Quiz II (CS 205 - Fall 2019) (Solutions)

Name:

NetID:

Section No.:

For each of the following problems, use the space provided below the problem statement to write down your answer. Write clearly and concisely. There are 3 problems in total.

1. (**10 pts**) Prove the following statement: If in a class of 100 students the average score on a math test is 50 then there must be at least one student in the class who scored greater than or equal to 50 on the test.
   Solution: Let us assume for the sake of contradiction that every student scored strictly less than 50 on the exam. This would mean that the sum of the scores of all the students is strictly less than $50 \times 100 = 5000$, and hence the average is strictly less than $\frac{5000}{100} = 50$, which would contradict the given fact that the average is 50. Thus, there must someone who scored at least 50.

2. (**10 + 10 = 20 pts**) For each of the following statements, state whether you think the statement is True or False. If you claim that a statement is True, you must supplement your answer with a proof, and if you claim that a statement is False, you must provide a *concrete* (i.e., provide actual numbers) counterexample to the statement.

   (a) If α is an irrational number then so is $\frac{1}{\alpha}$.
   Solution: The statement is True. We prove this by contradiction. Let us assume α is irrational but $\frac{1}{\alpha}$ is rational. If $\frac{1}{\alpha}$ is rational then there are non-zero integers p and q (obviously, q is never zero by definition, and here p cannot be zero because $\frac{1}{\alpha}$ is non-zero) such that

   $$\frac{1}{\alpha} = \frac{p}{q}.$$

   and so

   $$\alpha = \frac{q}{p},$$

   and this would mean α is rational, a contradiction to our assumption that α is irrational. So $\frac{1}{\alpha}$ must be irrational.

   (b) The product of two *distinct* positive irrational numbers is always irrational.
   Solution: False. To see this take $a = \sqrt{2}$, which is irrational, and take $b = \frac{1}{\sqrt{2}}$, which is also irrational by the previous part of the problem. Also, clearly $a \neq b$, and their product

   $$ab = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1,$$

   which is rational, and so this is a counterexample to the given statement.

3. (**20 pts**) Use induction to prove that $n^3 + 8n$ is divisible by 3 for all natural numbers $n \geq 0$.
Hint: You might have to use $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$.
Solution: Let $P(n)$ be the predicate

$$n^3 + 8n \text{ is divisible by 3.}$$

We want to prove $\forall n \geq 0\ P(n)$. We will use weak induction to prove this.
Base case: We will show that $P(0)$ is true. For $n = 0$,

$$n^3 + 8n = 0$$

which is divisible by 3 and so $P(0)$ is true.
Induction step: We want to prove that

$$\forall n \geq 0\ P(n) \to P(n+1).$$

Let $k \geq 0$ be any natural number. It suffices to show that

$$P(k) \to P(k+1).$$

Assume that $P(k)$ is true, i.e. $k^3 + 8k$ is divisible by 3 and so there is some integer m such that

$$k^3 + 8k = 3m \text{ (Induction hypothesis)}$$

We want to prove that $P(k+1)$ is true, i.e. $(k+1)^3 + 8(k+1)$ is divisble by 3. We can write

$$\begin{aligned}
&(k+1)^3 + 8(k+1) \\
&= k^3 + 1 + 3k(1+k) + 8k + 8 \\
&= (k^3 + 8k) + 3k(1+k) + 9 \\
&= 3m + 3k(1+k) + 9 \text{ (Using the induction hypothesis)} \\
&= 3(m + k(1+k) + 3) \qquad (1)
\end{aligned}$$

It is clear from (1) that $(k+1)^3 + 8(k+1)$ can be written as 3 times some integer, and so it is divisible by 3. Thus, $P(k+1)$ is true and this completes the proof of the induction step.