Midterm exam (Solutions)

CS 205: Discrete Structures I
Fall 2019

Total points: 100 (regular credit) + 40 (extra credit)

Duration: 1 hour

Name:

Section No.:

NetID:

INSTRUCTIONS:

1. There are 7 problems in all, 5 for regular credit and 2 for extra credit. You have 1 hour to attempt these problems. The points for each problem are specified along with the problem statement. To get full points for a problem, you must give details for all the steps involved in solving the problem AND arrive at the correct answer. Giving partial details or arriving at the wrong answer will result in a partial score.

2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork.

3. You may refer to physical copies of any books or lecture notes you want to during the exam. However, the use of any electronic devices will lead to cancellation of your exam and a zero score, with the possibility of the authorities getting involved.

4. Make sure you write your name, NetID, and section number in the space provided above.

5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the exam, and might even be reported to the authorities!

Regular credit: 100 pts, 7 problems

Problem 1. [20 pts]
Show that the following compound proposition is a tautology.

$$((r \to (q \to p)) \wedge (q \wedge r)) \to p.$$

If you choose to use truth tables be sure to include columns for all intermediate propositions that occur in the expression (i.e., columns for $(q \wedge r)$, $(q \to p)$, $r \to (q \to p)$, etc.)

Solution: There are two approaches to this problem:

1. **Approach I:** Draw the truth table with 8 rows and 7 columns, and show that the last column has all entries equal to true.

2. **Approach II:** The second approach is to realize that there is a valid argument embedded in this proposition. In particular, consider the argument with premises

   - $r \to (q \to p)$
   - $q \wedge r$

   and conclusion p. We can easily prove that this argument is valid:

   (a) $r \to (q \to p)$ (premise)
   (b) $q \wedge r$ (premise)
   (c) q (simplification on (b))
   (d) r (simplification on (b))
   (e) $q \to p$ (modus ponnens on (a) and (d))
   (f) p (modus ponnens on (e) and (c))

   Since the argument is valid, the proposition

   $$((r \to (q \to p)) \wedge (q \wedge r)) \to p$$

   is a tautology.

Problem 2. [$5 \times 4 = 20$ pts]
State which of the following statements are true and which of them are false. Give very short explanations for your answers.

1. If 2.5 is irrational then 72 is an odd number.
2. The sum of any three even numbers must be even.
3. $p \wedge (\neg p) \equiv \mathbf{F}$ only if $2 \times 2 = 5$.
4. $(p_1 \vee p_2) \rightarrow (p_1 \wedge p_2)$ is a tautology.
5. Assuming that the domain is the set of all integers, the expression $\forall x \; \exists y \; (x + y = 5)$ is true.

Solution:

1. True. $2.5 = \frac{5}{2}$ is rational so the hypothesis of the conditional is false and so the conditional must be true.
2. True. Consider any three even numbers $2k_1, 2k_2, 2k_3$, where k_1, k_2, k_3 are integers, and their sum is $2k_1 + 2k_2 + 2k_3 = 2(k_1 + k_2 + k_3)$. Since $k_1 + k_2 + k_3$ is an integer, it's clear that the sum is even.
3. False. The statement is equivalent to

   $$\textit{If } p \wedge (\neg p) \equiv \boldsymbol{F} \textit{ then } 2 \times 2 = 5.$$

   $p \wedge (\neg p) \equiv \mathbf{F}$ is true (you should know this by now!), so the hypothesis of the conditional is true. However, the consequence $2 \times 2 = 5$ is obviously false and so the conditional is false.

4. False. Consider the case when p_1 is true and p_2 is false. Then $p_1 \vee p_2$ is true but $p_1 \wedge p_2$ is false and so the conditional evaluates to false.
5. True. The statement translates to

   For every integer x there is an integer y such that $x + y = 5$.

   This is true since for a given integer x you can consider the integer $y = 5 - x$ (this is clearly integer since x and 5 are integers) and then $x + y = 5$.

More space for Problem 2:

Problem 3. [20 pts]
Consider an argument with premises

- $(p \to q) \land (r \lor s)$
- $\neg(q \lor s)$

and conclusion $r \land \neg p$. Show that it is a valid argument. Give all steps and mention the rules of inference being used.

Solution: Consider the following proof of validity:

1. $(p \to q) \land (r \lor s)$ (premise)
2. $\neg(q \lor s)$ (premise)
3. $\neg q \land \neg s$ (De Morgan's on 2)
4. $p \to q$ (Simplification on 1)
5. $\neg q$ (Simplification on 3)
6. $\neg p$ (Modus Tollens on 4 and 5)
7. $r \lor s$ (Simplification on 1)
8. $\neg s$ (Simplfication on 3)
9. r (Disjunctive Syllogism on 7 and 8)
10. $r \land \neg p$ (Conjunction of 9 and 6)

Since 10 is the conclusion, we have proved that the argument is valid.

Problem 4. [20 pts]
Prove that for all integers $n \geq 0$ $n^3 - n$ is divisible by 6. You must give a formal proof with all steps. Feel free to use the following facts if you need to:

$$\forall n \geq 0,\ n(n+1) = n^2 + n \textit{ is even.}$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b).$$

Solution: Let $P(n)$ be the predicate "$n^3 - n$ *is divisible by* 6". We want to prove that

$$\forall n \geq 0\ P(n).$$

Base case: We will show that $P(0)$ is true. This is easy because for $n = 0$, $n^3 - n = 0$, and 0 is divisible by 6.
Induction step: We want to show that $\forall n \geq 0\ P(n) \to P(n+1)$. Let $k \geq 0$ be any integer. It suffices to prove that $P(k) \to P(k+1)$. Assume that $P(k)$ is true, i.e.

$$k^3 - k \textit{ is divisible by } 6 \text{ (Induction hypothesis).}$$

We want to prove that $P(k+1)$ is true, i.e. $(k+1)^3 - (k+1)$ is also divisible by 6. We can write

$$\begin{aligned}
&(k+1)^3 - (k+1) \\
&= k^3 + 1 + 3k(1+k) - k - 1 \text{ (Using the given formula for } (a+b)^3\text{)} \\
&= (k^3 - k) + 3k(1+k) \text{ (By rearranging and canceling terms)} \\
&= (k^3 - k) + 3(2m) \text{ (Using the given fact that for all } n \geq 0\ n(n+1) \text{ is even)} \\
&= (k^3 - k) + 6m
\end{aligned}$$

Consider the last expression. The first term $k^3 - k$ is divible by 6 because of the induction hypothesis assumed earlier, and the second term is obviously divible by 6. Thus, their sum is also divisible by 6, and so $(k+1)^3 - (k+1)$ is divisible by 6, and $P(k+1)$ is true. This completes the induction step.

Problem 5. [20 pts]
Prove that if α is irrational then $\frac{1+\alpha}{1-\alpha}$ must also be irrational.

Solution: This can be proved using proof by contradiction or proof by contraposition. I will show the proof by contradiction here.
Let us assume that α is irrational but
$$\frac{1+\alpha}{1-\alpha}$$
is not, i.e. the latter is rational. This means that there are integers p and q ($q \neq 0$) such that
$$\frac{1+\alpha}{1-\alpha} = \frac{p}{q}$$

$$\begin{aligned}
&\implies (1+\alpha)q = (1-\alpha)p \\
&\implies q + \alpha q = p - \alpha p \\
&\implies \alpha \cdot (q+p) = p - q \qquad (1)
\end{aligned}$$

Let's analyze expression (1). There are two possible cases, both of which lead to contradiction:

- **Case 1: ($p + q = 0$)** If $p + q$ is 0, then the LHS of (1) becomes 0, and then we get
$$p = q.$$
The only way both $p = q$ and $p + q = 0$ are possible is if $p = q = 0$. However, $q = 0$ is a contradiction to the fact that
$$\frac{1+\alpha}{1-\alpha} = \frac{p}{q}$$
and $q \neq 0$ which itself follows from our assumption that $\frac{1+\alpha}{1-\alpha}$ is rational.

- **Case 2: ($p + q \neq 0$)** If $p + q$ is not 0, then we can write
$$\alpha = \frac{p-q}{p+q},$$
and so α is rational thus contradicting out assumption that α was irrational.

Since both cases lead to a contradiction, and no other cases are possible, it means our original assumption that $\frac{1+\alpha}{1-\alpha}$ can be rational when α is irrational is false. Hence, whenever α is irrational, it must follow that $\frac{1+\alpha}{1-\alpha}$ is also irrational.

Extra credit: 40 pts, 2 problems

Problem 8. [20 pts]
Assuming that both $\sqrt{2}$ and $\sqrt{3}$ are irrational, prove that both $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ must also be irrational.

Solution: Let us assume that $\sqrt{2}$ and $\sqrt{3}$ are irrational. There are 4 possible cases:

1. Both $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are rational.
2. $\sqrt{2}+\sqrt{3}$ is rational but $\sqrt{2}-\sqrt{3}$ is irrational.
3. $\sqrt{2}+\sqrt{3}$ is irrational but $\sqrt{2}-\sqrt{3}$ is rational.
4. Both $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are irrational.

We want to show that cases 1, 2 and 3 are not possible since they lead to contradictions and so, since these 4 cases cover all possibilities, it must be that case 4 is true. Let's prove that cases 1,2 and 3 lead to contradictions:

- **Case 1: (Both $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are rational)** This means that there are rational numbers a and b such that
$$\sqrt{2}+\sqrt{3}=a$$
$$\sqrt{2}-\sqrt{3}=b,$$
and adding the two we get $2\sqrt{2}=a+b$ and so $\sqrt{2}=\frac{a+b}{2}$ which would mean that $\sqrt{2}$ is rational which is a contradiction to the fact that $\sqrt{2}$ is irrational.

- **Case 2: ($\sqrt{2}+\sqrt{3}$ is rational but $\sqrt{2}-\sqrt{3}$ is irrational)** This means that there are integers p and q, such that $q \neq 0$ such that
$$\sqrt{2}+\sqrt{3}=\frac{p}{q}.$$
In fact, since the LHS is positive, even $p>0$. Now consider the product
$$(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})=(\sqrt{2})^2-(\sqrt{3})^3=2-3=-1,$$
$$\implies (\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})=-1,$$
$$\implies \frac{p}{q}(\sqrt{2}-\sqrt{3})=-1,$$
$$\implies (\sqrt{2}-\sqrt{3})=-\frac{q}{p},$$
which would mean that $\sqrt{2}-\sqrt{3}$ is rational, a contradiction to our assumption for this case.

- **Case 3: ($\sqrt{2}+\sqrt{3}$ is irrational but $\sqrt{2}-\sqrt{3}$ is rational)** Symmetric to case 2, can be shown to lead to a contradiction using similar proof as that of case 2.

This means that cases $1,2$ and 3 all lead to contradiction and so the only possibility is that case 4 is true, i.e. both $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$ are irrational.

Problem 9. [20 pts]
Use strong induction to prove that every positive integer $n \geq 1$ can be written as

$$n = 2^k \ell,$$

where $k \geq 0$ is some integer and ℓ is an odd integer.

Solution: Let $P(n)$ be the predicate

There is an integer $k \geq 0$ and an odd integer ℓ such that $n = 2^k \ell$.

We want to show that $\forall n \geq 1\ P(n)$. We will use strong induction.
Base case: 1 can be written as

$$1 = 2^0 \cdot 1,$$

and so $P(1)$ is true.
Induction step: We want to prove that

$$\forall n \geq 1,\ (P(1) \wedge P(2) \wedge \ldots \wedge P(n)) \rightarrow P(n+1).$$

Let $n \geq 1$ be any integer. It suffices to show that

$$(P(1) \wedge P(2) \wedge \ldots \wedge P(n)) \rightarrow P(n+1).$$

Let us assume that $P(1) \wedge P(2) \wedge \ldots \wedge P(n)$ is true, i.e. every integer m such that $1 \leq m \leq n$ can be written as

$$m = 2^k \ell,$$

for some integer $k \geq 0$ and an odd integer ℓ. We want to prove that $P(n+1)$ is true. We will use a case analysis:

- **Case I ($n+1$ is odd):** If $n+1$ is odd then,

  $$n + 1 = 2^k \cdot \ell,$$

  where $k = 0$ and $\ell = n + 1$ is an odd integer. Thus, in this case $P(n+1)$ is true.

- **Case II ($n+1$ is even):** If $n+1$ is even, then there is a positive integer n' such that

  $$n + 1 = 2n',$$

  Since

  $$n' = \frac{n+1}{2}$$

  and $n \geq 1$, it follows that $n' \geq 1$. Also it must be the case that $n' \leq n$ because otherwise we would have

  $$\begin{aligned} n' &= \frac{n+1}{2} > n \\ &\implies n + 1 > 2n \\ &\implies n < \frac{1}{2}, \end{aligned}$$

which is not possible since $n \geq 1$. Thus we can conclude that

$$1 \leq n' \leq n.$$

This means we can use the strong induction hypothesis, and say that there is an integer $k' \geq 0$ and an odd integer ℓ' such that

$$n' = 2^{k'}\ell',$$

and so

$$n + 1 = 2n' = 2(2^{k'}\ell') = 2^{k'+1}\ell'.$$

Thus, $n + 1$ can be written as $2^k\ell$ where $k = k' + 1 \geq 0$ is an integer and $\ell = \ell'$ is an odd integer and this proves $P(k + 1)$.

This completes the proof of the induction step.