Homework 2 (Solutions)

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Wednesday, Nov 6th 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 pts]
Without using induction, prove that $n^2 - n$ is even for all $n \geq 1$.
Hint: Consider using a case-analysis based proof; what happens when n is even/odd?

Solution:
Let $n \geq 1$ be an any integer. We will use a case analysis to prove the statement:

- **Case 1 (n is even)**: Since n is even it must be of the form $n = 2k$ for some integer k. Substituting this value of n in $n^2 - n$, we get

  $$n^2 - n = (2k)^2 - 2k = 4k^2 - 2k = 2k(2k - 1)$$

  Since $n^2 - n$ is of the form $2 \times$ *(some integer)* (since $k(2k - 1)$ is an integer), it must be even, and we are done proving this case.

- **Case 1 (n is odd)**: Since n is odd it must be of the form $n = 2k + 1$ for some integer k. Substituting this value of n in $n^2 - n$, we get

  $$n^2 - n = (2k + 1)^2 - (2k + 1) = 4k^2 + 1 + 4k - 2k - 1 = 4k^2 - 2k = 2k(2k - 1)$$

  Since $n^2 - n$ is of the form $2 \times$ *(some integer)* (since $k(2k - 1)$ is an integer), it must be even, and we are done proving this case.
  This completes the proof.

Problem 2. [10 pts]
Without using induction, use the result from Problem 1 to show that $n^2 - 1$ is a multiple of 8 whenever n is an odd integer greater than or equal to 1.

Solution:
Since $n \geq 1$ is given to be odd, we can write $n = 2k - 1$ for some integer $k \geq 1$. Substituting this value in $n^2 - 1$, we get

$n^2 - 1 = (2k-1)^2 - 1 = (2k-1+1)(2k-1-1)$ (Using the formula $a^2 - b^2 = (a+b)(a-b)$).

Thus,

$$n^2 - 1 = (2k - 1 + 1)(2k - 1 - 1) = (2k)(2k - 2) = 4k(k - 1).$$

Since $k \geq 1$, we know from Problem 1 that $k(k-1)$ is even and so $k(k-1) = 2m$ for some integer m. Thus, we get

$$n^2 - n = 4k(k - 1) = 4 \times (2m) = 8m.$$

Since $n^2 - n$ can be expressed as $8 \times$ *(some integer)*, it's clearly a multiple of 8 and this completes the proof.

Problem 3. [10 pts]
Let u, v, w, x, y be integers. Prove that if $u + 2v + 3w + 4x + 5y \geq 70$ then either

- $u \geq 2$, or
- $v \geq 3$, or
- $w \geq 4$, or
- $x \geq 5$, or
- $y \geq 6$.

Hint: $p \to q \equiv \neg q \to \neg p$

Let u, v, w, x, y be any integers. Let p be the proposition

$$u + 2v + 3w + 4x + 5y \geq 70$$

and q be the proposition

$$u \geq 2 \lor v \geq 3 \lor w \geq 4 \lor x \geq 5 \lor y \geq 6$$

We want to prove $p \to q$. Recall that this is equivalent to proving $\neg q \to \neg p$ and so we will focus on proving the latter.
$\neg p$ is the proposition

$$u + 2v + 3w + 4x + 5y < 70$$

and $\neg q$ is the proposition

$$\neg \left(u \geq 2 \lor v \geq 3 \lor w \geq 4 \lor x \geq 5 \lor y \geq 6\right)$$

$$\equiv \left(u < 2 \land v < 3 \land w < 4 \land x < 5 \land y < 6\right) \text{ (Using De Morgan's rule)}$$

To prove $\neg q \to \neg p$, let us assume that $\neg q$ is true and then using the assumption show that $\neg p$ must also be true.
If $\neg q$ is true then we know that

$$\left(u < 2 \land v < 3 \land w < 4 \land x < 5 \land y < 6\right)$$

and so

$$u + 2v + 3w + 4x + 5y < 2 + (2 \times 3) + (3 \times 4) + (4 \times 5) + (5 \times 6) = 70$$

and so

$$u + 2v + 3w + 4x + 5y < 70$$

This proves $\neg p$ is true. Thus we have showed that $\neg q \to \neg p$ is true and so $p \to q$ is also true and this completes the proof.

Problem 4. [10 pts]
Let $f(x) = ax^2 + bx + c$ be a quadratic function with rational coefficients (i.e., a, b and c are rational). Also suppose that $f(x)$ has two real roots α and β. Show that either α and β are both rational or α and β are both irrational.
Hint: If α and β are the two roots of $f(x)$ then we can factorize $f(x)$ as

$$f(x) = (x - \alpha)(x - \beta)$$

Solution:
Since α and β are roots of $f(x)$ which is a quadratic function, we have that

$$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

Since $f(x)$ is also equal to $ax^2 + bx + c$, we can conclude that

$$a = 1,\ b = -(\alpha + \beta),\ c = \alpha\beta.$$

Furthermore, a, b and c are all rational. For the sake of contradiction, assume first that α is rational but β is irrational. Since

$$b = -(\alpha + \beta)$$

we can write

$$\beta = -(\alpha + b).$$

But since α is rational and b is rational, it follows that their sum if also rational (recall the proof from the class; it's super easy), and multiplying by -1 does not change the rationality. This means that β is also rational which contradicts our initial assumption that α is rational and β is irrational. Thus, our initial assumption must have been wrong and it cannot be the case that α is rational and β is irrational.
By a similar argument and by exploiting symmetry, we can also prove that it cannot be the case that α is irrational and β is rational, and so the only two possiblities are that either both α and β are rational or both α and β are irrational.

Problem 5. [10 pts]
Let $a_1, a_2, \ldots, a_{101}$ be real numbers lying in the open interval $(0, 1)$. Show that there must be two numbers a_i and a_j among them such that

$$|a_i - a_j| < 0.01.$$

Hint: Divide $[0, 1]$ into 100 disjoint intervals: $(0, 0.01), [0.01, 0.02), [0.02, 0.03), \ldots, [0.99, 1)$.

Solution:
Let us divide $[0, 1]$ into 100 disjoint intervals: $(0, 0.01), [0.01, 0.02), [0.02, 0.03), \ldots, [0.99, 1)$. We claim that since there are 101 points, there must be an interval that contains at least two numbers from the list. We can prove this claim using a proof by contradiction. Suppose each of the 100 intervals had at most one number from the list. Since all numbers are given to lie in the range $(0, 1)$ and the intervals cover the entire range, it would imply there are only 100 numbers in the list, which is a contradiction to the fact that there are 101 numbers in total in the given list. Thus, it must be the case that one of the intervals contains at least two numbers from the list.
Now consider the interval that contains at least two numbers from the list, and let us assume that there are two numbers a_i and a_j that are contained in that interval. Given that all intervals are clopen (closed-open) or open intervals of size 0.01, it follows that

$$|a_i - a_j| < 0.01.$$

Problem 6. [10 pts]
Let a and b be two distinct rational numbers such that $a < b$. Show that there exists a rational number c such that $a < c < b$.

Solution:
Since a, b are both rational, it follows that $a + b$ is also rational. It also follows that $\frac{a+b}{2}$ is also rational. This is because if $a + b$ were equal to $\frac{p}{q}$ for some $q \neq 0$ then $\frac{a+b}{2}$ is just $\frac{p}{2q}$ and is also rational.
Now we will prove that

$$a < \frac{a+b}{2} < b.$$

To prove the first inequality, we will prove that

$$\frac{a+b}{2} - a > 0$$

Note that

$$\frac{a+b}{2} - a = \frac{a+b-2a}{2} = \frac{b-a}{2} > 0,$$

where the last inequality follows from the fact that $b > a$. Simlilary,

$$b - \frac{a+b}{2} = \frac{2b-a-b}{2} = \frac{b-a}{2} > 0,$$

and so

$$\frac{a+b}{2} < b.$$

Letting $c = \frac{a+b}{2}$, we see that $a < c < b$ and c is rational. This completes the proof.

Problem 7. [10 pts]
Prove or disprove: the sum of two positive irrational numbers is always irrational.
Hint: Feel free to use the fact that $\sqrt{2}$ is irrational.

Solution:
Let $a = 2 + \sqrt{2}$ and let $b = 2 - \sqrt{2}$.
First note that both a and b must be irrational. If not then it would contradict the irrationality of $\sqrt{2}$ (can you see why?).
Furthermore, clearly $a > 0$ and, since $\sqrt{2} < 2$, we also have that $b > 0$.
Now note that

$$a + b = 2 + \sqrt{2} + 2 - \sqrt{2} = 4,$$

and so $a + b$ is rational. Thus, we have given an example of two positive irrational numbers whose sum is rational and this disproves the statement.

Problem 8. [10 pts]
Recall that $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$. Prove using weak induction that $n! < n^n$ for all natural numbers $n \geq 2$. We will use weak induction to prove this.

Solution:
Let $P(n)$ be the predicate "$n! < n^n$". Then we want to show that $\forall n \geq 2\ P(n)$.
Base case: We prove that $P(2)$ is true.
$2! = 2$ and $2^2 = 4$, and thus

$$2! < 2^2$$

and so $P(2)$ is true.
Induction step: We want to prove that $\forall n \geq 2\ P(n) \rightarrow P(n+1)$.
Let $n \geq 2$ be any integer. It suffices to prove that $P(n) \rightarrow P(n+1)$.
To do this, we will assume $P(n)$ is true and then show that $P(n+1)$ follows from this assumption.
Assume $P(n)$ is true, i.e.

$$n! < n^n$$

This is the **induction hypothesis**. We now want to prove that $P(n+1)$ must also be true, i.e.

$$(n+1)! < (n+1)^{(n+1)}$$

Consider the LHS of the above inequality.

$$\begin{aligned}
(n+1)! \quad &= (n+1) \cdot n! \\
&< (n+1) \cdot n^n \text{ (Using induction hypothesis)} \\
&< (n+1) \cdot (n+1)^n \text{ (Since } 1 < n < n+1 \text{ and hence } n^n < (n+1)^n\text{)} \\
&= (n+1)^{(n+1)}
\end{aligned}$$

Thus, $(n+1)! < (n+1)^{(n+1)}$ and this shows that $P(n+1)$ is true. This completes the proof of the induction step.

Problem 9. [10 pts]
Use strong induction to prove that every natural number $n \geq 2$ can be written as

$$n = 2x + 3y,$$

where x and y are integers greater than or equal to 0.

Solution:
Let $P(n)$ be the predicate

$$\text{“}n \textit{ can be written as } 2x + 3y \textit{ for integers } x, y \geq 0\text{”}$$

We need to prove $\forall n \geq 2\ P(n)$. We use strong induction.
Base case: We will prove that $P(2)$ and $P(3)$ are true. This is easy because we can simply write

$$2 = 2(1) + 3(0),$$

and

$$3 = 2(0) + 3(1),$$

and this proves the base case.
Induction step: We will prove that

$$\forall n \geq 3\ P(2) \wedge P(3) \wedge \ldots \wedge P(n) \rightarrow P(n+1)$$

Let $n \geq 3$ be any integer. It suffices to show that

$$P(2) \wedge P(3) \wedge \ldots \wedge P(n) \rightarrow P(n+1)$$

To prove this, we will assume that $P(2) \wedge P(3) \wedge \ldots \wedge P(n)$ is true and show $P(n+1)$ is also true.
Assume $P(2), P(3), \ldots, P(n)$ are all true, i.e. for every $2 \leq k \leq n$, there are integers $x, y \geq 0$, such that k can be written as $2x + 3y$ (**Strong induction hypothesis**)
We need to show $P(n+1)$ is true, i.e. $n+1$ can be written as $2x' + 3y'$ for some integers x', y'.
Since $n \geq 3$, we know that $n - 1 \geq 2$, and using the strong induction hypothesis, we know that $P(n-1)$ must be true since $2 \leq n-1 < n$. This means that there are integers $x, y \geq 0$ such that

$$n - 1 = 2x + 3y \tag{1}$$

Now consider the expression $2(x+1) + 3y$. We can write

$$2(x+1) + 3y = 2x + 3y + 2 = (2x + 3y) + 2 = (n-1) + 2 = n + 1,$$

where the fact $2x + 3y = n - 1$ follows from Equation (1). Thus, if let $x' = x + 1$ and $y' = y$, we get that

$$n + 1 = 2x' + 3y'$$

and this proves that $P(n+1)$ is true and completes the proof of the induction step.

Problem 10. [10 pts]
Let $n \geq 2$ be any natural number and consider n lines in the xy plane. A point in the xy plane is called an *intersection point* if *at least* two lines pass through it. Use induction to show that the number of intersection points is at most $\frac{n(n-1)}{2}$.

Let $P(n)$ be the predicate

$$\textit{“If there are } n \textit{ distinct lines in the } xy \textit{ plane, there can be at most } \frac{n(n-1)}{2} \textit{ intersection points”}$$

We want to prove that $\forall n \geq 2\ P(n)$. We will use weak induction to prove this.
Base case: We will prove $P(2)$ is true. Suppose there are 2 distinct lines in the xy plane. Then since two lines can intersect in at most 1 point, we have that the number of intersection points is at most 1. Also,

$$\frac{2(2-1)}{2} = 1$$

and so $P(2)$ is true.
Induction step: We will prove that $\forall n \geq 2\ P(n) \to P(n+1)$.
Let $n \geq 2$ be any integer, then it suffices to show that $P(n) \to P(n+1)$.
Assume $P(n)$ is true, i.e

$$\textit{If there are } n \textit{ distinct lines in the } xy \textit{ plane, there can be at most } \frac{n(n-1)}{2} \textit{ intersection points}$$

This is the **Induction hypothesis**.
We now want to show that $P(n+1)$ is true, i.e. if there are $n+1$ distinct lines in the xy plane then there can be at most

$$\frac{(n+1)(n+1-1)}{2} = \frac{n(n+1)}{2}$$

intersection points.
Suppose there are $n+1$ distinct lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ on the xy plane. Let us remove the line ℓ_{n+1} for the time being from the xy plane. That leaves us with n distinct lines on the plane. Using the induction hypothesis, we know that these n lines can lead to at most $\frac{n(n-1)}{2}$ intersection points.
If we now add back the line ℓ_{n+1} to the xy plane, since this line intersects each of the other n lines at most one point, this will create at most n new intersection points. Thus, the total number of intersection points is

$$\frac{n(n-1)}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

Thus, $P(n+1)$ is true, and this completes the proof of the induction step.