Midterm exam (Soluions)

CS 205: Discrete Structures I
Fall 2019

(Sample)

Total points: 100 + 30 (extra credit)

Duration: 1 hour

Name:

Section No.:

NetID:

INSTRUCTIONS:

1. There are 9 problems in all, 7 for regular credit and 2 for extra credit. You have 1 hour to attempt these problems. The points for each problem are specified along with the problem statement. To get full points for a problem, you must give details for all the steps involved in solving the problem AND arrive at the correct answer. Giving partial details or arriving at the wrong answer will result in a partial score.

2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork.

3. You may refer to physical copies of any books or lecture notes you want to during the exam. However, the use of any electronic devices will lead to cancellation of your exam and a zero score, with the possibility of the authorities getting involved.

4. Make sure you write your name, NetID, and section number in the space provided above.

5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the exam, and might even be reported to the authorities!

Regular credit: 100 pts, 7 problems

Problem 1. [10 + 3 = 13 pts]
Write the truth table for the compound proposition:

$$((\neg r) \wedge (p \to q) \wedge (q \to r)) \to (\neg p).$$

What do you call this kind of proposition?

Solution: You can solve this problem in two different ways ways:

1. **(Approach 1)** Start out by writing the truth table for the given proposition. There will be 8 rows (since there are three variables) and the following columns:

    - the first column for the variable p,
    - the second one for q,
    - the third one for q,
    - one for $\neg r$,
    - one for $p \to q$,
    - one for $q \to r$,
    - one for $\neg p$,
    - one for $(\neg r) \wedge (p \to q) \wedge (q \to r)$,
    - and finally one for the given proposition: $((\neg r) \wedge (p \to q) \wedge (q \to r)) \to (\neg p)$.

    Once you are done writing the truth table, you will see that the values in the last column are all true, and this will help you answer the second part of the problem; this kind of proposition is called a tautology.
    Note that for this approach you MUST have all the columns mentioned above (and more if you need) in the truth table to show your work, otherwise if you just have the first three columns and the last column for the given proposition it might seem like you cheated!

2. **(Approach 2)** The second approach is much easier and shorter but requires you to use some of you knowledge about rules of inference and valid arguments.
    Let us consider the following argument with premises

    (a) $\neg r$
    (b) $p \to q$
    (c) $q \to r$

    and conclusion $\neg p$. Let us show that this argument is valid:

    (a) $\neg r$ (premise)
    (b) $p \to q$ (premise)
    (c) $q \to r$ (premise)

(d) $p \to r$ (using Hypothetical Syllogism on (b) and (c))

(e) $\neg p$ (using Modus Tollens on (d) and (a))

Since (e) is the conclusion we have successfully showed that the conclusion can be inferred from the premises and so the argument is valid.
Now recall that an argument with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid if and only if

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \to q$$

is a tautology. Since the argument considered above is valid it implies that the compound proposition

$$((\neg r) \wedge (p \to q) \wedge (q \to r)) \to (\neg p)$$

is a tautology. But this is exactly the proposition given in the problem and now we know it is a tautology. Thus, we can simply write its truth table out with 8 rows and 4 columns (one for p, one for q, one for r, and one for the given proposition) and easily fill in the table since we know that the last column contains all true values.
Note: if you follow this approach you don't need to provide intermediate columns in the truth table for propositions such as $\neg r$ or $\neg p$ since the proof of validity that we gave above already provides an explanation for how we ended up figuring out the truth table. BUT YOU STILL HAVE TO DRAW THE TRUTH TABLE.
As for the second part of the problem, we already mentioned that this proposition is a tautology.

Problem 2. [$5 \times 3 = 15$ pts]
State which of the following statements are True and which of them are False. Give very short explanations for your answers.

1. If $2 + 2 = 5$ then $1 < 2$.
2. If $1 < 2$ then $2 + 2 = 5$.
3. The product of two rational numbers is always rational.
4. $(p_1 \oplus p_2) \rightarrow (p_1 \vee p_2)$ is a tautology.
5. Assuming that the domain is natural numbers, $\forall x \, \exists y \, (x = y^2)$.

Solution:

1. True. Since $2 + 2 = 5$ is false, that makes the hypothesis of the conditional false and so the conditional is true.
2. False. Here $1 < 2$, the hypothesis of the conditional, is true, but the consequence (the proposition after the arrow in the conditional) is false, and this makes the conditional false.
3. True. If $\alpha = \frac{p}{q}$ and $\beta = \frac{r}{s}$ (such that $q \neq 0$ and $s \neq 0$) are any two rational numbers, then their product $\alpha\beta$ is $\frac{pr}{qs}$ and $qs \neq 0$, and so the product is always rational.
4. True. The hypothesis of this conditional is true if and only if exactly one of p_1 and p_2 is true. When exactly one of p_1 or p_2 is true, the consequence of this conditional $p_1 \vee p_2$ must be true. Thus, this means that whenever the hypothesis of the conditional is true the consequence is also true and so the proposition is a tautology.
5. False. The predicate formula translates to *"For every natural number x there is a natural number y such that $x = y^2$"*. Consider the natural number 5 and note that there is no natural number y such that $y^2 = 5$ and so this is a counterexample.

Problem 3. [15 pts]
Consider an argument with premises

- $(p \vee q) \to r$
- $(\neg p) \to (\neg a)$
- $b \vee a$
- $\neg b$

and conclusion r. Show that it is a valid argument. Give all steps and mention the rules of inference being used.

Solution:

1. $(p \vee q) \to r$ (premise)
2. $(\neg p) \to (\neg a)$ (premise)
3. $b \vee a$ (premise)
4. $\neg b$ (premise)
5. a (applying disjunctive syllogism on 3 and 4)
6. $\neg(\neg a)$ (this is equivalent to 5 via the double negation rule)
7. $\neg(\neg p)$ (using modus tollens on 2 and 6)
8. p (this is equivalent to 7 via the double negation rule)
9. $p \vee q$ (addition of q to 8)
10. r (modus ponnens on 1 and 9)

Since 10 is the conclusion, this shows that we can derive the conclusion from the premises, and so the argument is valid.

Problem 4. [15 pts]
Prove that, for all integers $n \geq 0$, $n(n+1)(n+2)$ is divisible by 3. You must give a formal proof with all steps.

Solution: Let $P(n)$ be the predicate

$$n(n+1)(n+2) \text{ is divisible by } 3.$$

We want to prove $\forall n \geq 0\ P(n)$. We will use weak induction to prove this.
Base case: We will show that $P(0)$ is true. For $n = 0$,

$$n(n+1)(n+2) = 0$$

which is divisible by 3 and so $P(0)$ is true.
Induction step: We want to prove that

$$\forall n \geq 0\ P(n) \to P(n+1).$$

Let $k \geq 0$ be any natural number. It suffices to show that

$$P(k) \to P(k+1).$$

Assume that $P(k)$ is true, i.e. $k(k+1)(k+2)$ is divisible by 3 and so there is some integer m such that

$$k(k+1)(k+2) = 3m \text{ (Induction hypothesis)}$$

We want to prove that $P(k+1)$ is true, i.e. $(k+1)(k+2)(k+3)$ is divisble by 3. We can write

$$\begin{aligned}
&(k+1)(k+2)(k+3) \\
&= (k+1)(k+2)k + (k+1)(k+2)3 \\
&= k(k+1)(k+2) + 3(k+1)(k+2) \\
&= 3m + 3(k+1)(k+2) \text{ (Using the induction hypothesis)} \\
&= 3(m + (k+1)(k+2)) \qquad (1)
\end{aligned}$$

It is clear from (1) that $(k+1)(k+2)(k+3)$ can be written as 3 times some integer (in this case the integer is $m + (k+1)(k+2)$), and so $(k+1)(k+2)(k+3)$ is divisible by 3. Thus, $P(k+1)$ is true and this completes the proof of the induction step.

Problem 5. [10 pts]
Prove that if x^{2019} is odd then x must be odd.

Solution: By contraposition, proving the given statement is equivalent to proving that if x is even then x^{2019} must also be even. We will prove the latter statement.
Let x be any even integer. Then there is some integer m such that $x = 2m$. This means that

$$x^{2019} = (2m)^{2019} = 2^{2019}m^{2019} = 2 \cdot (2^{2018}m^{2019}),$$

and so this means that x^{2019} can be written as 2 times some integer and hence it must be even. This completes the proof.

Problem 6. [15 pts]
Consider an argument with premises:

- $((\exists x\ P(x)) \wedge (\exists x\ Q(x))) \rightarrow p$
- $P(c)$ for some c
- $\neg p$

and conclusion

$$\forall x\ (\neg Q(x)).$$

Show that this argument is valid and give a proof thereof showing all the steps and the rules of inference used in each step.

Solution:

1. $((\exists x\ P(x)) \wedge (\exists x\ Q(x))) \rightarrow p$ (premise)
2. $P(c)$ for some c (premise)
3. $\neg p$ (premise)
4. $\neg((\exists x\ P(x)) \wedge (\exists x\ Q(x)))$ (applying modus tollens to 1 and 3)
5. $\neg(\exists x\ P(x)) \vee \neg(\exists x\ Q(x))$ (By applying De Morgan's rule on 4)
6. $\exists x\ P(x)$ (Applying Existential Generalization to 2)
7. $\neg(\neg(\exists x\ P(x)))$ (this is equivalent to 6 via double negation rule)
8. $\neg(\exists x\ Q(x))$ (By applying disjunctive syllogism on 5 and 7)
9. $\forall x\ \neg Q(x)$ (Applying De Morgan's rule for predicates to 8)

Since 9 is the conclusion, we have proved that the conclusion can be derived from the premises using rules of inference and so the argument is valid.

Problem 7. [8 + 9 pts]

In this problem you will prove that there are no positive integers $x, y, z \geq 1$ that satisfy the equation $x^3 + y^3 + z^3 = 28$.

1. (**8 pts**) First prove that if $x^3 + y^3 + z^3 = 28$ then $(x \geq 3) \vee (y \geq 3) \vee (z \geq 3)$.
2. (**9 pts**) Now using the statement proved in the previous part and the fact that $x, y, z \geq 1$ conclude that there are no integers $x, y, z \geq 1$ that satisfy the given equation.

Solution:

1. By contraposition, proving the given statement is equivalent to proving that if

   $$(x < 3) \wedge (y < 3) \wedge (z < 3)$$

   then

   $$x^3 + y^3 + z^3 \neq 28.$$

   We will focus on proving the latter. Let us assume that $(x < 3) \wedge (y < 3) \wedge (z < 3)$. Since x, y, z are integers, the condition $(x < 3) \wedge (y < 3) \wedge (z < 3)$ is equivalent to $(x \leq 2) \wedge (y \leq 2) \wedge (z \leq 3)$, and so

   $$\begin{aligned} x^3 + y^3 + z^3 &\leq 2^3 + 2^3 + 2^3 = 3(2^3) = 3 \cdot 8 = 24 \\ \implies x^3 + y^3 + z^3 &\neq 28. \end{aligned}$$

   This proves the statement.

2. Suppose there is an integer solution $x = \alpha,\ y = \beta,\ z = \gamma$ to the given equation such that $\alpha, \beta, \gamma \geq 1$. By part one of this problem, we know that since

   $$\alpha^3 + \beta^3 + \gamma^3 = 28,$$

   then either $\alpha \geq 3$ or $\beta \geq 3$ or $\gamma \geq 3$. Let us assume that $\alpha \geq 3$ and show that this leads to a contradiction. If $\alpha \geq 3$, then

   $$\alpha^3 \geq 27 \tag{2}$$

   Also since $\beta, \gamma \geq 1$, we have that

   $$\beta^3 \geq 1 \tag{3}$$
   $$\gamma^3 \geq 1 \tag{4}$$

   Combining $(2), (3)$ and (4) implies that

   $$\alpha^3 + \beta^3 + \gamma^3 \geq 27 + 1 + 1 = 29,$$

   which contradicts the fact that $x = \alpha,\ y = \beta,\ z = \gamma$ is a solution to

   $$x^3 + y^3 + z^3 = 28.$$

   Thus, $\alpha \geq 3$ is not possible. Similarly, by a symmetrical argument, we can show neither $\beta \geq 3$ nor $\gamma \geq 3$ are possible, and so the statement

   $$(\alpha \geq 3) \vee (\beta \geq 3) \vee (\gamma \geq 3)$$

is false. But this contradicts part 1 of the problem which says that if α, β, γ is a solution to the given equation then one of α, β or γ must be at least 3. This just means that our original assumption that there is an integer solution $x = \alpha,\ y = \beta,\ z = \gamma$ such that $\alpha, \beta, \gamma \geq 1$ must be wrong, and hence there is no solution satisfying the given constraints.

Extra credit: 30 pts, 2 problems

Problem 8. [15 pts]
Consider a rectangular board of size 60 inches $\times$ 10 inches. Suppose that 745 darts are thrown at the board and all of them end up landing on the board. Prove that there must be two darts that end up landing within 1.5 inches of each other. You may assume that $\sqrt{2} = 1.414$.

Solution: It is easy to see that we can divide the entire surface area of the board into 600 squares of dimension 1 inch $\times$ 1 inch. This can be done by imagining equally-spaced (1 inch apart) horizontal and vertical lines on the board. (**Note: You don't need to go into too much detail here. If you just say a few words to explain this that will be enough!**)
We say that a dart lands in a square if the dart either lands on the boundary of the square or in the interior of the square. Since there are 745 darts that landed on the board, and there are 600 squares, it must be the case that there is a square in which at least two darts landed. We can prove this fact by contradiction: if at most 1 dart landed in every square then the maximum number of darts that could have landed on the board would be at most 600 which is much less than the number of darts we actually expect on the board, i.e. 745.
Now consider some square in which two darts landed (if more than two darts landed, just focus on any two of them). Let the points at which the two darts landed be p_1 and p_2. Since p_1 and p_2 are within the square (including the boundary), the farthest apart they can be is if they are placed at the end points of a diagonal of the square. Since the length of the diagonal of a 1 inch $\times$ 1 inch square is $\sqrt{2}$ inches, this means that the distance between the landing points p_1 and p_2 of the two darts can be at most $\sqrt{2} = 1.414 < 1.5$ inches.
Thus, we have proved that there are two darts that land within 1.5 inches of each other.

Problem 9. [15 pts]
Prove that for every pair of rational numbers a and b such that $0 < a < b < 1$ there is an irrational number c such that $a < c < b$.

Solution: Since a and b are rational, $b - a$ must be rational. Furthermore, we also have that $b - a > 0$. Consider the number c given by

$$c = a + \frac{b-a}{\sqrt{2}}.$$

We will first prove that c is an irrational number. This is easy to see; since $\sqrt{2}$ is irrational and $b - a$ is a rational number not equal to zero, it must be the case that $\frac{b-a}{\sqrt{2}}$ is irrational. Then, since a is rational and $\frac{b-a}{\sqrt{2}}$ is irrational, their sum

$$c = a + \frac{b-a}{\sqrt{2}}$$

must be irrational, and so c is irrational.

Next, we will show that $a < c < b$. First, let us observe that $a < c$. To see this, note that since $b - a > 0$,

$$\frac{b-a}{\sqrt{2}} > 0$$

and so

$$c = a + \frac{b-a}{\sqrt{2}} > a.$$

To see that $c < b$, note that

$$\begin{aligned}
& c < b && (5)\\
\Leftrightarrow\quad & a + \frac{b-a}{\sqrt{2}} < b && (6)\\
\Leftrightarrow\quad & \frac{b-a}{\sqrt{2}} - (b-a) < 0 && (7)\\
\Leftrightarrow\quad & (b-a)\left(\frac{1}{\sqrt{2}} - 1\right) < 0 && (8)
\end{aligned}$$

Now observe that since $(b - a) > 0$ and $\frac{1}{\sqrt{2}} < 1$, (8) is true, and so because of the chain of equivalences between $(5), (6), (7)$ and (8), even (5) must be true and so $c < b$.
This finishes the proof.