A template for strong induction

This is a general template you should follow when writing strong induction proofs. We will see how the proof looks like for a particular example:
Show that every positive integer n can be written as a sum of distinct powers of two, e.g.,

$$3 = 2^1 + 2^0,$$

$$4 = 2^2,$$

$$6 = 2^2 + 2^1,$$

$$117 = 2^6 + 2^5 + 2^4 + 2^2 + 2^0.$$

Steps of a strong induction proof:

- Let $P(n)$ be the predicate

  n can be written as a sum of distinct powers of two.

  Then we want to prove that

  $$\forall n \geq 1 \; P(n).$$

- **Base case**: Here we want to prove that $P(1)$ is true, i.e.

  1 can be written as a sum of distinct powers of two.

  This is easy to show by simply writing

  $$1 = 2^0.$$

- **Induction step**: We now want to prove that

  $$\forall n \geq 1 \; P(1) \wedge P(2) \wedge \ldots \wedge P(n) \to P(n+1).$$

  Let k be any natural number greater than or equal to 1. It is enough to show that

  $$P(1) \wedge P(2) \wedge \ldots \wedge P(k) \to P(k+1)$$

  Let us assume $P(1), P(2), \ldots, P(k)$ are all true, i.e. for every $1 \leq m \leq k$,

  m can be written as a sum of distinct powers of two.

  Using the induction hypothesis, we want to show that $P(k+1)$ is also true. To show $P(k+1)$ is true, we would have to prove that

  $k+1$ can be written as a sum of distinct powers of two.

  We will prove this using a proof-by-case-analysis:

– **Case 1 ($k + 1$ is odd)**: If $k + 1$ is odd then k must be even. By induction hypothesis, we know that $P(k)$ is true and so k can be expressed as a sum of distinct powers of two, i.e. there is an integer $\ell \geq 0$, and ℓ *distinct* natural numbers $a_1, a_2, \ldots a_\ell$ such that

$$k = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_\ell}. \tag{1}$$

Notice that if k is even then all of $a_1, a_2, \ldots, a_\ell$ must be strictly greater than 0. This is because the presence of a $2^0 = 1$ when writing k as a sum of distinct powers of two would make k odd (Do you see why?). Thus,

$$a_1, a_2, \ldots, a_\ell > 0.$$

Now, consider the following sum of powers of two

$$\begin{aligned} &2^{a_1} + 2^{a_2} + \ldots + 2^{a_\ell} + 2^0 \qquad (2) \\ &= (2^{a_1} + 2^{a_2} + \ldots + 2^{a_\ell}) + 1 \\ &= k + 1 \text{ (Using Equation (1))} \end{aligned}$$

Also note that since $a_1, \ldots, a_\ell > 0$, we have that all the powers of two involved in (2) are distinct. Thus, we have expressed $k + 1$ as a sum of distinct powers of two. This finishes the proof for the case when $k + 1$ is odd.

– **Case 2 ($k + 1$ is even)**: Since $k + 1$ is even, there is a positive integer m such that

$$k + 1 = 2m.$$

Since $m = \frac{k+1}{2}$ and $k \geq 1$, we can conclude that $m \geq 1$. Furthermore, we can also conclude that $m \leq k$ because if $m > k$ then it would imply that

$$\frac{k+1}{2} > k \implies k < \frac{1}{2},$$

which cannot be since we are only looking at values of $k \geq 1$. So we can conclude that

$$1 \leq m \leq k.$$

This means that, by induction hypothesis, $P(m)$ is true and m can be expressed as a sum of distinct powers of two, i.e., there is an integer $\ell \geq 0$, and distinct natural numbers $a_1, a_2, \ldots, a_\ell$ such that

$$m = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_\ell}.$$

Since $k + 1 = 2m$, we can write

$$\begin{aligned} k + 1 &= 2m = 2(2^{a_1} + 2^{a_2} + \ldots + 2^{a_\ell}) \\ \implies k + 1 &= 2^{a_1+1} + 2^{a_2+1} + \ldots + 2^{a_\ell+1}. \end{aligned}$$

So we have expressed $k + 1$ as a sum of powers of two but we don't know if they are *distinct* powers of two. To conclude that the powers are distinct, recall that, $a_1, \ldots, a_\ell$ are all distinct and so $a_1 + 1, a_2 + 1, \ldots, a_\ell + 1$ must also be all distinct (do you see why?). This means we have written $k + 1$ as a sum of distinct powers of two and this completes the proof of the case when $k + 1$ is even.

Thus, we have shown that in both cases $k + 1$ can be written as a sum of distinct powers of two and so $P(k + 1)$ is true and this completes the proof of the induction step.