Homework 4

CS 205: Discrete Structures I
Fall 2019

Due: At the beginning of the lecture on Wednesday, Dec 11th, 2019

Total points: 100

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name, NetID, and Section No. in the space provided above.
3. After you are done writing the solutions, staple the sheets in the correct order and bring them to class on the day of the submission (See above). No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 + 5 + 10 = 25 pts]
Let n be an **odd** positive integer. In this problem, we will use P_n to denote the set pow($\{1, 2, \ldots, n\}$), i.e. the powerset of $\{1, 2, \ldots, n\}$. Additionally, we define O_n as follows

$$O_n := \{S \in P_n | \ |S| \text{ is odd}\},$$

i.e. the set of all odd size subsets of $\{1, 2, \ldots, n\}$, and E_n as

$$E_n := \{S \in P_n | \ |S| \text{ is even}\},$$

i.e. the set of all even size subsets of $\{1, 2, \ldots, n\}$.

1. Consider the following function $f : O_n \to E_n$:

$$f(A) = \bar{A},$$

for any $A \in O_n$. Prove that f is surjective using the fact that n is odd.

2. Prove that the function f defined in the previous part is injective.
Hint: Problem 1 on HW 3.

3. Now conclude that
$$|O_n| = |E_n| = 2^{n-1}.$$
Hint: First prove $|O_n| = |E_n|$, then use the fact that $E_n \cup O_n = P_n$, and that E_n and O_n are disjoint.

Problem 2. [15 + 5 pts = 20 pts]
Answer the following questions. Provide all essential details in your solution:

1. Let A, B and C be arbitary sets, and let $f : A \to B$ and $g : B \to C$ be bijective functions. Prove that $(g \circ f)(x) = g(f(x))$ is also bijective.

2. Let Y be a countably infinite set and X be another set such that there is bijection between X and Y. Prove that X must also be countably infinite by showing that there is a bijection between X and $\mathbb{N}$.

Problem 3. [4 + 1 + 10 + 12 + 13 = 40 pts]
Consider the following infinite sequence $a_0, a_1, \ldots a_n \ldots$:

- $a_0 = 1$
- $a_n = a_{n-1} + \frac{(-1)^n}{n+1}$ for $n \geq 1$.

1. Compute the first five terms of this sequence, i.e. a_0, a_1, a_2, a_3, a_4. Show your calculations.

2. Over the next few parts we will prove that $a_n \leq 1$ for all $n \geq 0$ using strong induction. Let $P(n)$ denote the predicate

$$a_n \leq 1.$$

Our goal is to show that $\forall n \geq 0\ P(n)$.
We will have two base cases: $n = 0$ and $n = 1$. Argue that the base cases are true.

3. We will now prove that

$$\forall n \geq 1 \ (P(0) \wedge \ldots \wedge P(n)) \rightarrow P(n+1).$$

   Let $k \geq 1$ be an arbitrary integer. It suffices to show that

$$P(0) \wedge \ldots \wedge P(k) \rightarrow P(k+1).$$

   Let us assume that $P(0), P(1), \ldots, P(k)$ are all true, i.e. $a_j \leq 1$ for all $j \in \{0, 1, \ldots, k\}$. This is our induction hypothesis.
   We now want to show that $P(k+1)$ is true. Prove that $P(k+1)$ is true if k is even using the fact that $P(k)$ is true.

4. Now consider the case when k is odd. Since $k \geq 1$, we can conclude that $k - 1 \geq 0$ and so we know from the induction hypothesis that $P(k-1)$ is true, i.e. $a_{k-1} \leq 1$. Write a_{k+1} in terms of a_{k-1} using the fact that $(-1)^k = -1$ and $(-1)^{k+1} = 1$ (since k is odd).

5. Use the expression obtained in the previous part along with the fact that $a_{k-1} \leq 1$ to conclude that $a_{k+1} \leq 1$ and so $P(k+1)$ is true. .

Problem 4. [5 + 10 = 15 pts]
Let S denote the set of all possible *finite* binary strings, i.e. strings of finite length made up of only 0s and 1s, and no other characters. E.g., 010100100001 is a finite binary string but $100ff101$ is not because it contains characters other than $0, 1$.

1. Give an informal proof arguing why this set should be countable. Even though the language of your proof can be informal, it must clearly explain the reasons why you think the set should be countable.
   Hint: Try to argue that it is possible to arrange the elements of S into an infinite sequence.

2. Now assume that we have proved that S is indeed countable. Use this fact to prove

(formally, this time) that the following set, denoted by T, is also countable:

$$T = \{X \subseteq \mathbb{N} | \ |X| \text{ is finite and is a prime number}\}.$$

Hint: You might want to first argue that the set

$$\{X \subseteq \mathbb{N} | \ |X| \text{ is finite}\}$$

is countable.