Some comments about existential and universal instantiation

1. Let us consider the following example. Suppose the premises are

    - $(\forall x\ Q(x)) \to p$
    - $\neg p$

    and the conclusion is $\exists x\ \neg Q(x)$. A lot of you end up doing the following:

    (a) $(\forall x\ Q(x)) \to p$ (premise)
    (b) $\neg p$ (premise)
    (c) $Q(c) \to p$ for any c (Universal instantiation on (a))
    (d) $\neg Q(c)$ for any c (using modus tollens on (b) and (c))
    (e) $\exists \neg Q(x)$ (applying existential generalization on (d)).

    This proof is wrong. Statement (c) cannot be inferred from statement (a) using universal instantiation. Let's see why. Before we go on, let's understand the meaning of inference or inferring one statement from another. Basically inferring a statement p from a statement q using rules of inference means that statement q follows/is true whenever statement p is true. Let's try to see if we can infer statement (c) from statement (a).

    Statement (a) is saying (in English): *"If every element in the domain makes $Q(x)$ true then p is true"*, while statement (c) is saying, *"For any c in the domain, if $Q(x)$ is true for the element c then p is true"*. Do you see they mean completely different things? One is saying that if *everything* in the domain makes Q(x) true then p follows, the other is saying if for any *one* element $Q(x)$ is true then p is true. In the latter, all that is needed for p to be true is one element that makes $Q(x)$ true, which is totally different from saying everything in the domain must be true for p to be true. So, clearly you cannot infer (c) from (a), i.e. (a) being true does not mean (c) should be true!

    To avoid issues like this, let's develop a rule of thumb: only apply universal/existential to premises that look like the following:

    $$\forall x\ \ (\ldots\ldots)\,,$$

    or

    $$\exists x\ \ (\ldots\ldots)\,.$$

    Basically, apply instantiation to statements/premises where everything is inside the scope of one quantifier. Here is the correct proof:

    (a) $(\forall x\ Q(x)) \to p$ (premise)
    (b) $\neg p$ (premise)
    (c) $\neg(\forall x\ Q(x))$ (using modus tollens on (a) and (b))
    (d) $\exists x\ \neg Q(x)$ (applying De Morgan's on (c))

2. Let's discuss another issue now. Consider the following set of premises:

    - $\forall x\ (P(x) \to Q(x))$

- $\exists x\ P(x)$.

and conclusion $\exists x\ Q(x)$. A lot of you end up doing the following:

(a) $\forall x\ (P(x) \to Q(x))$ (premise)

(b) $\exists x\ P(x)$. (premise)

(c) $P(c) \to Q(c)$ for any c (applying universal instantiation to (a))

(d) $P(c)$ for some c (applying existential instantiation to (b))

(e) $Q(c)$ for some c (using modus ponnens on (c) and (d))

(f) $\exists x\ Q(x)$ (applying existential generalization on (e))

This proof of validity is wrong. Do you see why? Well, when you apply universal instantiation to (a), it gives you $P(c) \to Q(c)$ for *any* c. Note that this c represents some arbitrary element. Now you proceed to apply existential instantiation to (b) and get $P(c)$ for some c. The problem is that the element from the domain that the existential instantiation gives you need not be the same element you got when you use universal generalization in the previous step, i.e. it need not be c!

Instead, you want to do the following:

(a) $\forall x\ (P(x) \to Q(x))$ (premise)

(b) $\exists x\ P(x)$. (premise)

(c) $P(c)$ for some c (applying existential instantiation to (b))

(d) $P(c) \to Q(c)$ for the same c as in (c)) (applying universal instantiation to (a) for the c in (c))

(e) $Q(c)$ for c in (c)(using modus ponnens on (c) and (d))

(f) $\exists x\ Q(x)$ (applying existential generalization on (e))

Notice that we first used existential instantiation to get a c (some element in the domain for which $P(x)$ is true) and then we invoked universal instantiation to instantiate the statement $P(c) \to Q(c)$ for the same c that we got from the previous step. The problem with trying to do the universal instantiation before the existential one is that you don't know which element to instantiate the universal statement for!