Final exam

CS 205: Discrete Structures I
Fall 2019

Wednesday, 18th December, 2019

Total points: 130 (regular credit) + 70 (extra credit)

Duration: 3 hours

Name:

NetID:

Section No.:

INSTRUCTIONS:

1. The exam has two parts: Part I and Part II. Part I constitutes the regular credit portion of the exam worth 130 points, and Part II is for extra credit (worth 70 points). Part I contains 12 problems and Part II contains 3. You have 180 minutes (3 hours) to solve the problems.
2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork if needed.
3. You may refer to physical copies of any books or lecture notes during the exam. However, the use of any electronic devices will lead to the cancellation of your exam and a zero score, with the possibility of the authorities getting involved.
4. Make sure you write your name, NetID, and section number in the space provided above.
5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the exam, and might even be reported to the authorities!

Part I (Regular credit)

Total points: 130
Number of problems: 12

Problem 1. [5 + 5 = 10 pts]
Let $P(x)$ be the predicate "*x is prime*". Translate the following predicate formula into English:

$$\forall x \in \mathbb{N} \ (P(x) \to \neg P(x+1)) .$$

Is this predicate formula True or False? Give a very short explanation.

Problem 2. [5 parts $\times$ 2 pts per part = 10 pts]
For each of the following statements, state where you think the statement is True or False. You do NOT need to explain your answers.

1. $\{1, 2\} \in \{1, 2, 3\}$
2. $\{1, 2\} \subseteq \text{pow}(\{1, 2, 3\})$
3. $\{1, \{2\}\} \in \{1, 2, 3, \{1\}, \{2\}\}$
4. $\{1, \{2\}\} \subseteq \{1, 2, 3, \{1\}, \{2\}\}$
5. $\emptyset \in \text{pow}(\emptyset)$

Problem 3. [5 + 5 = 10 pts]
Give a short proof of the following statement: every subset S of $\{1, 2, \ldots, 100\}$ such that $|S| = 51$ contains at least one odd and one even number. Is the statement still true if we replaced the condition "$|S| = 51$" with "$|S| = 50$"? Justify your answer.

Problem 4. [4 pts $\times$ 2.5 pts per part = 10 pts]
Let A be the set $\{1, 2, 3\}$. Let B be the power-set of A. Let C be the set $\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. Find $B, B \cap C$, $A \times C$, and $B - C$.

Problem 5. [10 pts]
Are there any integers $x, y \in \mathbb{Z}$ that satisfy the following equation?

$$2x^2 + 3y^2 = 15.$$

Supplement your answer with a short proof.

Problem 6. [5 + 5 = 10 pts]
Which of the following functions are injective, and which of them are surjective? You do NOT need to explain your answers.

1. $f : \{1, 2, 3\} \to \{a, b\}$ such that $f(1) = a$, $f(2) = a$ and $f(3) = b$.
2. $g : \{a, b, c, d, e\} \to \{a, b, c, d, e\}$ such that $g(a) = b$, $g(b) = c$, $g(c) = d$, $g(d) = e$, and $g(e) = a$.
3. $f : \{(1, 1), (1, 2), (2, 1), (2, 2)\} \to \{1, 2, 3, 4\}$ such that $f((a, b)) = a + b$.
4. $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) = n + 1$.
5. $f : \{\{1\}, \{1, 2\}, \{1, 2, 3\}\} \to \{1, 2, 3\}$ such that $f(S) = |S|$.

Problem 7. [10 pts]
Sheila is trying to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite and so she writes the following procedure/algorithm to print the elements of $\mathbb{N} \times \mathbb{N}$:

```
Let i = 0
While TRUE:
    Let j = 0
    While TRUE:
        print (i, j)
        j = j + 1
    i = i + 1
```

Is her procedure/algorithm enough to prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite? Why or why not? Give a short explanation.

Problem 8. [10 pts]
Let S, T be two sets such that $S \subseteq T$. Prove that if S is uncountable then T must also be uncountable.

Problem 9. [10 pts]
Let A and B be two subsets of a universe U where $|U| = 100$. Suppose that $|\bar{A} \cap \bar{B}| = 20$ and $|A - B| = 30$. Furthermore, let $f : A \to B$ be a bijection. Find $|A \cap B|$. Show all the steps involved in obtaining the answer, providing an explanation for each step.

Problem 10. [10 pts]
Let n be an odd number. Let S_1 be the set of all binary strings of length n that have more ones than zeros, and let S_2 be the set of all binary strings of length n that have more zeros than ones. Prove that $|S_1| = |S_2| = 2^{n-1}$.
Hint: Show that there is a bijection $f : S_1 \to S_2$. Recall that the total number of binary strings of length n is 2^n.

Problem 11. [5 + 10 = 15 pts]
Recall that the Fibonacci numbers are an infinite sequence $f_0, f_1, \ldots$ such that

$$f_0 = 0$$

$$f_1 = 1$$

and $\forall n \geq 2$.

$$f_n = f_{n-1} + f_{n-2}.$$

In this problem, our goal is to prove that $\forall n \geq 0$

$$f_n < 2^n.$$

Let $P(n)$ be the predicate "$f_n < 2^n$". Then our goal is to show that

$$\forall n \geq 0 \ P(n).$$

We will use strong induction to prove this.

1. Write down the base case for the induction proof. In particular, show that $P(0)$ and $P(1)$ are true.

2. We will now prove the induction step for strong induction, i.e

$$\forall n \geq 1 \ \ (P(0) \wedge P(1) \wedge \ldots \wedge P(n)) \rightarrow P(n+1).$$

   Let $k \geq 0$ be an arbitrary integer. It suffices to prove that

$$(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1).$$

   Let us assume that $P(0), P(1), \ldots, P(k)$ are all true, i.e. for all $0 \leq j \leq k$, $f_j < 2^j$. This is our induction hypothesis. We want to show that $P(k+1)$ is true, i.e.

$$f_{k+1} < 2^{k+1}.$$

   Since $k \geq 1$, we have that $k+1 \geq 2$ and so we can apply the recurrence for $n = k+1$ to get that

$$f_{k+1} = f_k + f_{k-1}.$$

Now use this recurrence along with the induction hypothesis to conclude that $f_{k+1} < 2^{k+1}$ and hence $P(k+1)$ is true.

Problem 12. [3 + 6 + 6 = 15 pts]
Let $\mathbb{Q}^+$ be the set of positive rational numbers and $\mathbb{Q}^-$ be the set of negative rational numbers. Let $\mathbb{Z}^+$ denote the set of positive integers. (Note: when we say "positive", 0 is not included)

1. Show that there is a bijection between $\mathbb{Q}^+$ and $\mathbb{Q}^-$.

2. Recall that we proved in class that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Assuming that there is a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Q}^+$, prove that $\mathbb{Q}^+$ is also countably infinite.

3. Using parts 1 and 2 of this problem conclude that $\mathbb{Q}$ is countably infinite.

Part II (Extra credit)

Total points: 70
Number of problems: 3

Problem 13. [20 pts]
Let S be a nonempty subset of the real numbers. Suppose that for every nonempty subset $T \subseteq S$ we have that the product of elements in T is negative. Prove that it must be the case that $|S| = 1$.

Hint: Use proof by contradiction: suppose there was such a set S with $|S| > 1$. Now let $T \subseteq S$ be an arbitrary subset of S such that $|T| > 1$ (such a subset exists because $|S| > 1$). The product of all the elements in T must be negative. Now use this along with the fact that $|T| > 1$ to obtain a contradiction.

Problem 14. [10 + 5 + 10 = 25 pts]
A set S is called magical if it satisfies the following conditions:

- every element of S is a set, i.e.
$$\forall X \in S,\ X \textit{ is a set.}$$

- every element of S is a subset of S, i.e.
$$\forall X \in S,\ X \subseteq S.$$

In this problem, we will prove using weak induction that there are infinitely many magical sets. In particular, we will prove that for every $n \geq 1$, there is a magical set S_n of size n. Let $P(n)$ be the predicate

$$\textit{“There is a magical set } S_n \textit{ of size } n\textit{”.}$$

Our goal is to show that $\forall n \geq 1,\ P(n)$.

1. First we will show the base case $P(1)$ is true, i.e. there is a magical set S_1 of size 1. This means our magical set S_1 should be of the form
$$S_1 = \{X\}$$
where X is some set such that $X \subseteq S_1$. What should X be? Complete the base case using the X you come up with.

2. We will now prove the induction step. Our goal is to show that $\forall n \geq 1,\ P(n) \to P(n+1)$. Let $k \geq 1$ be an arbitrary integer. It suffices to show that $P(k) \to P(k+1)$. Let us assume that $P(k)$ is true, i.e. there is a magical set S_k of size k. We want to use $P(k)$ to prove that $P(k+1)$ is also true, i.e. there is a magical set S_{k+1} of size $k+1$.
The way we will do this is to somehow use the magical set S_k to get a larger set S_{k+1}. In particular, we will define as S_{k+1} as
$$S_{k+1} := S_k \cup \{S_k\}.$$

Use this recurrence to compute S_2 and S_3, and verify that they are indeed magical sets of sizes 2 and 3 respectively.

3. Now prove that S_{k+1} defined as above using the recurrence is indeed a magical set of size $k+1$. You can use the fact that S_k is a magical set of size k to prove this.

Problem 15. [10 + 5 + 5 + 5 = 25 pts]
In this problem, we will prove that the set of real numbers $\mathbb{R}$ is uncountable. We will do this in a series of steps. Complete all the steps.

1. An infinite binary sequence $\{a_n\}$ is an infinite sequence such that for all $n \in \mathbb{N}$, $a_n \in \{0, 1\}$, i.e. every position contains either a 0 or a 1. Let B be the set of *all possible* infinite binary sequences. Give a bijection between B and pow($\mathbb{N}$).

2. Let B' be the set B with the all-zeros sequence (i.e., the sequence $\{a_n\}$ such that $a_n = 0$ for all $n \in \mathbb{N}$) removed. Prove using Part 1 that B' must be uncountable.

3. Let $A \subseteq \mathbb{R}$ be the set of all real numbers between 0 and 1 (not including 0 and 1) such that only the digits 0 and 1 appear in them appear after the decimal point (e.g., 0.0001, 0.101010101 . . ., etc.). Give a bijection between A and B', and conclude that A is uncountable.

4. Use the fact that A is countable to conclude that $\mathbb{R}$ is also uncountable.