Homework 1 (Solutions)

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Monday, Oct 21st 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!

2. Make sure you write your name, NetID, and Section No. in the space provided above.

3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!

4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 pts]

For what values of n is the proposition $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow (p_1 \oplus p_2 \oplus \ldots \oplus p_n)$ a tautology? Provide an explanation for your answer.

Hint: Recall that $\oplus$ satisfies associativity and so the order in which you evaluate doesn't matter! Also try writing down the truth table for $(p_1 \oplus p_2 \oplus \ldots \oplus p_n)$ for small values of n and see if you can observe a pattern!

Observe that if any of the n variables $p_1, p_2, \ldots, p_n$ is False, then the hypothesis (the left side) of the conditional becomes False because it is a logical AND of all those variables. In such cases, the conditional will evaluate to True since a conditional is True if the hypothesis (left side) is false irrespective of the truth value of the consequence (the right side).

The only remaining case is when all n variables are True. In this case, the hypothesis (left side of conditional) is True, and thus the conditional is True if and only if the consequence is True. Thus, we want to find the values of n for which the consequence $p_1 \oplus p_2 \oplus \ldots \oplus p_n$ is True whenever all of $p_1, p_2, \ldots, p_n$ are True. In other words, we want to understand for what values of n does the expression

$$True \oplus True \oplus \ldots \text{ (n times) } \cdots \oplus True$$

becomes True.

Let's work out the result for a few values of n. For $n = 1$, obviously the result would just be $True$. For $n = 2$, we would have $True \oplus True$ which by the definition of the XOR logical operator, is False. For $n = 3$, we have $True \oplus True \oplus True$. We can evaluate this left to right, in order, as follows (we can do this because of associativity of the XOR operator): the first two True values are XORed together and become False, and then we XOR the False with the last True, to get a True.

Similarly, you can observe that whenever n is even, the final result is False, whereas the final result is True when n is odd. Thus, for the case when $p_1, \ldots, p_n$ are all True, the only values of n for which the conditional is True are the odd values. This means that the given proposition is a tautology if and only if n is odd.

Problem 2. [3 parts, 10 pts + 10 pts + 10 pts]

Let $D(x, y)$ be the predicate defined on natural numbers x and y as follows: $D(x, y)$ is true whenever y divides x, otherwise it is false. Additionally, $D(x, 0)$ is false no matter what x is (since dividing by zero is a no-no!).

Let $P(x)$ be the predicate defined on natural numbers that is true if and only if x is a prime number.

1. Write $P(x)$ as a predicate formula involving quantifiers, logical connectives, and the predicate $D(x, y)$. Assume the domain to be natural numbers.

   Hint 1: n is prime if and only if the only numbers that divide it are 1 and n.

   Hint 2: You might have to use conditionals.

$$P(x) = \forall y \ (D(x, y) \rightarrow (y = 1 \ \lor y = x))$$

(English translation: *"For all y in the domain, if y divides x then either $y = 1$ or $y = x$"*, which is the exact condition for x being prime.)

2. Consider the proposition *"There are infinitely many prime numbers"*. Express the proposition as a predicate formula using quantifiers, logical connectives and the predicate $P(x)$. Assume the domain to be natural numbers.

Note that you don't need to use the answer from the previous part in this problem; you can write your answer in terms of $P(x)$.

Hint: For inspiration, consider the following game: you pick any natural number x (as large as you want) and I have to find a number y such that y is larger than x and is also a prime number (I win if I can find such a y, otherwise you win!). Now notice that the proposition is in some sense equivalent to saying that I can always win the game!

$$\forall x \; \exists y \; (P(y) \; \wedge \; (y > x))$$

3. Write the negation of the predicate formula obtained in part 2. Make sure you take the negation all the way in so that it sits right next to $P(x)$ in the final expression.

$$\exists x\ \forall y\ (\neg P(y)\ \lor\ (y \leq x))$$

Problem 3. [10 pts]
Consider the following compound proposition:

$$((\neg x_1) \wedge x_4 \wedge x_3 \wedge x_2) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4)).$$

Simplify it as much as possible using propositional equivalences. Show all the steps, and for every step mention the equivalence that you are using.

Hint: Use the distributive property in a smart way. I guarantee you that this problem can be solved in four or five steps and has a very clean solution if you approach it the right way!

$((\neg x_1) \wedge x_4 \wedge x_3 \wedge x_2) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4))$

Using commutavity in first expression,

$\equiv ((\neg x_1) \wedge x_2 \wedge x_3 \wedge x_4) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4))$

Applying distributive law on first two expressions,

$\equiv (((\neg x_1) \vee x_1) \wedge (x_2 \wedge x_3 \wedge x_4)) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4))$

Using $p \vee \neg p \equiv True$,

$\equiv ((True) \wedge (x_2 \wedge x_3 \wedge x_4)) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4))$

Using $True \wedge p \equiv p$,

$\equiv (x_2 \wedge x_3 \wedge x_4) \vee \neg((\neg x_2) \vee x_3 \vee (\neg x_4))$

Using De Morgan's law on the last expression,

$\equiv (x_2 \wedge x_3 \wedge x_4) \vee (x_2 \wedge (\neg x_3) \wedge x_4)$

Using distributivity,

$\equiv (x_2 \wedge x_4) \wedge (x_3 \vee (\neg x_3))$

Using $p \vee \neg p \equiv True$,

$\equiv (x_2 \wedge x_4) \wedge (True)$

Using $p \wedge True \equiv p$,

$\equiv (x_2 \wedge x_4)$

Problem 4. [2 parts, 10 pts + 10 pts]
Consider the predicate formulas $\forall x\,(P(x) \to Q(x))$ and $(\forall x\,P(x)) \to (\forall x\,Q(x))$. Assume that all quantifiers are over the same domain.

1. Suppose that an argument has $\forall x\,(P(x) \to Q(x))$ as the only premise and $(\forall x\,P(x)) \to (\forall x\,Q(x))$ as the conclusion. Then is it a valid argument? If yes, prove its validity using rules of inference (mention all the rules you use), if no, then give a counterexample and explain.
   Hint: See Exercise 11 at the end of Section 1.6 in the textbook. Think about rules of inference for quantifiers.

   Exercise 11 at the end of Section 1.6 from the textbook tells us that if we have an argument of the form

$$p_1$$
$$p_2$$
$$\vdots$$
$$p_n$$
$$----$$
$$\therefore p \to q$$

   Then this argument is valid if and only if the following argument is valid where the hypothesis of the conditional in the conclusion is made into a premise:

$$p_1$$
$$p_2$$
$$\vdots$$
$$p_n$$
$$p$$
$$----$$
$$\therefore q$$

   We have the following argument whose validity we want to prove:

$$\forall x\,(P(x) \to Q(x))$$
$$----$$
$$\therefore (\forall x\,P(x)) \to (\forall x\,Q(x))$$

   Using the result from the exercise, we know that showing the validity of the above

argument is equivalent to showing the validity of:

$$\forall x\, (P(x) \rightarrow Q(x))$$
$$\forall x\, P(x)$$
$$- - - -$$
$$\therefore (\forall x\, Q(x))$$

We will now prove the validity of the latter.

(a) $\forall x\, (P(x) \rightarrow Q(x))$ (Premise)

(b) $\forall x\, P(x)$ (Premise)

(c) $P(c)$ for an arbitrary c (Using Universal Instantiation on (b))

(d) $P(c) \rightarrow Q(c)$ for the same c as in (c) (Using Universal Instantiation on (a))

(e) $Q(c)$ (Using Modus Ponens on (c) and (d))

(f) $P(c) \wedge Q(c)$ (Using conjunction on (c) and (e))

(g) $\forall x\, (P(x) \wedge Q(x))$ (Using Universal Generalization on (f). Can do this because the variable c is arbitrary and the steps of the proof work no matter what the value of c is)

Thus, the original argument is valid.

2. Now consider the reverse: an argument with $(\forall x\ P(x)) \to (\forall x\ Q(x))$ as the premise and $\forall x\ (P(x) \to Q(x))$ as the conclusion. Is this argument valid? If yes, prove its validity using rules of inference (mention all the rules you use), if no, then give a counterexample and explain.

Let the domain be the set of natural numbers, and let $P(x)$ be the predicate *"x is even"* and $Q(x)$ the predicate *"x is divisible by 4"*.
Then the proposition $(\forall x\ P(x))$ is False because not all natural numbers are even. This means that the premise

$$(\forall x\ P(x)) \to (\forall x\ Q(x))$$

is True because its hypothesis (the left side) is False.
On the other hand, let us look at the conclusion

$$\forall x\ (P(x) \to Q(x))$$

The conclusion, translated to English, says *"For every natural number x, if x is even then x is divisible by 4"*. This is clearly false since the number 10 is a counterexample: it is even but it is not divisible by 4.
Thus, the argument is invalid.

Problem 5. [3 parts, 10 pts + 10 pts + 10 pts]
Consider the following propositions assigned to propositional variables:

$$p = \text{``There is an algorithm that can solve SAT in polynomial time.''},$$

$$q = \text{``There is a randomized algorithm that can solve SAT in polynomial time.''},$$

$$r = \text{``}P = NP\text{''},$$

$$s = \text{``The polynomial hierarchy collapses.''}.$$

1. Express each of the following compound propositions as propositional formulae using conditionals and the above variables:

   (a) *"A sufficient condition for $P \neq NP$ is that there is no algorithm that solves SAT in polynomial time".*

   (b) *"There is a randomized algorithm that can solve SAT in polynomial time only if the polynomial hierarchy collapses."*

   (c) *"Whenever there is an algorithm that can solve SAT in polynomial time it follows that there is a randomized algorithm that can solve SAT in polynomial time."*

   (d) *"A necessary condition for $P = NP$ is that the polynomial hierarchy collapses"*

   Answers:

   (a) $\neg p \rightarrow \neg r$

   (b) $q \rightarrow s$

   (c) $p \rightarrow q$

   (d) $r \rightarrow s$

2. The answers to part 1 should let you write each of the propositions in English in the form *"if ... then ..."*. Use this form to then write down the contrapositive, converse, and inverse in English of each of the propositions.

Answers:

(a) i. *"If there is no algorithm that solves SAT in polynomial time then $P \neq NP$"*

ii. Contrapositive: *"If $P = NP$ then there is an algorithm that solves SAT in polynomial time"*

iii. Converse: *"If $P \neq NP$ then there is no algorithm that solves SAT in polynomial time"*

iv. Inverse: *"If there is an algorithm that solves SAT in polynomial time then $P = NP$"*

(b) i. *"If there is a randomized algorithm that can solve SAT in polynomial time then the polynomial hierarchy collapses."*

ii. Contrapositive: *"If the polynomial hierarchy does not collapse then there is no randomized algorithm that can solve SAT in polynomial time."*

iii. Converse: *"If the polynomial hierarchy collapses then there is a randomized algorithm that can solve SAT in polynomial time."*

iv. Inverse: *"If there is no randomized algorithm that can solve SAT in polynomial time then the polynomial hierarchy does not collapse."*

(c) i. *"If there is an algorithm that can solve SAT in polynomial time then there is a randomized algorithm that can solve SAT in polynomial time."*

ii. Contrapositive: *"If there is no randomized algorithm that can solve SAT in polynomial time then there is no algorithm that can solve SAT in polynomial time."*

iii. Converse: *"If there is a randomized algorithm that can solve SAT in polynomial time then there is an algorithm that can solve SAT in polynomial time."*

iv. Inverse: *"If there is no algorithm that can solve SAT in polynomial time then there is no randomized algorithm that can solve SAT in polynomial time."*

(d) i. *"If $P = NP$ then the polynomial hierarchy collapses."*

ii. Contrapositive: *"If the polynomial hierarchy does not collapse then $P \neq NP$"*

iii. Converse: *"If the polynomial hierarchy collapses then $P = NP$"*

iv. Inverse: *"If $P \neq NP$ then the polynomial hierarchy does not collapse."*

3. Consider an argument in which the propositions $(a), (b)$ and (c) from part 1 are the premises, and proposition (d) is the conclusion. Show that this argument is valid using rules of inference. Show all the steps and mention the rule being used in every step.

Using Part 1., we know the premises are

1. $\neg p \to \neg r$

2. $q \to s$

3. $p \to q$

and the conclusion is $r \to s$. We want to show that the argument is valid, i.e. we can infer the conclusion from the premises. We use rules of inference:

1. $\neg p \to \neg r$ (premise)

2. $q \to s$ (premise)

3. $p \to q$ (premise)

4. $r \to p$ (contrapositive of 1)

5. $r \to q$ (hypothetical syllogism on 4 and 3)

6. $r \to s$ (hypothetical syllogism on 5 and 2)

6 is the conclusion of the argument we are considering and thus the argument is valid.