Quiz I [Version 1] (CS 205 - Fall 2019) [Solutions]

Name:

NetID:

Section No.:

For each of the following problems, use the space provided below the problem statement to write down your answer. Write clearly and concisely. There are 4 problems in total.

1. (**20 pts**) Consider an argument whose premise is $\exists x\ (P(x) \to Q(x))$ and conclusion is $(\forall x\ \neg Q(x)) \to \neg(\forall x\ P(x))$. Prove that this is a valid argument using rules of inference. Show all the steps and mention the rule you use in each step.
   (Hint: What can you do when the conclusion of an argument is of the form $p \to q$?)

   Proving that the given argument is valid is equivalent to proving that the argument

   $$\begin{array}{l} \exists x\ (P(x) \to Q(x)) \\ \forall x\ \neg Q(x) \\ ---- \\ \therefore \neg(\forall x\ P(x)) \end{array}$$

   So we will focus on proving that the latter argument is valid.

   (a) $\exists x\ (P(x) \to Q(x))$ (Premise)
   (b) $\forall x\ \neg Q(x)$ (Premise)
   (c) $P(c) \to Q(c)$ for some c (Existential Instantiation on (a))
   (d) $\neg Q(c)$ for the same c as in (c) (Universal Instantiation on (b))
   (e) $\neg P(c)$ (Modus Tollens on (c) and (d))
   (f) $\exists x\ \neg P(x)$ (Existential Generalization on (e). Can do this because $\neg P(c)$ is true for some c in the domain).
   (g) $\neg(\forall x\ P(x))$ (De Morgan's law for quantifiers on (f))

   Since (g) is the conclusion, we have proved that the argument is valid.

2. (**5 + 5 = 10 pts**) Write the following propositions in the *"If... then ..."* form:

   (a) *"A sufficient condition for $\mathcal{NP}$ to be contained in $\mathcal{BPP}$ is that SAT can be solved in randomized polynomial time".*

   Answer: *"If SAT can be solved in randomized polynomial time then $\mathcal{NP}$ is contained in $\mathcal{BPP}$".*

   (b) *"A necessary condition for Graph isomorphism to be $\mathcal{NP}$-complete is that the polynomial hierarchy collapses".*

   Answer: *"If Graph isomorphism is $\mathcal{NP}$-complete then the polynomial hierarchy collapses".*

3. (**10 pts**) Suppose the domain of discourse is the set of integers and let $P(x)$ be the predicate *"x is a perfect square"*. Express $P(x)$ as a predicate formula using logical connectives, quantifiers, and other mathematical symbols (if needed).
(Recall: x is a perfect square if x can be written as $x = n^2$ for some integer n.)

$$P(x) = \exists y \ (x = y^2).$$

4. (**10 pts**) Is the proposition $p_1 \wedge p_2 \wedge \ldots \wedge p_{1001} \wedge (p_1 \oplus p_2 \oplus \ldots \oplus p_{1001})$ a contradiction? Provide a short explanation for your answer.
(Recall: A contradiction is a proposition that is always false, regardless of what truth values are assigned to the variables.)

If any of the variables is False, then clearly the proposition becomes False because of the conjunction. Thus the only interesting case where the truth value of the proposition is not obvious is when all the variables are True. In this case, what will determine the truth value of the proposition is whether

$$(p_1 \oplus p_2 \oplus \ldots \oplus p_{1001})$$

is true when all variables are true.
Recall (from the HW) that an XOR of an odd number of True values is True, and since the number of True values in the above expression will be odd (1001), that means that the expression will evaluate to True. Thus, in the case when all the variables are True, the proposition is True, and so it is not a contradiction.