x) = ce^{-x^2}$.

3.2.2 Computing a Particular Solution

We will now see how to compute a particular solution of the equation $y' + \alpha(x)y = \gamma(x)$. In other words, we aim to find a specific solution among the infinite set of solutions to the differential equation under study. Since we are seeking a particular solution, we denote it by y_p.

How to compute y_p ?

Let's illustrate this with the example $y' + 2xy = 3x$. We have previously seen that ce^{-x^2} is a solution to $y' + 2xy = 0$. Moreover, the equation

$$(H): \quad y' + 2xy = 0$$

closely resembles the equation

$$(E): \quad y' + 2xy = 3x.$$

This suggests that the solutions of (E) will resemble those of (H).

The right-hand side of equation (H) is the constant zero, and its solutions are of the form ce^{-x^2}, where c is a constant.

The right-hand side of equation (E) is a non-zero function. We will therefore look for y_p in the form $y_p(x) = F(x)e^{-x^2}$, where F is a function to be determined.

Why does this approach work ?

We have set $y_p(x) = F(x)e^{-x^2}$. This gives :

$$y_p'(x) = F'(x)e^{-x^2} + F(x)(-2x)e^{-x^2}.$$

Moreover, y_p is a solution to the equation $y' + 2xy = 3x$, so $y_p' + 2xy_p = 3x$. Substituting y_p and y_p' with the expressions above, we get :

$$F'(x)e^{-x^2} - 2xF(x)e^{-x^2} + 2xF(x)e^{-x^2} = 3x.$$

Thus, $F'(x)e^{-x^2} = 3x$. Dividing both sides by e^{-x^2}, we obtain :

$$F'(x) = 3xe^{x^2}.$$

Therefore,

$$F(x) = \int 3xe^{x^2}dx.$$

This approach works in general :

Theorem 3.2.2 *Using the notation from the previous theorem :*

A particular solution of the differential equation $y' + \alpha(x)y = \gamma(x)$ is given by :

$$y_p(x) = F(x)e^{-A(x)}, \quad \text{where } F(x) = \int \gamma(x)e^{A(x)}dx.$$

Note the sign in front of $A(x)$.

Returning to our example :

$$F(x) = \int 3xe^{x^2}dx = \left[\frac{3}{2}2xe^{x^2}\right] = \frac{3}{2}\int 2xe^{x^2}dx = \frac{3}{2}e^{x^2}.$$

We used the formula $\int u'e^u dx = e^u$ with $u = x^2$.

Therefore, in our example :

$$y_p(x) = F(x)e^{-x^2} = \frac{3}{2}e^{x^2}e^{-x^2} = \frac{3}{2}.$$

3.2.3 General Solution

We will now see how to use the results from the two previous sections to provide the general solution of the differential equation.

Let y_G denote the general solution of the differential equation. In our example, this means :

$$y'_G(x) + 2xy_G(x) = 3x. \qquad (E_1)$$

On the other hand, we know that y_p is also a solution to this differential equation, so :

$$y'_p(x) + 2xy_p(x) = 3x. \qquad (E_2)$$

Subtracting (E_2) from (E_1), we obtain :

$$y'_G(x) - y'_p(x) + 2xy_G(x) - 2xy_p(x) = 0.$$

$$(y_G(x) - y_p(x))' + 2x(y_G(x) - y_p(x)) = 0.$$

This implies that $y_G - y_p$ is a solution to $(H) : y' + 2xy = 0$. Therefore, $y_G - y_p = y_0$ according to Theorem (3.2.1). Thus, $y_G = y_0 + y_p$.

This reasoning generalizes to :

Theorem 3.2.3 *Using the previous notation :*

The general solution of the differential equation $y' + \alpha(x)y = \gamma(x)$ is given by :

$$y_G = y_0 + y_p.$$

In our example, we obtain :

$$y_G(x) = ce^{-x^2} + \frac{3}{2}.$$

3.2.4 Tips

Superposition Principle

Exercise 3. *The equation $y' + 2xy = 3x$ has a particular solution $y_1(x) = \frac{3}{2}$. The equation $y' + 2xy = \cos(x)e^{-x^2}$ has a particular solution $y_2(x) = \sin(x)e^{-x^2}$. Find a particular solution for the equation $y' + 2xy = 3x + \cos(x)e^{-x^2}$.*

Solution :

We have $y'_1(x) + 2xy_1(x) = 3x$ and $y'_2(x) + 2xy_2(x) = \cos(x)e^{-x^2}$. Adding these two equations, we get :

$$y_1'(x) + 2xy_1(x) + y_2'(x) + 2xy_2(x) = 3x + \cos(x)e^{-x^2}$$

$$y_1'(x) + y_2'(x) + 2xy_1(x) + 2xy_2(x) = 3x + \cos(x)e^{-x^2}$$

$$(y_1(x) + y_2(x))' + 2x(y_1(x) + y_2(x)) = 3x + \cos(x)e^{-x^2}.$$

We conclude that $y_1 + y_2$ is a particular solution of $y' + 2xy = 3x + \cos(x)e^{-x^2}$.

Theorem 3.2.4 (Superposition Principle) *Let y_1 be a solution to $y' + \alpha(x)y = \gamma_1(x)$. Let y_2 be a solution to $y' + \alpha(x)y = \gamma_2(x)$. Then $y_1 + y_2$ is a solution of $y' + \alpha(x)y = \gamma_1(x) + \gamma_2(x)$.*

First-Order Linear Differential Equation with Constant Coefficients

When the function $\alpha(x)$ is constant, there are some tricks to find a particular solution without computing integrals. The idea is to look for a particular solution having the same form as $\gamma(x)$.

Proposition 3.2.5 *Let $y' + \alpha y = \gamma(x)$ be a differential equation with $\alpha \in \mathbb{R} \setminus \{0\}$. If $\gamma(x)$ is a polynomial, i.e., of the form :*

$$\gamma(x) = p_n x^n + \cdots + p_1 x + p_0,$$

where $p_n, \ldots, p_0 \in \mathbb{R}$, then one can look for $y_p(x)$ in the form :

$$y_p(x) = q_n x^n + \cdots + q_1 x + q_0,$$

with $q_n, \ldots, q_0 \in \mathbb{R}$.

Exercise 4. *Using the above proposition, solve :*

$$y' + 5y = x^2 + 3.$$

Proposition 3.2.6 *If $y' + \alpha y = \gamma(x)$ is a differential equation with $\alpha \in \mathbb{R}$, and if $\gamma(x)$ is of the form :*

$$\gamma(x) = (p_n x^n + \cdots + p_1 x + p_0)e^{\lambda x},$$

where $\lambda, p_n, \ldots, p_0 \in \mathbb{R}$, then :

– *If $y_0(x) \neq ce^{\lambda x}$, then we may seek $y_p(x)$ in the form :*

$$y_p(x) = (q_n x^n + \cdots + q_1 x + q_0)e^{\lambda x},$$

with $q_n, \ldots, q_0 \in \mathbb{R}$.

– *If $y_0(x) = ce^{\lambda x}$, then we may seek $y_p(x)$ in the form :*

$$y_p(x) = x(q_n x^n + \cdots + q_1 x + q_0)e^{\lambda x},$$

with $q_n, \ldots, q_0 \in \mathbb{R}$.

Exercise 5. *Using the previous proposition, solve the differential equation :*

$$y' + 5y = (x^2 + x + 1)e^{-3x}.$$

Proposition 3.2.7 *Let $y' + \alpha y = \gamma(x)$ be a differential equation where $\alpha \in \mathbb{R}$. If $\gamma(x)$ is of the form :*

$$\gamma(x) = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))P(x),$$

where $A, B \in \mathbb{R}$ and $P(x)$ is a polynomial, then we seek $y_p(x)$ in the form :

$$y_p(x) = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))Q(x),$$

where $Q(x)$ is a polynomial of the same degree as P.

Exercise 6. *Using the previous proposition, solve the differential equation :*

$$y' + 5y = e^{-3x}(53\cos(7x) + 53\sin(7x)).$$

3.3 Exercises

Exercise 1. Provide a differential equation for which the function $\cos(x)e^{x^2}$ is a solution.

Exercise 2. Is $\frac{1}{4} - \frac{1}{2}x + \frac{1}{2}x^2 + e^{-2x}$ a solution to the equation $y' + 2y = x^2$? Justify your answer.

Exercise 3. Solve :

1. $y' + \cos(x)y = 0$.
2. $xy' - 7y = 0$, where $x > 0$.

Exercise 4. Solve :

1. $xy' + 2y = x^4$, where $x > 0$.
2. $xy' + 2y = e^{x^2}$, where $x > 0$.

3. $xy' + 2y = x^4 + e^{x^2}$, where $x > 0$.

4. $(1 + e^x)y' + e^x y = 1 + e^x$, where $x \in \mathbb{R}$.

5. Provide the solution to the differential equation $(1+e^x)y'+e^x y = 1+e^x$ satisfying $y(0) = \frac{5}{2}$.

Exercise 5. Solve :

1. $y' - 2y = 2x + 1$.

2. $y' - 2y = 3e^{5x}$.

3. $y' - 2y = \cos(3x) - 5\sin(3x)$.

4. $y' - 2y = 17e^{2x}$.

5. $y' - 2y = \cos(x)e^{2x}$.

Chapitre 4

Second-order linear differential equations with constant coefficients

4.1 Generalities

In this chapter, we study second-order differential equations, that is, of the form $f(x, y, y', y'') = 0$.

In general, we do not know how to solve such equations. As with first-order equations, it can be shown that some second-order differential equations have solutions that cannot be expressed using standard elementary functions. We will therefore limit our study to equations of the following type :

$$(E): \quad y'' + \alpha y' + \beta y = \gamma(x), \quad \text{where } \alpha, \beta \in \mathbb{R}, \text{ and } \gamma(x) \text{ is a given function.}$$

This type of equation appears in physics when studying oscillatory phenomena (e.g., the motion of a mass on a spring, a pendulum, or the voltage in an RLC electrical circuit).

We already know how to solve a special case of second-order differential equations. When both α and β are zero, solving the differential equation amounts to computing two successive antiderivatives.

Exercise 1. *Solve the differential equation : $y'' = x^5$.*

Solution : We have $y'(x) = \int y''(x)\, dx = \int x^5\, dx = \frac{x^6}{6} + c_1$, with $c_1 \in \mathbb{R}$.

We then deduce :

$$y(x) = \int y'(x)\, dx = \int \left(\frac{x^6}{6} + c_1\right) dx = \frac{x^7}{42} + c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

This example illustrates a more general phenomenon. It can be shown (under assumptions that are always satisfied in this course) that the solution of a second-order differential equation depends on two parameters.

Reminder : For first-order equations, we only had one constant of integration.

To illustrate second-order differential equations graphically, we can use the following trick : Set $y' = z$, so that $y'' = z'$. This gives :

$$z' = y'' = -\alpha y' - \beta y + \gamma(x) = -\alpha z - \beta y + \gamma(x).$$

So we get :

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ -\alpha z - \beta y + \gamma(x) \end{pmatrix}.$$

This means that for every point $(x, y, z) \in \mathbb{R}^3$, we can compute a direction vector of a curve in space.
We will not explore this perspective further in this course.

4.2 Solving the equation

As in the first-order case, the general solution of a second-order differential equation is the sum of the solution of the associated homogeneous equation and a particular solution. The proof is identical.

Theorem 4.2.1 *The general solution y_G of the differential equation*

$$y'' + \alpha y' + \beta y = \gamma(x),$$

is given by

$$y_G = y_0 + y_p,$$

where y_0 is the solution to the associated homogeneous equation :

$$(H): \quad y'' + \alpha y' + \beta y = 0,$$

and y_p is a particular solution.

4.2.1 Solving the associated homogeneous equation

In the previous chapter, we saw that the solution to the first-order differential equation $y' + \alpha y = 0$ is $y_0 = ce^{-\alpha x}$, with $c \in \mathbb{R}$. Let us now see whether second-order equations also admit solutions of this type.

Assume $y(x) = e^{rx}$, where $r \in \mathbb{R}$. Then $y'(x) = re^{rx}$, and $y''(x) = r^2 e^{rx}$. This leads to :

$$y''(x) + \alpha y'(x) + \beta y(x) = r^2 e^{rx} + \alpha r e^{rx} + \beta e^{rx} = (r^2 + \alpha r + \beta) e^{rx} = 0.$$

Since the exponential function is never zero, we obtain :

Theorem 4.2.2 *The function e^{rx} is a solution of (H) if and only if r is a solution of :*

$$r^2 + \alpha r + \beta = 0,$$

This equation is called the characteristic equation associated with (H).

We've seen that the solution of a second-order differential equation depends on two parameters. This means we need two linearly independent functions to describe the full solution space.

Let $\Delta = \alpha^2 - 4\beta$ be the discriminant of the characteristic equation.

• If $\Delta > 0$:

We have two distinct real roots r_1 and r_2 :

$$r_1 = \frac{-\alpha - \sqrt{\Delta}}{2}, \quad r_2 = \frac{-\alpha + \sqrt{\Delta}}{2}.$$

By Theorem (4.2.2), $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ are solutions. Therefore, all solutions of (H) are given by :

$$y_0(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

• If $\Delta = 0$:

We have a single root r_0 :

$$r_0 = \frac{-\alpha}{2}.$$

Then $e^{r_0 x}$ is a solution, but we need a second linearly independent solution. It turns out $xe^{r_0 x}$ is also a solution.

Indeed, let $y(x) = xe^{r_0 x}$. Then :

$$y'(x) = (1 + r_0 x)e^{r_0 x}, \quad y''(x) = (2r_0 + r_0^2 x)e^{r_0 x}.$$

So :

$$y'' + \alpha y' + \beta y = (2r_0 + r_0^2 x)e^{r_0 x} + \alpha(1 + r_0 x)e^{r_0 x} + \beta x e^{r_0 x} = \left((r_0^2 + \alpha r_0 + \beta)x + (2r_0 + \alpha)\right) e^{r_0 x}.$$

Since r_0 is a solution of the characteristic equation and $r_0 = \frac{-\alpha}{2}$, the expression simplifies to 0, confirming that $xe^{r_0 x}$ is a solution.

Therefore, when $\Delta = 0$, the general solution is :

$$y_0(x) = (c_1 + c_2 x)e^{r_0 x}.$$

- If $\Delta < 0$:

We have two complex conjugate roots : $r_1 = a + ib$, $r_2 = a - ib$, where

$$a = \frac{-\alpha}{2}, \quad b = \frac{\sqrt{|\Delta|}}{2}.$$

The solutions from Theorem (4.2.2) are $e^{(a+ib)x}$ and $e^{(a-ib)x}$, and the general solution is :

$$y_0(x) = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}.$$

To obtain real-valued solutions, we use Euler's formula :

$$\frac{1}{2}e^{(a+ib)x} + \frac{1}{2}e^{(a-ib)x} = e^{ax}\cos(bx),$$

and similarly,

$$\frac{1}{2i}e^{(a+ib)x} - \frac{1}{2i}e^{(a-ib)x} = e^{ax}\sin(bx).$$

Thus, when $\Delta < 0$, the general real solution is :

$$y_0(x) = e^{ax}\left(c_1\cos(bx) + c_2\sin(bx)\right).$$

Summary :

Theorem 4.2.3 *Let $\Delta = \alpha^2 - 4\beta$ be the discriminant of the characteristic equation $r^2 + \alpha r + \beta = 0$.*

1. If $\Delta > 0$*, the solutions of* (H) *are :*

$$y_0(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x},$$

where

$$r_1 = \frac{-\alpha - \sqrt{\Delta}}{2}, \quad r_2 = \frac{-\alpha + \sqrt{\Delta}}{2},$$

and $c_1, c_2 \in \mathbb{R}$*.*

2. If $\Delta = 0$*, the solutions of* (H) *are :*

$$y_0(x) = (c_1 + c_2 x)e^{r_0 x}, \quad \textit{where } r_0 = \frac{-\alpha}{2}, \textit{ and } c_1, c_2 \in \mathbb{R}.$$

3. If $\Delta < 0$*, the solutions of* (H) *are :*

$$y_0(x) = e^{ax}\left(c_1 \cos(bx) + c_2 \sin(bx)\right),$$

where

$$a + ib = \frac{-\alpha \pm i\sqrt{|\Delta|}}{2}, \quad \textit{and } c_1, c_2 \in \mathbb{R}.$$

Exercise 2. *Solve the differential equation :* $y'' + y' - 2y = 0$.

Exercise 3. *Solve the differential equation :* $y'' - 6y' + 9y = 0$.

Exercise 4. *Solve the differential equation :* $y'' - 4y' + 29y = 0$.

Computing a particular solution

As seen in the previous chapter, when coefficients are constant, we can seek a particular solution with the same form as the nonhomogeneous term $\gamma(x)$. This technique still applies here.

Theorem 4.2.4 *Assume that* $\gamma(x)$ *is :*

- *a polynomial, i.e.,* $\gamma(x) = p_n x^n + \cdots + p_1 x + p_0$*,*
- *an exponential, i.e.,* $\gamma(x) = e^{ax}$*,*
- *a linear combination of cosine and sine, i.e.,*

$$\gamma(x) = A\cos(\omega x) + B\sin(\omega x),$$

- *or a product of such functions.*

If $\gamma(x)$ *is not of the form* $P(x)y_0(x)$ *(with* $P(x)$ *a polynomial), then we can search for a particular solution* $y_p(x)$ *of the same form as* $\gamma(x)$*.*

If $\gamma(x) = P(x)y_0(x)$, we instead look for $y_p(x)$ as :

- $xQ(x)y_0(x)$ *when* $\Delta \neq 0$,
- $x^2Q(x)y_0(x)$ *when* $\Delta = 0$.

Exercise 5. *Solve the differential equation :*

$$y'' + y' - 2y = x^2 + 2x + 3.$$

Exercise 6. *Solve the differential equation :*

$$y'' + y' - 2y = (6x + 4)e^{-2x}.$$

4.3 Method of variation of parameters

4.3.1 Introduction

The method of variation of parameters is a technique used to determine a particular solution of a non-homogeneous linear differential equation. It is particularly useful when the method of undetermined coefficients does not apply.

4.3.2 Second-order linear differential equation

Consider the following non-homogeneous linear differential equation :

$$y'' + ay' + by = g(x) \tag{4.1}$$

where a and b are constants, and $g(x)$ is a given function.

The associated homogeneous equation is :

$$y'' + ay' + by = 0 \tag{4.2}$$

The general solution of this homogeneous equation is :

$$y_h(x) = C_1y_1(x) + C_2y_2(x) \tag{4.3}$$

where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions.

4.3.3 Principle of the method

We look for a particular solution of the form :

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{4.4}$$

where $u_1(x)$ and $u_2(x)$ are functions to be determined.

Differentiating :

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \tag{4.5}$$

To simplify, we impose the condition :

$$u_1'y_1 + u_2'y_2 = 0 \tag{4.6}$$

This prevents the appearance of second derivatives. Substituting y_p' and y_p'' into the original equation, we obtain the system :

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(x) \end{cases} \tag{4.7}$$

We solve this system to find $u_1'(x)$ and $u_2'(x)$, then integrate to find $u_1(x)$ and $u_2(x)$. Finally, we obtain the particular solution $y_p(x)$.

4.3.4 Example application

Solve the equation :

$$y'' - y = e^x \tag{4.8}$$

Step 1 : Solve the Homogeneous Equation

The associated homogeneous equation is :

$$y'' - y = 0 \tag{4.9}$$

The characteristic equation is $r^2 - 1 = 0$, so $r = \pm 1$. The general solution is :

$$y_h = C_1e^x + C_2e^{-x} \tag{4.10}$$

Step 2 : Apply the Method

Assume :

$$y_p(x) = u_1(x)e^x + u_2(x)e^{-x} \tag{4.11}$$

Solving the associated system gives :

$$u_1 = \frac{e^x}{2}, \quad u_2 = -\frac{e^x}{2} \tag{4.12}$$

Hence the particular solution is :

$$y_p = \frac{e^{2x}}{2} - \frac{1}{2} \tag{4.13}$$

The general solution is :

$$y = C_1e^x + C_2e^{-x} + \frac{e^{2x}}{2} - \frac{1}{2} \tag{4.14}$$

4.4 Equations with separable variables

4.4.1 Definition and method

A differential equation is said to be **separable** if it can be written in the form :

$$\frac{dy}{dx} = f(x)g(y) \tag{4.15}$$

We separate the variables :

$$\frac{dy}{g(y)} = f(x)dx \tag{4.16}$$

Then we integrate both sides.

Example 1 : Simple equation

Solve :

$$\frac{dy}{dx} = xy \tag{4.17}$$

Separate the variables :

$$\frac{dy}{y} = xdx \tag{4.18}$$

Integrating :

$$\ln|y| = \frac{x^2}{2} + C \tag{4.19}$$

Hence the general solution is :

$$y = Ce^{x^2/2} \tag{4.20}$$

Example 2 : Equation with initial condition

Solve :

$$\frac{dy}{dx} = \frac{x}{1+y^2} \tag{4.21}$$

with initial condition $y(0) = 1$.

Step 1 : Separate the variables

$$(1+y^2)dy = xdx \tag{4.22}$$

Step 2 : Integrate

$$\int (1+y^2)dy = \int xdx \tag{4.23}$$

$$y + \frac{y^3}{3} = \frac{x^2}{2} + C \tag{4.24}$$

Step 3 : Apply the initial condition

When $x = 0, y = 1$:

$$1 + \frac{1}{3} = C \Rightarrow C = \frac{4}{3} \tag{4.25}$$

The final solution is :

$$y + \frac{y^3}{3} = \frac{x^2}{2} + \frac{4}{3} \tag{4.26}$$

1. The Bernouilli equation

It takes the general form

$$y' = a(x)y + b(x)y^\alpha, \qquad \alpha \in \mathbb{R} \ .$$

The solution of such an equation requires the following steps :

Let's first put $u = y^{1-\alpha}$. The equation becomes $\frac{1}{1-\alpha}u' = a(x)u + b(x)$.

Finally, this last equation is solved by the usual techniques for solving 1st order linear DE.

Exemple 4.4.1 *Solve on* $]-\frac{\pi}{2}, \frac{\pi}{2}[$, $y'cos(x) + ysin(x) + y^3 = 0$. *Let's put* $u = y^{-2}$

2. Riccati's equation

It is in the form

$$y' = A(x)y + B(x)y^2 + C(x).$$

Search for a particular solution y_p.

Change variables by posing $y = y_p + \frac{1}{u}$. We thus obtain an equation of order 1 given by :

$$u' + 2[y_p B(x) + A(x)]u = -B(x).$$

Exemple 4.4.2 $y'(y-1)(xy-y-x)$.

For $y=1$, the solution is obvious.

For $y \neq 1$, let $u = \frac{1}{y-1}$.

4.5 Conclusion

The method of variation of parameters is a powerful technique for solving non-homogeneous differential equations. Separable differential equations form a simpler class and can be solved directly through integration.

4.6 Exercises

Exercise 1. *Solve the following differential equations :*

1. $y'' - 3y' - 4y = (12x+7)e^{2x}$,
2. $y'' - 3y' - 4y = (-50x+5)e^{-x}$,
3. $y'' - 4y' + 4y = 9e^{-x}$,
4. $y'' - 4y' + 4y = 6e^{2x}$,
5. $y'' - 4y' + 4y = 9e^{-x} + 6e^{2x}$,
6. $y'' + 4y' + 13y = 40\sin(3x)$,
7. $y'' + y = 4\cos(x) - \sin(x)$,
8. $y'' + 4y = 2x^2$.

Exercise 2. *The goal of this exercise is to show why, when $\Delta = 0$, the "additional" solution to $e^{\alpha x}$ is of the form $xe^{\alpha x}$. We will explore this using the example :*

$$y'' - 8y' + 16y = 0.$$

1. Solve the differential equation $y'' - 8y' + 16y = 0$.
2. Consider the differential equation $(E_h) : y'' + (8+h)y' + (16+4h)y = 0$.

   Solve this equation when $h > 0$.
3. Deduce that the function $y_h(x) = \frac{1}{h}e^{(4+h)x} - \frac{1}{h}e^{4x}$ is a solution to (E_h).
4. As $h \to 0$, what equation does (E_h) approach ?
5. As $h \to 0$, what is the limit of $y_h(x)$?

Chapitre 5

Complex functions

5.1 Introduction

In this chapter, we study complex functions with complex variables, i.e.

$$f : U \subset \mathbb{C} \longrightarrow V \subset \mathbb{C}$$
$$z \longmapsto f(z)$$

Let $z = x + iy$, $x, y \in \mathbb{R}$ can be written. We can therefore identify $\mathbb{C}$ with $\mathbb{R}^2$.
•All complex functions $f(z)$ allows to define two real functions of two real variables P and Q, such that

$$f(z) = P(x, y) + iQ(x, y)$$

. • Reciprocally, giving two real functions of two variables allows us to define a complex function with complex variables.

Exemple 5.1.1 *Let the function f below be :*

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto f(z) = sin(z)$$

Find P and Q such that $f(z) = P(x, y) + iQ(x, y)$.

5.2 Study of complex functions with complex variables

5.2.1 Limits and continuity

Definition 5.2.1 ***i.*** *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z_0, l \in \mathbb{C}$; we say that f admits a limit l in z_0 if :*

$$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0, \quad tel \quad que \quad \forall z \in \mathbb{C}, \quad z \neq z_0 \quad alors \quad |z - z_0| < \eta_\varepsilon \Rightarrow |f(z) - l| < \varepsilon$$

ii. *Let $z_0 \in D_f$. We say that f is continuous in z_0 if*

$$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0, \quad tel \quad que \quad \forall z \in D_f, \quad z \neq z_0 \quad alors \quad |z - z_0| < \eta_\varepsilon \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

We note $\lim_{z \to z_0} f(z) = f(z_0)$

Proposition 5.2.2

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto f(z) = P(x,y) + iQ(x,y) \end{aligned}$$

with $z = x + iy$, $l = a + ib$ and $z_0 = x_0 + iy_0$.
i. $\lim_{z \to z_0} f(z) = l \Leftrightarrow \lim_{(x,y) \to (x_0,y_0)} P(x,y) = a$ *and* $\lim_{(x,y) \to (x_0,y_0)} Q(x,y) = b$.
ii. *f is continuous in z_0 if and only if P and Q are continuous in (x_0, y_0).*

Remarque 5.2.3 *The properties concerning the sum, product and quotient of limits and those concerning continuous functions are stated and proved as in the case of real functions with real variables.*

Exemple 5.2.4 *Let $f(z) = \frac{Re(z)}{z}$. Calculate $\lim_{z \to 0} f(z)$.*

5.2.2 Derivative

Definition 5.2.5 *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $D_f \subset D(z_0, R)$. (Reminder :$\subset D(z_0, R)$: an open disc of centre z_0 and radius R) f is said to be derivable at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is finite. We note $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ and is called the derivative of f in z_0 as in the case of real functions.*

5.2.3 The Euler formulas

$$\begin{cases} e^{it} = cost + isint \quad (1) \\ e^{-it} = cost - isint \quad (2) \end{cases}$$

Summing (1) and (2), we have :

$$cos(t) = \frac{e^{it} + e^{-it}}{2}$$

Subtracting equations (1) and (2), we obtain

$$sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

5.2.4 Equation of circle (in $\mathbb{C}$)

Let $z, z_0 \in \mathbb{C}$. Then the equation of the circle with centre z_0 and radius r is written

$$z = z_0 + re^{it}, \quad t \in [0, 2\pi]$$

Preuve : *To be demonstrated in class*

5.3 Holomorphic functions

Definition 5.3.1 *Let U be an open of $\mathbb{C}$, $z_0 \in U$ and $f : U \longrightarrow \mathbb{C}$.*
If f is differentiable at the point z_0, we say that f is holomorphic in z_0.
We say that f is holomorphic on U if it is holomorphic at any point z_0 in U.

Proposition 5.3.2 ***i.*** *Let $f, h : U \subset \mathbb{C} \longrightarrow \mathbb{C}$, holomorphic in $z_0 \in U$. Then $f+h, fh, \frac{f}{h}, (h(z_0) \neq 0)$ are holomorphic in z_0.*
ii. *Let $f : U \longrightarrow f(U) \subset V$ and $g : V \longrightarrow \mathbb{C}$ or U, V are open of C. If f is holomorphic in $z_0 \in U$ and g holomorphic in $f(z_0) \in V$ then $g \circ f$ is holomorphic in $z_0 \in U$.*
iii. *Cauchy-Riemann conditions Let $f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$, $z \in \mathbb{C}$ such that $= x+iy$ and $z_0 = x_0+iy_0$ such that f is holomorphic in z_0.*
Suppose that $f(z) = P(x, y) + iQ(x, y)$, then :

$$\begin{cases} \frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0) = f'(z_0) \\ \frac{\partial P}{\partial y}(x_0, y_0) = -\frac{\partial Q}{\partial x}(x_0, y_0) = if'(z_0) \end{cases} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$$

Remarque 5.3.3 ***f holomorphic in $z_0 \Rightarrow$ Cauchy Riemann condition***

Proposition 5.3.4 *Let $f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$, $z_0 = x_0 + y_0$ and $f(x, y) = P(x, y) + iQ(x, y)$. Then the following propositions are equivalent :*
i. *f is holomorphic at the point z_0 ;*

ii. *Considered as a function of two real variables, f is differentiable at (x_0, y_0) and f satisfies the Cauchy-Riemann conditions.*

Definition 5.3.5 ***Notion of harmonic function*** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, a function of several real variables.*
We say that f is harmonic if it is of class c^2 and its Laplacian is zero. ***Remember :*** *The Laplacian of f, denoted Δf is defined by :*

$$\Delta f = \sum_{i=i}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Proposition 5.3.6 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, holomorphic on U such that $f(x, y) = P(x, y) + iQ(x, y)$, then P and Q are harmonic.*

Preuve : *Exercise*

$$\boxed{\text{End}}$$

Analysis II tutorials

Exercise : 1 Revision exercises :

Give the limited expansion to order n of the functions in x_0.

a) $f(x) = ln\frac{sinx}{x}$, $x_0 = 0$ et $n = 5$.

b) $f(x) = exp(1 - cosx)$, $x_0 = 0$ et $n = 4$.

c) $f(x) = (cosx)^{sinx}$, $x_0 = 0$ et $n = 5$.

d) $f(x) = \frac{1}{x}ln(coshx)$, $x_0 = 0$ et $n = 4$.

e) $f(x) = arccos(\frac{1+x}{2+x})$, $x_0 = 0$ et $n = 2$.

f) $f(x) = ln\frac{sinx}{x}$, $x_0 = 0$ et $n = 5$.

g) $f(x) = \frac{ln(x)}{(1-x)^2}$, $x_0 = 1$ et $n = 3$.

Exercise : 2

1. Represent the following sets :

a) $\{(x, y) \in \mathbb{R}^2 | x \geq 1, y \geq 0, x + y - 3 \leq 0\}$

b) $\{(x, y) \in \mathbb{R}^2 | x^2 + v^2 \leq 4, 3y + x - 2 \leq 0, 3x + y - 12 \leq 0\}$

2. Determine and represent the definition set of the functions :

a) $f(x, y) = ln(\frac{4-x^2-y^2}{1+y^2})$.

b) $g(x) = \frac{ln(x)}{\sqrt{x(y+1)}}$

3. Determine the set of definition and then determine the first partial derivatives of the functions defined by :

a) $f(x, y) = ln(1 + \frac{x}{y})$.

b) $g(x, y) = \frac{ln(x)}{x^2+y^2-9}$

Exercise : 3

Determine and represent the largest possible domain of definition for the following functions :

1. $f(x, y) = \frac{\sqrt{x}y}{x^2+y^2}$,
2. $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$,
3. $f(x, y) = ln(xy)$,
4. $f(x, y) = xln(y^2 - x)$,
5. $f(x, y) = \sqrt{4x - x^2 + 4y - y^2}$.

Exercice : 4

For each of the following functions : calculate : $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$, $\frac{\partial^2 f}{\partial x^2}(x,y)$, $\frac{\partial^2 f}{\partial y^2}(x,y)$, $\frac{\partial f}{\partial x \partial y}(x,y)$

1. $f(x,y) = x^2 - 6xy - 6y^2 + 2x + 24y$,
2. $f(x,y) = x^2 + 2y^2 - \frac{x^3}{y}$,
3. $f(x,y) = e^{2x^2+xy+7x+y^2}$,
4. $f(x,y) = sin(xy)$,
5. $f(x,y) = ln(x+y)$.

Exercise : 5

Determine the local extrema of the functions and the saddle points :

1. $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$,
2. $f(x,y) = -5x^2 + 4xy - y^2 + 16x + 10$,
3. $f(x,y) = x^2 - y^2 + 4x - 4y - 8$,
4. $f(x,y) = xe^{-x^2-y^2}$,
5. $f(x,y) = (x^2 + y^2)^{\frac{1}{3}} + 1$,
6. $f(x,y) = x^2 + y^4$.

Exercise : 6

In order to treat a bacterial infection, the joint use of two chemical compounds is used. Studies have shown that in the laboratory the duration of infection can be modelled by

$$D(x,y) = x^2 + 2y^2 - 18x - 24y + 2xy + 120$$

where x is the dosage in mg of the first compound and y the dosage in mg of the second. How can the duration of the infection be minimised ?

Exercise : 7

Let f be the function defined by :

$$f(z) = \begin{cases} \frac{\bar{z}^2}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Show that f is continuous everywhere in $\mathbb{C}$ but is not analytic at any point on $\mathbb{C}$.

Exercise : 8

Let $g : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ defined by :

$$g(x+iy) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Is the function g holomorphic on $\mathbb{C} \setminus \{0\}$?

Exercise : 9

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ defined by :

$$g(x+iy) = e^{2x}[cos(2y) + isin(2y)] - iy + x$$

Is the function g holomorphic on $\mathbb{C}$?

Exercise : 10

Let $z = x + iy$ where x and y are two real numbers and let the function

$f = (-e^x siny + 3) + i(e^x cosy + 5)$.

Show that f is analytic (holomorphic) in $\mathbb{C}$.

Exercise : 11

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ defined by :

$$g(x+iy) = x^2 - y^2 - 2ixy + 2x + 2iy$$

Is the function g holomorphic ?

Exercise : 12

Let $z = x + iy$ and V fonction defined by :

$$V : (x, y) \longrightarrow xy^2 - \frac{1}{3}x^3$$

1. Show that V is harmonic.

2. Find a function U such that the complex function $f(x, y) = U(x, y) + iV(x, y)$ be holomorphic.

Exercise : 13

Soit $z = x + iy$ où x et y sont deux réels et soit la fonction

$$f(z) = ax + iy + ie^z$$

1. Mettre $f(z)$ sous la forme de $U(x, y) + iV(x, y)$.

2. Déterminer la constante a pour que la fonction $f(z)$ soit holomorphe.

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