Quiz IV (CS 205 - Fall 2019) (Solutions)

Name:

NetID:

Section No.:

For each of the following problems, use the space provided below the problem statement to write down your answer. Write clearly and concisely. There are 3 problems in total.

1. (**10 pts**) Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, i.e. the set of positive integers, and let $\mathbb{Q}^-$ be the set of all negative rational numbers (0 is not included). Show that there is a surjective function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^-$. You must prove that the function you state as an example is surjective. Is the function you provided as an example also injective? Why or why not?
Solution: Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^-$ defined as follows:

$$f((a,b)) = \frac{-a}{b}.$$

To see why f is surjective, let $\alpha \in \mathbb{Q}^-$ be any positive rational number. Then by the definition of rational numbers, and from the fact that $\alpha < 0$, there must be integers $p > 0$ and $q > 0$ such that

$$\alpha = \frac{-p}{q}.$$

If we consider the point $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ in the domain of f, then

$$f((p,q)) = \frac{-p}{q} = \alpha.$$

The function is not injective because $f((1, 2)) = -0.5 = f((2, 4))$.

2. (**10 + 10 = 20 pts**) For each of the following statements, state whether you think the statement is True or False and provide an explanation for your answer.

(a) Let A, B, C be finite sets such that there is an injective function $f : B \to A$ and a surjective function $g : B \to C$. Then $|A| \geq |B| \geq |C|$.
Solution: True. Since f is injective, it follows that $|B| \leq |A|$, and since g is surjective, it must be the case that $|B| \geq |C|$. Combining the two inequalities, we get

$$|A| \geq |B| \geq |C|.$$

(b) Let $A = \{0, 3, 6, 9, 12 \ldots\}$, i.e. the set of *all* nonnegative multiples of 3, and $B = \{0, 4, 8, 12, 16 \ldots\}$, i.e. the set of *all* nonnegative multiples of 4. Then $|A| \neq |B|$, i.e. there is no bijection between the two sets.
Solution: False. Consider the function $f : A \to B$ defined as

$$f(x) = \frac{4x}{3}.$$

f is injective since if $x_1 \neq x_2$ then $f(x_1) = \frac{4x_1}{3} \neq \frac{4x_2}{3} = f(x_2)$. To prove that f is surjective, consider an arbitrary multiple of four $n \in B$ from the codomain. Then $\frac{n}{4}$ is a natural number, and $\frac{3n}{4}$ is a multiple of 3 and hence an element of the domain of f. Furthermore $f(\frac{3n}{4}) = \frac{4}{3} \cdot \frac{3n}{4} = n$, and so every point in the codomain is in the range of f.

3. (**20 pts**) Consider the infinite sequence given by the following recurrence:

$$a_0 = 0$$

$$a_n = a_{n-1} - 2n + 1 \text{ for } n \geq 1.$$

Compute the first few terms of the sequence using the recurrence. Observe a pattern in the values and try to guess a formula for a_n (the formula should be purely in terms of n). Use induction to prove that the formula you guessed is correct.
Solution:

$$a_0 = 0$$

$$a_1 = a_0 - 2(1) + 1 = 0 - 1 = -1$$

$$a_2 = a_1 - 2(2) + 1 = -1 - 3 = -4$$

$$a_2 = a_3 - 2(3) + 1 = -4 - 5 = -9$$

One can observe that the guess $a_n = -n^2$ fits the first few terms. We will now show it fits all the $n \geq 0$ using weak induction. Let $P(n)$ be the predicate "$a_n = -n^2$". Our goal is to show that $\forall n \geq 0\ P(n)$.
Base case: $a_0 = 0$ (given) and $-n^2 = 0^2 = 0$, so for the case when $n = 0$, we have that $a_n = -n^2$. Thus, $P(0)$ is true.
Induction step: We will now prove that $\forall n \geq 0\ P(n) \rightarrow P(n+1)$. Let $n \geq 0$ be arbitrary. It suffices to show that $P(n) \rightarrow P(n+1)$. Let $P(n)$ be true, i.e. $a_n = -n^2$ (this is the induction hypothesis). Since $n \geq 0$, $n+1 \geq 1$ and so we can use the recurrence to write a_{n+1} in terms of a_n:

$$a_{n+1} = a_n - 2(n+1) + 1 = a_n - 2n - 1$$

Substituting the value of a_n into the above expression:

$$a_{n+1} = -n^2 - 2n - 1 = -(n^2 + 2n + 1) = -(n+1)^2$$

and this proves that $P(n+1)$ is true. This completes the proof.