A template for weak induction

This is a general template you should follow when writing weak induction proofs. We will see how the proof looks like for a particular example:
Show that if $x \geq -1$ then for all natural numbers $n \geq 1$ we have

$$(1+x)^n \geq 1 + nx$$

Steps of a weak induction proof:

- Let $P(n)$ be the predicate
$$(1+x)^n \geq 1 + nx.$$
Then we want to prove that if $x \geq -1$ then
$$\forall n \geq 1 \; P(n).$$
For the rest of the proof assume $x \geq -1$.

- **Base case**: Here we want to prove that $P(1)$ is true, i.e.
$$(1+x)^1 \geq (1 + 1 \cdot x).$$
There is nothing to prove here since the LHS and RHS of the statement are both equal to $(1+x)$.

- **Induction step**: We now want to prove that
$$\forall n \geq 1 \; P(n) \rightarrow P(n+1).$$
Let k be any natural number greater than or equal to 1. It is enough to show that
$$P(k) \rightarrow P(k+1).$$
Let us assume $P(k)$ is true, i.e.
$$(1+x)^k \geq 1 + kx \text{ (Induction hypothesis)} \tag{1}$$
Using the induction hypothesis, we want to show that $P(k+1)$ is also true. To show $P(k+1)$ is true, we would have to prove that
$$(1+x)^{k+1} \geq 1 + (k+1)x. \tag{2}$$
Since $(x+1) \geq 0$ (because $x \geq -1$), we can multiply both sides of inequality (1) (the induction hypothesis) by $(1+x)$ without changing the direction of the inequality:
$$\begin{aligned}
&(1+x)(1+x)^k \geq (1+kx)(1+x) \\
&\Rightarrow (1+x)^{k+1} \geq 1 + kx + x + kx^2 \\
&\Rightarrow (1+x)^{k+1} \geq 1 + (k+1)x + kx^2
\end{aligned} \tag{3}$$

Since $kx^2 \geq 0$, we have that

$$1 + (k+1)x + kx^2 \geq 1 + (k+1)x \tag{4}$$

Combining equation 3 and 4 we get

$$(1+x)^{k+1} \geq 1 + (k+1)x,$$

which is exactly the statement (2), i.e. $P(k+1)$. Thus, we have shown that $P(k+1)$ is true under the assumption that $P(k)$ true, and so $P(k) \to P(k+1)$ for any $k \geq 1$. This completes the proof.