Homework 3 (Solutions)

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Monday, Dec 2nd 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 + 10 = 20 pts]
Prove that for any two sets X and Y in some universe U,

$$X = Y \text{ if and only if } \bar{X} = \bar{Y}.$$

Use the previous statement to prove that if A and B are subsets of a universe U then

$$\bar{A} \cap \bar{B} = \emptyset \text{ if and only if } A \cup B = U.$$

Solution: Let us first prove that if $X = Y$ then $\bar{X} = \bar{Y}$. Suppose for the sake of contradiction that $X = Y$ but $\bar{X} \neq \bar{Y}$. Without loss of generality we can assume that there is some element $a \in U$ such that $a \in \bar{X}$ but $a \notin \bar{Y}$. This means $a \notin X$ and $a \in \bar{\bar{Y}} = Y$ (both of which follow from the definition of the complement operator) which would mean that $a \notin X$ but $a \in Y$ which is a contradiction since $X = Y$ and so $a \in X$ if and only if $a \in Y$ and so our assumption that $\bar{X} \neq \bar{Y}$ is wrong.
Let us now prove the other direction, i.e. if $\bar{X} = \bar{Y}$ then $X = Y$. Again, for the sake of contradiction assume that $\bar{X} = \bar{Y}$ but $X \neq Y$. Then without loss of generality there is an $a \in X$ such that $a \notin Y$. This is equivalent to saying $a \in \bar{\bar{X}}$ and $a \in \bar{Y}$, or $a \notin \bar{X}$ and $a \in \bar{Y}$ which would be a contradiction since $\bar{X} = \bar{Y}$. So our assumption that $X \neq Y$ is false.
Combining the above proofs we can conclude that $X = Y$ if and only if $\bar{X} = \bar{Y}$.

For the second part of the problem, let $X = A \cup B$ and let $Y = U$. Then using the fact that that $X = Y$ if and only if $\bar{X} = \bar{Y}$ we can conclude that

$$A \cup B = U \leftrightarrow \overline{A \cup B} = \bar{U}$$

$$A \cup B = U \leftrightarrow \bar{A} \cap \bar{B} = \emptyset,$$

where the the second statement follows from the first by using De Morgan's law for sets and the fact that $\bar{U} = \emptyset$.

Problem 2. [4 parts $\times$ 10 pts = 40 pts]
Answer the following questions showing all the steps/work involved (in other words, if you only write the answer without showing any work at all, you are at the risk of getting a zero):

1. What is $\text{pow}(\text{pow}(\text{pow}(\emptyset)))$, where $\text{pow}(S)$ denotes the power set of S?

2. If $A = \{n \in \mathbb{N} |\ 1 \leq n^2 < 90\}$ and $B = \{n^2 |\ n \in \mathbb{N} \wedge (1 \leq n \leq 3)\}$ then is it true that $A - B = A \oplus B$?

3. Describe the following set (by listing all its elements within { and }):

$$\{(a, b) \in \mathbb{Z} \times \mathbb{N} |\ a^2 + b^2 < 5\}.$$

4. For all $n \in \mathbb{N}, n \geq 1$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. What is the image of the function $f : [20] \to [20]$ that is defined as follows?

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{otherwise} \end{cases}$$

Solution:

1. $\text{pow}(\emptyset) = \{\emptyset\}$, and so

$$\text{pow}(\text{pow}(\emptyset)) = \text{pow}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\},$$

and thus

$$\text{pow}(\text{pow}(\text{pow}(\emptyset))) = \text{pow}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

2. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{1, 4, 9\}$, and so

$$B - A = \emptyset.$$

Recalling that $A \oplus B = (A - B) \cup (B - A)$, we can conclude that in our case,

$$A \oplus B = (A - B) \cup \emptyset = (A - B),$$

and so the statement is true.

3. Since we are looking at the set of all pairs (a, b) in $\mathbb{Z} \times \mathbb{N}$ that satisfy $a^2 + b^2 < 5$, we know that $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Clearly $-2 \leq a \leq 2$, and whenever $a = -2$ or $a = 2$ we have that $b = 0$, and whenever $a = -1$ or $a = 1$ we have that $b \in \{0, 1\}$, and finally when $a = 0$ then b can be $0, 1$ or 2, this means the set is

$$\{(-2, 0), (2, 0), (-1, 1), (-1, 0), (1, 1), (1, 0), (0, 0), (0, 1), (0, 2)\}.$$

4. Note that

$$f(1) = f(2) = 1$$
$$f(3) = f(4) = 2$$
$$f(5) = f(6) = 3$$
$$f(7) = f(8) = 4$$
$$f(9) = f(10) = 5$$
$$f(11) = f(12) = 6$$
$$f(13) = f(14) = 7$$
$$f(15) = f(16) = 8$$
$$f(17) = f(18) = 9$$
$$f(19) = f(20) = 10$$

and so

$$Image(f) = Range(f) = [10]$$

Problem 3. [20 pts]
Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective. You must explain why your example is surjective and why it is not injective.
Hint: To show that a function $f : \mathbb{N} \to \mathbb{N}$ is surjective, you need to show that for all $y \in \mathbb{N}$ there is some $x \in \mathbb{N}$ such that $f(x) = y$.
To show that a function is *not* injective, simply show that there are two points $x_1 \neq x_2$ in the domain such that $f(x_1) = f(x_2)$.

Solution: Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined as follows

$$f(x) = |x - 1|.$$

Clearly, f is not injective because

$$f(0) = |0 - 1| = 1,$$

and

$$f(2) = |2 - 1| = 1.$$

Let us prove that f is surjective. Let $y \in \mathbb{N}$ be any natural number. Then consider the value $y + 1$; it is in $\mathbb{N}$ and hence in the domain of f, and furthermore

$$f(y + 1) = |y + 1 - 1| = y,$$

where the second equality follows from the fact that $y \geq 0$. This proves that f is surjective. Thus, $f(x) = |x - 1|$ is surjective but not injective.

Problem 4. [20 pts]
Let $P(x)$ be the predicate *"x is divisible by 4"* and $Q(x)$ be the predicate *"x is divisible by 11"*. Let U, the universe, be the set $\{n \in \mathbb{N} \mid 1 \le n \le 1000\}$. What is the cardinality of the following set?

$$\{x \in U \mid P(x) \vee Q(x)\}.$$

Hint: Observe that

$$\{x \in U \mid P(x) \vee Q(x)\} = \{x \in U \mid P(x)\} \cup \{x \in U \mid Q(x)\}.$$

Solution: Let $A = \{x \in U \mid P(x)\}$ and $B = \{x \in U \mid Q(x)\}$. Then, it's easy to see that

$$|A| = \frac{1000}{4} = 250,$$

since A contains all multiples of 4 between 1 and 1000 and there are 250 such multiples. The largest multiple of 11 between 1 and 1000 is 990 which is 11×90 and so is the 90^{th} multiple of 11 if we consider 11 itself as the first multiple. This means there are 90 multiples of 11 between 1 and 1000 and so

$$|B| = 90.$$

Recall that

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

and so we need to find the $|A \cap B|$. Note that $A \cap B$ is the set

$$\{x \in U | P(x) \wedge Q(x)\},$$

i.e. the set of numbers between 1 and 1000 that are multiples of both 4 and of 11, or in other words, the set of numbers that are multplies of 44. The smallest multiple of 44 between 1 and 1000 is 44 and the largest multiple in that range is $968 = 22 \times 44$. Thus, there are 22 multiples of 44 between 1 and 1000, and so

$$|A \cap B| = 22$$

This means that

$$|\{x \in U \mid P(x) \vee Q(x)\}| = |\{x \in U \mid P(x)\} \cup \{x \in U \mid Q(x)\}|$$

$$= |A \cup B| = |A| + |B| - |A \cap B| = 250 + 90 - 22 = 318.$$