Quiz II (CS 205 : Fall 2019) (Solutions)

Name:

NetID:

Section No.:

For each of the following problems, use the space provided below the problem statement to write down your answer. Write clearly and concisely. There are 3 problems in total.

1. (**20 pts**) Use induction to prove that $n^3 - 7n$ is divisible by 3 for all natural numbers $n \geq 0$.
   Hint: You might have to use $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$.
   Solution: Let $P(n)$ be the predicate

   $$n^3 - 7n \text{ is divisible by } 3.$$

   We want to prove $\forall n \geq 0 \; P(n)$. We will use weak induction to prove this.
   Base case: We will show that $P(0)$ is true. For $n = 0$,

   $$n^3 - 7n = 0$$

   which is divisible by 3 and so $P(0)$ is true.
   Induction step: We want to prove that

   $$\forall n \geq 0 \; P(n) \rightarrow P(n+1).$$

   Let $k \geq 0$ be any natural number. It suffices to show that

   $$P(k) \rightarrow P(k+1).$$

   Assume that $P(k)$ is true, i.e. $k^3 - 7k$ is divisible by 3 and so there is some integer m such that

   $$k^3 - 7k = 3m \text{ (Induction hypothesis)}$$

   We want to prove that $P(k+1)$ is true, i.e. $(k+1)^3 - 7(k+1)$ is divisble by 3. We can write

   $$\begin{aligned}
   &(k+1)^3 - 7(k+1) \\
   &= k^3 + 1 + 3k(1+k) - 7k - 7 \\
   &= (k^3 - 7k) + 3k(1+k) - 6 \\
   &= 3m + 3k(1+k) - 6 \text{ (Using the induction hypothesis)} \\
   &= 3(m + k(1+k) - 2) \qquad (1)
   \end{aligned}$$

   It is clear from (1) that $(k+1)^3 - 7(k+1)$ can be written as 3 times some integer, and so it is divisible by 3. Thus, $P(k+1)$ is true and this completes the proof of the induction step.

2. (**10 pts**) Prove the following statement: If the average high temperature in New Brunswick over the past 365 days was 53° F, then there must have been a day (among the past 365 days) on which the high temperature was at least 53° F.
   Solution: Let us assume for the sake of contradiction that the high temperature every day over the past 365 days was strictly less than 53° F. This would mean that the sum of the high temperatures of the past 365 days is strictly less than 53×365, and hence the average is strictly less than $\frac{53\times365}{365} = 53$, which would contradict the given fact that the average high temperature is 53. Thus, there must be at least one day when the temperature was at least 53° F.

3. (**10 + 10 = 20 pts**) For each of the following statements, state whether you think the statement is True or False. If you claim that a statement is True, you must supplement your answer with a proof, and if you claim that a statement is False, you must provide a *concrete* (i.e., provide actual numbers) counterexample to the statement.

   (a) If α is an irrational number and β is a rational number then $\alpha\beta$ must be irrational.
   Solution: This is false. Let $\alpha = \sqrt{2}$ (irrational) and $\beta = 0$ (rational) then $\alpha\beta = 0$ which is rational, and so this is a counterexample.

   (b) The ratio of two *distinct* positive irrational numbers is always irrational.
   Solution: This is False. Consider two numbers, $\sqrt{2}$ and $2\sqrt{2}$. Both are irrational since we know that $\sqrt{2}$ is irrational, and they are clearly distinct and positive, but their ratio is
   $$\frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2},$$
   which is a rational number. Thus, this is a counterexample.