Homework 1

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Monday, Oct 21st 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 pts]
For what values of n is the proposition $(p_1 \land p_2 \land \ldots \land p_n) \rightarrow (p_1 \oplus p_2 \oplus \ldots \oplus p_n)$ a tautology? Provide an explanation for your answer.
Hint: Recall that $\oplus$ satisfies associativity and so the order in which you evaluate doesn't matter! Also try writing down the truth table for $(p_1 \oplus p_2 \oplus \ldots \oplus p_n)$ for small values of n and see if you can observe a pattern!

Problem 2. [3 parts, 10 pts + 10 pts + 10 pts]
Let $D(x, y)$ be the predicate defined on natural numbers x and y as follows: $D(x, y)$ is true whenever y divides x, otherwise it is false. Additionally, $D(x, 0)$ is false no matter what x is (since dividing by zero is a no-no!).
Let $P(x)$ be the predicate defined on natural numbers that is true if and only if x is a prime number.

1. Write $P(x)$ as a predicate formula involving quantifiers, logical connectives, and the predicate $D(x, y)$. Assume the domain to be natural numbers.
   Hint 1: n is prime if and only if the only numbers that divide it are 1 and n.
   Hint 2: You might have to use conditionals.

2. Consider the proposition *"There are infinitely many prime numbers"*. Express the proposition as a predicate formula using quantifiers, logical connectives and the predicate $P(x)$. Assume the domain to be natural numbers.
   Note that you don't need to use the answer from the previous part in this problem; you can write your answer in terms of $P(x)$.
   Hint: For inspiration, consider the following game: you pick any natural number x (as large as you want) and I have to find a number y such that y is larger than x and is also a prime number (I win if I can find such a y, otherwise you win!). Now notice that the proposition is in some sense equivalent to saying that I can always win the game!

3. Write the negation of the predicate formula obtained in part 2. Make sure you take the negation all the way in so that it sits right next to $P(x)$ in the final expression.

Problem 3. [10 pts]
Consider the following compound proposition:

$$((\neg x_1) \wedge x_4 \wedge x_3 \wedge x_2) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \vee \neg ((\neg x_2) \vee x_3 \vee (\neg x_4)) \, .$$

Simplify it as much as possible using propositional equivalences. Show all the steps, and for every step mention the equivalence that you are using.
Hint: Use the distributive property in a smart way. I guarantee you that this problem can be solved in four or five steps and has a very clean solution if you approach it the right way!

More space for problem 3:

Problem 4. [2 parts, 10 pts + 10 pts]
Consider the predicate formulas $\forall x\,(P(x) \to Q(x))$ and $(\forall x\ P(x)) \to (\forall x\ Q(x))$. Assume that all quantifiers are over the same domain.

1. Suppose that an argument has $\forall x\,(P(x) \to Q(x))$ as the only premise and $(\forall x\ P(x)) \to (\forall x\ Q(x))$ as the conclusion. Then is it a valid argument? If yes, prove its validity using rules of inference (mention all the rules you use), if no, then give a counterexample and explain.
   Hint: See Exercise 11 at the end of Section 1.6 in the textbook. Think about rules of inference for quantifiers.

2. Now consider the reverse: an argument with $(\forall x\ P(x)) \rightarrow (\forall x\ Q(x))$ as the premise and $\forall x\,(P(x) \rightarrow Q(x))$ as the conclusion. Is this argument valid? If yes, prove its validity using rules of inference (mention all the rules you use), if no, then give a counterexample and explain.

Problem 5. [3 parts, 10 pts + 10 pts + 10 pts]
Consider the following propositions assigned to propositional variables:

$$p = \text{“}\textit{There is an algorithm that can solve SAT in polynomial time.}\text{”},$$

$$q = \text{“}\textit{There is a randomized algorithm that can solve SAT in polynomial time.}\text{”},$$

$$r = \text{“}P = NP\text{”},$$

$$s = \text{“}\textit{The polynomial hierarchy collapses.}\text{”}.$$

1. Express each of the following compound propositions as propositional formulae using conditionals and the above variables:

   (a) *“A sufficient condition for $P \neq NP$ is that there is no algorithm that solves SAT in polynomial time”.*

   (b) *“There is a randomized algorithm that can solve SAT in polynomial time only if the polynomial hierarchy collapses.”*

   (c) *“Whenever there is an algorithm that can solve SAT in polynomial time it follows that there is a randomized algorithm that can solve SAT in polynomial time.”*

   (d) *“A necessary condition for $P = NP$ is that the polynomial hierarchy collapses”*

2. The answers to part 1 should let you write each of the propositions in English in the form *"if . . . then . . . "*. Use this form to then write down the contrapositive, converse, and inverse in English of each of the propositions.

3. Consider an argument in which the propositions $(a), (b)$ and (c) from part 1 are the premises, and proposition (d) is the conclusion. Show that this argument is valid using rules of inference. Show all the steps and mention the rule being used in every step.