Barcelona School of Economics

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# 2nd Problem Set

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# 3.2. Downsian competition model

## 3.2.a Derive the policy preferences.

The utility function for agent $i$ is given as:

$$w^i = c^i + \alpha^i V(y),$$

Where:

- $c^i = 1 - q$
- $y = q$
- $W(q; \alpha^i) = (1 - q) + \alpha^i V(q)$
- $\frac{\partial W}{\partial q} = -1 + \alpha^i V'(q)$
- $\alpha^i V'(q) = 1$

The government provides $y$ by taxing $q$ on each individual, with:

$$c^i = 1 - q.$$

The relationship between public good provision and taxes is:

$$y = q.$$

Substituting these into the utility function, we get:

$$w^i = (1 - q) + \alpha^i V(q).$$

Then, the utility function becomes:

$$W(q; \alpha^i) = (1 - q) + \alpha^i V(q).$$

To find the agent's preferred level of $q$, we take the derivative of $W(q; \alpha^i)$ with respect to $q$:

$$\frac{\partial W}{\partial q} = -1 + \alpha^i V'(q).$$

The agent's preferred policy satisfies the first-order condition:

$$\alpha^i V'(q) = 1.$$

This condition shows that each agent prefers the level of public good $q$ where their marginal utility from the public good equals the marginal cost of providing it (the tax rate $q$). The social planner maximizes the average utility across all agents:

$$\text{Maximize} \int W(q;\alpha^i)\,dF(\alpha^i).$$

The average utility is:

$$\bar{W}(q) = (1-q) + \alpha V(q),$$

Where $\alpha = \mathbb{E}[\alpha^i]$ (the mean of $\alpha^i$). The first-order condition for the social optimum is:

$$\frac{\partial \bar{W}}{\partial q} = -1 + \alpha V'(q) = 0.$$

Rearranging, the socially optimal level of $q$ satisfies:

$$\alpha V'(q) = 1.$$

This is similar to the individual condition but uses the mean $\alpha$ of the distribution instead of the individual $\alpha^i$. In the Downsian competition, two candidates $P = A, B$ announce policy platforms $q^A$ and $q^B$. The goal of each candidate is to maximize their expected utility, which depends on their probability of winning the election. The expected utility for candidate $P$ is:

$$p_P R,$$

Where $R$ is an exogenous rent. And both candidates select policies that maximize the utility of the median voter. Let the median value of $\alpha^i$ be $\alpha_m$. The preferred policy of the median voter satisfies:

$$\alpha_m V'(q) = 1.$$

Thus, both candidates will announce:

$$q^A = q^B = q^* = y*,$$

where $q^*$ satisfies:

$$\alpha_m V'(q^*) = 1.$$

This results in policy convergence at the median voter's preferred policy.

### 3.2.b Determining candidates' probability of winning

Given the assumption that $\alpha^i = \alpha$, all voters have identical preferences. The utility function for each voter is:

$$W(q) = (1 - q) + \alpha V(q).$$

The preferred policy of the voters is determined by solving the first-order condition:

$$\frac{\partial W(q)}{\partial q} = -1 + \alpha V'(q) = 0.$$

This implies:

$$\alpha V'(q) = 1.$$

The equilibrium policy $q^*$ is the same for all voters because preferences are identical. To maximize their chances of winning, both candidates $A$ and $B$ will announce $q^*$ as their platform:

$$q^A = q^B = q^*.$$

Since both candidates propose the same policy (equilibrium), voters are indifferent between them. Therefore, the probability of winning for each candidate is:

$$P_A = P_B = \frac{1}{2}.$$

In conclusion, the candidates' platforms converge to the voters' preferred policy $q^*$, because all voters have the same preferences, there is no incentive for candidates to differentiate their platforms.

### 3.2.c Probabilities of winning in heterogeneous case

When agents are heterogeneous, each agent $i$'s preferences depend on their individual parameter $\alpha^i$. The utility of agent $i$ from a candidate's proposed policy $q$ is:

$$W(q; \alpha^i) = (1 - q) + \alpha^i V(q).$$

An agent votes for candidate $A$ if the utility from $q^A$ exceeds that from $q^B$, i.e.:

$$W(q^A; \alpha^i) > W(q^B; \alpha^i).$$

If:

$$W(q^A; \alpha^i) = W(q^B; \alpha^i),$$

The agent splits their vote equally between the two candidates. Thus, the voting behavior of an agent $i$ can be summarized as:

$$\begin{cases} 0 & \text{if } W(q^A;\alpha^i) < W(q^B;\alpha^i), \\ \frac{1}{2} & \text{if } W(q^A;\alpha^i) = W(q^B;\alpha^i), \\ 1 & \text{if } W(q^A;\alpha^i) > W(q^B;\alpha^i). \end{cases}$$

In conclusion, when agents are heterogeneous, the Downsian model ensures that candidates strategically choose policies to appeal to the median voter. The equilibrium occurs when both candidates announce the same platform, $q^A = q^B = q^*$. In this equilibrium, candidates converge to the same policy, reflecting the preferences of the median voter.

### 3.2.d Model discussion

The model predicts that representative democracies tend to implement policies favoring the median voter, which can create both stability and predictability in economic outcomes. However, it also highlights potential biases, as the preferences of non-median voters are not directly represented in the implemented policy, specially if the voters are heterogeneous.

# 3.3. A simple model of probabilistic voting

## 3.3.a An interpretation of $\sigma^i$

The parameter $\sigma^i$ represents the individual ideological bias of voter $i$ toward politician $B$. It captures non-policy factors that influence voter preferences, such as cultural alignment, personal beliefs, or other idiosyncratic factors that cause a voter to prefer candidate $B$ over $A$. A positive $\sigma^i$ indicates a bias in favor of $B$, while a negative $\sigma^i$ implies a bias against $B$. However, an agent $i$ is indifferent between voting for $A$ and $B$ if their utilities for both candidates are equal. The utilities for $A$ and $B$ are:

$$W(q^A, \alpha^i) \quad \text{and} \quad W(q^B, \alpha^i) + \sigma^i + \delta,$$

Where $\delta$ represents the popularity of politician $B$, which is common across all voters. The indifference condition is:

$$W(q^A, \alpha^i) = W(q^B, \alpha^i) + \sigma^i + \delta.$$

Rearranging for $\sigma^i$, we find:

$$\sigma^i = W(q^A, \alpha^i) - W(q^B, \alpha^i) - \delta.$$

This equation describes the ideological bias $\sigma^i$ required for voter $i$ to be indifferent between voting for $A$ and $B$.

## 3.3.b Platforms

Politicians $A$ and $B$ simultaneously announce their platforms $q^A$ and $q^B$. Each voter chooses the candidate that provides them with the highest utility. The equilibrium in this model occurs when neither politician has an incentive to deviate unilaterally from their chosen platform. Given the voter utility functions:

$$W(q^A, \alpha^i) \quad \text{and} \quad W(q^B, \alpha^i) + \sigma^i + \delta,$$

the policy that maximizes voter support will reflect the preferences of the median voter. Thus, both candidates select the platform:

$$q^A = q^B = q^*,$$

where $q^*$ satisfies the median voter's first-order condition:

$$\alpha_m V'(q^*) = 1.$$

The implemented policy is $q^*$, ensuring that the marginal benefit of the public good equals its marginal cost, as perceived by the median voter.

### 3.3.c Heterogeneous agents

When agents are heterogeneous, their preferences for the public good differ and implies that the median voter's preferences will still determine the equilibrium policy because candidates focus on maximizing vote shares. The equilibrium policy $q^*$ remains the one that satisfies the median voter's condition:

$$\alpha_m V'(q^*) = 1.$$

And, again both candidates select the platform:

$$q^A = q^B = q^*,$$

However, heterogeneity in $\alpha^i$ implies that the implemented policy will generally not reflect the preferences of individuals at either end of the distribution. Instead, the equilibrium reflects a compromise policy, aligning with the median voter.

### 3.3.d Comparisson with problem 2.

Compared to Problem 2, where voter preferences were homogeneous ($\alpha^i = \alpha$ for all $i$), the inclusion of heterogeneity in Problem 3 introduces additional complexity. Despite this, the fundamental result of policy convergence remains unchanged, as candidates continue to focus on the median voter.

A key difference, however, is the probabilistic element introduced by $\sigma^i$ and $\delta$, which captures ideological biases, this means that even if policies are identical, factors unrelated to policy can influence election outcomes. In

Problem 2, policies perfectly aligned with the average voter's preferences, as there was no heterogeneity. In contrast, Problem 3 shows that heterogeneity leads to policies aligning with the median rather than the average voter.

# 3.4. Probabilistic voting in groups of voters

## 3.4.a Indifferent voter and candidate $A$'s vote share

The population consists of three groups of voters $J = \{R, M, P\}$, where each group $J$ has an intrinsic parameter $\alpha^J$. The proportion of agents in group $J$ is $\lambda^J$, and the preferences of agent $i$ in group $J$ are:

$$W(q, \alpha^J) \quad \text{for candidate } A,$$

$$W(q, \alpha^J) + \sigma^{iJ} + \delta \quad \text{for candidate } B.$$

The indifferent voter between $A$ and $B$ in group $J$ satisfies:

$$W(q^A, \alpha^J) = W(q^B, \alpha^J) + \sigma^{iJ} + \delta.$$

Rearranging for $\sigma^{iJ}$, the condition is:

$$\sigma^{iJ} = W(q^A, \alpha^J) - W(q^B, \alpha^J) - \delta.$$

The vote share of candidate $A$ in group $J$, denoted $\pi_A^J$, is the proportion of voters in $J$ for whom $\sigma^{iJ}$ is less than this threshold. Assuming $\sigma^{iJ}$ is uniformly distributed on:

$$\left[-\frac{1}{2\phi^J}, \frac{1}{2\phi^J}\right],$$

Cumulative distribution function (CDF) for $\sigma^{iJ}$ is:

$$F(\sigma^{iJ}) = \frac{\sigma^{iJ} + \frac{1}{2\phi^J}}{\frac{1}{\phi^J}} = \phi^J \sigma^{iJ} + \frac{1}{2}.$$

Thus, the vote share of $A$ in group $J$ is:

$$\pi_A^J = \phi^J \left(W(q^A, \alpha^J) - W(q^B, \alpha^J) - \delta\right) + \frac{1}{2}.$$

Candidate $A$'s total vote share is the weighted sum across all groups:

$$\pi_A = \sum_{J=1}^{3} \lambda^J \pi_A^J.$$

Thus, his probability of winning is represents by:

$$p_A = \frac{\psi}{\phi} \sum_J \lambda^J \phi^J \left[W(q^A; \alpha^J) - W(q^B; \alpha^J)\right] + \frac{1}{2}$$

### 3.4.b Characterizing politician's optimal platform

To maximize their probability of winning, both politicians choose platforms $q^A$ and $q^B$ to maximize their vote shares. Also, the optimal platform of candidate B is one that produce zero probability of win to the candidate A:

$$\frac{\psi}{\phi}\sum_J \lambda^J \phi^J W_{q^A}(q^A;\alpha^J) = 0$$

The equilibrium occurs when neither candidate can increase their vote share by unilaterally changing their platform. Each candidate's optimal platform converges to a weighted average of group preferences.This weighted condition implies that the implemented policy reflects a compromise among groups.

### 3.4.c General common-knowledge distribution $G^J(\cdot)$

$\sigma^{iJ}$ is drawn from a distribution $G^J(\cdot)$ rather than a uniform distribution, the vote share for candidate $A$ in group $J$ becomes:

$$\pi_A^J = G^J\left(W(q^A,\alpha^J) - W(q^B,\alpha^J) - \delta\right).$$

The total vote share of $A$ is still:

$$\pi_A = \sum_{J=1}^{3} \lambda^J \pi_A^J,$$

but the functional form of $\pi_A^J$ now depends on $G^J(\cdot)$. While this changes the specific calculations, the qualitative results remain the same: candidates choose platforms that maximize their weighted vote shares. The equilibrium policy $q^*$ continues to reflect a compromise among groups, as it balances their sizes and sensitivities.

### 3.4.d Distribution $L(\cdot)$ for $\delta$

If $\delta$ is drawn from a distribution $L(\cdot)$ with density $l(\cdot)$, the threshold for indifference across all groups becomes:

$$\delta = W(q^A,\alpha^J) - W(q^B,\alpha^J) - \sigma^{iJ}.$$

The probability of voting for candidate $A$ depends on the joint distribution of $\sigma^{iJ}$ and $\delta$. However, restricting $\delta$ to a uniform distribution does not qualitatively alter the results, as the equilibrium remains focused on maximizing weighted vote shares.

# 3.5. Lobbying

## 3.5.a Indifferent agent and vote share of $A$

The utility for voters depends on the platforms $q^A$ and $q^B$, as well as the contributions from groups. Let $\delta$ represent the popularity of candidate $B$, given by:

$$\delta = \tilde{\delta} + h \cdot (C_B - C_A),$$

Where $C_P = \sum_J O^J \alpha^J C_P^J$ is the total contribution received by candidate $P$. The indifferent agent in group $J$ satisfies:

$$W(q^A, \alpha^J) = W(q^B, \alpha^J) + \sigma^{iJ} + \delta.$$

Rearranging, we find:

$$\sigma^{iJ} = W(q^A, \alpha^J) - W(q^B, \alpha^J) - \delta.$$

The vote share of $A$ in group $J$, denoted $\pi_A^J$, is:

$$\pi_A^J = F\left(W(q^A, \alpha^J) - W(q^B, \alpha^J) - \delta\right).$$

Assuming $\sigma^{iJ}$ is uniformly distributed, the total vote share of $A$ is:

$$\pi_A = \sum_{J=1}^{3} \lambda^J \pi_A^J.$$

The probability of winning for $A$ depends on whether $\pi_A > 1/2$, and its representation by:

$$p_A = \frac{\psi}{\phi} \sum_J \lambda^J \phi^J \left[W(q^A; \alpha^J) - W(q^B; \alpha^J) + h(C_A - C_B)\right] + \frac{1}{2}$$

## 3.4.b Objective function of groups

Each agent in group $J$ aims to maximize their expected utility, given by:

$$p_A W(q^A; \alpha^J) + (1 - p_A) W(q^B; \alpha^J) - D(C_A^J + C_B^J),$$

Where the cost of contribution is:

$$D(C_A^J + C_B^J) = \frac{1}{2}\left((C_A^J)^2 + (C_B^J)^2\right).$$

Taking the derivative of this expression with respect to $C_A^J$ and $C_B^J$, we obtain the following first-order conditions:

$$\frac{h\lambda^J\phi^J}{\phi}\left[W(q^A,\alpha^J)-W(q^B,\alpha^J)\right]-C_A^J-C_B^J\leq 0,$$

$$-\frac{h\lambda^J\phi^J}{\phi}\left[W(q^A,\alpha^J)-W(q^B,\alpha^J)\right]-C_B^J-C_A^J\leq 0.$$

If $W(q^A,\alpha^J)<W(q^B,\alpha^J)$, the first derivative with respect to $C_A^J$ is negative, implying no contributions to $A$. Similarly, if $W(q^B,\alpha^J)<W(q^A,\alpha^J)$, the first derivative with respect to $C_B^J$ is negative, implying no contributions to $B$. Therefore, the optimal contribution of group $J$ to politician $P$ is:

$$C_P^J=\max\left\{0,\frac{h\lambda^J\phi^J}{\phi}\left[W(q^P,\alpha^J)-W(q^{P'},\alpha^J)\right]\right\},$$

Where $P'$ represents the rival of politician $P$. It is never optimal for a group to finance both politicians, as this would not maximize utility. Instead, the group rewards the politician who offers the highest welfare gain.

### 3.5.c Platforms and contributions

If all groups contribute ($O^J=1$ for all $J$), the total contributions influence the popularity $\delta$, altering the probabilities of winning. The equilibrium platforms $q^A$ and $q^B$ still satisfy:

$$\sum_{J=1}^{3}\lambda^J\phi^J V'(q^*)=1,$$

But the contributions modify the effective popularity weights.If no group contributes ($O^J=0$ for all $J$), $\delta$ simplifies to $\tilde{\delta}$, and the equilibrium platforms revert to those derived in Problem 4. Contributions in equilibrium depend on the willingness of groups to finance candidates. The distribution of $\phi^J$ across groups affects the marginal impact of contributions.

### 3.5.d Uneven contributions

When some groups opt not to contribute ($O^J=0$ for some $J$), the remaining groups have greater influence over the election outcome. The equilibrium platforms $q^A$ and $q^B$ will shift toward the preferences of contributing groups, as their contributions increase the popularity impact $\delta$. This creates an incentive for groups with higher $\alpha^J$ or $\lambda^J$ to organize and contribute, as their influence on policy outcomes is amplified.