Acadmic year : 2024-2025

. . . .

Pathways : CS

. . . .

First year / Semester II

Linear Algebra

Course volume : 20H

Tutorial : 20H

Teacher : Hamed OUEDRAOGO

Email : **ouedraogohamed557@gmail.com**

Date : March 2025

Contents

Preambule

Bienvenue dans ce cours d'algèbre linéaire, une discipline mathématique fondamentale qui joue un rôle central dans de nombreux domaines de l'informatique moderne. À travers ce cours, nous explorerons les structures et outils mathématiques qui sont à la base d'algorithmes, de l'intelligence artificielle, de la vision par ordinateur, de la cryptographie, et bien d'autres domaines encore.

Pour les étudiants en informatique, l'algèbre linéaire est un pilier incontournable. Elle intervient dans le traitement d'images et de signaux, l'apprentissage automatique (machine learning), les moteurs de rendu 3D, les simulations physiques, la compression de données, et l'analyse de graphes. Des concepts tels que les matrices, les transformations linéaires, les espaces vectoriels ou encore les valeurs propres sont omniprésents dans les algorithmes utilisés en intelligence artificielle, en big data, et en sécurité informatique.

Au-delà de ces applications concrètes, l'algèbre linéaire offre un langage puissant et universel pour modéliser et résoudre des problèmes complexes de manière élégante et efficace. Elle favorise également une rigueur et une abstraction précieuses dans la conception d'algorithmes et de systèmes informatiques.

Dans ce cours, nous établirons une base solide en algèbre linéaire, avec un accent particulier sur les concepts et les méthodes qui trouvent des applications directes en informatique. L'objectif est de vous doter des outils mathématiques essentiels à votre réussite académique et à votre développement en tant qu'informaticien(ne) compétent(e), prêt(e) à relever les défis du monde numérique d'aujourd'hui.

Preambule :

Welcome to this course on Linear Algebra, a foundational branch of mathematics that underpins many areas of computer science and modern technology. Throughout this course, we will delve into essential mathematical structures that shape the digital world around us.

For computer science students, linear algebra is a vital tool. It plays a central role in areas such as machine learning, computer graphics, computer vision, data science, signal processing, and cryptography. Concepts like matrices, vector spaces, linear transformations, eigenvalues and eigenvectors are fundamental in designing efficient algorithms and understanding how information is represented, processed, and transformed.

Whether it's training neural networks, optimizing search engines, compressing media, or simulating physical systems in games, linear algebra provides the mathematical foundation. Its abstract language allows us to model complex systems in a compact and elegant way, making it an indispensable asset in the toolkit of any computer scientist.

In this course, we will focus on building a strong foundation in linear algebra, with an emphasis on concepts and techniques that are directly applicable to computer science. Our goal is to equip you with the mathematical tools necessary for academic success and for tackling real-world challenges in today's data-driven and algorithm-powered world.

Chapter 1

Notion of $\mathbb{K}$- Vector spaces ($\mathbb{K}$ being a Commutative field)

1.1 Vector space and sub-vector space

Let $\mathbb{K}$ be a commutative field (usually $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty set with an internal operation $(+)$:

$$\begin{aligned} E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned}$$

and an external operation noted $(\cdot)$:

$$\begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\lambda, y) &\longmapsto \lambda.y \end{aligned}$$

Definition 1.1.1 *A vector space over the field $\mathbb{K}$ or an $\mathbb{K}$-vector space is a triplet $(E, +, .)$ such that :*

(1) $(E, +)$ *is a commutative group.*

(2) $\forall \lambda \in \mathbb{K}, \forall x, y \in E, \lambda.(x + y) = \lambda.x + \lambda.y$

(3) $\forall \lambda, \mu \in \mathbb{K}, \forall x \in E, (\lambda + \mu).x = \lambda.x + \mu.x$

(4) $\forall \lambda, \mu \in \mathbb{K}, \forall x \in E, (\lambda.\mu).x = \lambda(\mu.x)$

(5) $\forall x \in E, 1_{\mathbb{K}}.x = x$

The elements of the vector space are called vectors and those of $\mathbb{K}$ *are called scalars.*

Proposition 1.1.2 *If* E *is* $\mathbb{K}$*- vector space, then we have the following properties :*

(1) $\forall x \in E, 0_{\mathbb{K}}.x = 0_E$

(2) $\forall x \in E,\ -1_{\mathbb{K}}.x = -x$

(3) $\forall \lambda \in \mathbb{K},\ \lambda.0_E = 0_E$

(4) $\forall \lambda \in \mathbb{K}, \forall x, y \in E,\ \lambda.(x-y) = \lambda.x - \lambda.y$

(5) $\forall \lambda \in \mathbb{K}, \forall x \in E, x.\lambda = 0_E \Leftrightarrow x = 0_E \vee \lambda = 0_{\mathbb{K}}$

Example 1.1.3 *(1)* $(\mathbb{R}, +, .)$ *is an* $\mathbb{R}$*- vector space,* $(\mathbb{C}, +, .)$ *is a* $\mathbb{C}$*- vector space.*

(2) If we consider $\mathbb{R}^2$ *with the following two operations :*

$(+) : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad$ $(.) : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$((x, y); (x', y')) \longrightarrow (x + x', y + y'),$ $\qquad$ $(\lambda, (x, y)) \longrightarrow (\lambda.x, \lambda.y)$

we can easily show that $(\mathbb{R}^2, +, .)$ *is an* $\mathbb{R}$*- vector space.*

Definition 1.1.4 *Let* $(E, +, .)$ *be an* $\mathbb{K}$*- vector space and let* F *be a non-empty subset of* E*,* F *is said to be a sub-vector space if* $(F, +, .)$ *is also an* $\mathbb{K}$*- vector space.*

Remark 1.1.5 *When* $(F, +, .)$ *is* $\mathbb{K}$*- a vector subspace of* $(E, +, .)$*, then* $0_E \in F$*.*
If $0_E \notin F$ *then* $(F, +, .)$ *cannot be an* $\mathbb{K}$*- sub vector space of* $(E, +, .)$*.*

Theorem 1.1.6 *Let* $(E, +, \cdot)$ *be an* $\mathbb{K}$*- vector space and* $F \subset E$*,* F *non-empty we have the following equivalences :*

(1) F *is a sub-vector space of* E

(2) F *is stable by addition and multiplication i.e. :*

$$\forall x, y \in F, \forall \lambda \in \mathbb{K}, x + y \in F, \lambda.x \in F$$

(3) $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda.x + \mu.y \in F,$ *hence*

F *is a vector subspace* $\Leftrightarrow \begin{cases} F \neq \emptyset, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda.x + \mu.y \in F \end{cases}$

(4) $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda.x + \mu.y \in F,$ *hence*

F *is a vector subspace* $\Leftrightarrow \begin{cases} 0_E \in F, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda.x + \mu.y \in F \end{cases}$

Example 1.1.7 *(1)* $\{0_E\}, E$ *are sub-vector spaces of E, called trivial vector spaces.*

(2) $F = \{(x, y) \in \mathbb{R}^2 / x + y = 0\}$ *is a sub-vector space because ;*

• $0_E = O_{\mathbb{R}^2} = (0, 0) \in F \Rightarrow F \neq \emptyset$.

• $\forall (x, y), (x', y') \in F, \lambda, \mu \in \mathbb{R}$, *let us show that* $\lambda(x, y) + \mu(x', y') \in F$, *i.e :* $(\lambda x + \mu x', \lambda y + \mu y') \in F$, *we have :* $\lambda x + \mu x', \lambda y + \mu y' = \lambda(x + y) + \mu(x' + y') = \lambda.0 + \mu.0 = 0$, *because* $\forall (x, y) \in F \Rightarrow x + y = 0$ *and* $\forall (x', y') \in F \Rightarrow x' + y' = 0$. *Hence* F *is a sub-vector spaces of* $\mathbb{R}^2$.

(3) $F = \{(x + y + z, x - y, z)/x, y, z \in \mathbb{R}\}$ *is a sub-vector spaces of* $\mathbb{R}^3$, *evidence to be provided.*

Theorem 1.1.8 *The intersection of a non-empty family of subvector spaces is a subvector space.*

Remark 1.1.9 *The union of two sub-vector spaces is not necessarily a sub-vector space.*

Example 1.1.10 $E_1 = \{(x, 0) \in \mathbb{R}^2\}, E_2 = \{(0, y) \in \mathbb{R}^2\}$, $E_1 \cup E_2$ *is not a sub-vector space because* $U_1 = (1, 0) \in E_1, U_2 = (1, 1) \in E_2$ *and* $U_1 + U_2 = (1, 1) \notin E_1 + E_2$ *as* $(1, 1) \notin E_1 \wedge (1, 1) \notin E_2$.

1.2 Sum of two sub-vector spaces

Let E_1, E_2 be two sub-vector spaces of an $\mathbb{K}$-vector space E, the sum of two vector spaces, E_1 and E_2, is called vector spaces, E_1 and E_2 and note $E_1 + E_2$ the following set :

$$E_1 + E_2 = \{u \in E / \exists u_1 \in E_1, \exists u_2 \in E_2 / u = u_1 + u_2\}.$$

Proposition 1.2.1 *The sum of two subvector spaces of* E_1 *and* E_2 *(of the same* $\mathbb{K}$*-vector space) is a sub-vector space of* E *containing* $E_1 \cup E_2$, *i.e.,* $E_1 \cup E_2 \subset E_1 + E_2$.

1.3 Direct sum of two sub-vector spaces

We will say that the sum $E_1 + E_2$ is direct if $\forall U = U_1 + U_2$, there exists a unique vector $U_1 \in E_1$, a unique vector $U_2 \in E_2$ / $U = U_1 + U_2$, we note $E_1 \oplus E_2$.

Theorem 1.3.1 *Let E_1, E_2 be two subspaces of the same $\mathbb{R}$-vector space E, the sum $E_1 + E_2$ is direct if $E_1 \cap E_2 = \{0_E\}$.*

1.3.1 Additional sub-space

Let E_1 and E_2 be two sub-vector spaces of the same $\mathbb{K}$-vector space E, E_1 and E_2 are said to be supplementary if $E_1 \oplus E_2 = E$.

Example 1.3.2 $E_1 = \{(x, 0) \in \mathbb{R}^2\}, E_2 = \{(0, y) \in \mathbb{R}^2\}$, $E_1 \oplus E_2 = \mathbb{R}^2$, *$E_1$ and E_2 are supplementary.*

1.4 Generating families, free families and bases

In the following, we will denote the vector space $(E, +, .)$ by E.

Definition 1.4.1 *Let E be a vector space and $e_1, e_2 \cdots$, be elements of E.*
(1) We say that $\{e_1, e_2 \cdots, e_n\}$ are free or linearly independent, if $\forall \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{K}$: $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, a unique solution.
Otherwise, they are said to be related.
(2) We say that $\{e_1, e_2, \cdots, e_n\}$ is a generating family of E, or that E is generated by $\{e_1, e_2, \cdots, e_n\}$ if, $\forall x \in E, \exists \lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n \in \mathbb{K}/x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \cdots + \lambda_n e_n$.
(3) If $\{e_1, e_2, \cdots, e_n\}$ is a free and generating family of E, then $\{e_1, e_2, \cdots, e_n\}$ is called a base of E.

Remark 1.4.2 *In a vector space E, every non-zero vector is free*

Theorem 1.4.3 *If $\{e_1, e_2, \cdots, e_n\}$ and $\{e'_1, e'_2, \cdots, e'_m\}$ are two bases of the vector space E, then $n = m$. In other words, if a vector space has a base then all bases of E have the same number of elements (or cardinal), this number does not depend on the base but only on the space E. Hence the following definition.*

Definition 1.4.4 *Let E be a $\mathbb{K}$- vector space of basis $B = \{e_1, e_2, \cdots, e_n\}$, then $dim(E) = Card(B) = n$. where $dim(E)$: is the dimension of E and $Card(B)$: is the cardinal of B.*

Remark 1.4.5 *So to find a basis for a vector space is to find a family of vectors in E, which form a free and generating family of E, the number of elements of this family is $dimE$.*

Example 1.4.6 *Let us look for a basis of* $\mathbb{R}^3$ *, we must find a family of vectors in* $\mathbb{R}^3$ *which generates* $\mathbb{R}^3$ *and which is free.*

Theorem 1.4.7 *Let* E *be a vector space of dimension* n *:*
(1) If $\{e_1, e_2 \cdots, e_n\}$ *is base of* $E \Leftrightarrow \{e_1, e_2 \cdots, e_n\}$ *is generative* $\Leftrightarrow \{e_1, e_2 \cdots, e_n\}$ *is free.*
(2) If $\{e_1, e_2 \cdots, e_p\}$ *are* p *vectors in* E*, with* $p > n$*, then* $\{e_1, e_2 \cdots, e_p\}$ *cannot be free, moreover if* $\{e_1, e_2 \cdots, e_p\}$ *is generative, then there are* $(p - n)$ *vectors among* $\{e_1, e_2 \cdots, e_p\}$ *form a base* E*.*
(3) If $\{e_1, e_2 \cdots, e_p\}$ *are* p *vectors in* E*, with* $p < n$*, then* $\{e_1, e_2 \cdots, e_p\}$ *cannot be generative moreover if* $\{e_1, e_2 \cdots, e_p\}$ *is free, then there exists* $(n - p)$ *vector among* $\{e_{p+1}, e_{p+2}, \cdots, e_n\}$ *in* E *such that* $\{e_1, e_2 \cdots, e_p, e_{p+1}, \cdots, e_n\}$ *is a base for* E*.*
(4) If F *is a sub-vector space of* E *then* $dim F \leq n$*, and in addition if* $dim F = n \Leftrightarrow E = F$

Example 1.4.8 *(1) To show that* $\{(1,1,1),(1,1,0),(0,1,-1)\}$ *is a basis of* $\mathbb{R}^3$*, it is sufficient to show that it is free or generative because* $dim\mathbb{R}^3 = 3$*,* $\{e_1, e_2, e_3\}$ *is free.*
(2) Let us look for a basis for $F = \{(x + y, x - z, -y - z)/x, y, z \in \mathbb{R}\}$*.*

1.5 Notion of Linear Application

1.5.1 Generalities

Definition 1.5.1 *(1) Let* $(E, +, .)$ *and* $(F, +, .)$ *be two* $\mathbb{K}$*- vector spaces and let* f *be an application from* E *into* F*, we say that* f *is a linear application if and only if :*

$$\forall x, y \in E, \forall \lambda, \in \mathbb{K}, \quad f(x + y) = f(x) + f(y) \quad \text{and} \quad f(\lambda.x) = \lambda.f(x) \quad ,$$

or equivalently :

$$\forall x, y \in E, \forall \lambda, \mu \in \mathbb{K}, \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y).$$

(2) If furthermore f *is bijective, then we say that* f *is an isomorphism of* E *in* F*.*
(3) A linear application of $(E, +, .)$ *in* $(E, +, .)$ *is said to be an endomorphism.*
(4) An isomorphism of $(E, +, .)$ *into* $(E, +, .)$ *is also called an automorphism of* E *into* E*.*

Example 1.5.2 *Show that the following applications are linear applications :*

(1) $f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, (2) $f_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x, y) \longmapsto x - y$, $(x, y, z) \longmapsto (-x + y, x - 5z, y)$.

Remark 1.5.3 *It is easy to show that the sum of two linear applications is a linear application, so the product of a linear application by a scalar and the compound of two linear applications is a linear application.*

Proposition 1.5.4 *Let f be a linear application of E in F.*

(1) $f(O_E) = O_F$ *;*

(2) $\forall x \in E, \quad f(-x) = -f(x)$.

Proof: *(1)* $f(O_E) = f(O_E + O_E) = f(O_E) + f(O_E) = O_F + O_F = O_F \Rightarrow f(O_E) = O_F$

(2) $f(-x) + f(x) = f(-x + x) = f(O_E) = O_F \Rightarrow f(-x) = -f(x)$.

Definition 1.5.5 *Let f be a linear application of E in F.*

(1) We call the image of f and note Imf the set defined as follows

$$Imf = \{y \in F / \exists x \in E : f(x) = y\}.$$

(2) We call the kernel of f and note kerf the set defined as follows:

$$kerf = \{x \in E / f(x) = 0\}.$$

Sometimes kerf is noted as $f^{-1}(\{0\})$.

Proposition 1.5.6 *If f is a linear application of E in F, then if $dimImf = n < +\infty$, then n is called the rank of f and we note $rk(f)$.*

Imf and $kerf$ are subvector spaces of E.

Example 1.5.7 *1. Determine the kernel of the function g defined by :*

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x, y) \longmapsto x - 3y$

2. Determine the image of the function h defined by :

$h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$(x, y, z) \longmapsto (-x + y, x - z, y)$.

Proposition 1.5.8 *Let f be a linear application of E in F we have the following equivalences :*
(1) f is surjective $\Leftrightarrow Imf = F$.
(2) f is injective $\Leftrightarrow kerf = \{0_E\}$.

Example 1.5.9 *In the example $Imh = \mathbb{R}^3$ so h is surjective, let us show that h is injective because $kerh = \{(0,0,0)\}$, hence h is bijective.*

1.5.2 Linear application on finite dimensional spaces

Proposition 1.5.10 *Let E and F be two $\mathbb{K}$ vector spaces and f, g be two linear applications of E in F. If E is of finite dimension n and $\{e_1, e_2, \cdots, e_n\}$ a basis of E , then*

$$\forall k \in \{1, 2, \cdots, n\}, f(e_k) = g(e_k) \Leftrightarrow \forall x \in E, f(x) = g(x).$$

Proof: *The implication $\Leftarrow$ is evident.*
For $\Rightarrow$ we have E is generated by $\{e_1, e_2, \cdots, e_n\}$, so $\forall x \in E, \exists \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{K}, x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$, as f and g are linear, then
$f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + \cdots + \lambda_n f(e_n)$,
$g(x) = g(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n) = \lambda_1 g(e_1) + \lambda_2 g(e_2) + \cdots + \lambda_n g(e_n)$,
so if we suppose that $\forall k \in \{1, 2, \cdots, n\}$, $f(e_k) = g(e_k)$ so we deduce that $\forall x \in E, f(x) = g(x)$.

Remark 1.5.11 *For two linear applications f and g of E into F to be equal it is sufficient that they coincide on the base of the $\mathbb{K}$- vector space E.*

Example 1.5.12 *Let g be an application of $\mathbb{R}^2$ in $\mathbb{R}^2$ such that $g(1,0) = (2,1)$, $g(0,1) = (-1,-1)$ then let us determine the value of g at all points of $\mathbb{R}^2$, indeed we have : $\forall (x,y) \in \mathbb{R}^2$, $(x,y) = x(1,0) + y(0,1)$ $g(x,y) = g(x(1,0) + y(0,1)) = xg(1,0) + yg(0,1) = x(2,1) + y(-1,-1) = (2x-y, x-y)$ so $g(x,y) = (2x-y, x-y)$.* $\square$

Theorem 1.5.13 *Let f be a linear application of E in F with dimension of E is finite, we have :*

$$dimE = dimker(f) + dimIm(f).$$

Example 1.5.14 *Let f be the application defined previously by :*

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x, y) \longmapsto x + 2y.$

It was shown earlier that $dimKer(f) = 1$. Since $dim\mathbb{R}^2 = 2 \Rightarrow dimIm(f) = dim\mathbb{R}^2 - dimker(f) = 2 - 1 = 1.$

Proposition 1.5.15 *Let f be a linear application of E in F with $dimE = dimF = n$. Then we have the following equivalences :*

f is isomorphism $\Leftrightarrow$ f is surjective $\Leftrightarrow$ f is injective $\Leftrightarrow$ $dimIm(f) = dimF \Leftrightarrow Imf = F \Leftrightarrow dimkerf = 0 \Leftrightarrow kerf = \{0_E\}$.

From this proposition, we deduce that if f is an isomorphism of E in F with finite $dimE$ then necessarily $dimE = dimF$ in other words if $dimE \neq dimF$ then f cannot be an isomorphism.

Example 1.5.16 *(1) The application f is not an isomorphism because $dim\mathbb{R}^2 \neq dim\mathbb{R}$*

(2) Let $g(x, y) = (2x-y, x-y)$, g defined from $\mathbb{R}^2$ into $\mathbb{R}^2$, we have, $dim\mathbb{R}^2 = dim\mathbb{R}^2 = 2$, so g is an isomorphism because $dimkerg = 0$ in fact : $kerg = \{(x, y) \in \mathbb{R}^2/(2x-y, x-y) = (0, 0)\} = \{(0, 0)\}$, it's even an automorphism.

1.6 Exercises

1.6.1 Exercise 1

Consider in $\mathbb{R}^3$, the subset F defined by : $F = \{(x, y, z) \in \mathbb{R}^3/2x + y - z = 0\}$.

(1) Show that F is a sub-vector space of $\mathbb{R}^3$.

(2) Give a basis of F, what is its dimension ?

(3) Is F equal to $\mathbb{R}^3$?

1.6.2 Exercise 2

Consider in $\mathbb{R}^3$, the subset F defined by : $F = \{(x-y, 2x+y+4z, 3y+2z)/x, y, z \in \mathbb{R}^3\}$.

(1) Show that F is a subvector space of $\mathbb{R}^3$.

(2) Give a basis of F, what is its dimension ?

(3) Is F equal to $\mathbb{R}^3$?

1.6.3 Exercise 3

Consider in $\mathbb{R}^4$, the subset F defined by : $F = \{(x, y, z, t) \in \mathbb{R}^4/(x + z = 0) \wedge (y + t = 0)\}$

(1) Show that F is a subvector space of $\mathbb{R}^4$.

(2) Give a base of F, deduce its dimension.

1.6.4 Exercise 4

(1) Show that the family $\{(1, 2), (-1, 1)\}$ is generative of $\mathbb{R}^2$.

(2) Which are the free families among the following families: $F_1 = \{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$, $F_2 = \{(0, 1, 1, 0), (1, 1, 1, 0), (2, 1, 1, 0)\}$.

(3) Show that the family $\{(1, 2), (-1, 1)\}$ is a base of $\mathbb{R}^2$, and that the family $F_1 = \{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$ is a base of $\mathbb{R}^3$.

1.6.5 Exercise 5

Let f be defined from $\mathbb{R}^2$ into $\mathbb{R}^2$ by : $f(x, y) = (x + y, x - y)$.

(1) Show that f is linear.

(2) Determine $ker f$, and $Im f$ and give their dimensions, is f bijective ?

(3) Determine $f \circ f$.

1.6.6 Exercise 6

Let f be defined from $\mathbb{R}^2$ into $\mathbb{R}^2$ by : $f(x, y) = (2x - 4y, x - 2y)$.

(1) Show that f is linear.

(2) Determine $ker f$, and $Im f$ and give their dimensions, is f bijective ?

Chapter 2

Notion of matrix associated with a linear application and algebraic calculus on m atrices

Let $\mathbb{K}$ be a commutative field.

Let E and F be two $\mathbb{K}$-vector spaces of finite dimension n and m, f a linear application of E in F, let $B = \{e_1, e_2, \cdots, e_n\}$ a base of E, $B' = \{e'_1, e'_2, \cdots, e'_m\}$ a base of F, the vectors $f(e_1), f(e_2), \cdots, f(e_n)$ are vectors in F as $\{e'_1, e'_2, \cdots, e'_m\}$ is a base of F, then $f(e_1), f(e_2), \cdots, f(e_n)$ are written as linear combinations of the vectors of the base $B' = \{e'_1, e'_2, \cdots, e'_m\}$.

We have for all $j = 1, \cdots, n$:

$$f(e_j) = a_{1j}e'_1 + a_{2j}e'_2 + a_{3j}e'_3 + \cdots a_{mj}e'_m$$

$$\begin{array}{c} \\ \left(\begin{array}{cccc} f(e_1) & f(e_2) & \cdots & f(e_n) \\ & & & \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right) \end{array} \begin{array}{c} e'_1 \\ e'_2 \\ \vdots \\ \vdots \\ e'_m \end{array}$$

The following table:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is called the matrix associated with f relative to the bases B and B'. The matrix (a_{ij}) where i denotes the row index and j the column index is noted.
We now introduce the notion of matrix and the algebraic operations of matrices.

2.1 Vector space of matrices

Definition 2.1.1 *A rectangular array A of elements of $\mathbb{K}$ with n rows and p columns is called a matrix in $\mathbb{K}$ of type (n,p).*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix}.$$

We note a_{ij} the element which is in the row number i and the column j and we note the matrix A by $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq p}$.
The set of matrices of type (n,p) is noted $\mathcal{M}_{(n,p)}(\mathbb{K})$.
(1) For $n = 1$, we say that A is a row matrix, $A = (a_{11}, a_{12}, \cdots, a_{1p})$.
(2) For $p = 1$ we say that A is a column matrix, $A = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \dots \\ a_{1p} \end{pmatrix}$.
(3) For $n = p$, we say that A is a square matrix of order n and we note $A \in \mathcal{M}_{(n)}(\mathbb{K})$.

Example 2.1.2 $A = \begin{pmatrix} 5 & 3 & 8 & 0 \\ 2 & 1 & 0 & 5 \\ 6 & 3 & 7 & 9 \end{pmatrix}$, *A is a matrix of type* $(3,4)$.

$B = \begin{pmatrix} 7 & 4 \\ 8 & 5 \\ 2 & 0 \end{pmatrix}$, *B is a matrix of type* $(3,2)$.

$C = \begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix}$, *C is a square matrix of order* 2.

Definition 2.1.3 *Let* $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq p}$, *and* $B = (b_{ij})_{1\leq i\leq n, 1\leq j\leq p}$ *two matrices of types* (n,p),

(1) We say that $A = B$ *if* $\forall i = 1, \cdots, n$, $\forall j = 1, \cdots, p$; $a_{ij} = b_{ij}$.

(2) The transpose of the matrix A is a matrix noted A^t *defined by :*

$$A^t = (a_{ji})_{1\leq j\leq p, 1\leq i\leq n}$$

in other words A^t *is the matrix of type* (p,n) *obtained by replacing the lines by the columns and the columns by the lines and we have :* $(A^t)^t = A$.

Example 2.1.4 *Give the transpositions of the matrices* A, B *and* C *from the previous example.*

Theorem 2.1.5 *By fitting the set* $\mathcal{M}_{(n,p)}(\mathbb{K})$ *with the following operations:*

$$(+) : \mathcal{M}_{(n,p)}(\mathbb{K}) \times \mathcal{M}_{(n,p)}(\mathbb{K}) \longrightarrow \mathcal{M}_{(n,p)}(\mathbb{K})$$

$$\left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix} \right) \rightarrow \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1p}+b_{1p} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2p}+b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \dots & a_{np}+b_{np} \end{pmatrix}$$

and

$$(+) : \mathbb{K} \times \mathcal{M}_{(n,p)}(\mathbb{K}) \longrightarrow \mathcal{M}_{(n,p)}(\mathbb{K})$$

$$\left(\lambda, \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix} \right) \rightarrow \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1p} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{np} \end{pmatrix}.$$

Then $(\mathcal{M}_{(n,p)}(\mathbb{K}), +, \cdot)$ is $\mathbb{K}$- vector space of dimension $n \times p$, knowing that the neutral element of the addition is the null matrix :

$$\begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ . & . & .. & . \\ 0 & 0 & .. & 0 \end{pmatrix}.$$

2.2 Product of two matrices

Definition 2.2.1 *Let $A \in \mathcal{M}_{(n,p)}(\mathbb{K})$ and $B \in \mathcal{M}_{(p,m)}(\mathbb{K})$, we define the product of the matrix A by B as being a matrix $C = (c_{ij})_{1 \leq i \leq p, 1 \leq j \leq m} \in \mathcal{M}_{(n,m)}(\mathbb{K})$, with*

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{31}b_{3j} + \cdots + a_{ip}b_{pj}.$$

Remark 2.2.2 *(1) The element c_{ij} of the matrix C is calculated by adding the product of the elements of the row i of the matrix A by the elements of the column j of the matrix B.*
(2) The product of two matrices can only be done if the number of columns of the matrix A corresponds to the number of rows of the matrix B.

Example 2.2.3 *calculate $A.B$ knowing that $A = \begin{pmatrix} 5 & 3 & 4 \\ -8 & 1 & 2 \\ 4 & 0 & -7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 5 & 2 \\ 2 & 0 & -3 & 1 \\ -3 & 0 & -2 & 2 \end{pmatrix}$.*
Also calculate $B.A$, what do you notice?

Remark 2.2.4 *The product of two matrices is not commutative.*

2.3 Square matrix

Definition 2.3.1 *Let A be a square matrix of order n, $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$,*
(1) The sequence of elements $\{a_{11}, a_{22}, \cdots, a_{nn}\}$ is called the main diagonal of A.
(2) The trace of A is the number $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$.
(3) A is said to be a diagonal matrix if $a_{ij} = 0, \forall i \neq j$; that is to say that the elements of A are all zero except the main diagonal.
(4) A is said to be an upper (respectively inferior) triangular matrix if $a_{ij} = 0, \forall i > j$,

(respectively $i < j$), i.e the elements which are below (respectively above) the diagonal are null.
(5) A is said to be symmetric if $A = A^t$.

Example 2.3.2 $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{pmatrix}$, *$A$ is a diagonal matrix.*

$B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -3 & 7 \end{pmatrix}$, *$B$ is an inferior triangular matrix.*

$C = \begin{pmatrix} 1 & 4 & 7 \\ 0 & 8 & 4 \\ 0 & 0 & 5 \end{pmatrix}$, *$C$ is an upper triangular matrix.*

$D = \begin{pmatrix} 4 & 5 & 8 \\ 5 & -1 & 7 \\ 8 & 7 & 2 \end{pmatrix}$, *$D$ is a symmetric matrix.*

Proposition 2.3.3 *The product of matrices is an internal operation in $\mathcal{M}_{(n,p)}(\mathbb{K})$ and it has a neutral element the matrix named identity matrix noted I_n defined by :*

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0.. & 0 \\ 0 & 1 & 0 & 0.. & 0 \\ 0 & 0 & 1 & 0.. & 0 \\ 0 & 0 & 0 & 1.. & 0 \\ . & . & . & .. & 0 \\ 0 & 0 & 0 & ..0 & 1 \end{pmatrix}.$$

Definition 2.3.4 *Let $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$ we say that A is invertible if there exists a matrix $B \in \mathcal{M}_{(n,n)}(\mathbb{K})$ such that $A.B = B.A = I_n$.*

Example 2.3.5 *Show that the matrix* $A = \begin{pmatrix} -1 & 3 \\ 5 & 0 \end{pmatrix}$ *is invertible assuming*

$B = \begin{pmatrix} a & d \\ b & c \end{pmatrix}$.

2.4 Determinant of a matrix

Definition 2.4.1 *Let* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *a matrix in* $\mathcal{M}_{(2,2)}(\mathbb{K})$, *the determinant of* A *is the real number given by :* $a_{11}a_{22} - a_{12}a_{21}$. *It is denoted det*$(A)$ *or* $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Example 2.4.2 *Calculate the determinant of the following matrices ;*

$$A = \begin{pmatrix} -11 & 8 \\ 22 & \frac{1}{2} \end{pmatrix}, B = \begin{pmatrix} -1 & 5 \\ 3 & 5 \end{pmatrix}, C = \begin{pmatrix} \frac{-1}{3} & \frac{5}{2} \\ -7 & 9 \end{pmatrix}.$$

Definition 2.4.3 *Similarly, we define the determinant of a matrix,*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{M}_{(3,3)}(\mathbb{K}) \text{ by}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 2.4.4 *Calculate the determinant of the following matrices :*

$$A = \begin{pmatrix} -1 & 8 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & -5 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}, C = \begin{pmatrix} \frac{-1}{3} & \frac{5}{2} & 2 \\ -7 & 9 & -2 \\ \frac{8}{3} & \frac{-1}{3} & 5 \end{pmatrix}.$$

Proposition 2.4.5 *To calculate the determinant of a matrix* A *one can expand* A *in any row or column.*
Following this proposal it is better to choose the line or column containing the most of zeros.

Definition 2.4.6 *Similarly, we define the determinant of a matrix,*

$$A= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathcal{M}_4(\mathbb{K})$$

by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^{1+1}a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + (-1)^{1+2}a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+(-1)^{1+3}a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + (-1)^{1+4}a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

Definition 2.4.7 *Let $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq n}$, the determinant of A following the j-th column is :*

$det(A) = (-1)^{1+j}a_{1j}D_{1j} + (-1)^{2+j}a_{2j}D_{2j} + \cdots + (-1)^{n+j}a_{nj}D_{nj}$ *,*$j = 1, \cdots, n$.

The determinant of A following the i-th line is :

$det(A) = (-1)^{i+1}a_{i1}D_{i1} + (-1)^{i+2}a_{i2}D_{i2} + \cdots + (-1)^{i+n}a_{in}D_{in}$, $i = 1, \cdots, n$. *Where D_{ij} represents what we call the minor determinant of the term a_{ij} , the determinant of order $n-1$ obtained from $det(A)$ by deleting the i-th row and the j-th column.*

Proposition 2.4.8 *Let $A \in \mathcal{M}_n(\mathbb{K})$ we have :*

(1) $det(A) = det(A^t)$.

(2) $det(A) = 0$ if two lines of A are equal (or two columns).

(3) $det(A) = 0$ if two rows of A are proportinals (or two columns are).

(4) $det(A) = 0$ if a row is a linear combination of two other rows of A (same for the columns).

(5) $det(A)$ does not change if we add to a row a linear combination of other rows (same for the columns).

(6) If $B \in \mathcal{M}_n(\mathbb{K})$, then $det(A.B) = det(A).det(B)$.

Example 2.4.9 *(1)* $|A| = \begin{vmatrix} 8 & 5 & 7 \\ 2 & 1 & 0 \\ 8 & 5 & 7 \end{vmatrix} = 0$ *because line 1 is equal to line 3,* $L_1 = L_3$.

$$|B| = \begin{vmatrix} -7 & 5 & 7 & -6 \\ 2 & -1 & 0 & -2 \\ -8 & -5 & 7 & 9 \\ 14 & -10 & -14 & 12 \end{vmatrix} = 0 \text{ because } L_1 = -2 \times L_4.$$

$$|C| = \begin{vmatrix} -7 & 5 & 7 & -5 \\ 2 & 5 & 0 & -5 \\ 8 & -5 & 7 & 5 \\ 14 & -5 & -14 & 5 \end{vmatrix} = 0 \text{ because } C_2 = -C_4.$$

Definition 2.4.10 *Let* $V_1, V_2, V_3, \cdots, V_n$, *n be vectors of* $\mathbb{R}^n$ *we call the determinant of vectors* $(V_1, V_2, \cdots, V_n)$ *and we note it* $det(V_1, V_2, \cdots, V_n)$ *the determinant whose columns are the vectors* $V_1, V_2, \cdots, V_n$.

Example 2.4.11 *Let* $v_1 = (2, 5, 8)$, $v_2 = (1, 7, 3)$ *and* $v_3 = (-1, 6, \frac{1}{2})$, *then*

$$det(V_1, V_2, V_3) = \begin{vmatrix} 2 & 1 & -1 \\ 5 & 7 & 6 \\ 8 & 3 & \frac{1}{2} \end{vmatrix}. \text{ Do the math!}$$

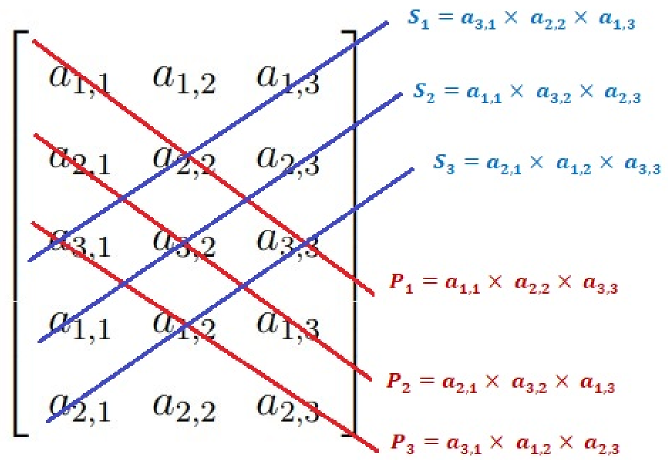
Proposition 2.4.12 *Let* $V_1, V_2, ..., V_n$, *n be vectors of* $\mathbb{R}^n$. *we* $(V_1, V_2, \cdots, V_n)$ *is a base of* $\mathbb{R}^n \Leftrightarrow det(V_1, V_2, \cdots, V_n) \neq 0$.

Example 2.4.13 *Let* $v_1 = (2, 5, 8)$, $v_2 = (1, 7, 3)$ *and* $v_3 = (-1, 6, \frac{1}{2})$, *form a base of* $\mathbb{R}^3$, *because* $det(V_1, V_2, V_3) \neq 0$.

2.4.1 Sarrus' rule

The Sarrus rule (named after Pierre-Frédéric Sarrus) is a visual procedure for remembering the formula for calculating 3rd-order determinants. Sarrus' rule consists in writing the three columns of the matrix and repeating, in order, the first two rows below the matrix. All that's then needed are the products of the coefficients on each diagonal, and the sum if the diagonal is descending, or the difference if the diagonal is ascending.

Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$. We transform A into $\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.



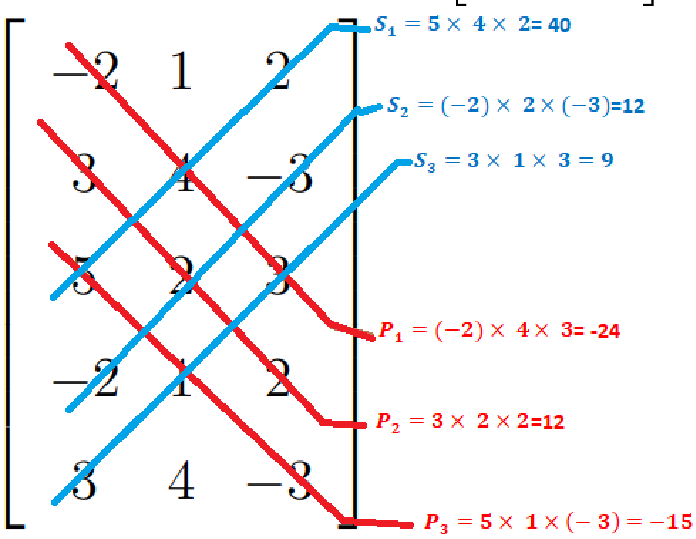
$P = P_1 + P_2 + P_3$

$S = S_1 + S_2 + S_3$

$\boxed{\text{Det}(A) = P - S}$

Example 2.4.14 *Determine the determinant of the following matrix :*

$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & -3 \\ 5 & 2 & 3 \end{bmatrix}$. *We have :* $\begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & -3 \\ 5 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & 4 & -3 \end{bmatrix}$



$P = P_1 + P_2 + P_3 = -24 + 12 - 15 = -27$

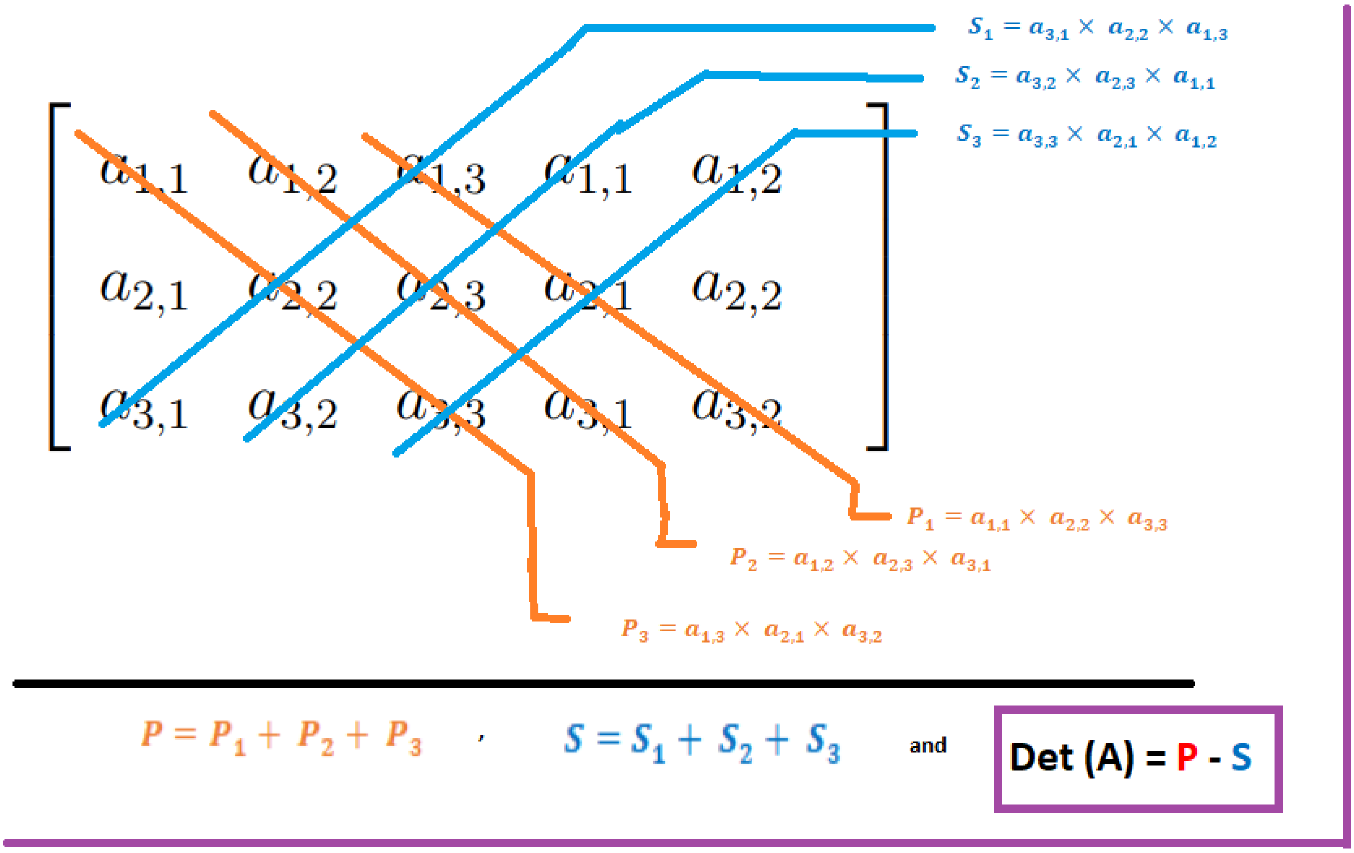
$S = S_1 + S_2 + S_3 = 40 + 12 + 9 = 61$

So,
det(A)= P - S = -27 - 61 = -88

Remark 2.4.15 *Note that with Sarrus' method, instead of adding the first two rows of the A matrix, we can add the first two columns of the A matrix. The rest is similar to the case described above.*

We have :

Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$. *We transform A into* $\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,1} & a_{3,2} \end{bmatrix}$



Example 2.4.16 *Use this second technique to determine the previous A matrix.*

Where $A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & -3 \\ 5 & 2 & 3 \end{bmatrix}$.

2.4.2 Rank of a matrix

Definition 2.4.17 *Let A be in $\mathcal{M}_{(n,p)}(\mathbb{K})$, we call the rank of A and note $rk(A)$ the order of the largest square matrix B taken (extracted) in A such that $detB \neq 0$.*

Example 2.4.18 $A = \begin{pmatrix} -1 & 5 \\ 3 & 0 \end{pmatrix}$, $|A| = -15$, $rk(A) = 2$.

$B = \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix}$, $|B| = 0$, $rk(A) = 1$.

$M = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$, $rk(M) < 4$, *(or* $rk(M) \leq 3$*) the largest square matrix contained in* M *is of order 3, in this example we have : 4 possibility :*

$M_1 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, *and*

$M_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$detM_1 = detM_2 = detM_3 = detM_4 = 0$ *, so the* $rk(M) < 3$ *and we have*

$$\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = -1 \neq 0 \Leftrightarrow rk(M) = 2.$$

Theorem 2.4.19 *The rank of a matrix is equal to the maximum number of linearly independent rows (or columns) that are linearly independent.*

Definition 2.4.20 *Let* $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{M}_n(\mathbb{K})$*, the scalar :*

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

is called the cofactor of index i *and* j *of* A*. Where* A_{ij} *is the matrix deduced from* A *by deleting the i-th row and the* j *column.*
The matrix $C = (c_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ *is called the cofactor matrix and the matrix* C^t *is called the comatrix of* A*.*

Example 2.4.21 *Let a matrix* $M = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$.

Let's calculate the factors of A *:*

$$c_{11}=(-1)^{1+1}det(M_{11})=(-1)^2\begin{vmatrix}-1 & 1\\ 2 & 2\end{vmatrix}=-4;$$

$$c_{12}=(-1)^{1+2}det(M_{12})=(-1)^3\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}=-2;$$

$$c_{13}=(-1)^{1+3}det(M_{13})=(-1)^4\begin{vmatrix}1 & -1\\ 0 & 2\end{vmatrix}=2;$$

$$c_{21}=(-1)^{2+1}det(M_{21})=(-1)^3\begin{vmatrix}0 & 3\\ 2 & 2\end{vmatrix}=6;$$

$$c_{22}=(-1)^{2+2}det(M_{22})=(-1)^4\begin{vmatrix}1 & 3\\ 0 & 2\end{vmatrix}=2;$$

$$c_{23}=(-1)^{2+3}det(M_{23})=(-1)^5\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}=-2;$$

$$c_{31}=(-1)^{3+1}det(M_{31})=(-1)^4\begin{vmatrix}0 & 3\\ -1 & 1\end{vmatrix}=3;$$

$$c_{32}=(-1)^{3+2}det(M_{32})=(-1)^5\begin{vmatrix}1 & 3\\ 1 & 1\end{vmatrix}=2;$$

$$c_{33}=(-1)^{3+3}det(M_{33})=(-1)^6\begin{vmatrix}1 & 0\\ 1 & -1\end{vmatrix}=-1.$$

so the cofactor matrix is given by :

$$C=\begin{pmatrix}-4 & -2 & 2\\ 6 & 2 & -2\\ 3 & 2 & -1\end{pmatrix}$$

and the co-matrix is

$$C^t=\begin{pmatrix}-4 & 6 & 3\\ -2 & 2 & 2\\ 2 & -2 & -1\end{pmatrix}.$$

Theorem 2.4.22 *Let $A\in\mathcal{M}_n(\mathbb{K})$, we have :*

$$A \quad is \quad invertible \quad \Leftrightarrow det(A) \neq 0,$$

and in this case the inverse matrix of A is given by :

$$A^{-1} = \frac{1}{det(A)} C^t.$$

Where C^t is the co-matrix of A.

Example 2.4.23 *Let a matrix* $M = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$, $det(M) = 2 \neq 0$ *so it is invertible, in addition* $M^{-1} = \frac{1}{2} C^t = \frac{1}{2} \begin{pmatrix} -4 & 6 & 3 \\ -2 & 2 & 2 \\ 2 & -2 & -1 \end{pmatrix} \Leftrightarrow M^{-1} = \begin{pmatrix} -2 & 3 & \frac{3}{2} \\ -1 & 1 & 1 \\ 1 & -1 & \frac{-1}{2} \end{pmatrix}.$

We can verify that $A^{-1}A = I_3 = AA^{-1}$.

2.5 Relationship between a linear application and its associated matrix

Definition 2.5.1 *The matrix $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is called the matrix of f in the bases B and B' and is sometimes noted $\mathcal{M}_{(B,B')}(f)$. If $E = F$ and $B = B'$, we say that A is said to be the matrix of f in the base B and is denoted $\mathcal{M}_B(f)$.*

Example 2.5.2 *Find the matrices in the canonical bases of the functions f and g :*

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + 3y + z, 2x - y); \\ g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (2x - 3y, -x - 2y). \end{aligned}$$

Proposition 2.5.3 *Let E and F be two $\mathbb{K}$- vector spaces of finite dimensions n and m, $B = (e_1, e_2, \cdots, e_n)$ a base for E and $B' = (v_1, v_2, \cdots, v_m)$ a base for F, then giving a matrix $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ gives a unique linear application f from E into F the matrix according to the bases, B and B' is A.*

Example 2.5.4 *Let a matrix* $A = \begin{pmatrix} -1 & 5 \\ 3 & 0 \end{pmatrix}$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ A *is the matrix of* f *in the canonical base of* $\mathbb{R}^2$ *defined by* (e_1, e_2)*, Determine the expression for* f.

Remark 2.5.5 *If* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *have their canonical bases then the linear application* f *from* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *associated to a matrix* $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ *is given by :*

$$\forall (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \quad f(x_1, x_2, \cdots, x_n) = A. \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 2.5.6 *Let* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $M = \begin{pmatrix} 5 & 3 \\ 7 & 1 \end{pmatrix}$*, we have :*

$$f(x, y) = M. \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 7 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix} = (5x + 3y, 7x + y).$$

Theorem 2.5.7 *Let* E*,* F *and* G *be* $\mathbb{K}$*- vector spaces provided respectively by the bases* B*,* B' B''*,* $f : E \longrightarrow F$*,* $g : F \longrightarrow G$*, two linear applications, then :*

$$\mathcal{M}_{(B,B'')}(g \circ f) = \mathcal{M}_{(B',B'')}(g)\mathcal{M}_{(B,B')}(f).$$

Example 2.5.8

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + 3y + z, 2x - y); \\ g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (2x - 3y, -x - 2y). \end{aligned}$$

Determine $\mathcal{M}_{(\mathbb{R}^3,\mathbb{R}^2)}(g \circ f)$.

Theorem 2.5.9 *Let* $f ; E \longrightarrow F$*,* B *is a base for* E *and* B' *is a base for* F*, then we have :*

$$f \quad bijective \Leftrightarrow det\mathcal{M}_{(B,B')}(f) \neq 0$$

and in this case we have

$$\mathcal{M}_{(B,B')}(f^{-1}) = (\mathcal{M}_{(B,B')}(f))^{-1}.$$

Example 2.5.10

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + 3y, 2x - y).$$

Let show that f is bijective and calculate its inverse.

Proposition 2.5.11 *If $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ associated to a linear application f of E in F the matrix following the bases, B of E and B' of F, then :*

$$rk(A) = rk(f), \quad rk(A) = rk(A^t).$$

Theorem 2.5.12 ***Rank theorem :*** *Let E, F be two vector spaces with E of finite dimension. Let also $u \in \mathcal{L}(E, F)$, then :*

$$dim(E) = dimker(u) + rk(u).$$

Remark 2.5.13 *The rank theorem gives an indirect way to compute the rank of a linear application : we determine the kernel of the application, and a base of the kernel, which gives the dimension of the kernel, and so we immediately determine the rank by this theorem.*

Proposition 2.5.14 *Let $\phi : V \longrightarrow W$ a linear application with $dimV = n$ and $dimW = p$. We have :*
(i) $rk(\phi) = p \Leftrightarrow \phi$ is surjective.
(ii) $rk(\phi) = n \Leftrightarrow kerf\phi = \{\overrightarrow{0}\} \Leftrightarrow \phi$ is injective.
(iii) ϕ is bijective $\Leftrightarrow n = p$ and $kerf\phi = \{\overrightarrow{0}\}$.

2.6 Matrices and Base Changes

Definition 2.6.1 *Let E be a vector space and let $B = (e_1, e_2, \cdots, e_n)$ and $B' = (e'_1, e'_2, \cdots, e'_n)$ be two bases for E, the matrix of passage from the base B' to the base B is by definition the matrix $\mathcal{M}_{(B,B')}(\mathbb{I}d_E)$ where $\mathbb{I}d_E$ is the identity application :*

$$\begin{aligned} \mathbb{I}d_E : E &\longrightarrow E \\ x &\longmapsto x. \end{aligned}$$

The base vectors of B can be expressed in B' according to the relations :

$$(S) : \begin{cases} e_1 = a_{11}e'_1 + a_{12}e'_2 + \ldots + a_{1n}e'_n \\ e_2 = a_{21}e'_1 + a_{22}e'_2 + \ldots + a_{2n}e'_n \\ e_3 = a_{31}e'_1 + a_{32}e'_2 + \ldots + a_{3n}e'_n \\ \vdots \\ e_n = a_{n1}e'_1 + a_{n2}e'_2 + \ldots + a_{nn}e'_n \end{cases}.$$

The square matrix P defined below is called the transition matrix from B' to B :

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Example 2.6.2 *Let $B' = \{e'_1, e'_2, e'_3\}$, with $e'_1 = (1, -1, 1)$, $e'_2 = (1, 1, -2)$, $e'_3 = (1, 0, 3)$ and $B = \{e_1, e_2, e_3\}$ the canonical base for $\mathbb{R}^3$:*

$$\begin{aligned} \mathbb{I}d_{\mathbb{R}} : \mathbb{R}^3_B &\longrightarrow \mathbb{R}^2_{B'} \\ (x, y, z) &\longmapsto (x, y, z). \end{aligned}$$

Determine $\mathcal{M}_{(B,B')}(\mathbb{I}d_{\mathbb{R}^3})$.

Proposition 2.6.3 *The matrix of passage from a base B to a base B' is the matrix inverse of the matrix of transition from B' to B :*

$$\mathcal{M}_{(B',B)}(\mathbb{I}d_{\mathbb{R}^3}) = (\mathcal{M}_{(B,B')}(\mathbb{I}d_{\mathbb{R}^3}))^{-1}.$$

Remark 2.6.4 *Let E be a v.s and let $B = (e_1, e_2, \cdots, e_n)$ and $B' = (e'_1, e'_2, \cdots, e'_n)$ be two bases for E. The base vectors of B' can be expressed in B according to the relations :*

$$(S) : \begin{cases} e'_1 = b_{11}e_1 + b_{12}e_2 + ... + b_{1n}e_n \\ e'_2 = b_{21}e_1 + b_{22}e_2 + ... + b_{2n}e_n \\ e'_3 = b_{31}e_1 + b_{32}e_2 + ... + b_{3n}e_n \\ :::::::::: \\ e'_n = b_{n1}e_1 + b_{n2}e_2 + ... + b_{nn}e_n \end{cases}.$$

The square matrix P^{-1} is called the transition matrix from B to B', it's defined by :

$$P^{-1} = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & ::: & \vdots \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix}.$$

Example 2.6.5 *In our previous example we have : $\mathcal{M}_{(B,B')}(\mathbb{I}d_{\mathbb{R}^3}) = A$, so $\mathcal{M}_{(B',B)}(\mathbb{I}d_{\mathbb{R}^3}) = A^{-1}$.*

Theorem 2.6.6 *Let $f : E \longrightarrow F$, B_1, B'_1 be two bases for E, B_2, B'_2 be bases for F.*
If P denotes the matrix of transition from B_1 to B'_1 and Q denotes the matrix of transition from B_2 to B'_2 then :

$$\mathcal{M}_{(B'_1,B'_2)}(f) = Q^{-1}\mathcal{M}_{(B_1,B_2)}(f)P.$$

Definition 2.6.7 *Two matrices A and B of the same format (n, p) are said to be equivalent if and only if there exist two matrices P and Q (of format (n, n) and (m,m) respectively) such that :*

$$B = Q^{-1}AP.$$

Theorem 2.6.8 *Two matrices of the same size are equivalent if, and only if, they have the same rank.*

Definition 2.6.9 *Two matrices $A, B \in \mathcal{M}_n(\mathbb{K})$ are similar if there is an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that :*

$$A = PAP^{-1}.$$

Property 2.6.10 *(i)It is an equivalence relationship.*
(ii) Two matrices are similar if and only if they represent the same endomorphism in two

bases taken simultaneously as the starting and ending base.

(iii) Similar matrices are equivalent. (The converse is false.)

2.7 Diagonalization

Definition 2.7.1 *Let $A \in \mathcal{M}_{(\mathbb{K})}$ and let $\lambda \in \mathbb{K}$, we say that λ is an eigenvalue of A if there exists a column vector $\nu \neq 0$ such that $A\nu = \lambda\nu$.*

The vector ν is called the eigenvector associated with the value λ.

Example 2.7.2 *Let $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, we have $\lambda_1 = 1$ and $\lambda_2 = 2$ are eigenvalues of A.*

Proposition 2.7.3 *Let $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$, $\lambda \in \mathbb{K}$ is an eigenvalue of A if and only if $P_A(\lambda) = det(A - \lambda\mathbb{I}d_n) = 0$.*

$P_A(\lambda)$ is called the characteristic polynomial of A (It is a polynomial in λ) with :

$P_A(\lambda) = det(A - \lambda I_n)$.

Remark 2.7.4 *The eigenvalues of a square matrix are the roots of its characteristic polynomial.*

Example 2.7.5 *1. The characteristic polynomial of :*

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is :* $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - cd = \lambda^2 - (a+d)\lambda + ad - bc.$

2. Calculate the eigenvalues of the following matrices : $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Theorem 2.7.6 ***Sum and product of eigenvalues***

The trace of A is equal to the sum of the eigenvalues of A and the determinant of A is the product of the eigenvalues of A.

Proof: *In the case $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $det(A) = ad - bc$, $tr(A) = a + d$ and $det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda_1\lambda - \lambda_2\lambda + \lambda_1\lambda_2$.*

Then by identification we have $\lambda_1 + \lambda_2 = a + d = tr(A)$ and $\lambda_1\lambda_2 = ad - bc = det(A)$.

The general case is demonstrated in a similar way.

Theorem 2.7.7 ***Caylay-Hamilton theorem***

For the characteristic polynomial of a matrix A, if substituting λ by the matrix A, we obtain a matrix expression which is the matrix of zeros.

Example 2.7.8 *Let* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, *so* $det(A-\lambda I) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - 5\lambda - 2$. *The Caylay-Hamilton Theorem states that $A^2 - 5A - 2I$ must be the zero matrix, i.e $A^2 - 5A - 2I = 0$ (This can be easily verified).*

What is the purpose of this theorem?

It helps to calculate :

1. *the inverse matrix A^{-1}: Since $A^2 - 5A - 2I = 0$, we have $A^2 - 5A = 2I$ and $A(A-5I) = 2I$, so $A.\frac{1}{2}(A-5I) = I$, then $A^{-1} = \frac{1}{2}(A-5I)$.*

2. *The powers : $A^3 = A^2.A = (5A+2I)A = 5A^2 + 2A = 5(5A+2I) + 2A = 27A + 10I$ and $A^4 = \cdots = 145A + 52I$.*

Definition 2.7.9 *Let $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$, and $\lambda \in \mathbb{K}$ an eigenvalue of A, the set E_λ defined by :*

$$E_\lambda = \{\nu \in \mathbb{R}^n \quad or \quad \mathbb{C}^n \quad /A\nu = \lambda\nu\},$$

is called the eigenspace associated with the eigenvalue then E_λ is a subvector space of E.

Example 2.7.10 *Calculate the eigenspaces associated with the eigenvalues of the matrices A and B in the previous example **2.7.5** .*

Definition 2.7.11 *A matrix is said to be $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$ is diagonalizable if there exists an invertible matrix P is a diagonal matrix D such that : $A = PDP^{-1}$. (where P is the passing matrix.*

Theorem 2.7.12 *Let $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$ and $\lambda_1, \cdots, \lambda_p \in \mathbb{K}$ the eigenvalues of A of order of respective multiplicities $m_1, \cdots, m_p$, then if :*

(1) $dimE_{\lambda_i} = m_i$, $i = 1, 2, \cdots, p$

or

(2) $dimE_{\lambda_1} + dimE_{\lambda_2} \cdots dimE_{\lambda_p} = n$.

Then the matrix A is diagonalizable and the diagonal matrix D associated to A is given by :

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0.. & 0 \\ 0 & \lambda_1 & 0 & 0.. & 0 \\ 0 & 0 & \lambda_2 & 0.. & 0 \\ . & . & . & .. & 0 \\ 0 & 0 & 0 & \lambda_p.. & 0 \\ . & . & . & .. & 0 \\ 0 & 0 & 0 & ..0 & \lambda_p \end{pmatrix}$$

each λ_i is repeated m_i times, the matrix P is formed of the eigenvectors.

Remark 2.7.13 *If the matrix $A \in \mathcal{M}_{(n,n)}(\mathbb{K})$ has n distinct eigenvalues then A is diagonalizable and the diagonal matrix D associated with A is :*

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0.. & 0 \\ 0 & \lambda_2 & 0 & 0.. & 0 \\ 0 & 0 & \lambda_3 & 0.. & 0 \\ . & . & . & .. & 0 \\ . & . & . & .. & 0 \\ 0 & 0 & 0 & ..0 & \lambda_n \end{pmatrix}.$$

Example 2.7.14 *Consider the matrix* $B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, *admits the eigenvalues* $\lambda_1 = 2$ *double and* $\lambda_2 = 1$ *single, then the diagonal matrix D is given by* $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and the pass matrix is given by* $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

What is the use of diagonalizing a matrix? I.e expressing A in the form PDP^{-1} with D diagonal?

In particular, it serves to facilitate the calculation of a power of the matrix, for example $A^3 = PD^3P^{-1}$.

What is the purpose of calculating powers of a matrix? (Open question.)
(It is used, for example, to calculate the cumulative interest.)

Theorem 2.7.15 ***Diagonalizability criteria***

Theorem 2.7.16 1. Easy version
If all roots of the characteristic polynomial of A are simple, then A is diagonalizable. (Otherwise, A may or may not be diagonalizable.)

Theorem 2.7.17 Difficult version
If A is a real and symmetric matrix, then all eigenvalues of A are real and A is diagonalizable.

Example 2.7.18 *For each of the following three matrices, determine whether it is is diagonalizable, and diagonalize it if possible :*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 5-1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 8 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 2 & 8 & 1 \end{pmatrix}.$$

2.8 Trigonalization

Two examples to be discussed during the course.

2.9 Systems of linear equations

In this part of the course the field $\mathbb{K}$ will designate $\mathbb{R}$ (the set of real numbers) or $\mathbb{C}$ (the set of complex numbers).
A system of n linear equations with p unknowns and coefficients in $\mathbb{K}$ is any system of the form :

$$(S): \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1p}x_p & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2p}x_p & = b_2 \\ \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{np}x_p & = b_n \end{cases}$$

where $(x_{ij})_{j=1,\cdots,p}$ are the unknowns, the $(a_{ij}), b_j \in \mathbb{K}$.

1. **Matrix form of the system :**

Let's put $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq p}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$, the system (S) becomes; $AX = B$.

If f is a linear application of $\mathbb{K}^p$ in $\mathbb{K}^n$ such that A is the matrix associated with f in the canonical bases and if we note by $X = (x_1, \cdots, x_p)$ and $b = (b_1, \cdots, b_n)$, the system (S) becomes $f(X) = B$.

2. **Solution of the system**

Definition 2.9.1 *We call a solution of the system (S) any element $X = (x_1, \cdots, x_p)$ satisfying the n equations of (S) this is equivalent to finding a vector X such that $AX = B$ or an element $X \in \mathbb{K}^p$ such that $f(X) = B$.*

Example 2.9.2 $\begin{cases} 2x - 5y + z & = \frac{-1}{3} \\ -x + 3y + 2z & = -2 \\ x - 7y - 3z & = 4 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & -5 & 1 \\ -1 & 3 & 2 \\ 1 & -7 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\frac{-1}{3}, -2, 4).$

We note $A = \begin{pmatrix} 2 & -5 & 1 \\ -1 & 3 & 2 \\ 1 & -7 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *and* $B = (\frac{-1}{3}, -2, 4)$.

3. **Rank of a linear system**

The rank of a linear system is the rank of the matrix $(a_{ij})_{1\leq i\leq n, 1\leq j\leq p}$. If r is the rank of the linear system (S), then $r \leq n$ and $r \leq p$.

2.9.1 Cramer system

Definition 2.9.3 *The system (S) is said to be Cramer if $n = p = r$, i.e, (S) is a system of n equations with n unknowns and such that :*

$$detA \neq 0.$$

Theorem 2.9.4 *Every Cramer system has a solution given by :*

$$X = A^{-1}B.$$

Example 2.9.5 *Resolve :* $\begin{cases} x - y &= 0 \\ x + y &= 1 \end{cases}.$

Theorem 2.9.6 *In a Cramer system, the solution is given by the formulas :*

$$x_i = \frac{detA_i}{detA}, \quad i = 1, \cdots, n.$$

Where A_i is the reduced matrix of A, replacing the column i by the vector B.

Example 2.9.7 *Resolve :* $\begin{cases} 2x + 2y + z &= 1 \\ 2x + y - z &= 2 \\ 3x + y + z &= 3 \end{cases}.$

1. Cases where $n = p$ and $r < n$:

If we now consider a system of n equations with n unknowns, but $rkA < n$ i.e $detA = 0$, in this case we extract a matrix M from A knowing that it is the largest invertible square matrix i.e $detM \neq 0$ contained in A and of order r, this is what is called a submatrix, the unknowns associated with M become principal unknowns and the $(n-r)$ other unknowns become parameters or what we call arbitrary values and we consider the following system :

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1r}x_r &= b_1 - (a_{1r+1}x_{r+1} + ... + a_{1n}x_n) = b'_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2r}x_r &= b_2(a_{2r+1}x_{r+1} + ... + a_{2n}x_n) = b'_2 \\ \vdots & \vdots \\ a_{r1}x_1 + a_{r2}x_2 + ... + a_{rr}x_r &= b_n(a_{rr+1}x_{r+1} + ... + a_{rn}x_n) = b'_r \end{cases}$$

the latter is a system of cramers, so it admits a single solution $(x_1, \cdots, x_r)$ which depends on $(x_{r+1}, \cdots, x_n)$. If this solution verifies the remaining $(n-r)$ equations, then the global system admits an infinite number of solutions. If on the other hand $(x_1, \cdots, x_r)$ does not satisfy a single equation among the $(n - r)$ remaining equations then the global system has no solution.

Example 2.9.8 *Resolve :* $\begin{cases} 3x - y + 2z &= 3 \\ 2x + 2y + z &= 2 \\ x - 3y + z &= 1 \end{cases}.$

2. Case where $n \neq p$:

If the number of equations is not equal to the number of unknowns, then we first look for the rank of A and proceed as before. If M is a matrix contained in A and of order r and $detM \neq 0$ then we consider the system of r equations with r unknowns corresponding to M which is a Cramer system.
If the solution verifies the remaining equations then the global system admits an infinity of solutions otherwise it admits no solution.

Example 2.9.9 *Resolve :* $\begin{cases} 3x - y &= 4 \\ 2x + 2y &= 3 \\ x - 5y &= -5 \end{cases}$.

2.10 Gauss pivot method

The Gauss pivot method allows the general solution of systems of linear equations n equations and p unknowns. It is used in particular for their numerical solution with the help of a computer program. It can be used to solve systems with a large number of unknowns and equations (several hundred or even several thousand).
In all cases, the Gaussian pivot method makes it possible to determine whether the system has solutions or not (and in particular whether it is a Cramer system when n = p).
If the system has solutions, the pivot method can be used to calculate them. In particular, if $n = p$ and if the system has a unique solution (Cramer's system), it can be calculated much more economically (in number of solutions) than by Cramerís formulas. When the solution of the system is not unique, the pivot method allows to express the solutions using the principal unknowns.

2.10.1 Study of an example

Solve the system below $\begin{cases} x + y + 2z &= -1 \\ 2x - y + 2z &= -4 \\ 4x + y + 4z &= -2 \end{cases}$.

2.10.2 Procedure of the Gaussian pivot method

1. Start-up

In the general case, we consider a linear system (S) with n equations and p unknowns $x_1; x_2; \cdots ; x_p$:

$$(S)\ \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = b_n \end{cases}.$$

As usual, we note :

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \in \mathcal{M}_{n,p}(\mathbb{K}),\ B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{K}),\ X = \begin{pmatrix} x_1 \\ \vdots \\ x_1 \end{pmatrix} \in \mathcal{M}_{p,1}(\mathbb{K})$$

respectively the matrix associated with the system, the column vector associated with the second member, and the column vector of the unknowns. Thus the solution of (S) is equivalent to finding X such that : $AX = B$.

In practice, the system is arranged as a matrix without the unknowns. The augmented matrix associated with the system is

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} & b_n \end{pmatrix} \in \mathcal{M}_{n,p+1}(\mathbb{K}).$$

We then operate only on the lines of A_0. The pivot method consists first of all in bringing the system to a triangular system, and this only by elementary operations on the rows.

It is assumed that the first column is not identically zero (otherwise the unknown x_1 does not appear!), so even if we swap rows, we assume that $a_{11} \neq 0$. This coefficient a_{11} is called pivot, the unknown x_1 is called a principal unknown.

By elementary operations on the lines, we "put" 0 under the pivot :

$$\left(\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1p} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2p} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} & b_n \end{array}\right) \underset{\substack{L_2 \leftarrow L_2 - \frac{a_{21}}{a_{11}} L_1 \\ \vdots \\ L_n \leftarrow L_n - \frac{a_{n1}}{a_{11}} L_1}}{\longrightarrow} \left(\begin{array}{ccccc} \boxed{a_{11}} & a_{12} & \cdots & a_{1p} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2p} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{np} & b'_n \end{array}\right) = F.$$

Two cases can then arise, depending on the matrix :

$$A' = \begin{pmatrix} a'_{22} \cdots a'_{2p} \\ \vdots \qquad \vdots \\ a'_{n2} \cdots a'_{np} \end{pmatrix}.$$

1.a. First case

Let us first assume that $A' = 0$ (null matrix). Then, if one of the b'_i is different from 0; the system has no solution because the last row of the matrix F above represents the equations

$$\begin{cases} 0x_2 + 0x_3 + \cdots + 0x_p = b'_2 \\ \qquad\qquad \vdots \\ 0x_2 + 0x_3 + \cdots + 0x_p = b'_n \end{cases} \qquad (1).$$

On the other hand, if $b'_2 = b'_3 = b'_4 = b'_n = 0$; there are solutions. The first line of F allows us to express x_1 (main unknown) as a function of $x_2; \cdots ; x_n$ (so-called secondary unknowns). Each value of the secondary unknowns gives a solution of the system. The rank of the system is 1 : it is equal to the number of principal unknowns and the rank of the matrix A of the system.

The relations $b'_2 = b'_3 = b'_4 = b'_n = 0$ are called compatibility relations. If they are not verified, the system has no solution.

Example 2.10.1 *Let a be a parameter. Consider the system :* $(S) : \begin{cases} x + 2y - z &= 1 \\ 2x + 4y - 2z &= 2 \\ -x - 2y + z &= a \end{cases}$

1.b. Second case

Now suppose that the matrix A' defined in (1) is not the null matrix

First sub-case :

The first column of A' is non-zero. If we swap the rows (which only means swapping the equations), we can assume that $a_{22} \neq 0$. This coefficient becomes the second pivot, x_2 is said to be the main unknown. Using this pivot, we "put" zeros under a_{22} :

$$\begin{pmatrix} \boxed{a_{11}} & \cdots & \cdots & a_{1p} & b_1 \\ 0 & a'_{22} & \cdots & a'_{2p} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{np} & b'_n \end{pmatrix} \xrightarrow[\vdots]{L_3 \leftarrow L_3 - \frac{a'_{32}}{a'_{22}} L_2} \begin{pmatrix} \boxed{a_{11}} & \cdots & \cdots & \cdots & a_{1p} & b_1 \\ 0 & \boxed{a'_{22}} & \cdots & \cdots & a'_{2p} & b'_2 \\ & 0 & a''_{33} & \cdots & a''_{3p} & b''_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & a''_{n3} & \cdots & a''_{np} & b''_n \end{pmatrix}.$$

We then note :

$$A'' = \begin{pmatrix} a''_{33} & \cdots & a''_{3p} \\ \vdots & & \vdots \\ a''_{n3} & \cdots & a''_{np} \end{pmatrix}$$

and we repeat the process by proceeding on A'' as we did on A'.

Second sub-case :

The first column of A' is zero. Then x_2 is said to be a secondary unknown. If the second column of A' is zero, x_3 is also called secondary unknown. As A' is not identically zero, one of these columns is not zero, and even if we is not zero, and even if we swap the rows, we can assume that

$$A' = \begin{pmatrix} 0 & \cdots & 0 & a'_{2j} & \cdots & a'_{2p} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a'_{nj} & \cdots & a'_{np} \end{pmatrix},$$

with $a'_{2j} \neq 0$. The coefficient a'_{2j} is then the second pivot, x_j is a main unknown and $x_2; x_3; x_{j-1}$ are called secondary unknowns. Using this pivot, we 'put' 0's under a_{2j}.

$$\begin{pmatrix} \boxed{a_{11}} & \cdots & \cdots & \cdots & a_{1p} & b_1 \\ 0 & 0 \cdots 0 & \boxed{a'_{2j}} & \cdots & a'_{2p} & b'_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 \cdots 0 & a'_{nj} & \cdots & a'_{np} & b'_n \end{pmatrix} \xrightarrow[\vdots]{L_3 \leftarrow L_3 - \frac{a'_{3j}}{a'_{2j}} L_2} \begin{pmatrix} \boxed{a_{11}} & \cdots & \cdots & \cdots & a_{1p} & b_1 \\ 0 & 0 \cdots 0 & \boxed{a'_{2j}} & \cdots & a'_{2p} & b'_2 \\ \vdots & \vdots & 0 & & & b''_3 \\ \vdots & \vdots & \vdots & (A'') & & \vdots \\ 0 & 0 \cdots 0 & 0 & & & b''_n \end{pmatrix}.$$

We then repeat the process with the A'' matrix.

Conclusion : At the end of the process, we obtain a matrix of the following form, the so-called step form :

$$
\left(\begin{array}{ccccccccc|c}
\boxed{a_{11}} & \times & \cdots & & \cdots & \cdots & & \cdots & a_{1p} & \beta_1 \\
0 & 0 & \cdots & 0 & \boxed{a'_{2j}} & \cdots & & \cdots & a'_{2p} & \beta_2 \\
\vdots & & & & 0 & & & & & \vdots \\
\vdots & & & & & 0 & \boxed{a^{(r)}_{rk}} & \times & \cdots & \beta_r \\
\vdots & & & & & & 0 & \cdots & 0 & \vdots \\
\vdots & & & & & & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & & \cdots & 0 & \cdots & 0 & \beta_n
\end{array}\right).
$$

The rank of the system is $r =$ number of pivots. The principal unknowns are the unknowns that correspond to the pivots (x_j is a principal unknown if column j contains a pivot). There are $r = rk(S)$ principal unknowns. The other unknowns are called secondary.

In the case $n = p = r$; the system is a Cramer system and the Gaussian pivot method gives the unique solution of the system.

In the general case, the compatibility relations are $b_{r+1} = \cdots = b_n = 0$. The system has solutions only if these relations are verified. In this case, we express the principal unknowns in terms of the secondary unknowns.

Example 2.10.2 *Consider the system of 3 equations with 4 unknowns :*

$$
(S): \begin{cases} x + 2y + z + t & = 1 \\ x + y + z - t & = 2 \\ 2x + y + z & = 3 \end{cases}.
$$

Let's solve the system.

Example 2.10.3 *Let a be a real parameter, and let the system :*

$$
(S): \begin{cases} x + y + 2z - t - u & = 1 \\ x + y + z & = 3 \\ 2x + 2y - z + 4t + 4u & = 2 \\ 3x + 3y + z - 6t - 6u & = 93 \\ x + y & = a \end{cases}.
$$

Let's solve the system

2.11 Exercises

2.11.1 Exercise 1

Let the matrix A be defined by : $A = \begin{pmatrix} 5 & 6 - 3 \\ -18 - 19 & 9 \\ -30 - 30 & 14 \end{pmatrix}$.

a. Is A invertible? If so, determine its inverse A^{-1}.

b. Calculate $A^2 - A - 2I_3 = 0$, where I_3 is the identity matrix.

2.11.2 Exercise 2

Let the matrix B be defined by : $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

1 Find $a, b \in \mathbb{R}$ such that $A^2 = a.I_n + b.A$.

2 Deduce that A is invertible and give its inverse.

2.11.3 Exercise 3

Let C be the matrix associated to the application f defined on $\mathbb{R}^3$ in the canonical base of $\mathbb{R}^3$.

$C = \begin{pmatrix} 1 - 1 & 5 \\ 3 \quad 0 & 2 \\ 1 \quad 1 & 4 \end{pmatrix}$.

1. Determine the application f.

2. Determine $kerf$ and Imf and their dimension, is f bijective?

3. Let $S = \{v_1 = (1,1,1), v_2 = (1,0,1), v_3 = (2,-1,0)\}$.

a) Show that S is a base of $\mathbb{R}^3$.

b) Give the matrix associated with f following the base S.

2.11.4 Exercise 4

Let the matrix M be defined by : $M = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$.

1. Determine the eigenvalues of A.
2. Show that A is diagonalizable.
3. Determine P, calculate A^k.

2.11.5 Exercise 5

Solve the following system (using the two methods seen in the course):

$$(S_1) : \begin{cases} x + y + z & = 3 \\ 2x + y + z & = 2 \\ x + 5y + z & = 1 \end{cases} ; \qquad (S_2) : \begin{cases} 3x + y - 2z + 3t & = 0 \\ -x + 2y - 4z + 6t & = 2 \\ 2x - y + 2z - 3t & = 0 \end{cases} .$$

Chapter 3

Linear Algebra 2 tutorials

Exercise 1

Solve in $\mathbb{R}$ the following system of linear equations: $(S):\begin{cases} x+y+z &= 1 \\ x+5y+5z &= 5 \\ 2x+5y+5z &= 5 \\ -y-z &= m \end{cases}$.

Exercise 2

Solving by the Gaussian pivot method :

$$(S):\begin{cases} x-y+z+t &= 0 \\ 3x-3y+3z+2t &= 0 \\ x-y+z &= 0 \\ 5x+5y+5z+7t &= 0 \end{cases} \qquad (P):\begin{cases} x+y+mz &= m \\ x+y-z &= 1 \\ x+my-mz &= 1 \end{cases}.$$

Exercise 3

Let $v_1 = (1,1,0), v_2 = (4,1,4)$ and $v_3 = (2,-1,4)$ be the vectors Is the family free?

Exercise 4

Say whether the following matrices are diagonalizable on $\mathbb{R}$, on $\mathbb{C}$ and if so diagonalize them :

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix};\ B = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix};\ C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix};\ D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Exercise 5

Solve the following system using the pivot method :

$$(S_1) : \begin{cases} x + 2y - 3z &= -1 \\ 8x + 2y - 2z &= 9 \\ 3x - y + 2z &= 7 \end{cases} ; (S_2) : \begin{cases} 2x + y - 2z &= 10 \\ x + y + 4z &= -9 \\ 7x + 5y + z &= 14 \end{cases} ; (S_3) : \begin{cases} x - 3y + 7z &= -4 \\ x + 2y - 3z &= 6 \\ 7x + 4y - z &= 22 \end{cases} .$$

Exercise 6

Determine the values of a $\beta \in \mathbb{R}$ for which the system :

$$(S_\beta) : \begin{cases} x + y - z &= 1 \\ x + 2y + \beta z &= 2 \\ 2x + \beta y + 2z &= 3 \end{cases}$$

a) Has no solution.

b) Has an infinite number of solutions.

c) Has a unique solution.

Exercise 7

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x, y, z) \longmapsto (x + y, x + z)$.

1. Check that f is a linear application.
2. Determine $ker f$ the kernel of f, then give a base for $ker f$ and deduce $dim(ker f)$.
3. is f injective?
4. Give $dim(Im f)$; then give a base for $Im f$.
5. is f surjective.

Exercise 8

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x, y, z) \longmapsto (x + y, x + z, x + y + z)$

1. Determine $ker f$ the kernel of f and deduce $dim(ker f)$.
2. is f injective? is f overjective? is f bijective?
3. Give $dim(Im f)$; then give a base for $Im f$.

Exercise 9

1. Check that $(1, 1)$ and $(2, 3)$ generate $\mathbb{R}^2$. Conclude!
2. Verify that $\{(1, 2, 3), (0, 1, -1), (2, 0, 1)\}$ is a free family in $\mathbb{R}^3$. Conclude!

Exercise 10

Let the following sets be : $E_1 = \{(x,y,z) \in \mathbb{R}^3; x+y+z=0\}$ and $E_2 = \{(x,y,z) \in \mathbb{R}^3; x-y=x+z=0\}$.

1. Show that E_1 and E_2 are sub-vector spaces of $\mathbb{R}^3$.
2. Give a base for E_1 and a base for E_2; and derive $dimE_1$ and $dimE_2$.

Exercise 11

Let $E_1 = \{(x,y,z) \in \mathbb{R}^3; x=y=z\}$ and $E_2 = \{(x,y,z) \in \mathbb{R}^3; x=0\}$.

1. Show that E_1 and E_2 are subvector spaces of $\mathbb{R}^3$.
2. Show by two methods that $\mathbb{R}^3 = E_1 \oplus E_2$.

Exercise 12

Among the sets F say which are sub-vector spaces of E :

1. $E = \mathbb{R}^3$; $F = \{(x,y,z) \in \mathbb{R}^3; x+y+3z=0\}$
2. $E = \mathbb{R}^2$; $F = \{(x,y) \in \mathbb{R}^2; x+3y=3\}$
3. $E = \mathbb{R}^2$; $F = \{(x,y) \in \mathbb{R}^2; x^2+y^2=4\}$
4. $E = \mathbb{R}[X]$; $F = \{P \in \mathbb{R}[X], deg(P)=4\}$
5. $E = \mathbb{R}[X]$; $F = \{P \in \mathbb{R}[X], deg(P) \leq 4\}$
6. $E = \{f : \mathbb{R} \longrightarrow \mathbb{R}, application\}$, $F = \{f \in E; pair\}$

Exercise 13

Let A, B and C be the following matrices :

$$A = \begin{pmatrix} 0 & 1-1 \\ -3 & 4-3 \\ -1 & 1 \quad 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -6-1 \end{pmatrix}, C = \begin{pmatrix} 0-3 \\ 2 \quad 1 \\ 8-7 \end{pmatrix}.$$

1. Calculer $B+C$, $B-C$, $B+2C$, $2B-3C$.
2. Calculer AB, AC, A^2, A^3.
3. Calculer t_A ; t_A, $t_{(AB)}$.

Exercise 14

1. Calculate the determinant of the following matrices :

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & 1-1 \\ -3 & 4-3 \\ -1 & 1 \quad 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1-1 \\ -3 & 4-3 \\ -1 & 1 \quad 0 \end{pmatrix}, C = \begin{pmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{pmatrix}$$

$$, B = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 3 & 9 & 0 \\ 2 & 1 & 5 & 4 \\ -2 & 1 & -3 & -4 \end{pmatrix}.$$

2. Give A^{-1} and B^{-1}.

Exercise 14

Let $M = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

1. Calculate $M^3 - 2M^2 + 2M$.
2. Deduce from the above that the matrix M is invertible; then give M^{-1}.
3. Find M^{-1} by using the co-matrix.

Exercise 15

Let $N = \begin{pmatrix} \alpha & -1 & 0 \\ -2 & \alpha & -2 \\ 0 & -1 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{R}$ a parameter.

1. Discuss according to the values of α the invertibility of N.
2. Where possible, calculate N^{-1}.

Exercise 16

Let $A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

1. Check that A is invertible, and give A^{-1}.
2. Deduce the solution of the system $(S) : \begin{cases} -x + y - z & = 10 \\ x + y + z & = -4 \\ x - 2y + 4z & = 6 \end{cases}$.

Exercise 17

Let a matrix $A = \begin{pmatrix} 1 & a & a^2 \\ 1 & a & b^2 \\ 1 & c & c^2 \end{pmatrix}$, where a, b and c are non zero.

Say for which value(s) of a, b and c the matrix A is invertible.

Exercise 18

Let $A = \begin{pmatrix} 1 - 3 & 6 \\ 6 - 8 & 12 \\ 3 - 3 & 4 \end{pmatrix}$.

1. Calculate A^2, then find two real α, β such that $A^2 = \alpha A + \eta I$, where $I = I_3$.

2. Deduce from the above that A is invertible, and give A^{-1} .

3. Find A^{-1} by using the co-matrix.

Exercise 19

Consider the endomorphism f of $\mathbb{R}^3$ defined by $f : (x, y, z) \longrightarrow (3x - z, 2x + 4y + 2z, -x + 3z)$.

1. Determine the matrix $A = Mat(f)_B$ of f in the canonical base of $\mathbb{R}^3$.

2. Determine the characteristic polynomial of f. Deduce the eigenvalues of f.

3. Determine a base for each eigenspace of f. Is the endomorphism f diagonalizable?

4. Find a matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix. where D is a diagonal matrix which we will explain. Make it explicit.

5. Determine the matrix A^n for all $n \geq 1$.

Bibliography

[1] E. Azouly, J. Avignant, G. Auliac, *Problèmes Corrigés de mathématiques, DEUG MIAS/SM, Ediscience (Dunod pour la nouvelle édition) Paris 2002.*,

[2] E. Azouly, J. Avignant, G. Auliac, *les mathématiques en Licence, Tome 1 : Cours+ exercices corrigés , Ediscience.*

[3] E. Azouly, J. Avignant, G. Auliac, *les mathématiques en Licence, Tome 2 : Cours+ exercices corrigés , Ediscience.*

[4] *Cours d'algèbre. Hermann, 1966.*

[5] *Mortad, Exercices Corrigés d'Algèbre, Première Année L.M.D., Edition "Dar el Bassair"(Alger-Algérie),2012*