Final exam (Solutions)

CS 205: Discrete Structures I
Fall 2019

Wednesday, 18th December, 2019

Total points: 130 (regular credit) + 70 (extra credit)

Duration: 3 hours

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. The exam has two parts: Part I and Part II. Part I constitutes the regular credit portion of the exam worth 130 points, and Part II is for extra credit (worth 70 points). Part I contains 12 problems and Part II contains 3. You have 180 minutes (3 hours) to solve the problems.
2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork if needed.
3. You may refer to physical copies of any books or lecture notes during the exam. However, the use of any electronic devices will lead to the cancellation of your exam and a zero score, with the possibility of the authorities getting involved.
4. Make sure you write your name, NetID, and section number in the space provided above.
5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the exam, and might even be reported to the authorities!

Part I (Regular credit)

Total points: 130
Number of problems: 12

Problem 1. [5 + 5 = 10 pts]
Let $P(x)$ be the predicate "*x is prime*". Translate the following predicate formula into English:

$$\forall x \in \mathbb{N} \ \left(P(x) \to \neg P(x+1)\right).$$

Is this predicate formula True or False? Give a very short explanation.

Solution: One possible translation is

"If a natural number is prime then the number that comes after it is not prime".

This statement is False because 2 is a natural number that is prime and yet the number that comes after it, i.e. 3, is also prime.

Problem 2. [5 parts $\times$ 2 pts per part = 10 pts]
For each of the following statements, state where you think the statement is True or False. You do NOT need to explain your answers.

1. $\{1, 2\} \in \{1, 2, 3\}$ False
2. $\{1, 2\} \subseteq \text{pow}(\{1, 2, 3\})$ False
3. $\{1, \{2\}\} \in \{1, 2, 3, \{1\}, \{2\}\}$ False
4. $\{1, \{2\}\} \subseteq \{1, 2, 3, \{1\}, \{2\}\}$ True
5. $\emptyset \in \text{pow}(\emptyset)$ True

Problem 3. [5 + 5 = 10 pts]
Give a short proof of the following statement: every subset S of $\{1, 2, \dots, 100\}$ such that $|S| = 51$ contains at least one odd and one even number. Is the statement still true if we replaced the condition "$|S| = 51$" with "$|S| = 50$"? Justify your answer.

Solution: First note that $\{1, \dots, 100\}$ contains exactly 50 even numbers and 50 odd numbers. We will prove the first statement by contradiction. Let us suppose for the sake of contradiction that there is a subset $S \subseteq \{1, \dots, 100\}$ such that $|S| = 51$ and S either contains only even numbers or contains only odd numbers. If S contains only even numbers then it would imply that S contins 51 even numbers which would be a contradiction to the fact that $S \subseteq \{1, \dots, 100\}$ and so S can contain at most 50 even numbers.
Similarly, if S contains only odd numbers, it would imply that S contains 51 odd numbers again a contradiction to the fact that S can contain at most 50 odd numbers since it is a subset of $\{1, \dots, 100\}$. Since both possibilities lead to a contradiction is must be the case that our initial assumption is false and that it is indeed the case that for every subset $S \subseteq \{1, \dots, 100\}$ with $|S| = 51$, S must contain at least one even and one odd number.

If we replace the condition $|S| = 51$ with the condition $|S| = 50$ the statement is no longer true: consider the case when S is the set of all even numbers in $\{1, \dots, 100\}$ then $|S| = 50$ and S does not contain an odd number.

Problem 4. [4 pts $\times$ 2.5 pts per part = 10 pts]
Let A be the set $\{1,2,3\}$. Let B be the power-set of A. Let C be the set $\{\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$. Find $B, B \cap C$, $A \times C$, and $B - C$.

$$B = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$B \cap C = \{\{1,2\}, \{1,2,3\}\}$$

$$\begin{aligned} A \times C = \quad & \{(1,\{1,2\}), (1,\{1,2,3\}), (1,\{1,2,3,4\}), (2,\{1,2\}), \\ & (2,\{1,2,3\}), (2,\{1,2,3,4\}), (3\{1,2\}), (3,\{1,2,3\}), (3,\{1,2,3,4\})\} \end{aligned}$$

$$B - C = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$$

Problem 5. [10 pts]
Are there any integers $x, y \in \mathbb{Z}$ that satisfy the following equation?

$$2x^2 + 3y^2 = 15.$$

Supplement your answer with a short proof.

Solution: Clearly, if $|x| > 2$ or $|y| > 2$ then $2x^2 + 3y^2 > 15$ and so we only need to focus on the cases when $|x|, |y| \in \{0, 1, 2\}$. We will do a case analysis based on $|x|$:
Case 1 ($|x| = 0$)**:** In this case, the equation reduces to $3y^2 = 15$ which is the same as $y^2 = 5$. The latter has no integer solutions.
Case 2 ($|x| = 1$)**:** In this case, the equation reduces to $3y^2 = 13$ which is the same as $y^2 = \frac{13}{3}$ and the latter has no integer solutions.
Case 3 ($|x| = 2$)**:** In this case, the equation reduces to $3y^2 = 7$ which is the same as $y^2 = \frac{7}{3}$ and the latter has no integer solutions.
Since none of the cases lead to an integer solution, we can conclude that the equation has no integer solutions.

Problem 6. [5 + 5 = 10 pts]
Which of the following functions are injective, and which of them are surjective? You do NOT need to explain your answers.

1. $f : \{1, 2, 3\} \to \{a, b\}$ such that $f(1) = a$, $f(2) = a$ and $f(3) = b$.
2. $g : \{a, b, c, d, e\} \to \{a, b, c, d, e\}$ such that $g(a) = b$, $g(b) = c$, $g(c) = d$, $g(d) = e$, and $g(e) = a$.
3. $f : \{(1, 1), (1, 2), (2, 1), (2, 2)\} \to \{1, 2, 3, 4\}$ such that $f((a, b)) = a + b$.
4. $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) = n + 1$.
5. $f : \{\{1\}, \{1, 2\}, \{1, 2, 3\}\} \to \{1, 2, 3\}$ such that $f(S) = |S|$.

Solution:
Injective functions: $2., 4., 5.$
Surjective functions: $1., 2., 5.$

Problem 7. [10 pts]
Sheila is trying to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite and so she writes the following procedure/algorithm to print the elements of $\mathbb{N} \times \mathbb{N}$:

Let $i = 0$
While TRUE*:*
    Let $j = 0$
    While TRUE*:*
        print (i, j)
        $j = j + 1$
    $i = i + 1$

Is her procedure/algorithm enough to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite? Why or why not? Give a short explanation.

Solutions: The program first enters the inner loop with $i = 0$ and keeps looping and incrementing the value of j, starting with $j = 0$. Thus, the program will only print pairs of the form $(0, j)$ for $j \in \mathbb{N}$, and never get to, for example, pairs like $(1, 0), (1, 1)$ etc.
In order to use an algorithm/procedure to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite, the procedure/algorithm should print every pair in $\mathbb{N} \times \mathbb{N}$ at least once. However, Sheila's program, as argued above, does not do so and hence it is not enough to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Problem 8. [10 pts]
Let S, T be two sets such that $S \subseteq T$. Prove that if S is uncountable then T must also be uncountable.

Solution: Since S is uncountable, it cannot obviously be finite. So S is an infinite set and thus so is T. Now suppose for the sake of contradiction that T was countably infinite. Then, since subsets of countably infinite sets are countable, it would imply that S is countably infinite, which would be a contradiction to the fact that S is uncountable. This means T cannot be countably infinite and hence must be uncountable.

Problem 9. [10 pts]
Let A and B be two subsets of a universe U where $|U| = 100$. Suppose that $|\bar{A} \cap \bar{B}| = 20$ and $|A - B| = 30$. Furthermore, let $f : A \to B$ be a bijection. Find $|A \cap B|$. Show all the steps involved in obtaining the answer, providing an explanation for each step.

Solution: Using De Morgan's law we know that

$$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$$

and so we have that

$$|\overline{A \cup B}| = 20 \implies |A \cup B| = |U| - 20 = 80.$$

Also, we know that

$$|A - B| = 30$$

and since $|A - B| = |A| - |A \cap B|$, we have that

$$|A| - |A \cap B| = 30.$$

Similarly, $|B - A| = |B| - |A \cap B|$, and since there is a bijection between A and B, we know that $|A| = |B|$ which implies that

$$|B - A| = |B| - |A \cap B| = |A| - |A \cap B| = 30.$$

Now recall that

$$|A \cup B| = |A - B| + |B - A| + |A \cap B|.$$

Since $|A \cup B| = 80$, and $|A - B| = |B - A| = 30$, we get that

$$|A \cap B| = 80 - 60 = 20.$$

Problem 10. [10 pts]
Let n be an odd number. Let S_1 be the set of all binary strings of length n that have more ones than zeros, and let S_2 be the set of all binary strings of length n that have more zeros than ones. Prove that $|S_1| = |S_2| = 2^{n-1}$.
Hint: Show that there is a bijection $f : S_1 \to S_2$. Recall that the total number of binary strings of length n is 2^n.

Solution: Consider the function $f : S_1 \to S_2$ that given a binary string $x \in S_1$ that has more ones than zeros, transforms into a string in S_2 by flipping all the bits, i.e. by replacing the ones in x by zeros and replacing the zeros in x by ones. Clearly, this transformation results in a string in S_2 since the transformed string will have more zeros than ones.

It also not hard to see that f is injective. If we take two distinct strings $x_1, x_2 \in S_1$ then both must differ in at least one position. Let us assume that x_1 and x_2 differ in position i, for some $1 \leq i \leq n$. Then since f simply flips the bits, $f(x_1)$ and $f(x_2)$ will still differ in position i, and so $f(x_1) \neq f(x_2)$.
f can also be shown to be surjective. Let $y \in S_2$ be an arbitrary binary string which has more than zeros than ones. Then consider the string x obtained by flipping all the bits in y, i.e. replacing the ones in y with zeros, and the zeros with ones. Then clearly, $x \in S_1$ since it has more ones than zeros, and also $f(x) = y$, and so since y was an arbitrary element of S_2, we have shown that every element in the codomain of f is also in the range of f.

Since there is a bijection between S_1 and S_2, we have that $|S_1| = |S_2|$. Now let S be the set of all binary strings of length n. Since n is odd, every string in S either has more ones than zeros, or has more zeros than ones, i.e. there cannot be a string in S that has equal number of ones and zeros as that would make the length of the string even which is not the case here. So,

$$S = S_1 \cup S_2$$

and since S_1 and S_2 are disjoint sets

$$|S| = |S_1| + |S_2|$$

which implies that $|S_1| = |S_2| = 2^{n-1}$ since $|S| = 2^n$.

Problem 11. [5 + 10 = 15 pts]
Recall that the Fibonacci numbers are an infinite sequence $f_0, f_1, \ldots$ such that

$$f_0 = 0$$

$$f_1 = 1$$

and $\forall n \geq 2$.

$$f_n = f_{n-1} + f_{n-2}.$$

In this problem, our goal is to prove that $\forall n \geq 0$

$$f_n < 2^n.$$

Let $P(n)$ be the predicate "$f_n < 2^n$". Then our goal is to show that

$$\forall n \geq 0 \ P(n).$$

We will use strong induction to prove this.

1. Write down the base case for the induction proof. In particular, show that $P(0)$ and $P(1)$ are true.
   Solution: Clearly, $f_0 = 0 < 2^0 = 1$ and $f_1 = 1 < 2^1 = 2$ and so $P(0)$ and $P(1)$ are true.

2. We will now prove the induction step for strong induction, i.e

   $$\forall n \geq 1 \ \ (P(0) \wedge P(1) \wedge \ldots \wedge P(n)) \rightarrow P(n+1).$$

   Let $k \geq 1$ be an arbitrary integer. It suffices to prove that

   $$(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1).$$

   Let us assume that $P(0), P(1), \ldots, P(k)$ are all true, i.e. for all $0 \leq j \leq k$, $f_j < 2^j$. This is our induction hypothesis. We want to show that $P(k+1)$ is true, i.e.

   $$f_{k+1} < 2^{k+1}.$$

   Since $k \geq 1$, we have that $k+1 \geq 2$ and so we can apply the recurrence for $n = k+1$ to get that

   $$f_{k+1} = f_k + f_{k-1}.$$

   Now use this recurrence along with the induction hypothesis to conclude that $f_{k+1} < 2^{k+1}$ and hence $P(k+1)$ is true.
   Solution: From the induction hypothesis, we know that $P(k)$ and $P(k-1)$ are true, and so

   $$f_k < 2^k$$

   $$f_{k-1} < 2^{k-1}.$$

   Since

   $$f_{k+1} = f_k + f_{k-1}$$

   we have that

   $$f_{k+1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}.$$

Problem 12. [3 + 6 + 6 = 15 pts]
Let $\mathbb{Q}^+$ be the set of positive rational numbers and $\mathbb{Q}^-$ be the set of negative rational numbers. Let $\mathbb{Z}^+$ denote the set of positive integers. (Note: when we say "positive", 0 is not included)

1. Show that there is a bijection between $\mathbb{Q}^+$ and $\mathbb{Q}^-$.
   Solution: Consider the function $f : \mathbb{Q}^+ \to \mathbb{Q}^-$ defined as $f(x) = -x$. If $x_1 \neq x_2$ are two distinct points in the domain then it follows that $-x_1 \neq -x_2$ and so $f(x_1) \neq f(x_2)$. This proves that f is injective.
   Now let $y \in \mathbb{Q}^-$ be an arbitrary point in the codomain. Consider $x = -y$. Clearly, $x \in \mathbb{Q}^+$, and also
   $$f(x) = -x = y.$$
   Since y was an arbitrary point in the codomain, this establishes that every point in the codomain is also in the range of f, and so f is surjective. Hence f is a bijection.

2. Recall that we proved in class that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Assuming that there is a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Q}^+$, prove that $\mathbb{Q}^+$ is also countably infinite.
   Solution: Note that $\mathbb{Z}^+ \times \mathbb{Z}^+ \subset \mathbb{N} \times \mathbb{N}$ and clearly $\mathbb{Z}^+ \times \mathbb{Z}^+$ is also infinite. Since infinite subsets of countably infinite sets are countably infinite, this implies that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite. Thus, there is some bijection $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{N}$.
   We are also told that there is a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Q}^+$. Let the bijection be $g : \mathbb{Q}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$. We know that the composition of two bijections is also a bijection, and thus $(f \circ g) : \mathbb{Q}^+ \to \mathbb{N}$ is also a bijection, and thus $\mathbb{Q}^+$ is countably infinite.

3. Using parts 1 and 2 of this problem conclude that $\mathbb{Q}$ is countably infinite.
   Solution: Note that
   $$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+.$$
   Since the $\mathbb{Q}$ is the union of three countable sets it must also be countable. Furthermore, $\mathbb{Q}$ is also infinite and thus it is countably infinite.

Part II (Extra credit)

Total points: 70
Number of problems: 3

Problem 13. [20 pts]
Let S be a nonempty subset of the real numbers. Suppose that for every nonempty subset $T \subseteq S$ we have that the product of elements in T is negative. Prove that it must be the case that $|S| = 1$.

Hint: Use proof by contradiction: suppose there was such a set S with $|S| > 1$. Now let $T \subseteq S$ be an arbitrary subset of S such that $|T| > 1$ (such a subset exists because $|S| > 1$). The product of all the elements in T must be negative. Now use this along with the fact that $|T| > 1$ to obtain a contradiction.
Solution: We will use proof by contradiction: suppose there was such a set S with $|S| > 1$. Now let $T \subseteq S$ be an arbitrary subset of S such that $|T| > 1$ (such a subset exists because $|S| > 1$). The product of all the elements in T must be negative. There are only two possibilities for T, both of which lead to a contradiction:
Case 1 (T contains no negative elements): If T does not contain any negative elements, then product of all the elements in T will be positive which is a contradiction.
Case 2 (T contains at least one negative element): Let $x \in T$ be a negative element (such an element exists because T contains at least one negative element), i.e. $x < 0$. Consider the set $T' = T - \{x\}$. Since $|T| > 1$, $|T'| > 0$ and so T' is a nonempty subset. Since every nonempty subset of S has the property that the product of its elements are negative, it must be the case that the product of all the elements of T' is negative. Let us assume that the product of the elements of T' is a number $y < 0$. Since $T = T' \cup \{x\}$, the product of all the elements in T must be xy which must be positive since $x < 0$ and $y < 0$ but this is a contradiction because we have found a nonempty subset of S (namely T) the product of whose elements is positive!
Since both possiblities lead to a contradiction, it must be the case that our initial assumption that $|S| > 1$ is false, and so $|S| \leq 1$. But since S is nonempty, it follows that $|S| = 1$.

Problem 14. [10 + 5 + 10 = 25 pts]
A set S is called magical if it satisfies the following conditions:

- every element of S is a set, i.e.
$$\forall X \in S,\ X \textit{ is a set.}$$
- every element of S is a subset of S, i.e.
$$\forall X \in S,\ X \subseteq S.$$

In this problem, we will prove using weak induction that there are infinitely many magical sets. In particular, we will prove that for every $n \geq 1$, there is a magical set S_n of size n. Let $P(n)$ be the predicate

$$\textit{“There is a magical set } S_n \textit{ of size } n\textit{”}.$$

Our goal is to show that $\forall n \geq 1,\ P(n)$.

1. First we will show the base case $P(1)$ is true, i.e. there is a magical set S_1 of size 1. This means our magical set S_1 should be of the form

$$S_1 = \{X\}$$

where X is some set such that $X \subseteq S_1$. What should X be? Complete the base case using the X you come up with.
Solution: Consider $X = \emptyset$ and so

$$S_1 = \{\emptyset\}.$$

Clearly, S_1 only contains sets as elements, and also since $\emptyset \subseteq S_1$ it is also true that every element of S_1 is also a subset of S_1. Thus, $S_1 = \{\emptyset\}$ is magical and this proves the base case.

2. We will now prove the induction step. Our goal is to show that $\forall n \geq 1,\ P(n) \to P(n+1)$. Let $k \geq 1$ be an arbitrary integer. It suffices to show that $P(k) \to P(k+1)$. Let us assume that $P(k)$ is true, i.e. there is a magical set S_k of size k. We want to use $P(k)$ to prove that $P(k+1)$ is also true, i.e. there is a magical set S_{k+1} of size $k+1$.
The way we will do this is to somehow use the magical set S_k to get a larger set S_{k+1}. In particular, we will define as S_{k+1} as

$$S_{k+1} := S_k \cup \{S_k\}.$$

Use this recurrence to compute S_2 and S_3, and verify that they are indeed magical sets of sizes 2 and 3 respectively.
Solution:

$$S_2 = S_1 \cup \{S_1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}.$$

Clearly, both the elements of S_2 are sets themselves, and also since $\emptyset \subseteq S_2$ and $\{\emptyset\} \subseteq S_2$, S_2 is magical.

$$S_3 = S_2 \cup \{S_2\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Clearly, all three elements of S_3 are sets themselves, and also since $\emptyset \subseteq S_3$, $\{\emptyset\} \subseteq S_3$, and $\{\emptyset, \{\emptyset\}\} \subseteq S_3$, it follows that S_3 is magical.

3. Now prove that S_{k+1} defined as above using the recurrence is indeed a magical set of size $k+1$. You can use the fact that S_k is a magical set of size k to prove this.
Solution: Let us assume that $S_k = \{X_1, X_2, \ldots, X_k\}$, where $X_1, \ldots, X_k$ are sets that satisfy $X_i \subseteq S_k$ for all $1 \leq i \leq k$. Using the recurrence, we can define S_{k+1} as

$$S_{k+1} = S_k \cup \{S_k\} = \{X_1, X_2, \ldots, X_k, \{X_1, X_2, \ldots, X_k\}\}.$$

Clearly, S_{k+1} has $k+1$ elements. Also all the elements of S_{k+1} are sets. As argued before, for every $1 \leq i \leq k$, we have that $X_i \subseteq S_k$, and from the definition of S_{k+1} it is clear that $S_k \subseteq S_{k+1}$ (S_{k+1} is basically just S_k with another set added to it), and this means that

$$X_i \subseteq S_k \subseteq S_{k+1} \implies X_i \subseteq S_{k+1}.$$

This show that, for every $1 \leq i \leq k$, $X_i \subseteq S_{k+1}$. Since S_{k+1} contains $X_1, X_2, \ldots, X_k$ as elements, it is clear that

$$\{X_1, X_2, \ldots, X_k\} \subseteq S_{k+1}.$$

Thus, we have established that all the elements of S_{k+1} are also its subsets and so S_{k+1} is magical set of size $k+1$.

Problem 15. [10 + 5 + 5 + 5 = 25 pts]
In this problem, we will prove that the set of real numbers $\mathbb{R}$ is uncountable. We will do this in a series of steps. Complete all the steps.

1. An infinite binary sequence $\{a_n\}$ is an infinite sequence such that for all $n \in \mathbb{N}$, $a_n \in \{0,1\}$, i.e. every position contains either a 0 or a 1. Let B be the set of *all possible* infinite binary sequences. Give a bijection between B and pow($\mathbb{N}$).
   Solution: We can define a function $f : B \to \text{pow}(\mathbb{N})$ as follows: given an infinite binary sequence $\{a_n\}$, f transforms it into a subset $f(\{a_n\})$ of $\mathbb{N}$ by including the natural number n in $f(\{a_n\})$ if and only if $a_n = 1$.

   To see why f is injective, consider two distinct infinite binary sequences $\{a_n\}$ and $\{b_n\}$. Since the two sequences are distinct they must differ at some position. Say they differ at some position $i \in \mathbb{N}$, i.e. $a_i \neq b_i$. Let us assume $a_i = 1$ and $b_i = 0$. This means that $i \in f(\{a_n\})$ but $i \notin f(\{b_n\})$ and so $f(\{a_n\}) \neq f(\{b_n\})$. The case when $a_i = 0$ and $b_i = 1$ can be argued in a similar way.

   To see that f is surjective, an arbitrary subset S of $\mathbb{N}$ can be converted into an infinite binary sequence $\{a_n\}$ by setting $a_n = 1$ if and only if $n \in S$, and it follows then that
   $$f(\{a_n\}) = S.$$

2. Let B' be the set B with the all-zeros sequence (i.e., the sequence $\{a_n\}$ such that $a_n = 0$ for all $n \in \mathbb{N}$) removed. Prove using Part 1 that B' must be uncountable.
   Solution: Clearly B' is an infinite set. Let us assume for the sake of contradiction that B' is countably infinite. Then B is also countably infinite since it is the union of a countably infinite set B' with a finite set (the set containing only the all-zeros sequence). Part 1 would then imply that since there is a bijection between B and pow($\mathbb{N}$), the latter is also countably infinite, which is a contradiction since we know that pow($\mathbb{N}$) is uncountable.

   Thus our initial assumption must be wrong and so B' is an uncountable set.

3. Let $A \subseteq \mathbb{R}$ be the set of all real numbers between 0 and 1 (not including 0 and 1) such that only the digits 0 and 1 appear in them appear after the decimal point (e.g., 0.0001, 0.101010101 . . ., etc.). Give a bijection between A and B', and conclude that A is uncountable.
   Solution: We can define a function $f : B' \to A$ that converts an infinite binary sequence $\{a_n\}$ into the number $f(\{a_n\})$ defined as
   $$f(\{a_n\}) = 0.a_0a_1a_2a_3 \ldots.$$
   Since B' does not contain the all zeros sequence, $f(\{a_n\}$ always results in a number in A. The bijectivity of f basically follows from its definition: if two sequences differ at a position then the numbers they are converted into by f will also differ, and for every number $a \in A$ we can easily find a sequence $\{a_n\}$ such that $f(\{a_n\}) = a$ by setting a_n equal to the $(n+1)^{th}$ digit of a after the decimal point.

4. Use the fact that A is countable to conclude that $\mathbb{R}$ is also uncountable.
   Solution: Since $A \subseteq \mathbb{R}$, and A is uncountable, it must be the case that $\mathbb{R}$ is also uncountable. This follows from the statement of Problem 8.