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First year / Semester I

Analysis 1 course

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General introduction

Welcome to the Analysis 1 course, an essential component of your mathematical training for the first year, Semester I, in the Computer Science (CS) program. This course is designed to introduce you to the fundamental concepts of real analysis, a branch of mathematics that is crucial for your academic and professional development, particularly in fields such as computer science, applied sciences, and engineering. Analysis 1 will provide you with a rigorous understanding of the foundational concepts that underpin many branches of mathematics, statistics, and theoretical computer science. The course is structured around core concepts such as the properties of real numbers, bounds, intervals, and numerical sequences. These concepts are vital as they form the groundwork for more advanced theories like continuity, differentiability, and infinite series.

Course Objectives

Throughout this semester, we will:

Explore Real Numbers: We will begin by studying the real numbers in depth, focusing on their definition, properties, and central role in analysis. You will learn to manipulate concepts such as upper and lower bounds, majorization, and minorization, understanding their significance in various mathematical contexts.

Study Intervals: We will develop the concepts of intervals and their characterization, introducing key notions such as the density of rationals in the reals and the fundamental Archimedean property. These tools are crucial for building the foundation of real analysis and topology.

Analyze Numerical Sequences: The course will also focus on numerical sequences, starting with fundamental concepts like convergence, limits, and asymptotic behavior. You will work with convergent and divergent sequences, which are essential for understanding series and limits in more advanced contexts.

Deepen Topological Concepts: A significant part of the course will be dedicated to the topology of $\mathbb{R}$ (the set of real numbers), where we will cover notions such as adherence, convergence, and continuity in the context of sequences and series.

Course Methodology and Structure

The course will be structured through lectures, tutorial sessions, and practical exercise classes. Each session is designed to help you develop a deep understanding of both the theoretical aspects of analysis and the practical skills needed to apply these concepts to real-world problems. The lectures will provide the foundational theory, while the tutorial sessions and exercises will give you hands-on experience solving problems.

Tutorial sessions are an important opportunity for you to ask questions, discuss challenges, and collaborate with peers on specific exercises. These sessions are key for reinforcing your ability to think critically and reason mathematically, which will be crucial throughout your studies and in your future career.

Importance of the Course

This course is foundational to your academic journey. The rigor and logic that underlie real analysis are transferable skills that will serve you throughout your university education and beyond. The concepts you will learn in this course will be applied in numerous fields within computer science, such as algorithms, graph theory, numerical analysis, cryptography, and more.

The analytical skills you develop here will not only help you solve complex mathematical problems but also equip you with the logical reasoning needed to tackle challenges in many scientific and technological domains. This type of reasoning is essential in a wide range of applications, from developing software algorithms to conducting advanced research.

Chapter 1

Real numbers

1.1 Introduction

Up to now, you have encountered different types of numbers: first the natural integers, from early childhood, and then in secondary school the relative integers and the rationals. You have noted $\mathbb{N}$ the set of natural integers, $\mathbb{Z}$ that of the relative integers, and $\mathbb{Q}$ that of the rationals. By identifying the natural integers with the positive relative integers, you have written $\mathbb{N}$, then by identifying the relative integers with the rational fractions whose denominator is 1, you have written also wrote: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

In $\mathbb{Q}$ you can add and subtract, multiply and divide. You can also compare any two rational numbers. You know how to situate these numbers on a straight line: all you have to do is choose an origin (which will represent the number 0), a unit of length and a direction of travel (generally from left to right). This is called the "numerical line": the rational number x is represented by the point of abscissa x on the line.

The following question then arises: does any point on the number line have a rational number as its abscissa ? The answer is no: we can construct a square whose side has length 1; the diagonal of this square has length l which verifies $l^2 = 1^2 + 1^2 = 2$ (Pythagorean Thesis). It to determine a point of abscissa l. Now $l \notin \mathbb{Q}$ because :

Proposition 1.1.1 *There is no rational number whose square is* 2.

Proof: *Let us reason by the absurd: let us assume the opposite.*

There is an irreducible fraction $\frac{p}{q}$ *such that* $2 = \frac{p^2}{q^2}$. *But then* $p^2 = 2q$, *so* p^2 *is even. Since only even numbers have an even square,* p *is even and is written* $p = 2k$. *Therefore*

$2q^2 = 4k^2$ so q^2 is even, and q is even. This is absurd since the fraction $\frac{p}{q}$ is irreducible.
$\square$

We then intuitively introduce the set of real numbers (which we note $\mathbb{R}$) as the set of abscissa of all the points of the numerical line. Thus $\mathbb{Q}$. The elements of $\mathbb{R} - \mathbb{Q}$ (such as $l = \sqrt{2}, e$ or π) are called irrational numbers. The set $\mathbb{R}$ is identified with the numerical line and is called "point" or "real number".
In $\mathbb{R}$ you can add, subtract, multiply and divide, and also compare any two real numbers. This geometric representation of real numbers is very useful, but to do Analysis rigorously it is necessary to specify the fundamental properties of $\mathbb{R}$.

1.2 Major, minor, maximum, minimum, upper bound, lower bound

Definition 1.2.1 *Let A be a non-empty part of $\mathbb{R}$, and m a real number. We say that*
1. m is a majorant of A when $a \leq m$ for all $a \in A$.
2. m is a minorant of A when $m \leq a$ for all $a \in A$.

Remark 1.2.2 ***Examples***
1. If m is a majorant of A, then any real $m' \geq m$ is also a majorant of A.
If m is a minorant of A, then any $m' \leq m$ is also a minorant of A.
2. The set $A = [3, +\infty[$ has no majorant, 3 is a minorant of A.
Let $I =]4, 11[$: 4 is a minorant of I, 11 is a majorant of I. It is surely clear to you that the set $\mathbb{N}$ has no majorant in $\mathbb{R}$. A minorant of $\mathbb{N}$ is, for example, 0.

Definition 1.2.3 *Let A be a non-empty part of $\mathbb{R}$.*
1. If there exists M as a majorant of A such that $M \in A$, then M is unique. We say that M is the largest element, or the maximum, of A (we note $M = max(A)$).
2. If there exists m as a minorant of A such that $m \in A$, then m is unique. We say that m is the smallest element, or the minimum, of A (we note $m = min(A)$).

Example 1.2.4 *1. $max([1, 2]) = 2$ since 2 is a majorant of $[1, 2]$ and 2 is a majorant of $[1, 2]$. Similarly, $min([1, 2]) = 1$.*
2. The set $A =]1, 3[$ is major and minor, but has no largest or smallest element: indeed,

let $x_0 \in A$, then x_0 is not a major or minor of A because $1 < \frac{1+x_0}{2} < x_0 < \frac{3+x_0}{2} < 3$.

Let $A = \{\frac{1}{n}, n \in \mathbb{N}^*\}$.

We have $max(A) = 1$: 1 is a majorant of A and $1 \in A$.

Although A is minorized (for example by 0), it does not admit a smallest element: let $n_0 \in \mathbb{N}^*$, $\frac{1}{n_0} \in A$ is not a minorant of A because $\frac{1}{n_0} < \frac{1}{n_0}$ et $\frac{1}{n_0+1} \in A$.

Definition 1.2.5 *Let A be a non-empty part of $\mathbb{R}$, and b a real number.*

1. If there exists a real number b verifying

(a) b is a majorant of A.

(b) If m is a majorant of A, we have $b \leq m$.

then b is unique, we say that b is the upper bound of A (we note $b = sup(A)$).

In summary: $sup(A)$ is the smallest majorant of A.

2. If there exists a real b verifying (a) b is a minorant of A.

(b) If m is a minorant of A, we have $m \leq b$.

then b is unique, we say that b is the lower bound of A (we note $b = inf(A)$).

To summarise: $inf(A)$ is the greatest of the minorants of A.

Remark 1.2.6 ***Example***

1. It is easy to show that :

If $max(A)$ exists, then $sup(A)$ exists and $sup(A) = max(A)$.

If $sup(A)$ exists and $sup(A) \in A$, then $max(A)$ exists and $max(A) = sup(A)$.

We have analogous results for min(A) and inf(A)

2. Let $A =]1, 3[$. We have $sup(A) = 3$ because 3 is a majorant of A and, as we saw previously, $x_0 < 3$ is no longer a majorant of A. Similarly, $inf(A) = 1$.

Proposition 1.2.7 ***(Characterization of the upper bound)***

Let A be a non-empty part of $\mathbb{R}$, and b a real number. The following two statements are equivalent

1. b is the upper bound of A.

2. b is a majorant of A and, for any $\epsilon > 0$, there exists at least one element of A in the interval $[b - \epsilon, b]$.

Proof: *If b is the upper bound of A, it is a majorant of A, and for any $\epsilon > 0$, $b - \epsilon$ is not a majorant of A: there exists an element x of A which is greater than $b - \epsilon$. Since b*

is a majorant of A, we also have $x \leq b$, so $x \in [b - \epsilon, b]$.

Conversely, if 2 is true, then b is indeed the smallest majorant of A. $\square$

Proposition 1.2.8 ***(Characterization of the lower bound)*** *Let A be a non-empty part of $\mathbb{R}$, and b a real. The following two statements are equivalent*

1. b is the lower bound of A.

2. b is a minorant of A and, for any $\epsilon > 0$, there exists at least one element of A in the interval $[b, b + \epsilon]$.

Example 1.2.9 *Let $A = \{\frac{1}{n}, n \in \mathbb{N}^*\}$.*

Let us use the characterization of the lower bound to show that $inf(A) = 0$:

• *0 is a minorant of A.*

Let $\epsilon > 0$ and show that there is at least one element of A in the interval $[0, 0 + \epsilon]$:

Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq \frac{1}{\epsilon}$ (the existence of n_0 is a consequence of the Archimedean Property: see in the following), we have $\frac{1}{n_0} \in [0, \epsilon]$.

1.3 The property of the upper bound

Of course, if $A \subset \mathbb{R}$ admits no majorant, A has no upper bound either: this is the only case where a (non-empty) part of $\mathbb{R}$ has no upper bound, as the following fundamental result states:

Theorem 1.3.1 ***(The upper bound property)***

Let A be a non-empty part of $\mathbb{R}$.

1. If A is increased, then A has an upper bound.

2. If A is minor, then A has a lower bound.

We will admit this Theorem, whose proof uses the rigorous construction of real numbers from rationals.

The upper bound property marks the essential difference between $\mathbb{Q}$ and $\mathbb{R}$ because it is not true in $\mathbb{Q}$: A non-empty major part of $\mathbb{Q}$ does not, in general, admit an upper bound in $\mathbb{Q}$.

Example : Indeed, let us consider $A = \{x \in \mathbb{Q}, x^2 \leq 2\}$.

◦ A is not empty (for example $1 \in A$) A is increased by 2 (if $x > 2$, then $x^2 > 4$ and $x \notin A$) A has no upper bound in $\mathbb{Q}$.

1.4 Interval in $\mathbb{R}$.

Definition 1.4.1 *The intervals of* $\mathbb{R}$ *are the sets :*

$[a,b] = \{x \in \mathbb{R}; a \leq x \leq b\}$,

$[a,b[= \{x \in \mathbb{R}; a \leq x < b\}$,

$]a,b] = \{x \in \mathbb{R}; a < x \leq b\}$,

$]a,b[= \{x \in \mathbb{R}; a < x < b\}$,

$[a,+\infty[= \{x \in \mathbb{R}; a \leq x\}$,

$]a,+\infty[= \{x \in \mathbb{R}; a < x\}$,

$]-\infty,b] = \{x \in \mathbb{R}; x \leq b\}$,

$]-\infty,b[= \{x \in \mathbb{R}; x < b\}$.

Definition 1.4.2 *A set C is said to be convex when, for all x and y of C, the interval (segment) $[x,y]$ is completely in C, that is : $\forall x,y \in C$, $\forall t \in [0,1]$ $tx + (1-t)y \in C$*

Proposition 1.4.3 *The intervals of X are the convex parts of $\mathbb{R}$, i.e. the parts P of $\mathbb{R}$ such that, whatever $x,y \in P$, $x < y$, we have $[x,y] \subset P$.*

Proof: *The intervals are obviously convex. Conversely, if P is a convex part convex part of $\mathbb{R}$, let $a = infP$, $b = supP$. Then $]a,b[\subset P$ and the only question is whether or not a,b are infinite and are in P. There are eight possible cases:*

1. $-\infty < a$, $b < +\infty$, $a \in P$, $b \in P$, then $P = [a,b]$;

2. $-\infty < a$, $b < +\infty$, $a \in P$, $b \notin P$, then $P = [a,b[$;

3. $-\infty < a$, $b < +\infty$, $a \notin P$, $b \in P$, then $P =]a,b]$;

4. $-\infty < a$, $b < +\infty$, $a \notin P$, $b \notin P$, so $P =]a,b[$;

5. $a = -\infty, b < +\infty$, $b \in P$, so $P = [-\infty,b]$;

6. $a = -\infty$, $b < +\infty$, $b \notin P$, so $P =]-\infty,b[$;

7. $a > -\infty$, $b = +\infty$, $a \in P$, so $P = [a,+\infty]$;

8. $a > -\infty$, $b = +\infty$, $a \notin P$, so $P =]a,+\infty[$.

1.4.1 Characterisation of intervals

Proposition 1.4.4 *Let A be a non-empty part of $\mathbb{R}$.*

The following statements are equivalent:

1. A is an interval.

2. For all $\alpha \leq \beta$, $\alpha,\beta \in A$, the interval $[\alpha,\beta]$ is included in A.

Proof: *The implication* $1 \Rightarrow 2$ *is obvious.*

Let us show that $2 \Rightarrow 1$. *Let us first assume that* A *is increased and decreased.*

Then we know that $a = inf(A)$ *and* $b = sup(A)$ *exist. We have* $A \subset [a, b]$.

Moreover, $]a, b[\subset A$. *Indeed, if* $a < x < b$, *there exists an element* $\beta \in A$ *in the interval* $[x, b]$ *(because* x *is not a majorant of* A*). In the same way, there exists alpha* $\in A$ *in* $[a, x]$ *(since* x *is not a minorant of* A*). Therefore,* $x \in [\alpha, \beta] \subset A$ *by hypothesis 2.*

Since $A \subset [a, b]$ *and* $]a, b[\subset A$, A *is one of the* 4 *intervals of bounds* a *and* b. *You are invited to complete the demonstration if* A *is not major or minor.* $\square$

1.4.2 $\overline{\mathbb{R}}$

Definition 1.4.5 *The set* $\overline{\mathbb{R}}$ *is* $\mathbb{R}$ *to which we add the elements* $+\infty$ *and* $-\infty$.

$$\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

. *It is provided with a relation of total order inherited from the one on* $\mathbb{R}$ *by posing in addition for all* $x \in \overline{\mathbb{R}}$, *we have :* $-\infty \leq x \leq +\infty$.

We have the following properties:

- $+\infty$ is the greatest element of $\overline{\mathbb{R}}$;
- $-\infty$ is the smallest element of $\overline{\mathbb{R}}$;

i any part of $\overline{\mathbb{R}}$ has an upper bound in $\overline{\mathbb{R}}$;

Any part of $\overline{\mathbb{R}}$ has a lower bound in $\overline{\mathbb{R}}$.

1.5 Archimedes' property

It is surely clear to you that :

For any real number x, we can find a natural number n such that $x < n$.

This property is called the Archimedean property.

It is very easy to show that $\mathbb{Q}$ verifies this property. Indeed, let r be $\in \mathbb{Q}$, if $r > 0$, then $n = 1$ fits. If $r > 0$, then it is written $r = \frac{l}{m}$, with $l, m \in \mathbb{N}^*$ and $n = l + 1$ fits.

We will show, from the upper bound property, that $\mathbb{R}$ also verifies the Archimedean Property:

Proposition 1.5.1 *For all* $x \in \mathbb{R}$, *there exists* $n \in \mathbb{N}$ *such that* $x < n$.

Proof: *By the absurd. Let us suppose that there exists $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}$ is a non-empty and major part (by x) of $\mathbb{R}$ and it thus admits an upper bound $s = sup(\mathbb{N})$.*
In particular $n+1 \leq s$, for all $n \in \mathbb{N}$, hence $n \leq s-1$ for all $n \in \mathbb{N}$. Nonsense: $s-1$ is a majorant of $\mathbb{N}$ strictly less than $s = sup(\mathbb{N})$. $\square$

Proposition 1.5.2 *$\mathbb{R}$ is Archimedean, i.e.: for all $x > 0$, for all $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof: *Let us suppose that it is false: there exist $x > 0$ and $y \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $nx \leq y$. Then the set $A = \{nx; n \in \mathbb{N}\}$ is a major part (byy) of $\mathbb{R}$, so it has an upper bound $a \in \mathbb{R}$. In particular, for all $n \in \mathbb{N}$ we $a(n+1)x \leq a$ and thus $nx \leq a - x$. Therefore $a - x$ is a majorant of A and $a - x < a$. This is impossible because a is the upper bound of A.* $\square$

1.5.1 Decimal development of a real

An immediate application of Archimedes' Property is to allow the integer part of a real to be defined.

Proposition 1.5.3 *For any real x, there exists a single relative integer m such that $m \leq x < m+1$. Let us note $m = E(x)$: it is the integer part of the real x.*

Proof: *Let $x \in \mathbb{R}^+$, and A be the part of $\mathbb{R}$ defined by $A = \{p \in \mathbb{N}, x \geq p\}$. A is not empty since $0 \in A$. In addition, thanks to the Archimedean property, there is an integer n such that $x < n$: all the elements of A are therefore less than n : A is a finite set, and therefore has a maximum element m. By definition of A we then have $m \leq x < m+1$ since $m+1 \notin A$.*
For $x < 0$, we apply what precedes to $-x$: in total, we have then shown that for all $x \in \mathbb{R}$, there exists an integer m such that $m \leq x < m+1$.
It remains to be seen that there cannot be a second one: suppose we also have $m_0 \leq x < m_0+1$. We would have $-m_0-1 < -x \leq -m_0$. and therefore $m-m_0-1 < 0 < m-m_0+1$, which, for integers, leads to $m - m_0 = 0$. $\square$

We can be much more precise.

Proposition 1.5.4 *Let x be a real number. For all $n \in \mathbb{N}$ there exists a unique integer q_n such that $\frac{q_n}{10^n} \leq x < \frac{q_n+1}{10^n}$.*
The decimal rational $\frac{q_n}{10^n}$ (respectively, $\frac{q_n+1}{10^n}$) is called the approximate decimal value to the nearest 10^n by default (respectively, by excess) of the real x.

Proof: *The integer $q_n = E(10^n x)$ is suitable, and the only one.*

1.5.2 $\mathbb{Q}$ is dense in $\mathbb{R}$

We have now understood that $\mathbb{Q} \neq \mathbb{R}$. However, thanks to the Archimedean property, we show that these two sets are not very different:

Proposition 1.5.5 *$\mathbb{Q}$ is dense in $\mathbb{R}$: any non-empty interval $]a, b[$ of $\mathbb{R}$ contains at least one rational.*

Proof: *Since $b - a > 0$, the Archimedean property allows us to affirm that there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Let us then put $m = E(na)$: we have $m \leq na < m + 1$,, so $\frac{m}{n} \leq a \leq \frac{m+1}{n} \leq \frac{m}{n} + \frac{1}{n} < a + (b - a) = b$. The rational number $\frac{(m+1)}{n}$ therefore belongs to $]a, b[$.*

Proposition 1.5.6 *Given $x, y \in \mathbb{R}$, $x < y$ there is at least one rational and one irrational in the interval $]x, y[$.*

Proof: *Let $q = [\frac{1}{y-x}] + 1$ and $p = [qx]$. Then we have : $q > \frac{1}{y-x}$, $q \geq 1 > 0$ and $a \leq qx < p + 1$. We then deduce $a < \frac{p+1}{q} \leq x + \frac{1}{q} < x + (y - x) = y$.. So the rational $\frac{p+1}{q}$ is in the interval $]x, y[$. Can we also find an irrational in this interval? It's easy: we can find a rational r in the interval $]x + \sqrt{2}, y + \sqrt{2}[$, the number $r - \sqrt{2}$ is irrational and is in the interval $]x, y[$.* $\square$

1.6 Absolute value, distance

Definition 1.6.1 *The absolute value of a real number a is the number, denoted $|a|$, defined by :* $|a| = \begin{cases} a & , \ if \quad a \geq 0; \\ -a & , \ if \quad a \leq 0. \end{cases}$

Proposition 1.6.2 *Let $a, b \in \mathbb{R}$. We have :*

1) $|a| = |-a|$,

2) $|a| = 0 \Leftrightarrow a = 0$,

3) $a \neq 0 \Rightarrow |a| > 0$,

4) $|a| = |b| \Leftrightarrow a = b$ *or* $a = -b$,

5) $|a| = \sqrt{a^2}$,

6) $|a|^2 = a^2$,

7) $|ab| = |a||b|$,

8)$|\frac{a}{b}| = \frac{|a|}{|b|}$.

Definition 1.6.3 *The distance between the real numbers a and b (denoted $d(a,b)$) is defined by $d(a,b) = |a-b|$.*

Proposition 1.6.4 *Let a be a positive real. We have :*

1. $|x| \leq a \Leftrightarrow d(x,0) \leq a \Leftrightarrow -a \leq x \leq a \Leftrightarrow x \in [-a, a]$,
2. $|x| \geq a \Leftrightarrow d(x,0) \geq a \Leftrightarrow x \leq -a$ *or* $x \geq a \Leftrightarrow x \in]-\infty, -a] \cup [a, +\infty[$.

More generally :

Proposition 1.6.5 *Let $c, r \in \mathbb{R}$, $r > 0$. We have :*

$$|x-c| \leq r \Leftrightarrow d(x,c) \leq r \Leftrightarrow x \in [c-r, c+r]$$

Proposition 1.6.6 *Let a et b deux réels. We have :*

$$||a| - |b|| \leq |a+b| \leq |a| + |b|$$

This result is known as the triangular inequality.

1.7 Topological concepts

Definition 1.7.1 ***(Interior of a set)*** *Let $A \subset R$ and $a \in \mathbb{R}$. We will say that a is an interior point of A if there exists a positive real number δ, such that $]a-\delta, a+\delta[\subset A$. The set of interior points is named interior of A and is noted by int$A$ or $\overset{\circ}{A}$.*

Definition 1.7.2 ***(adhesion of a set)***

Let $A \subset R$ and $a \in \mathbb{R}$. We will say that a is an adherent point of A if $\forall \delta > 0$ we have:

$$]a-\delta, a+\delta[\cap A \neq \emptyset$$

. The set of adherent points is called adherence of A and is noted by $\bar{A}$.

Definition 1.7.3 ***(point of accumulation)*** *Let $A \subset \mathbb{R}$ and $a \in \mathbb{R}$. . We will say that a is a point of accumulation of A if $\forall \delta > 0$ we have:*

$$(]a-\delta, a[\cup]a, a+\delta[) \cap A \neq \emptyset$$

The set of accumulation points is called the derived set of A and is denoted by A'. We have: $\bar{A} = A \cup A'$.

Example 1.7.4 *Adhesion and accumulation do not coincide in general. Let the set $A = \{1\} \cup]2, 3[$. We verify that:*

$$\bar{A} = \{1\} \cup [2, 3], \quad A' = [2, 3]$$

Definition 1.7.5 ***(open)*** *Let $U \subseteq \mathbb{R}$. The set U is said to be open if, for every point $x \in U$, there exists $\epsilon > 0$ such that the interval $(x-\epsilon, x+\epsilon)$ is contained in U.*

Definition 1.7.6 ***(closed)*** *Let $C \subseteq \mathbb{R}$. The set C is said to be closed if the following conditions hold:*

- *C contains all of its limit points, i.e., if a sequence in C converges to a point x, then $x \in C$.*
- *The complement of C is open, i.e., $\mathbb{R} \setminus C$ is open.*

Definition 1.7.7 ***(Neighborhood of a point)***. *Let X be a topological space and $x \in X$. A set $V \subseteq X$ is a neighborhood of x if there exists an open set U such that $x \in U \subseteq V$.*

Definition 1.7.8 ***(Open set and neighborhood)***. *A set $U \subseteq X$ is said to be open if, for every point $x \in U$, there exists a neighborhood V of x such that $V \subseteq U$.*

Chapter 2

Numerical sequences

2.1 First concepts

2.1.1 What is a sequence of numbers?

A numerical sequence is an infinite list of numbers. More precisely

Definition 2.1.1 *A real numerical sequence is a real-valued function defined on an infinite part of* $\mathbb{N}$.

If $u : \mathbb{R} \longrightarrow \mathbb{R}$ *is a sequence, let un be the nth element of the list, i.e. the number* $u(n)$ *which is the image of* n *by* u.

We can also denote $(u_n)_{n\in\mathbb{N}}$, *or even simply* (u_n), *the sequence* u.

Remark 2.1.2 *Note the notations:* u_n *is a number, the general term of the sequence* (u_n).

Nor should we confuse the given sequence (u_n) *(the infinite list.) with the image in* $\mathbb{R}$ *of the application* u, *i.e. the set* $u(n), n \in \mathbb{N}\}$ *(the real numbers which appear in the list). the list). For example, for* $u_n = (-1)^n$, *this image is the subset* $\{-1, 1\}$ *of* $\mathbb{R}$*: there are many distinct sequences with this subset. there are many distinct sequences with this set as their image.*

2.1.2 Direction of variation

Definition 2.1.3 *A numerical sequence* (u_n) *is increasing if and only if, for all* n, $u_n \leq u_{n+1}$. *It is decreasing if and only if* $u_n \geq u_{n+1}$ *for all* n.

Remark 2.1.4 (Examples) *1. To study the monotonicity of a sequence, we are interested in the sign of the difference* $u_{n+1} - u_n$.
For example, the sequence $(u_n)_{n\in\mathbb{N}}$ *defined by* $u_n = 2^n - n$ *. (To be processed).*
2. To study the monotonicity of a sequence whose terms are all strictly positive, we can compare the quotient $\frac{u_{n+1}}{u_n}$ *with 1.*
For example, consider the sequence $(u_n)_{n\geq 0}$ *with general term* $u_n = \frac{3^n}{n!}$. *Study the variation of the sequence* (u_n).
3. If $f : [0, +\infty[\longrightarrow \mathbb{R}$ *is an increasing (respectively decreasing) function, then the sequence* $(f(n))_{n\geq 0}$ *is increasing (respectively decreasing).*
4. There are, of course, sequences which are neither increasing nor decreasing: this is the case, for example, of $((-1)^n)_{n\geq 0}$.

2.1.3 Increased, decreased or bounded sequences

Definition 2.1.5 *We say that a sequence* $(u_n)_{n\geq 0}$ *is increased if there is a real* M *greater than all the terms of the sequence: for all* $n \in \mathbb{N}$*, we have*

$$u_n \leq M$$

The sequence $(u_n)_{n\geq 0}$ *is said decreased if there is a real* m *less than all the terms in the sequence : for all*$n \in \mathbb{N}$*, we have*

$$m \leq u_n$$

When a sequence is increased and decreased, it is said to be bounded.

Remark 2.1.6 (Examples) *1. Note that a sequence is not major (resp. not minor) when, for any given real* A*, there is at least one term of the sequence greater (resp. smaller) than* A.
2. The sequence $((-1)^n)$ *is bounded.*

2.2 Sequences defined by recurrence

2.2.1 Recurrence principle

We will come across two different ways of defining a sequence, both of which are important to recognise: the study of a sequence differs considerably depending on whether the definition is of one type or the other. of one type or the other.

• A sequence can be defined explicitly, the general term un of the sequence being given as a function of n: for example $u_n = 2n$, $u_n = (-1)n$ or $u_n = n!$

• A sequence can also be defined by a recurring expression. In this case, the general term u_n is given as a function of u_{n-1} or of several of the terms that preceding it:

$$u_n = f(u_{n-1}, u_{n-2}, \cdots, u_{n-k})$$

with k fixed. For example, we can define the sequence (v_n) of odd numbers by recurrence: $v_n = v_{n-1} + 2$. It is then essential to set the initial conditions: here we need to specify $v_0 = 1$. If we had taken $v_0 = 0$, we would have obtained the sequence of even numbers.
Is this the correct definition of a sequence of numbers? To answer this question is to use the

Theorem 2.2.1 (Recurrence principle) *Let $P(n)$ be a property whose statement depends on the natural number n. Suppose that*
1. Initialisation: *There exists a natural number n_0 such that $P(n_0)$ is true.*
2. Heredity: *For any natural number $n \geq n_0$, we can show the implication*

$$P(n) \Rightarrow P(n+1)$$

Then the property $P(n)$ is true for any natural number $n \geq n_0$.

2.2.2 Graphical representation

The graphical representation of a real sequence defined explicitly (from a real function of real variable f) by $u_n = f(n)$ is very simple: just draw the representative curve of the function f, and indicate the image of the integers.
On the other hand, calculating a sequence defined by recurrence is a little trickier. Below

is the graphical representation of the sequence (u_n) given by :

$$u_0 = 1, \quad u_n = 1 + \frac{2}{u_{n-1}}, \quad n \geq 1.$$

Here, $u_n = f(u_{n-1})$ with $f(x) = 1 + \frac{2}{x}$ and we must begin by plotting together the graphs of $y = x$ and $y = f(x)$.

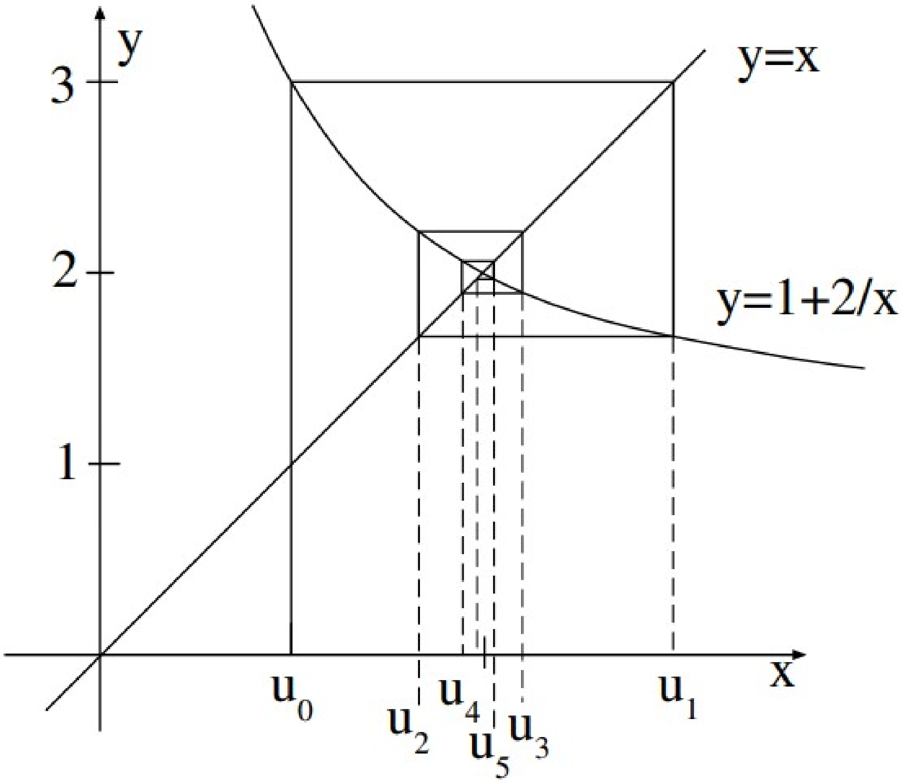

Figure 1: Graphical representation of a recurring sequence

The representation seems to indicate that the sequence (u_n) converges to 2 (the positive solution of the equation $f(x) = x$).

Example 2.2.2 (Exercise) *Draw together the graphs of* $x \longmapsto \sqrt{x}$ *and* $x \longmapsto x$*. Give the graphical representation of the sequence* (u_n) *defined by :*

$$u_0 = 4, \quad u_{n+1} = \sqrt{u_n}, \quad n \geq 0.$$

2.2.3 Fixed points, stable intervals, monotony

Let f be a real function of a given real variable. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by the recurrence relation

$$u_{n+1} = f(u_n)$$

for all $n \geq 0$ and u_0 given.

1. An interval I is said to be stable by f if it is contained in the definition set of f and $f(I) \subset I$.

If $u_0 \in I$, an interval stable by f, it is easy to show by recurrence that u_n is well defined and $u_n \in I$ for all $n \geq 0$.

2. If, in addition, $f(x) \geq x$ for all $x \in I$, then the sequence (u_n) is increasing since :

$$u_{n+1} = f(u_n) \geq u_n$$

Similarly, if $f(x) \leq x$ on I, the sequence (u_n) is decreasing.

3. If u_0 is a fixed point of $f(f(u_0) = u_0)$, then the sequence (u_n) is constant.

2.3 Convergence

2.3.1 Limit of a sequence

Definition 2.3.1 *A sequence (u_n) of real numbers is said to be limited by a given real l, or to tend towards l, or to converge towards l when*

$$for \quad all \quad \varepsilon > 0, \quad it \quad exists \quad N_\varepsilon \in \mathbb{N} \quad such \quad that \quad n \geq N_\varepsilon \Rightarrow |u_n - l| \leq \varepsilon.$$

We then note

$$\lim_{n\to\infty} u_n = l$$

Remark 2.3.2 *1. . The condition $|u_n - l| \leq \varepsilon$ means that $u_n \in [l - \varepsilon, l + \varepsilon]$.*

2. We have

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \lim_{n\to\infty} (u_n - l) = 0 \Leftrightarrow \lim_{n\to\infty} |u_n - l| = 0.$$

All we have to do is apply the definition by writing

$$|u_n - l| = |(u_n - l) - 0| = ||u_n - l| - 0|$$

3. Note that changing a finite number of terms in a sequence does not change anything in terms of its possible limit.

Proposition 2.3.3 *A sequence cannot have two distinct limits.*

Proof: *Exercise*

Definition 2.3.4 *When a sequence (u_n) tends towards a certain real l, it is said to be convergent. Otherwise it is said to be divergent.*

Among the divergent sequences, we distinguish those which tend towards infinity:

Definition 2.3.5 *The sequence (u_n) is said to tend towards $+\infty$ when :*

$$for \quad all \quad A \in \mathbb{R}, \quad it \quad exists \quad N_A \in \mathbb{N} \quad such \quad that \quad n \geq N_A \Rightarrow u_n \geq A.$$

We then note

$$\lim_{n\to\infty} u_n = +\infty$$

The definition of a sequence tending towards $-\infty$ is analogous.

Remark 2.3.6 *1. Note (exercise) the link between the limit of a sequence and the limits of the extracted sequences with even and odd indices:*

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \lim_{n\to\infty} u_{2k} = \lim_{n\to\infty} u_{2k+1} = l.$$

(similar result with $+\infty$ or $-\infty$ instead of l).
2. In particular, if $\lim_{n\to\infty} u_{2k} \neq \lim_{n\to\infty} u_{2k+1}$, then the limit of (u_n) does not exist (this is the case, for example, of the sequence $u_n = (-1)^n$.
3. Let f be a real function defined on a neighbourhood of ∞ and consider the sequence with general term $u_n = f(n)$. It is easy to show that

$$if \quad \lim_{n\to\infty} f(x) = l \quad (respectively \quad +\infty, \quad -\infty) \quad then \quad \lim_{n\to\infty} u_n = l \quad (respectively \quad +\infty, \quad -\infty).$$

2.3.2 Limit and order relation

Wide inequalities are preserved by crossing the boundary:

Proposition 2.3.7 *Let (u_n) and (v_n) be two numerical sequences, and l and l' be two real numbers.*
Suppose (u_n) tends to l and (v_n) tends to l'. If there exists n_0 such that, for all $n \geq n_0$, we have $u_n \leq v_n$, then $l \leq l'$.

Remark 2.3.8 *1. In general, strict inequalities are not preserved by passing to the limit: for example, we have for example $\frac{1}{n^2} < \frac{1}{n}$ for all $n \in \mathbb{N}^*$ but the two sequences tend towards 0.*
2. If $u_n < v_n$ for all $n \geq n_0$, (u_n) tends towards l and v_n tends towards l', then we can say in general that $l \leq l'$.

Proposition 2.3.9 *Any convergent sequence is bounded.*

Proof: *Exercise*

Note that the converse is false: the sequence (u_n) defined by $u_n = (-1)^n$ is bounded. is bounded but not convergent.

2.4 Limit calculation

The above definitions are quite difficult to handle. Here are a few theorems to show that a sequence converges to a real l.

2.4.1 Limits and operations

The table below summarises the main results concerning the limit of the sum, product and quotient of two sequences. sum, product and quotient of two sequences. They make it fairly easy to determine the possible limit of certain sequences.
Let (u_n) and (v_n) be two numerical sequences.

if SO

$\lim(u_n)$	$\lim(v_n)$	$\lim(u_n+v_n)$	$\lim(u_n v_n)$
$\ell \in \mathbb{R}$	$\ell' \in \mathbb{R}$	$\ell + \ell'$	$\ell\ell'$
$+\infty$	$\ell' \in \mathbb{R}$	$+\infty$	$\begin{cases} +\infty & ; \ \ell' > 0 \\ -\infty & ; \ \ell' < 0 \\ ? & ; \ \ell' = 0 \end{cases}$
$+\infty$	$+\infty$	$+\infty$	$+\infty$
$+\infty$	$-\infty$	?	$-\infty$
$-\infty$	$\ell' \in \mathbb{R}$	$-\infty$	$\begin{cases} -\infty & ; \ \ell' > 0 \\ +\infty & ; \ \ell' < 0 \\ ? & ; \ \ell' = 0 \end{cases}$
$-\infty$	$-\infty$	$-\infty$	$+\infty$

if SO

$\lim(u_n)$	$\lim(\frac{1}{u_n})$
$\ell \in \mathbb{R}, \ell \neq 0$	$1/\ell$
0	?
$0, \quad u_n > 0 \ \forall n$	$+\infty$
$0, \quad u_n < 0 \ \forall n$	$-\infty$
$+\infty$	0
$-\infty$	0

Figure 2: Limit of a sum, product and quotient

Remark 2.4.1 *In these tables, the presence of a ? indicates that no general result is possible, and that each of these indeterminate forms must be studied using other methods. other methods.*

Example 2.4.2 *For example if $u_n = -n^2$ and $v_n = n^3$, we see that $\lim\limits_{n\to\infty} u_n = -\infty$, and that $\lim\limits_{n\to\infty} v_n = +\infty$. The table does not therefore give a direct indication of the possible limit of the sequence (wn) given by $w_n = u_n + v_n$. However, it is enough to note that $w_n = n^2(n-1)$ and to conclude from the eighth line. conclude from the eighth line. (We say that we have removed the indeterminacy, a sport that you have no doubt already practised.)*

We have simply given the proof of the result concerning the limit of a sum. To demonstrate the others.

Proposition 2.4.3 *Let (a_n) and (b_n) be two numerical sequences.*
Let (c_n) be the sum of (a_n) and (b_n), i.e. the sequence with general term $c_n = a_n + b_n$. If (a_n) tends towards the real l and (b_n) tends towards the real l', then (c_n) tends towards $l + l'$.

Proof: *Exercise*

2.4.2 The gendarme theorem

Proposition 2.4.4 *If*

$$a_n \leq c_n \leq b_n$$

from a certain rank n_0 *and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in \mathbb{R} \quad \textit{then} \quad \lim_{n\to\infty} c_n = l$$

Proof: *Exercise*

Example 2.4.5 *The sequence with general term* $c_n = \frac{2+(-1)^n}{n^2}$ *tends to 0, because* $\frac{1}{n^2} \leq c_n \leq \frac{3}{n^2}$ *, if* $n \geq 1$ *and* $\lim\limits_{n\to\infty} \frac{1}{n^2} = \lim\limits_{n\to\infty} \frac{3}{n^2} = 0$.

Here is a consequence of the theorem

Proposition 2.4.6 *Let* (u_n) *be a numerical sequence. If there exists a sequence* (v_n) *which tends towards* 0 *and such that, for all* n *(or even from a certain rank):*

$$|u_n - l| \leq v_n$$

then (u_n) *tends towards* l.

Proof: *Exercise*

In other words, to show that a sequence converges to $l \in \mathbb{R}$ it is sufficient to increase its distance to l by a positive sequence which we know tends to 0. To be able to use this result reference sequences, of which there are three below:

Proposition 2.4.7 *Let* α *and* a *be two real numbers.*

1. The sequence (n^α) *tends towards zero if* $\alpha < 0$ *and towards* $+\infty$ *if* $\alpha > 0$.

2. The sequence (a^n) *tends to zero if* $|a| < 1$*, and to* $+\infty$ *if* $a > 1$.

3. The sequence $(n^\alpha a^n)$ *tends to zero if* $|a| < 1$ *for any* α.

Proof: *Exercise*

Proposition 2.4.8 *Let's assume that* $a_n \leq b_n$ *from a certain rank.*

1. *If* $\lim\limits_{n\to\infty} a_n = +\infty$ *then* $\lim\limits_{n\to\infty} b_n = +\infty$.

1. *If* $\lim\limits_{n\to\infty} a_n = -\infty$ *then* $\lim\limits_{n\to\infty} b_n = -\infty$

Example 2.4.9 *1. We show that* $\lim\limits_{n\to\infty} n + cosn = +\infty$ *noting that* $n + cosn \geq n - 1$ *and* $\lim\limits_{n\to\infty} n - 1 = +\infty$.

2. We have $\lim\limits_{n\to\infty} n(sin(n) - 3) = -\infty$ *from the inequality* $n(sin(n) - 3) \leq -2n$.

2.4.3 Case of sequences defined by recurrence

Proposition 2.4.10 *Let* (u_n) *be a numerical sequence which converges to a real* l.
If f *is a continuous function at point* l, *then the sequence* $(f(u_n))$ *converges to* $f(l)$.

Proof: *Exercise*

Remark 2.4.11 *1. We often write this proposition in the form of*

$$\lim_{n\to+\infty} f(u_n) = f(\lim_{n\to+\infty} u_n)$$

2. If, for example, f *is continuous on the right at* l *and* $\lim\limits_{n\to+\infty} u_n = l^+$, *then we also have*

$$\lim_{n\to+\infty} f(u_n) = f(l)$$

Cette proposition est souvent utilisée pour trouver les valeurs possibles de la limite d?une suite définie par récurrence :

Proposition 2.4.12 *Let's assume that 1.* I *is a closed interval (i.e.* I *is of the form* $[a,b]$, $[a,+\infty[,]-\infty,b]$ *or* $]-\infty,+\infty[$*).*
2. f *is a continuous function on* I *(if, for example,* $I = [a,b]$, *this means that* f *is continuous on the right at* a, *continuous on the left at* b, *and continuous at any* $x \in]a,b[$ *).*
3. The interval I *is stable by* $f (f(I) \subset I)$.
Consider the sequence $(u_n)_{n\geq 0}$ *defined by the recurrence relation* $u_{n+1} = f(u_n)$, *for all* $n \geq 0$, *and* $u_0 \in I$ *given. Then :*
a) The sequence is well defined and all its terms are in the interval I.
b) If (u_n) *converges, its limit* l *is a fixed point of* f *in* I $(f(l) = l, l \in I)$.

2.5 Convergence criteria

We now give some more difficult results, which allow us to determine whether a sequence converges without having any a priori idea of its limit. These statements can all be seen as consequences of consequences of the upper bound axiom.

2.5.1 Monotonous sequences

Theorem 2.5.1 *a) If (u_n) is an increasing sequence with a major then it converges, and its limit is the upper bound of the set $\{u_n, n \in \mathbb{N}\}$.*
b) If (u_n) is an increasing and non-majorised sequence, then

$$\lim_{n\to+\infty} u_n = +\infty$$

Similarly
a) If (u_n) is a decreasing and minorized sequence, then it converges, and its limit is the lower bound of the set $\{u_n, n \in \mathbb{N}\}$.
b) If (u_n) is a decreasing non-minority sequence, then

$$\lim_{n\to+\infty} u_n = -\infty$$

Proof: *Exercise*

Example 2.5.2 *Study the convergence of the following sequences*
1.

$$\forall n \geq 1, u_n = \sum_{p=1}^{n} \frac{1}{p^2}$$

2.

$$\forall n \in \mathbb{N}, \quad u_{n+1} = \sqrt{u_n}$$

2.5.2 Adjacent sequence

Definition 2.5.3 *We say that two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are adjacent if*
1. *The sequence (a_n) is increasing.*
2. *The sequence (b_n) is decreasing.*
3. $\lim\limits_{n\to+\infty} (a_n - b_n) = 0.$

Proposition 2.5.4 *Properties 1, 2 and 3 of the Definition imply that*

$$a_n \leq b_n, \quad \forall n \in \mathbb{N}.$$

Proof: *Since the sequence (a_n) is increasing, the sequence $(-a_n)$ is decreasing. So $(b_n - a_n)$ is a decreasing sequence (the sum of two decreasing sequences).*
Let's show that $b_n - a_n \geq 0$ for all $n \in \mathbb{N}$. By the absurd: Suppose there exists $p \in \mathbb{N}$ such that $b_p - a_p < 0$.
Then, if $n \geq p$ we have $b_n - a_n \leq b_p - a_p$ and $0 = \lim\limits_{n\to+\infty}(a_n - b_n) \leq b_p - a_p < 0$.
Absurd

Remark 2.5.5 *In the notations of the Definition, and given the Proposition, we have*

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

for all $n \in \mathbb{N}$.
Here is the geometric interpretation of the notion of adjacent sequences:
If the suites $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are adjacent (in the notation of the Definition), then the segments

$$I_n = [a_n, b_n]$$

are nested within each other

$$\cdots \subset I_{n+1} \subset I_n \subset \cdots \subset I_2 \subset I_1 \subset I_0$$

and its length tends towards zero.

The following result should come as no surprise

Proposition 2.5.6 *Two adjacent sequences converge to the same limit.*

Proof: *Let (a_n) and (b_n) be two adjacent sequences (in the notation of the Definition).*
Since $a_n \leq b_n \leq b_0$, for all $n \in \mathbb{N}$, the increasing sequence (a_n) is increased: it converges.
Let $l = \lim\limits_{n\to+\infty} a_n$.
Like $\lim\limits_{n\to+\infty}(a_n - b_n) = 0$, we have $\lim\limits_{n\to+\infty} b_n = l$, also (write $b_n = a_n + (b_n - a_n)$)

Remark 2.5.7 *1. The third condition of the Definition (on its own) is not generally sufficient to ensure the convergence of the sequences* (a_n) *and* (b_n) *:*
Take for example $a_n = \sqrt{n+1}$ *and* $b_n = \sqrt{n}$.
2. Still in the notations of Definition:
The adjacent sequences $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$ *provide a framework for their common limi* l
:
$a_n \leq l \leq b_n$ *for all* $n \in \mathbb{N}$ *and* $b_n - a_n$ *increases the error of the approximation:*

$$0 \leq l - a_n \leq b_n - a_n \quad \text{and} \quad 0 \leq bn - l \leq b_n - a_n$$

for all $n \in \mathbb{N}$ *.*
3. The proposition can be stated in terms of nested segments: Let be a sequence of nested segments -

$$\cdots \subset [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a_1, b_1] \subset [a_0, b_0]$$

whose length tends to zero.
Then there is one and only one real l *belonging to the intersection of all the segments*

$$\bigcap_{n=0}^{+\infty} [a_n, b_n] = \{l\}.$$

Example 2.5.8 *Let* $(a_n)_{n\geq 2}$ *and* $(b_n)_{n\geq 2}$ *be the sequences defined, for all* $n \geq 2$*, by :*

$$a_n = \sum_{k=2}^{n} \frac{1}{(k-1).k^2} \quad \text{and} \quad b_n = a_n - \frac{1}{n^2}.$$

Show that the sequences (a_n) *and* (b_n) *are adjacent.*

2.5.3 Cauchy criterion for sequences

Definition 2.5.9 *We say that a sequence* (u_n) *satisfies the Cauchy criterion or that* (u_n) *is a Cauchy sequence when*

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \quad \text{such} \quad \text{that} \quad n > N_\varepsilon, m > N_\varepsilon \Rightarrow |u_n - u_m| < \varepsilon$$

In other words, (u_n) is a Cauchy sequence when the distance between any two terms is as small as we want, even if it means considering only terms of sufficiently high rank . Here are a large number of examples of Cauchy sequences!

Proposition 2.5.10 *Any convergent sequence is a Cauchy sequence.*

Proof: *Let l be the limit of the sequence (u_n), and $\varepsilon > 0$.*
There exists $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$, then $|u_n - l| \leq \frac{\varepsilon}{2}$.
Therefore if $n, m \geq N_\varepsilon$, we have

$$|u_n - u_m| \leq |u_n - l| + |u_m - l| \leq \varepsilon$$

.

Proposition 2.5.11 *All Cauchy sequences are bounded.*

Proof: *There exists $N_1 \in \mathbb{N}$ such that if $n, m \geq N_1$, then $|u_n - u_m| \leq 1$.*
In particular, for all $n \geq N_1$, we have $u_{N_1} - 1 \leq u_n \leq u_{N_1} + 1$.
The sequence (u_n) is bounded from rank N_1, so it is bounded.

Here's another consequence of the upper bound property (we'll see that it's false in $\mathbb{Q}$):

Proposition 2.5.12 *Any Cauchy sequence is convergent in $\mathbb{R}$.*

Proof: *Homework*

Remark 2.5.13 (Examples) ***1.*** *To summarise A real sequence (u_n) is convergent if and only if it is a Cauchy sequence.*
In particular, if (u_n) is not a Cauchy sequence, then it diverges:
Let (u_n) be the sequence given by

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}, \quad n \geq 1.$$

We have

$$u_{2n} - u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} \geq n\frac{1}{2n} = \frac{1}{2}$$

This inequality shows that (u_n) is not a Cauchy sequence (take $\varepsilon = \frac{1}{4}$) and that, consequently, this sequence is divergent. Since (u_n) is increasing, it is not increased (in which case it would be convergent!): it therefore tends towards $+\infty$.
2. *We now give an example of a sequence of rational numbers which satisfies the Cauchy*

criterion, but whose limit (which exists in $\mathbb{R}$ according to the previous proposition!) is not a rational number. In other words, there are sequences of rational numbers which satisfy the Cauchy criterion but which are not convergent in $\mathbb{Q}$:
Let (u_n) be the sequence given by $u_0 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.

Chapter 3

Limites of functions and continuity

In this chapter, the key notion of a *continuous function* is introduced, followed by several important theorems about continuous functions.

3.1 Limits of Functions

In this chapter, we deal exclusively with functions taking values in the set of real numbers (that is, *real-valued functions*).

We aim to define what is meant by

$$f(x) \to b \quad \text{as} \quad x \to a.$$

Definition 4.1.1 (Cauchy)

Let $f : D(f) \to \mathbb{R}$ and let $c < a < d$ be such that $(c, a) \cup (a, d) \subseteq D(f)$. We say that $f(x) \to b$ as $x \to a$ and write $\lim_{x \to a} f(x) = b$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D(f)) \quad \big[0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon\big].$$

Example 4.1.1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Then for any $a \in \mathbb{R}$,

$$\lim_{x \to a} f(x) = a.$$

Proof. Let $a \in \mathbb{R}$ be arbitrary. Fix $\varepsilon > 0$. (We need to show that $0 < |x-a| < \delta \implies |x-a| < \varepsilon$. Hence we may choose $\delta = \varepsilon$. Choose $\delta = \varepsilon$. Then:

$$(\forall x \in \mathbb{R}) \quad \big[0 < |x-a| < \delta \implies |x-a| < \varepsilon\big].$$

Example 4.1.2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then for any $a \in \mathbb{R}$,

$$\lim_{x \to a} f(x) = a^2.$$

Proof. Let $a \in \mathbb{R}$ be arbitrary. Fix $\varepsilon > 0$. *(We need to show that* $0 < |x-a| < \delta \implies |x^2 - a^2| < \varepsilon$.*)*

We may assume that $|x-a| < 1$. Then $|x| \leq |x-a| + |a|$, and

$$|x^2 - a^2| = |x-a||x+a| \leq |x-a|(|x| + |a|) \leq |x-a|(1 + 2|a|).$$

Choose $\delta = \min\{1, \frac{\varepsilon}{1+2|a|}\}$. Then

$$(\forall x \in \mathbb{R}) \quad \big[0 < |x-a| < \delta \implies |x^2 - a^2| < \varepsilon\big].$$

□

The limit of a function may be defined in another way, based on the definition of a limit of a sequence.

Definition 4.1.2 (Heine)

Let f be such that $(c, a) \cup (a, d) \subseteq D(f)$. We say that

$$f(x) \to b \quad \text{as} \quad x \to a$$

if for any sequence $(x_n)_{n \in \mathbb{N}}$ such that:

(i) $(\forall n \in \mathbb{N})\,[x_n \in (c, d) \setminus \{a\}]$,

(ii) $x_n \to a \quad \text{as} \quad n \to \infty$, we have

$$\lim_{n\to\infty} f(x_n) = b.$$

In order to make use of the above two definitions of the limit of a function, we need first to make sure that they are compatible. This is the content of the next result.

Theorem 4.1.1

The Cauchy and Heine definitions for the limit of a function are equivalent.

Proof. (Cauchy implies Heine)
Assume that $\lim_{x\to a} f(x) = b$ in the sense of Definition (4.1.1). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying the conditions of Definition (4.1.2). We need to prove that

$$\lim_{n\to\infty} f(x_n) = b.$$

Fix $\varepsilon > 0$. Then:

$$(\exists\delta > 0)(\forall x \in D(f)) \quad \big[0 < |x-a| < \delta \implies |f(x) - b| < \varepsilon\big].$$

Fix δ as found above. Then:

$$(\exists N \in \mathbb{N})(\forall n \in \mathbb{N}) \quad \big[n > N \implies |x_n - a| < \delta\big].$$

Then by (4.1.1) and (4.1.2),

$$(\forall n > N) \quad |f(x_n) - b| < \varepsilon,$$

which proves that

$$\lim_{n\to\infty} f(x_n) = b.$$

(Heine $\implies$ Cauchy)
We argue by contradiction. Suppose that $f(x) \to b$ as $x \to a$ in the sense of Definition (4.1.2) but not in the sense of Definition (4.1.1). This means that

$$(\exists\varepsilon > 0)(\forall\delta > 0)(\exists x \in D(f)) \quad \big[0 < |x-a| < \delta \wedge |f(x) - b| \geq \varepsilon\big].$$

Take $\delta = \frac{1}{n}$. Find x_n such that

$$0 < |x_n - a| < \frac{1}{n} \quad \wedge \quad |f(x_n) - b| \geq \varepsilon.$$

We have that $x_n \to a$ as $n \to \infty$. Therefore, by Definition (4.1.2), $f(x_n) \to b$ as $n \to \infty$. This is a contradiction. $\square$

Definition 4.1.3

We say that A is the limit of the function $f : D(f) \to \mathbb{R}$ as $x \to \infty$, and write

$$\lim_{x \to \infty} f(x) = A,$$

if for every $\varepsilon > 0$, there exists $K \in \mathbb{R}$ such that $(K, \infty) \subseteq D(f)$ and

$$(\forall x \in D(f)) \quad (x > M) \implies |f(x) - A| < \varepsilon.$$

The limit

$$\lim_{x \to -\infty} f(x) = A$$

is defined similarly.

Example 4.1.3

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Proof. We have to prove that

$$(\forall \varepsilon > 0)(\exists M > 0)(\forall x \in D(f)) \quad \left[x > M \implies \left|\frac{1}{x}\right| < \varepsilon\right].$$

It is sufficient to choose M with $M > \frac{1}{\varepsilon}$, as such an M exists by the Archimedean Principle. $\square$

The process of taking limits of functions behaves in much the same way as the process of taking limits of sequences. For example, the limit of a function (if it exists) is unique — we prove this using the Heine definition.

Theorem 4.1.2 (Uniqueness of Limits)

Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} f(x) = B$. Then $A = B$.

Proof. Let $x_n \to a$ as $n \to \infty$. Then $f(x_n) \to A$ and $f(x_n) \to B$ as $n \to \infty$. From this, $A = B$. $\square$

Sometimes it is more convenient to use the Cauchy definition.

Theorem 4.1.3

Let $\lim_{x\to a} f(x) = A$. Let $B > A$. Then

$$(\exists\delta > 0)(\forall x \in D(f)) \quad \big[0 < |x - a| < \delta \implies f(x) < B\big].$$

Proof. Using $\varepsilon = B - A > 0$ in the Cauchy definition of the limit, we have

$$(\exists\delta > 0)(\forall x \in D(f)) \quad \big[0 < |x - a| < \delta \implies |f(x) - A| < B - A\big],$$

which implies that $f(x) > A - (B - A) = A$ and $f(x) < B$. $\square$

The following two theorems can be proved using the Heine definition for the limit of a function at a point, and corresponding properties for the limit of a sequence. The proofs are left as exercises.

Theorem 4.1.4

Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$. Then:

(i) $\lim_{x\to a}(f(x) + g(x)) = A + B$,

(ii) $\lim_{x\to a}(f(x) - g(x)) = A - B$.

If in addition $B \neq 0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Theorem 4.1.5

Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$. Suppose that

$$(\exists\delta > 0)\big[x \in \mathbb{R} \wedge 0 < |x - a| < \delta \implies f(x) \le g(x)\big],$$

and that $A \le B$; that is,

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Using Theorem 4.1.4, one can easily compute limits of some functions.

Example 4.1.4

$$\lim_{x\to\infty} \frac{x^2 - 7}{2x^2 - 1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{7}{x^2}}{\frac{2x^2}{x^2} - \frac{1}{x^2}} = \lim_{x\to\infty} \frac{1 - \frac{7}{x^2}}{2 - \frac{1}{x^2}} = \frac{1-0}{2-0} = \frac{1}{2}.$$

Definition 4.1.4 (One-sided limits)

(i) Let f be defined on an interval $(a, c) \subseteq \mathbb{R}$. We say that $f(x) \to b$ as $x \to a^+$ and write

$$\lim_{x\to a^+} f(x) = b,$$

if

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in (a, c)) \quad \big[0 < x - a < \delta \implies |f(x) - b| < \varepsilon\big].$$

(ii) Let f be defined on an interval $(c, a) \subseteq \mathbb{R}$. We say that $f(x) \to b$ as $x \to a^-$ and write

$$\lim_{x\to a^-} f(x) = b,$$

if

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in (c, a)) \quad \big[0 < a - x < \delta \implies |f(x) - b| < \varepsilon\big].$$

3.2 Continuous Functions

Definition 4.2.1

Let $a \in \mathbb{R}$. Let a function f be defined in a neighborhood of a. Then the function f is said to be *continuous at* a if

$$\lim_{x \to a} f(x) = f(a).$$

The above definition may be reformulated in the following way:

Definition 4.2.2

A function f is said to be *continuous at a point* $a \in \mathbb{R}$ if f is defined on an interval (c, d) containing a and

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in (c, d)) \quad \big[|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\big].$$

Note the difference between this definition and the definition of the limit: the function f needs to be defined at a.

Using the above definition, it is easy to specify what it means for a function f to be *discontinuous* at a point a. A function f is *discontinuous at* a if either:

- f is not defined in any neighborhood (c, d) containing a, or
- $$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in (c, d)) \quad \big[|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon\big].$$

An equivalent way to define continuity at a point is to use the Heine definition of the limit.

Definition 4.2.3

A function f is said to be *continuous at a point* $a \in \mathbb{R}$ if f is defined on an interval (c, d) containing a, and for any sequence $(x_n)_{n \in \mathbb{N}}$ such that:

(i) $(\forall n \in \mathbb{N}) \quad [x_n \in (c, d)]$,

(ii) $x_n \to a \,\text{as}\, n \to \infty$,

we have

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Examples

Example 4.2.1. Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$ for any $x \in \mathbb{R}$. Then f is continuous at any point in $\mathbb{R}$.

Example 4.2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for any $x \in \mathbb{R}$. Then f is continuous at any point in $\mathbb{R}$.

The following theorem follows easily from the definition of continuity and properties of limits.

Theorem 4.2.1

Let f and g be continuous at $a \in \mathbb{R}$. Then:

(i) $f + g$ is continuous at a,

(ii) $f \cdot g$ is continuous at a,

(iii) If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

It is a consequence of this last theorem and the preceding two examples that the rational function

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$$

is continuous at every point of its domain of definition.

Theorem 4.2.2

Let g be continuous at $a \in \mathbb{R}$ and f continuous at $b = g(a) \in \mathbb{R}$. Then $f \circ g$ is continuous at a.

Proof. Fix $\varepsilon > 0$. Since f is continuous at b,

$$(\exists \delta > 0)(\forall y \in D(f)) \quad \big[|y - b| < \delta \implies |f(y) - f(b)| < \varepsilon\big].$$

Fix this $\delta > 0$. From the continuity of g at a,

$$(\exists \gamma > 0)(\forall x \in D(g)) \quad \big[|x - a| < \gamma \implies |g(x) - g(a)| < \delta\big].$$

From the above, it follows that

$$(\exists \gamma > 0)(\forall x \in D(g)) \quad \big[|x - a| < \gamma \implies |f(g(x)) - f(g(a))| < \varepsilon\big].$$

This proves continuity of $f \circ g$ at a. $\square$

Another useful characterization of continuity of a function f at a point a is the following:

f is continuous at $a \in \mathbb{R}$ if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$

In other words, the one-sided limits exist, are equal, and equal the value of the function at a.

Theorem 4.2.3

Let f be continuous at $a \in \mathbb{R}$. Let $f(a) < B$. Then there exists a neighborhood of a such that $f(x) < B$ for all points x belonging to this neighborhood.

Proof. Take $\varepsilon = B - f(a)$ in Definition 4.2.2. Then

$$(\exists \delta > 0)(\forall x \in (c, d)) \quad \big[|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\big].$$

$\square$

Remark 4.2.1. A similar fact is true for the case $f(a) > B$.

Definition 4.2.4

Let $f : D(f) \to \mathbb{R}$, $D(f) \subseteq \mathbb{R}$. f is said to be bounded if $\mathrm{Ran}(f)$ is a bounded subset of $\mathbb{R}$.

Example 4.2.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}$. Then $\mathrm{Ran}(f) \subseteq (0, 1]$, so f is bounded.

Definition 4.2.5

Let $A \subseteq D(f)$. f is said to be bounded above on A if

$$(\exists K \in \mathbb{R})(\forall x \in A) \quad f(x) \leq K.$$

Remark 4.2.2

Boundedness below and boundedness on a set are defined analogously.

Theorem 4.2.4

If f is continuous at a, then there exists $\delta > 0$ such that f is bounded on the interval $(a - \delta, a + \delta)$.

Proof. Since $\lim f(x) = f(a)$,

$$(\exists \delta > 0)(\forall x \in D(f)) \quad \big[|x - a| < \delta \implies |f(x) - f(a)| < 1\big].$$

So on the interval $(a - \delta, a + \delta)$,

$$f(a) - 1 < f(x) < f(a) + 1.$$

□

3.3 Continuous Functions on a Closed Interval

In the previous section, we dealt with functions which were continuous at a point. Here we consider functions which are continuous at every point of an interval $[a, b]$. We say that such functions are continuous on $[a, b]$. It can be shown that these functions can all be interested here in the global behaviour of such functions. We first define this notion formally.

Definition 4.3.1

Let $[a, b] \subseteq \mathbb{R}$ be a closed interval and let $f : D(f) \to \mathbb{R}$ be a function with $[a, b] \subseteq D(f)$. We say that f is a continuous function if:

(i) f is continuous at every point of $[a, b]$, and

(ii) $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Our first theorem describes the intermediate-value property for continuous functions.

Theorem 4.3.1 (Intermediate Value Theorem)

Let f be a continuous function on a closed interval $[a, b] \subseteq \mathbb{R}$. Suppose that $f(a) < 0$ and $f(b) > 0$. Then

$$(\exists c \in (a, b)) \quad f(c) = 0.$$

Proof. If $f = f(a) = 0$, then there is nothing to prove. Fix $y \in (f(a), f(b))$. Let us introduce the set

$$A = \{x \in [a, b] : f(x) \leq y\}.$$

The set A is not empty, as $a \in A$ (since $f(a) < y$). Therefore, the supremum $c = \sup A$ exists and $c \in [a, b]$. Our aim is to prove that $f(c) = y$. We do this by... $\square$

Ruling out the possibilities, $f(x) < n$ and $f(x) > n$. Note that $x \in (a, b)$ by continuity of f at a and b.

First, let us assume that $f(x) < n$. Then, by Theorem 4.2.3, we have that

$$(\exists \delta > 0)(\forall x \in D(f)) \quad \big[x \in (c - \delta, c + \delta) \implies f(x) < n\big].$$

Therefore,

$$(\exists n \in \mathbb{R}) \quad \big[n \in (c-\delta, c+\delta) \wedge f(x) < n\big].$$

In other words,

$$(\exists n \in \mathbb{R}) \quad \big[n > a \wedge n \in A^c\big].$$

This contradicts the fact that n is an upper bound of A.

Next, let us assume that $f(x) > n$. Then, by the remark after Theorem 4.2.3, we have that

$$(\exists \delta > 0)(\forall x \in D(f)) \quad \big[n \in (c-\delta, c+\delta) \wedge A^c\big].$$

This contradicts the fact that $n = \sup A$. $\square$

This theorem establishes boundedness for continuous functions defined on a closed interval.

Theorem 4.3.2 (Boundedness Theorem)

Let f be continuous on $[a, b]$. Then f is bounded on $[a, b]$.

Proof. Let us introduce the set

$$A = \{x \in \mathbb{R} \mid f \text{ is bounded on } [a, x]\}.$$

Note that:

- $A \neq \emptyset$ as $a \in A$,
- A is bounded (by b).

This means that $s = \sup A$ exists. We prove:

Step 1: $s = b$

First note that $s > a$. For $s > b$, by (left-)continuity of f at x,

$$(\exists \delta > 0)(\forall x \in D(f)) \quad \big[s - \delta < x < s + \delta \implies |f(x) - f(s)| < x\big].$$

So f is bounded on $[a, b + \delta]$. For example, set $a > 2 + b + \delta$. This shows that $s \leq b$.

If $s < b$, then by Theorem 4.2.4,

$$(\exists \delta > 0) \quad \big[s \in (c - \delta, c + \delta)\big].$$

By definition of the supremum,

$$(\exists x_1 \in (c - \delta, c)) \quad \big[f \text{ is bounded on } [a, x_1]\big].$$

Also, from (4.3.3), it follows that

$$(\exists x_2 \in (c, c + \delta)) \quad \big[f \text{ is bounded on } [x_2, x]\big].$$

Therefore f is bounded on $[a, x_2]$ where $x_2 > c$, which contradicts the fact that c is the supremum of A. This proves that $c \notin A$.

(Note that this does not complete the proof, as the supremum may not belong to the set: it is possible that $c \notin A$.)

Step 2: $b \in A$.

From the continuity of f at b, it follows that

$$(\exists \delta > 0) \quad \big[f \text{ is bounded on } [b - \delta, b)\big].$$

By definition of the supremum,

$$(\exists x \in (b - \delta, b)) \quad \big[f \text{ is bounded on } [a, x]\big].$$

Therefore f is bounded on $[a, b]$. $\square$

The last theorem asserted that the range $\mathrm{Ran}(f)$ of a continuous function f restricted to a closed interval is a bounded subset of $\mathbb{R}$. Consequently, $\mathrm{Ran}(f)$ possesses a supremum and infimum. The next theorem asserts that f, in fact, takes values at each extreme. In other words, there exist points in $[a, b]$ where the function attains its maximum and minimum values.

Theorem 4.3.3

Let f be continuous on $[a,b] \subseteq \mathbb{R}$. Then

$$(\exists y \in [a,b]) \quad \bigl[y = \arg\max_{x\in[a,b]} f(x)\bigr].$$

(In other words, $f(y) = \max_{x\in[a,b]} f(x)$.) Similarly,

$$(\exists z \in [a,b]) \quad \bigl[z = \arg\min_{x\in[a,b]} f(x)\bigr].$$

Proof. Let us introduce the set of values attained by f on $[a,b]$, namely:

$$\mathrm{Ran}(f) = \bigl\{f(x) : x \in [a,b]\bigr\}.$$

Then $f \neq \emptyset$ and by Theorem 4.3.2, f is bounded. Therefore the supremum $c = \sup f$ exists. We prove that there exists $x_0 \in [a,b]$ such that $f(x_0) = c$. We do this by contradiction.

Suppose, on the contrary, that

$$(\forall x \in [a,b]) \quad f(x) < c.$$

Define

$$g(x) = \frac{1}{a - f(x)}, \quad x \in [a,b].$$

Since the denominator is never zero on $[a,b]$, g is continuous and, by Theorem 4.3.2, g is bounded on $[a,b]$.

At the same time, by definition of the supremum, for any $\varepsilon > 0$,

$$(\exists x \in [a,b]) \quad (f(x) > a - \varepsilon).$$

In other words, $a - f(x) < \varepsilon$. So $g(x) > \frac{1}{\varepsilon}$. This proves that

$$(\forall \varepsilon > 0)(\exists x \in [a,b]) \quad \bigl[g(x) > \frac{1}{\varepsilon}\bigr].$$

Therefore g is unbounded on $[a,b]$, which contradicts the above. $\square$

3.4 Inverse Functions

In this section, we discuss the inverse function of a continuous function. Recall that a function possesses an inverse if and only if it is a bijection.

It turns out that a continuous bijection defined on an interval $[a, b]$ is monotone (that is, either strictly increasing or strictly decreasing) (see Exercises). The notions of (strictly) increasing and (strictly) decreasing parallel the corresponding notions for sequences. Here is a precise definition.

Definition 4.4.1

A function f defined on $[a, b]$ is said to be *increasing* on $[a, b]$ if

$$(\forall x_1, x_2 \in [a, b]) \quad \big[x_1 < x_2 \implies f(x_1) \leq f(x_2)\big].$$

It is said to be *strictly increasing* on $[a, b]$ if

$$(\forall x_1, x_2 \in [a, b]) \quad \big[x_1 < x_2 \implies f(x_1) < f(x_2)\big].$$

The notions of *decreasing* and *strictly decreasing* functions are defined similarly.

Theorem 4.4.1

Let f be continuous and strictly increasing on $[a, b]$. Let $f(a) = c$, $f(b) = d$. Then there exists a function $g : [c, d] \to [a, b]$ which is continuous and strictly increasing such that

$$(\forall y \in [c, d]) \quad f(g(y)) = y.$$

Proof. Let $y \in [c, d]$. By Theorem 4.3.1,

$$(\exists x \in [a, b]) \quad [f(x) = y].$$

□

There is only one such value x since f is increasing. (Prove this.) The inverse function

g is defined by

$$x = g(y).$$

It is easy to see that g is strictly increasing. Indeed, let $y_1 < y_2$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$. Suppose also that $x_2 \leq x_1$. Since f is strictly increasing, it follows that $y_2 \leq y_1$. This is a contradiction.

Now let us prove that g is continuous. Let $y_0 \in f(c, d)$. Then

$$(\exists x_0 \in (c, d)) \quad [y_0 = f(x_0)],$$

or, in other words,

$$x_0 = g(y_0).$$

Let $\varepsilon > 0$. We assume also that ε is small enough such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq (c, d)$. Let $y_1 = f(x_0 - \varepsilon)$ and $y_2 = f(x_0 + \varepsilon)$. Since g is increasing, we have that

$$y_1 \leq y \leq y_2 \quad \Longrightarrow \quad x = g(y) \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

Take $\delta = \min\{y_2 - y_0, y_0 - y_1\}$. Then

$$|y - y_0| < \delta \implies |g(y) - g(y_0)| < \varepsilon.$$

Continuity at the endpoints of the interval can be established similarly. $\square$

Chapter 4

Differential Calculus

In this chapter, the notion of a *differentiable function* is developed, and several important theorems about differentiable functions are presented.

4.1 Definition of Derivative. Elementary Properties

Definition 5.1.1

Let f be defined in a δ-neighborhood $(a-\delta, a+\delta)$ of $a \in \mathbb{R}$ ($\delta > 0$). We say that f is *differentiable at* a if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists in $\mathbb{R}$. This limit, denoted by $f'(a)$, is called the *derivative of* f *at* a.

Example 5.1.1

Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$. Then f is differentiable at any $x \in \mathbb{R}$ and $f'(x) = 0$.

Proof.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0.$$

$\square$

Example 5.1.2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Then f is differentiable at any $x \in \mathbb{R}$ and $f'(x) = 1$.

Proof.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h) - a}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$\square$

Example 5.1.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f is differentiable at any $x \in \mathbb{R}$ and $f'(x) = 2x$.

Proof.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h}.$$

Simplifying:

$$= \lim_{h\to 0} \big(2a + h\big) = 2a.$$

$\square$

Proof.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0}(2a + h) = 2a.$$

Example 5.1.4

Let $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$. Then f is differentiable at any $x \in \mathbb{R}$ and $f'(x) = nx^{n-1}$.

This may be proved using mathematical induction together with the product rule (see below). It is left as an exercise.

Example 5.1.5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then f is differentiable at any $x \in \mathbb{R} \setminus \{0\}$. But f is not differentiable at 0.

Proof. If $x > 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

If $x < 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h)+x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

Therefore, the derivative does not exist at $x = 0$ as

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$$

□

Note that the function f in the above example is continuous at 0; thus, continuity does not imply differentiability. However, the converse is true.

Theorem 5.1.1

If f is differentiable at a, then f is continuous at a.

Proof.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Hence,

$$\lim_{h \to 0}(f(a+h) - f(a)) = \lim_{h \to 0} \left(h \cdot \frac{f(a+h) - f(a)}{h} \right).$$

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = f(a).$$

Therefore,

$$\lim_{x \to a} f(x) = f(a).$$

□

Remark 5.1.1

If f is differentiable at $a \in \mathbb{R}$, then there exists a function $o(x)$ such that

$$\lim_{x \to a} o(x) = 0$$

and

$$f(x) = f(a) + f'(a)(x-a) + o(x)(x-a).$$

Indeed, define

$$o(x) = \frac{f(x) - f(a)}{x - a} - f'(a).$$

Then $o(x) \to 0$ as $x \to a$, and $f(x) = f(a) + f'(a)(x-a) + o(x)(x-a)$.

This enables one to reinterpret the formula in the above Remark as follows. If f is differentiable at $a \in \mathbb{R}$, then one can write for the value of $f(a+h)$, that is "near" a:

$$f(a+h) = f(a) + hf'(a) + o(h)h,$$

where the notation $o(h)$ reads as "little o of h" and denotes any function which has the following property:

$$\lim_{h \to 0} o(h) = 0.$$

A more explicit expression for the quantity $o(h)h$ can be obtained, under the assumption of the existence of higher order derivatives of f using the Taylor theorem (see Section 5.3). The latter formula can be taken as useful for the definition of differentiability, as follows: f is differentiable at a if and only if $f(a+h)$ can be expressed as

$$f(a+h) = f(a) + hf'(a) + \phi(h)h,$$

where $\phi(h) \to 0$ as $h \to 0$. The quantity $\phi(h)$ being small means that $f(a+h)$ can be expressed by the absolute value of $o(h)h$, which is much smaller than that of $hf'(a)$ and becomes negligible as $h \to 0$. However, writing $o(h)h$ instead of $\phi(h)h$ stresses that the quantity $o(h)h$ vanishes faster than h, whereas $\phi(h)h$ does not necessarily vanish.

Theorem 5.1.2

If f and g are differentiable at a, then $f+g$ is also differentiable at a, and

$$(f+g)'(a) = f'(a) + g'(a).$$

The proof is left as an exercise.

Theorem 5.1.3 (Product Rule)

If f and g are differentiable at a, then $f \cdot g$ is also differentiable at a, and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{g(a+h)\big[f(a+h) - f(a)\big]}{h} + \lim_{h\to 0} \frac{f(a)\big[g(a+h) - g(a)\big]}{h} \\
&= \lim_{h\to 0} g(a+h) \cdot \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} f(a) \cdot \frac{g(a+h) - g(a)}{h}.
\end{aligned}$$

Since $g(a+h) \to g(a)$ as $h \to 0$, we conclude that

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

□

Theorem 5.1.4

If g is differentiable at a and $g(a) \neq 0$, then $\varphi = 1/g$ is also differentiable at a, and

$$\varphi'(a) = \frac{-g'(a)}{[g(a)]^2}.$$

Proof. The result follows from

$$\frac{\varphi(a+h)-\varphi(a)}{h} = \frac{g(a)-g(a+h)}{h\cdot g(a)g(a+h)}.$$

$\square$

Theorem 5.1.5 (Quotient Rule)

If f and g are differentiable at a and $g(a) \neq 0$, then $\varphi = f/g$ is also differentiable at a, and

$$\varphi'(a) = \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof. This follows from Theorems 5.1.3 and 5.1.4. $\square$

Theorem 5.1.6 (Chain Rule)

If g is differentiable at $a \in \mathbb{R}$ and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Proof. By the definition of the derivative and Remark 5.1.1, we have

$$\frac{f(g(x)) - f(g(a))}{x-a} = \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a},$$

where $g(x) \to g(a)$ as $x \to a$. Replace y and h in the above equality by $y = g(x)$ and $h = g(a)$, and divide both sides by $x - a$, to obtain

$$\frac{f(g(x)) - f(g(a))}{x-a} = f'(g(a)) \cdot g'(a).$$

By Theorem 5.1.1, g is continuous at a. Hence as $x \to a$, $g(x) \to g(a)$, and

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} = f'(g(a)) \cdot g'(a),$$

passing to the limit $x \to a$ in the above equality yields the required result. $\square$

Example 5.1.6

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x^2+1)^{10}$. Then f is differentiable at every point in $\mathbb{R}$, and

$$f'(x) = 20x(x^2+1)^9.$$

In the next theorem, we establish a relation between the derivative of an invertible function and the derivative of its inverse function.

Theorem 5.1.7

Let f be continuous and strictly increasing on (a,b). Suppose that, for some $x_0 \in (a,b)$, f is differentiable at x_0 and $f'(x_0) \neq 0$. Then the inverse function $g = f^{-1}$ is differentiable at $y_0 = f(x_0)$, and

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. According to Remark 5.1.1, we may write:

$$y - y_0 = \frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)} \cdot (g(y) - g(y_0)),$$

where $o(g(y) - g(y_0)) = 0$ in case $g(y) = g(y_0)$. However, as g is continuous at y_0, it follows that $g(y) \to g(y_0)$ as $y \to y_0$; hence $o(g(y) - g(y_0)) \to 0$ as $y \to y_0$. Therefore, we have:

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{f(g(y)) - f(g(y_0))}{y - y_0} \cdot \frac{1}{f'(g(y_0)) + o(1)} \quad \text{as } y \to y_0.$$

□

Example 5.1.7

Let $n \in \mathbb{N}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) = x^n$. Then the inverse function is $g(y) = y^{1/n}$. We know that $f'(x) = nx^{n-1}$. Hence, by the previous theorem:

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{ny^{(n-1)/n}}.$$

One-Sided Derivatives

In a manner similar to the definition of the one-sided limit, we may also define the *left* and *right* derivatives of f at a via:

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}, \quad f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}.$$

4.2 Theorems on Differentiable Functions

In this section, we investigate properties of differentiable functions on intervals.

Theorem 5.2.1

Let f be a function defined on (a,b). If f attains its maximum (or minimum) at $x_0 \in (a,b)$, and if f is differentiable at x_0, then $f'(x_0) = 0$.

Proof. We prove the theorem in the case that f attains its maximum at x_0 ($f(x_0)$); the proof is similar in the other case.
Let $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq (a,b)$. For $0 < h < \delta$, $f(x_0+h) \leq f(x_0)$ and:

$$\frac{f(x_0+h)-f(x_0)}{h} \leq 0,$$

so that

$$f'_+(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0.$$

For $-\delta < h < 0$, $f(a+h) \leq f(a)$ and

$$\frac{f(a+h)-f(a)}{h} \geq 0,$$

so that

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \geq 0 \quad \text{and} \quad f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \leq 0.$$

In sum,

$$0 \leq f'(a) = f'(a) \leq 0,$$

so $f'(a) = 0$. $\square$

Note that the converse statement is false. Here is a simple counter-example. Consider

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3.$$

Then $f'(0) = 0$, but f does not attain its maximum or minimum at 0 on any interval containing 0.

Theorem 5.2.2 (Rolle's Theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then

$$(\exists c \in (a, b)) \quad f'(c) = 0.$$

Proof. It follows from the continuity of f that it attains its maximum and minimum value on $[a, b]$. Suppose that f attains either its maximum or its minimum at an interior point $c \in (a, b)$. Then, by Theorem 5.2.1, $f'(c) = 0$, and the result follows. Now suppose that f attains both its maximum and its minimum value at the endpoints a or b. This means that f is constant on $[a, b]$, and in this case $f'(c) = 0$ for all $c \in (a, b)$. However, $f(a) = f(b)$, so the absolute maximum and minimum values of f must be equal. Therefore, it follows that $f'(c) = 0$ for all $c \in (a, b)$. $\square$

Theorem 5.2.3 (Mean Value Theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), then

$$(\exists c \in (a, b)) \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Set

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Then g is continuous on $[a, b]$ and differentiable on (a, b). Also,

$$g(a) = f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](a - a) = f(a),$$

and

$$g(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) = f(b).$$

Moreover, $g(a) = g(b)$.

By Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. Then:

$$g'(x) = f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right].$$

So:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Therefore, by Rolle's theorem,

$$(\exists c \in (a, b)) \quad [f'(c) = 0].$$

Corollary 5.2.1

If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant there.

Proof. Let a and b be any two points in the interval with $a \neq b$. Then, by the mean value theorem, there is a point $z \in (a, b)$ such that

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(z) = 0$ for all z in the interval, so

$$0 = \frac{f(b) - f(a)}{b - a}.$$

and consequently, $f(b) = f(a)$. Thus, the value of f at any two points is the same and f is constant on the interval. □

Corollary 5.2.2

If f and g are defined on the same interval and $f'(x) = g'(x)$ there, then $f(x) = g(x) + c$ for some number $c \in \mathbb{R}$.

The proof is left as an exercise.

Corollary 5.2.3

If $f'(x) > 0$ (resp. $f'(x) < 0$) for all x in some interval, then f is increasing (resp. decreasing) on the interval.

Proof. Consider the case $f'(x) > 0$. Let a and b be any two points in the interval, with $a < b$. By the mean value theorem, there is a point $z \in (a, b)$ such that

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(z) > 0$ for all z in the interval, so that

$$f(b) - f(a) > 0.$$

Since $b - a > 0$, it follows that $f(b) > f(a)$, which proves that f is increasing on the interval. The case $f'(x) < 0$ is left as an exercise. $\square$

Theorem 5.2.4 (Cauchy Mean Value Theorem)

If f and g are continuous on $[a, b]$ and differentiable on (a, b), then

$$(\exists c \in (a, b)) \quad \big[g(b)f'(c) - f(b)g'(c) = g(a)f'(c) - f(a)g'(c)\big].$$

(If $g(a) \neq g(c)$, and $g'(x) \neq 0$, the above equality can be rewritten as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$

Note that if $g(x) = x$, we obtain the mean value theorem.)

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then $h(a) = f(b)g(a) - f(a)g(b) = h(b)$, so that h satisfies Rolle's theorem. Therefore,

$$(\exists x_0 \in (a, b)) \quad h'(x_0) = [f(b) - f(a)]g'(x_0) - [g(b) - g(a)]f'(x_0).$$

$\square$

Theorem 5.2.5 (L'Hôpital's Rule)

Let f and g be differentiable functions in the neighborhood of $a \in \mathbb{R}$. Assume that:

(i) $g(x) \neq 0$ in some neighborhood of a,

(ii) $f(a) = g(a) = 0$, or

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. By the Cauchy mean value theorem,

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

for some $0 < \theta < 1$. Now pass to the limit $h \to 0$ to get the result. $\square$

L'Hôpital's rule is a useful tool for computing limits.

Example 5.2.1

Let $m, n \in \mathbb{N}, n > 0, a \neq 0$. Then

$$\lim_{x\to\infty} \frac{m^x}{n^x} = 0, \quad \text{and} \quad \lim_{x\to\infty} x^m e^{-nx} = 0.$$

4.3 Approximation by Polynomials. Taylor's Theorem

The class of polynomials consists of all those functions f which may be written in the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. In this section, we show that functions that are differentiable sufficiently many times can be approximated by polynomials. We use the notation $f^{(k)}(a)$ to stand for the k-th derivative of f at a.

Lemma 5.3.1

Suppose $f : \mathbb{R} \to \mathbb{R}$ is the polynomial function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. Then

$$a_k = \frac{1}{k!} f^{(k)}(0) \quad \text{for } k = 0, 1, \ldots, n.$$

Proof. a_k is computed by differentiating f k times and evaluating at 0.

Thus, for a polynomial of degree n, we have

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

More generally, if $x = a + h$, where a is fixed, we have

$$f(a+h) = f(a) + f'(a)h + \frac{f^{(2)}(a)}{2!}h^2 + \cdots + \frac{f^{(n)}(a)}{n!}h^n.$$

The next theorem describes how a sufficiently well-behaved function may be approximated

by a polynomial.

Theorem 5.3.1 (Taylor's Theorem)

Let $n > 0$ and $p \geq 1$. Suppose that f and its derivatives up to order n are continuous on $[a, b]$ and that $f^{(n)}$ exists on (a, b). Then there exists a number $c \in (a, b)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f^{(2)}(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_n,$$

where

$$R_n = \frac{h^n}{n!}(1-\theta)^n f^{(n)}(a+\theta h).$$

The polynomial

$$p(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

Tutorials of Analysis.I

Exercises I

Exercise 1 :

1. If a and b are positive or zero real numbers, show that: $\sqrt{a}+\sqrt{b} \leq 2\sqrt{a+b}$.

2. Show that $\sqrt{2} \notin \mathbb{Q}$.

3. Show that if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ then $r+x \notin \mathbb{Q}$ and if $r \neq 0$ then $r.x \notin \mathbb{Q}$.

4. Deduce that between two numbers there is always an irrational number.

Exercise 2 :

Find in the form $\frac{p}{q}$ rationals x whose periodic decimal development is given by :

a. $3.14\overline{14}$ b. $0.99\overline{9}$ c. $3.149\overline{9}$.

Exercise 3 :

The maximum of two numbers x, y (i.e. the larger of the two) is denoted $max(x,y)$. Similarly, the smaller of the two numbers x, y is denoted $min(x,y)$.

1. a)Show that :

$$max(x,y) = \frac{x+y+|x-y|}{2} \quad and \quad min(x,y) = \frac{x+y-|x-y|}{2}.$$

b) Find a formula for $max(x,y,z)$.

2. Let A and B be two non-empty parts of $\mathbb{R}$. We define $A+B = \{x+y/x \in A, y \in B\}$. Show that if A and B are majorities then $(A+B)$ is major and we have $sup(A+B) = sup(A)+sup(B)$.

Exercise 4 :

Determine (if they exist): majorants, minorants, upper bound, lower bound, largest element, smallest element of the following sets:

$$[0,1] \cap \mathbb{Q}; \quad]0,1[\cap \mathbb{Q}; \quad \mathbb{N}; \quad \{(-1)^n + \frac{1}{n} \quad /n \in \mathbb{N}^*\}.$$

Exercise 5 :

1) Give the meaning of $\mathbb{Q}$ is dense in $\mathbb{R}$.

2) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

3) Determine whether the sets are open, closed or neither:

$$\emptyset; \quad \mathbb{R}; \quad [2,3] \cup \{-1\}; \quad \bigcap_{n>0}]\frac{-1}{n}, \frac{1}{n}[; \quad \bigcup_{n>0} [0, 1 - \frac{1}{n}[.$$

4) Show that :

a. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b. $int(A \cup B) = int(A) \cap int(B)$.

Exercises II

Exercice 1 : square root calculations

Let $a \in \mathbb{R}_+^*$. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by :

$$u_0 \in \mathbb{R}_+^*, \quad u_{n+1} = \frac{1}{2}(u_n + \frac{a}{u_n}),$$

and we note $(v_n)_{n\in\mathbb{N}}$ the sequence defined by : $v_n = \frac{(u_n - \sqrt{a})}{(u_n + \sqrt{a})}$.

1. Show that, for any integer n, $v_{n+1} = v_n^2$.

2. Calculate v_n as a function of v_0 and show that $|v_0| < 1$. Deduce that v_n converges to 0.

3. Express u_n as a function of v_n and show that $lim\ u_n = \sqrt{a}$.

4. Calculate the first three terms of the sequence, for $u_0 = 1$ and $a = 2$.

Exercise 2 :

Consider the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ denoted for all $n \in \mathbb{N}^*$ by :

$$\begin{cases} u_n & = \sum_{k=1}^{n} \frac{1}{k^2} \\ v_n & = u_n + \frac{3}{n} \end{cases}$$

1. Study the monotony of sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$.

2. Show that for all n in $\mathbb{N}^*$, $u_n \leq v_n$.

3. Returning to the formal denition, show that the sequence $(v_n - u_n)_{n\in\mathbb{N}^*}$ converges to 0.

4. What have we just shown about the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$?

Exercise 3 :

For each of the following statements say whether it is true or false. Justify your answer in each case.

1. A sequence converges if and only if it is bounded.

2. An increasing real sequence with a majority converges.

3. A non-majoritized real sequence tends to $+\infty$.

4. A real sequence converging to 1 by lower values is increasing.

5. A positive real sequence which tends towards 0 is decreasing from a certain rank.

6. The sequence (a_n) converges to 0 if and only if the sequence $(|a_n|)$ converges to 0.

7. Let be two sequences (u_n) and (v_n) with $v_n = u_{n+1} - u_n$. Then (u_n) converges if and only if (v_n) converges to 0.

Exercise 4 :

What can be said of the sum of two convergent sequences?

Of two divergent sequences?

Of a convergent sequence and a divergent sequence?

Exercises III

Exercise 1 :

1. We note $f(x) = \sqrt{x-3} - \sqrt{x+5}$ for $x \geq 3$. What is the limit of f in $+\infty$?

2. We note $g(x) = \sqrt{x^2+2x+7} - (x+5)$ for $x \in \mathbb{R}$. What is the limit of g in $+\infty$? And in $-\infty$?

3. Calculate the limits of the following functions :

a. $\lim\limits_{x\to+\infty} \frac{ln(1+e^x)}{x}$; b. $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}$; c. $\lim\limits_{x\to+\infty} \frac{x}{(1+x)^n-1}$; d. $\lim\limits_{x\to+\infty} x(\sqrt{x^2+1}-x)$;
e. $\lim\limits_{x\to0}\frac{e^{-ax}-e^{-bx}}{x}$; f. $\lim\limits_{x\to+\infty} \sqrt{x^2+1} - \sqrt{x^2-1}$; h. $\lim\limits_{x\to-4} \frac{sin(\pi x)}{x^2-16}$.

Exercise 2 :

Calculer, en utilisant les limites connues des fonctions usuelles, les limites suivantes :

1. $\lim\limits_{x\to+\infty} \frac{x^2+5x+e^{\frac{1}{x}}}{(x+1)^3}$
2. $\lim\limits_{x\to+\infty} \sqrt{x^2+x+x} - x$
3. $\lim\limits_{x\to+\infty} \frac{(tanx)^2}{cos(2x)-1}$
4. $\lim\limits_{x\to1^+} (x^3-1)ln(2x^2+x-3)$.

Exercise 3 :

1) Calculate the n-th derivative of the functions :

a. $f(x) = x^2(1+x)^n$ b. $f(x) = x^3 ln(x)$ c. $g(x) = \frac{1}{1-x^2}$.

2) Show that the n-th derivative of the function defined by $h_n(x) = x^{n-1}e^{\frac{1}{x}}$ is defined by

$h_n^{(n)}(x) = (-1)^n x^{-(n+1)} e^{\frac{1}{x}}$.

Exercise 4 :

Using integration by parts, calculate the following integrals :

$$A = \int_0^{\pi} e^x sinx dx \qquad ; \qquad B = \int_0^1 x e^x dx.$$

Exercises IV

Exercise 1 :

Donner les développements limités d'ordre 3 en 0 des fonctions suivantes :

1. $x \mapsto e^x$; 2. $x \mapsto ln(x+1)$; 3. $x \mapsto cosx$; 4. $x \mapsto sinx$; 5. $x \mapsto tanx$.

Exercise 2 :

Let f(x) be a function from $\mathbb{R}$ to $\mathbb{R}$. Use the following mathematical symbols $(\forall, \exists, \Rightarrow)$ to write the following sentences :

1. f tends to 0 when x tends to ∞.
2. f does not tends to 3 when x tends to 1.
3. f tends to $+\infty$ when x tends to $+\infty$.

Exercise 3 :

For all $x \in \mathbb{R}$, we note $f(x) = x^6 + 2x^5 - 3x^4 + x^2 + 1$ and $g(x) = x^6 - x^5 + 3x^4 - x + 3$. Show that it exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$.

Exercise 4 :

Consider a function $f : [0, +\infty[\longrightarrow \mathbb{R}$ continues such that $\lim\limits_{+\infty} f = 0$. Show that it exists λ such that f is bounded on $[\lambda, +\infty[$. Deduce that f is bounded on $[0, +\infty[$. Is it true that there is $a \in [0, +\infty[$ such that $\sup\limits_{[0,+\infty[} f = f(a)$?

Exercise 5 :

Let $f : \mathbb{R} \to \mathbb{R}$ continues and $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Show that it exist an interval $[a, b]$ of $\mathbb{R}$ (with $a < b$) such that :

$$\forall x \in [a, b] : f(x) > \frac{f(x_0)}{3}.$$

Exercise 6 :

Consider a function $G :]0,1[\longrightarrow \mathbb{R}$ checking for a constant $\mu > 0$ the following property :

$$\forall (x,z) \in]0,1[\times]0,1[, \quad |G(x) - G(z)| \leq \mu\sqrt{|x-z|} \quad . \qquad \circledast \,.$$

1. Show that G is continuous on $]0,1[$
2. Show that G is bounded on $]0,1[$.
3. Give an example of a continuous function of $]0,1[$ in $\mathbb{R}$ and unbounded.
4. Give an example of function G non-constant checking $\circledast$.