Quiz IV (CS 205 - Fall 2019) (Solutions)

Name:

NetID:

Section No.:

For each of the following problems, use the space provided below the problem statement to write down your answer. Write clearly and concisely. There are 3 problems in total.

1. (**10 pts**) Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, i.e. the set of positive integers, and let $\mathbb{Q}^+$ be the set of all positive rational numbers (0 is not included). Show that there is a surjective function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$. You must prove that the function you state as an example is surjective. Is the function you provided as an example also injective? Why or why not?
   Solution: Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$ defined as follows:
   $$f((a, b)) = \frac{a}{b}.$$
   To see why f is surjective, let $\alpha \in \mathbb{Q}^+$ be any positive rational number. Then by the definition of rational numbers, and from the fact that $\alpha > 0$, there must be integers $p > 0$ and $q > 0$ such that
   $$\alpha = \frac{p}{q}.$$
   If we consider the point $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ in the domain of f, then
   $$f((p, q)) = \frac{p}{q} = \alpha.$$
   The function is not injective because $f((1, 2)) = 0.5 = f((2, 4))$.

2. (**10 + 10 = 20 pts**) For each of the following statements, state whether you think the statement is True or False and provide an explanation for your answer.

   (a) Let A, B, C be finite sets such that there is an injective function $f : A \to B$ and a surjective function $g : C \to B$. Then $|A| \leq |B| \leq |C|$.
   Solution: True. Since f is injective, it follows that $|A| \leq |B|$, and since g is surjective, it must be the case that $|B| \leq |C|$. Combining the two inequalities, we get
   $$|A| \leq |B| \leq |C|.$$

   (b) Let $E = \{0, 2, 4, \ldots\}$, i.e. the set of *all* even natural numbers, and $O = \{1, 3, 5, \ldots\}$, i.e. the set of *all* odd natural numbers. Then $|E| \neq |O|$, i.e. there is no bijection between the two sets.
   Solution: False. Consider the function $f : E \to O$ defined as
   $$f(x) = x + 1.$$
   f is injective since if $x_1 \neq x_2$ then $f(x_1) = x_1 + 1 \neq x_2 + 1 = f(x_2)$. To prove that f is surjective, consider an arbitrary odd number $n \in O$ in the codomain. Then $n - 1$ is an even number and hence an element of the domain of f, and furthermore $f(n - 1) = (n - 1) + 1 = n$, and so every point in the codomain is in the range of f.

3. (**20 pts**) Consider the infinite sequence given by the following recurrence:

$$a_0 = 0$$

$$a_n = a_{n-1} + 2n - 1 \text{ for } n \geq 1.$$

Compute the first few terms of the sequence using the recurrence. Observe a pattern in the values and try to guess a formula for a_n (the formula should be purely in terms of n). Use induction to prove that the formula you guessed is correct.

Solution:

$$a_0 = 0$$

$$a_1 = a_0 + 2(1) - 1 = 0 + 1 = 1$$

$$a_2 = a_1 + 2(2) - 1 = 1 + 3 = 4$$

$$a_2 = a_3 + 2(3) - 1 = 4 + 5 = 9$$

One can observe that the guess $a_n = n^2$ fits the first few terms. We will now show it fits all the $n \geq 0$ using weak induction. Let $P(n)$ be the predicate "$a_n = n^2$". Our goal is to show that $\forall n \geq 0\ P(n)$.

Base case: $a_0 = 0$ (given) and $n^2 = 0^2 = 0$, so for the case when $n = 0$, we have that $a_n = n^2$. Thus, $P(0)$ is true.

Induction step: We will now prove that $\forall n \geq 0\ P(n) \rightarrow P(n+1)$. Let $n \geq 0$ be arbitrary. It suffices to show that $P(n) \rightarrow P(n+1)$. Let $P(n)$ be true, i.e. $a_n = n^2$ (this is the induction hypothesis). Since $n \geq 0$, $n + 1 \geq 1$ and so we can use the recurrence to write a_{n+1} in terms of a_n:

$$a_{n+1} = a_n + 2(n+1) - 1 = a_n + 2n + 1$$

Substituting the value of a_n into the above expression:

$$a_{n+1} = n^2 + 2n + 1 = (n+1)^2$$

and this proves that $P(n+1)$ is true. This completes the proof.