Homework 2

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Wednesday, Nov 6th 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 pts]
Without using induction, prove that $n^2 - n$ is even for all $n \geq 1$.
Hint: Consider using a case-analysis based proof; what happens when n is even/odd?

Problem 2. [10 pts]
Without using induction, use the result from Problem 1 to show that $n^2 - 1$ is a multiple of 8 whenever n is an odd integer greater than or equal to 1.

Problem 3. [10 pts]
Let u, v, w, x, y be integers. Prove that if $u + 2v + 3w + 4x + 5y \geq 70$ then either

- $u \geq 2$, or
- $v \geq 3$, or
- $w \geq 4$, or
- $x \geq 5$, or
- $y \geq 6$.

Hint: $p \to q \equiv \neg q \to \neg p$

Problem 4. [10 pts]
Let $f(x) = ax^2 + bx + c$ be a quadratic function with rational coefficients (i.e., a, b and c are rational). Also suppose that $f(x)$ has two real roots α and β. Show that either α and β are both rational or α and β are both irrational.
Hint: If α and β are the two roots of $f(x)$ then we can factorize $f(x)$ as

$$f(x) = (x - \alpha)(x - \beta)$$

Problem 5. [10 pts]
Let $a_1, a_2, \ldots, a_{101}$ be real numbers lying in the open interval $(0, 1)$. Show that there must be two numbers a_i and a_j among them such that

$$|a_i - a_j| < 0.01.$$

Hint: Divide $[0, 1]$ into 100 disjoint intervals: $(0, 0.01), [0.01, 0.02), [0.02, 0.03), \ldots, [0.99, 1)$.

Problem 6. [10 pts]
Let a and b be two distinct rational numbers such that $a < b$. Show that there exists a rational number c such that $a < c < b$.

Problem 7. [10 pts]
Prove or disprove: the sum of two positive irrational numbers is always irrational.
Hint: Feel free to use the fact that $\sqrt{2}$ is irrational.

Problem 8. [10 pts]
Recall that $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$. Prove using weak induction that $n! < n^n$ for all natural numbers $n \geq 2$.

More space for Problem 8:

Problem 9. [10 pts]
Use strong induction to prove that every natural number $n \geq 2$ can be written as

$$n = 2x + 3y,$$

where x and y are integers greater than or equal to 0.

More space for Problem 9:

Problem 10. [10 pts]
Let $n \geq 2$ be any natural number and consider n lines in the xy plane. A point in the xy plane is called an *intersection point* if *at least* two lines pass through it. Use induction to show that the number of intersection points is at most $\frac{n(n-1)}{2}$.

More space for Problem 10: