

Homework 4 (Solutions)

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Wednesday, Dec 11th, 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 + 5 + 10 = 25 pts]

Let n be an **odd** positive integer. In this problem, we will use P_n to denote the set $\text{pow}(\{1, 2, \dots, n\})$, i.e. the powerset of $\{1, 2, \dots, n\}$. Additionally, we define O_n as follows

$$O_n := \{S \in P_n \mid |S| \text{ is odd}\},$$

i.e. the set of all odd size subsets of $\{1, 2, \dots, n\}$, and E_n as

$$E_n := \{S \in P_n \mid |S| \text{ is even}\},$$

i.e. the set of all even size subsets of $\{1, 2, \dots, n\}$.

1. Consider the following function $f : O_n \rightarrow E_n$:

$$f(A) = \bar{A},$$

for any $A \in O_n$. Prove that f is surjective using the fact that n is odd.

Solution: Let $B \in E_n$ be an arbitrary set. Then clearly $B \subseteq \{1, \dots, n\}$ and also $|B|$ is even. Let $A = \bar{B}$. So by definition A contains all those elements from $\{1, 2, \dots, n\}$ that are not in B , and so

$$|A| = n - |B|.$$

Since n is odd and $|B|$ is even, $n - |B|$ must be odd (since $\text{odd} - \text{even} = \text{odd}$), and so $|A|$ is odd. Thus, $A \in O_n$, and is in the domain of f .

Let us now see what $f(A)$ is:

$$f(A) = \bar{A} = B,$$

and so we have shown that there is an element in the domain of f (namely, the element A) that maps/transforms into the element $B \in E_n$. Since B was an arbitrary element of the codomain of f , we can conclude that f is surjective.

2. Prove that the function f defined in the previous part is injective.

Hint: Problem 1 on HW 3.

Solution: Let $A_1, A_2 \in O_n$ such that $A_1 \neq A_2$, i.e. A_1 and A_2 are distinct elements of the domain of f . To show f is injective, it suffices to show that $f(A_1) \neq f(A_2)$, i.e. prove that

$$\bar{A}_1 \neq \bar{A}_2$$

since $f(A_1) = \bar{A}_1$ and $f(A_2) = \bar{A}_2$.

Note that A_1 and A_2 are themselves sets, and in particular, they are subsets of $\{1, \dots, n\}$. Let $U = \{1, \dots, n\}$ be the universe. Problem 1 on HW 3 tells us that for any universe U and $X, Y \subseteq U$ it is the case that

$$X = Y \leftrightarrow \bar{X} = \bar{Y}.$$

Let $X = A_1$ and $Y = A_2$. Then we know that $X \neq Y$, and so from the fact we just stated above, it must be the case that

$$\begin{aligned}\bar{X} &\neq \bar{Y} \\ \implies \bar{A}_1 &\neq \bar{A}_2.\end{aligned}$$

This proves that f is injective.

3. Now conclude that

$$|O_n| = |E_n| = 2^{n-1}.$$

Hint: First prove $|O_n| = |E_n|$, then use the fact that $E_n \cup O_n = P_n$, and that E_n and O_n are disjoint.

Solution: Since $f : O_n \rightarrow E_n$ is both injective and surjective, it must be bijective. Since f is bijective, we can conclude that

$$|E_n| = |O_n|.$$

Note that since every subset $S \subseteq \{1, \dots, n\}$ is either of odd size or of even size, we have that

$$\forall S \in P_n, S \in O_n \vee S \in E_n.$$

In other words, by the definition of union of two sets,

$$P_n = E_n \cup O_n.$$

Also, there is no set S that is of both even size and odd size, and so

$$E_n \cap O_n = \emptyset.$$

This means that

$$|P_n| = |E_n \cup O_n| = |E_n| + |O_n| - |E_n \cap O_n| = |E_n| + |O_n|.$$

Recall that $|P_n| = 2^n$ (the cardinality of the power of set of a set of size n is 2^n). So

$$|E_n| + |O_n| = |P_n| = 2^n.$$

We also concluded earlier that $|E_n| = |O_n|$. Let $|E_n| = |O_n| = m$. Then

$$\begin{aligned}m + m &= 2^n \\ \implies m &= \frac{2^n}{2} = 2^{n-1} \\ |O_n| &= |E_n| = m = 2^{n-1}.\end{aligned}$$

Problem 2. [15 + 5 pts = 20 pts]

Answer the following questions. Provide all essential details in your solution:

1. Let A, B and C be arbitrary sets, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijective functions. Prove that $(g \circ f)(x) = g(f(x))$ is also bijective.

Solution: Let us first prove that $g \circ f : A \rightarrow C$ is injective. Let $a_1, a_2 \in A$ be distinct points in the domain, i.e. $a_1 \neq a_2$. Then, since f is injective, it follows that

$$f(a_1) \neq f(a_2).$$

Note that $f(a_1), f(a_2)$ are points in the set B , and we just concluded that they are distinct. Again, using the fact that g is injective, we can conclude that

$$g(f(a_1)) \neq g(f(a_2)).$$

This means that for distinct points a_1, a_2 in the domain A ,

$$(g \circ f)(a_1) = g(f(a_1)) \neq g(f(a_2)) = (g \circ f)(a_2),$$

and so $g \circ f$ is injective.

Let $c \in C$ be an arbitrary point in the codomain of $g \circ f$. Consider the function g . Since g is surjective, there is a $b \in B$ such that

$$g(b) = c.$$

Also, since f is surjective and there $b \in B$ is in the codomain of f , we have that there is a point $a \in A$ such that

$$f(a) = b.$$

It now follows that

$$(g \circ f)(a) = g(f(a)) = g(b) = c,$$

and so we have shown a point $a \in A$, i.e. in the domain of $(g \circ f)$ such that $(g \circ f)(a) = c$, and since $c \in C$ was arbitrary, this demonstrates that $g \circ f$ is surjective.

2. Let Y be a countably infinite set and X be another set such that there is bijection between X and Y . Prove that X must also be countably infinite by showing that there is a bijection between X and \mathbb{N} .

Solution: Since Y is countably infinite there exists some bijection $g : Y \rightarrow \mathbb{N}$. Now let us assume that there is a bijection between X and Y (this fact is given to us in the problem statement)

$$f : X \rightarrow Y.$$

From part 1, we can conclude that

$$(g \circ f) : X \rightarrow \mathbb{N}$$

is a bijection since both g and f are both bijections. Now, $g \circ f$ is a bijection between X and \mathbb{N} and so X is countably infinite.

Problem 3. [4 + 1 + 10 + 12 + 13 = 40 pts]

Consider the following infinite sequence $a_0, a_1, \dots, a_n, \dots$:

- $a_0 = 1$
- $a_n = a_{n-1} + \frac{(-1)^n}{n+1}$ for $n \geq 1$.

1. Compute the first five terms of this sequence, i.e. a_0, a_1, a_2, a_3, a_4 . Show your calculations.

Solution:

$$\begin{aligned}a_0 &= 1 \\a_1 &= a_0 + \frac{(-1)^1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \\a_2 &= a_1 + \frac{(-1)^2}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\a_3 &= a_2 + \frac{(-1)^3}{4} = \frac{5}{6} - \frac{1}{4} = \frac{7}{12} \\a_4 &= a_3 + \frac{(-1)^4}{5} = \frac{7}{12} + \frac{1}{5} = \frac{47}{60}\end{aligned}$$

2. Over the next few parts we will prove that $a_n \leq 1$ for all $n \geq 0$ using strong induction. Let $P(n)$ denote the predicate

$$a_n \leq 1.$$

Our goal is to show that $\forall n \geq 0 \ P(n)$.

We will have two base cases: $n = 0$ and $n = 1$. Argue that the base cases are true.

Solution: From Part 1, we can see that

$$a_0 = 1 \leq 1$$

$$a_1 = \frac{1}{2} \leq 1$$

and so $P(0)$ and $P(1)$ are true.

3. We will now prove that

$$\forall n \geq 1 \ (P(0) \wedge \dots \wedge P(n)) \rightarrow P(n+1).$$

Let $k \geq 1$ be an arbitrary integer. It suffices to show that

$$P(0) \wedge \dots \wedge P(k) \rightarrow P(k+1).$$

Let us assume that $P(0), P(1), \dots, P(k)$ are all true, i.e. $a_j \leq 1$ for all $j \in \{0, 1, \dots, k\}$. This is our induction hypothesis.

We now want to show that $P(k+1)$ is true. Prove that $P(k+1)$ is true if k is even

using the fact that $P(k)$ is true.

Solution: Since $P(k)$ is true,

$$a_k \leq 1.$$

Since k is even, we have that $k+1$ is odd and so $(-1)^{k+1} = -1$. Since $k \geq 0$, $k+1 \geq 1$ and so we can use the recurrence for $n = k+1$ to write

$$a_{k+1} = a_k + \frac{(-1)^{k+1}}{k+2} = a_k - \frac{1}{k+2}.$$

Since $a_k \leq 1$, it is clear from the above expression that $a_{k+1} < 1$. This is because we are subtracting a positive quantity from a_k to get a_{k+1} . This proves $P(k+1)$ for the case when k is even.

4. Now consider the case when k is odd. Since $k \geq 1$, we can conclude that $k-1 \geq 0$ and so we know from the induction hypothesis that $P(k-1)$ is true, i.e. $a_{k-1} \leq 1$. Write a_{k+1} in terms of a_{k-1} using the fact that $(-1)^k = -1$ and $(-1)^{k+1} = 1$ (since k is odd).

Solution: Since $k \geq 1$, $k+1 \geq 2$, and hence we can use the recurrence for $n = k+1$ to write

$$a_{k+1} = a_k + \frac{(-1)^{k+1}}{k+2} = a_k + \frac{1}{k+2},$$

where the second equation follows from the fact that $k+1$ is even and so $(-1)^{k+1} = 1$.

Also, $k \geq 1$, and so we can again apply the recurrence for $n = k$ to get

$$a_k = a_{k-1} + \frac{(-1)^k}{k+1} = a_{k-1} - \frac{1}{k+1},$$

where the last equation follows from the fact that k is odd and so $(-1)^k = -1$. Combining the two equations we obtained above, we can write a_{k+1} in terms of a_{k-1}

$$a_{k+1} = a_{k-1} - \frac{1}{k+1} + \frac{1}{k+2}.$$

5. Use the expression obtained in the previous part along with the fact that $a_{k-1} \leq 1$ to conclude that $a_{k+1} \leq 1$ and so $P(k+1)$ is true.

Solution: Since $k \geq 1$, we know that $k-1 \geq 0$ and so by the induction hypothesis $P(k-1)$ is true, and so

$$a_{k-1} \leq 1.$$

From the previous part, we know that

$$\begin{aligned} a_{k+1} &= a_{k-1} - \frac{1}{k+1} + \frac{1}{k+2} = a_{k-1} - \frac{k+2 - (k+1)}{(k+1)(k+2)} \\ &\implies a_{k+1} = a_{k-1} - \frac{1}{(k+1)(k+2)} \end{aligned}$$

Since $a_{k-1} \leq 1$ and we are subtracting a positive quantity from a_{k-1} to obtain a_{k+1} it must be the case that $a_{k+1} < 1$ and so this proves that $P(k+1)$ is true when k is odd.

Problem 4. [5 + 10 = 15 pts]

Let S denote the set of all possible *finite* binary strings, i.e. strings of finite length made up of only 0s and 1s, and no other characters. E.g., 010100100001 is a finite binary string but 100ff101 is not because it contains characters other than 0, 1.

1. Give an informal proof arguing why this set should be countable. Even though the language of your proof can be informal, it must clearly explain the reasons why you think the set should be countable.

Hint: Try to argue that it is possible to arrange the elements of S into an infinite sequence.

Solution: We will provide a procedure that prints every finite binary string and this will prove that the set of finite strings is countably infinite:

- We will have a main infinite loop with variable i that starts at 0 and then increments by 1 every time, iterating over all natural numbers.
- Inside the main loop, we will print all binary strings of length i exactly once. The number of binary strings of length i is finite and it's easy to write a procedure to print all of them once.

It's easy to see that the above procedure will print every finite binary string exactly once and also the procedure only prints finite binary strings. This implies that the set of finite binary strings is countable.

2. Now assume that we have proved that S is indeed countable. Use this fact to prove (formally, this time) that the following set, denoted by T , is also countable:

$$T = \{X \subseteq \mathbb{N} \mid |X| \text{ is finite and is a prime number}\}.$$

Hint: You might want to first argue that the set

$$\{X \subseteq \mathbb{N} \mid |X| \text{ is finite}\}$$

is countable.

Solution: Let

$$T' = \{X \subseteq \mathbb{N} \mid |X| \text{ is finite}\}.$$

It suffices to prove that T' is countably infinite because then since T is a subset of T' it will also be countably infinite. So we will focus on proving that T' is countably infinite.

Let S' be the set of finite binary strings that end with a '1'. Let's also add the string '0' to S . We will show that there is a bijection $f : T' \rightarrow S'$. f transforms a finite subset $X \subseteq \mathbb{N}$ into a finite binary string $f(X)$ as follows:

- f transforms \emptyset into the string '0'.

- Now suppose $X \neq \emptyset$. Let n be the largest number occurring in the set X . Then the length of the finite binary string $f(X)$ that X is transformed is $n + 1$. The bits of $f(X)$ are determined as follows: for every $0 \leq i \leq n$, if the number $i \in X$ then we set the $(i + 1)^{th}$ bit of $f(X)$ from the left to '1' otherwise we set it to '0'.
- For example, if $X = \{0, 3, 5\}$. Then since 5 is the largest number, the length of $f(X)$ is 6. Since $0 \in X$, we set the 1^{st} bit of $f(X)$ from the left to equal to 1. However, since 1 and 2 are not part of the X , we will set the 2^{nd} and 3^{rd} bit of $f(X)$ from the left to 0. Since $3 \in X$ we will set the 4^{th} bit of $f(X)$ to 1. Finally we set the 5^{th} bit of $f(X)$ equal to 0 (since $4 \notin X$), and the last bit of $f(X)$ is set to 1 since $5 \in X$. Thus,

$$f(X) = 100101.$$

It's not hard to see that f either outputs '0' (in the case when the input is the empty set), or always outputs a string that ends with a 1. It's also not hard to convince yourself that f is injective: if we have two distinct sets X_1, X_2 such that the maximum value in X_1 is different from that in X_2 then clearly $f(X_1) \neq f(X_2)$ simply because their lengths will not match, and otherwise if their maximum value is the same, there has to be some element i that is one set but not the other and that will lead to the $(i + 1)^{th}$ bit in $f(X_1)$ and $f(X_2)$ being different.

Similarly, it's not hard to see that f is surjective. The codomain S' of f contains '0' and only those binary strings that end up with a '1'. We know that $f(\emptyset) = '0'$, and given some string x that ends with a 1, we can just reverse the above procedure to transform the string x back into a finite subset X of \mathbb{N} such that $f(X) = x$.

This proves that f is a bijection and so S' and T' have the same cardinality. Also note that $S' \subset S$ and so S' is countably infinite (since S is countably infinite by part 1), which in turn implies that T' is countably infinite. This completes the proof.