MAE 3260 State of Space

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Final Groupwork

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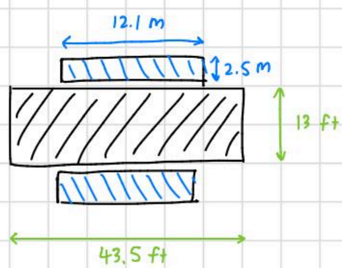
Introduction

For this project we aimed to develop a satellite model that is based off the Hubble Telescope. We developed a block diagram model, a state space model, governing equations, and a Matlab simulation for our system. This project was part of our system dynamics class in Undergrad for Mechanical Engineering, and demonstrates some of the core concepts we have learned thus far. Many of our references, which are listed at the end, are from NASA and helped guide our understanding throughout this project.

Model Development

The Hubble Space Telescope (HST) was modeled as a solid, uniform cylinder with two rectangular solar array panels mounted 4 ft from the centerline of the main body. Based on NASA geometric data, the cylindrical telescope body was approximated as having a diameter of 13 ft and a length of 43.5 ft. Each solar array was modeled using NASA-published dimensions of 12.1 m × 2.5 m [3].

Using standard moment-of-inertia expressions for a solid cylinder and rectangular plates, combined with the parallel-axis theorem to account for the 4-ft offset of each solar array, the total moment of inertia of the modeled system was calculated to be:



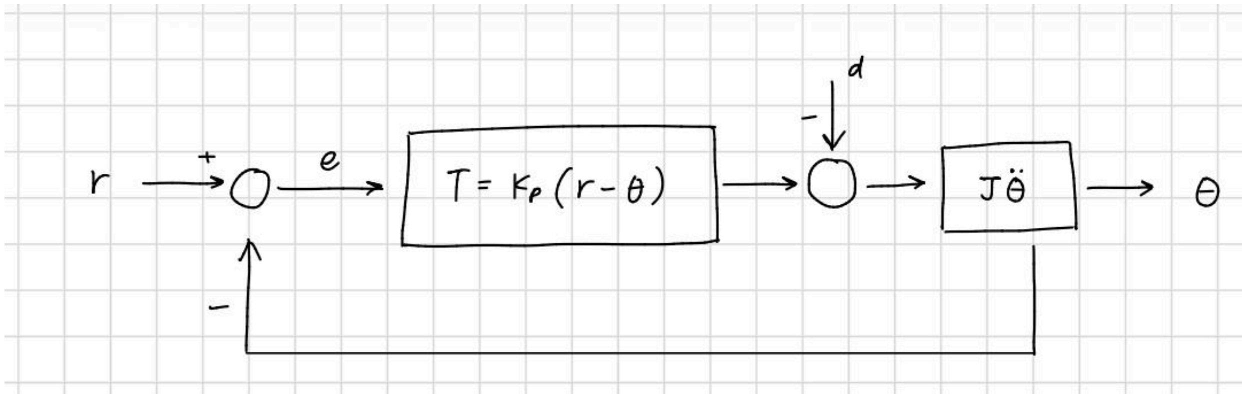
$$J_{cyl} = \frac{1}{2} m R^2 = \frac{1}{2}(122{,}000\ kg)\left(\left(\frac{13}{2} ft\right)\left(\frac{0.3048\ m}{ft}\right)\right)^2 \approx 23{,}943.44\ kg \cdot m^2$$

$$I_z = \frac{hw^3}{12} = \frac{(12.1\ m)(2.5\ m)^3}{12} = 15.7552\ kg \cdot m^2$$

$$J_{panel} = I_{cm} + MR^2 = 15.7552 + (160\ ft)\left(4\ ft\left(\frac{0.3048\ m}{ft}\right) + 1.25\ m\right)^2 = 991.267\ kg \cdot m^2$$

$$J_{tot} = (J_{panel})2 + J_{cyl} = (991.267)2 + 23{,}943.44 \approx \boxed{25{,}925.974\ kg \cdot m^2}$$

State space: for the sake of time and simplicity, we decided to implement a proportional controller, introducing only one extraneous variable to care about. Assuming our system is modeled as follows:



This model is similar to the one used in Lab 1, except for b. Based on our findings about the value for b, b = $\frac{1}{2}C_d A\rho_{atm}$. This means that the value of b depends on atmospheric pressure, which, for simplicity, we assume is near zero, resulting in a b that is approximately zero [2]. From this model, we also get the following:

$\tau_{CL} = \frac{J}{K}$ and $e_{ss} = \frac{1}{k}$

We can calculate the state space model as shown below:

$$\begin{cases} T = k_p(r-\theta) \\ J\ddot{\theta} = T-d \end{cases}$$

$$x = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} r \\ d \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} \theta \end{bmatrix}$$

$$\dot{x}_1 = \ddot{\theta} = \frac{T-d}{J} = \frac{k(r-\theta)-d}{J} = \frac{k}{J}(u_1 - x_2) - u_2 \qquad \dot{x}_2 = \dot{\theta} = x_1$$

$$\dot{x} = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -k/J \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix} + \begin{bmatrix} k/J & -1/J \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$y = \begin{bmatrix} \theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

STATE SPACE MODEL

$$\boxed{A = \begin{bmatrix} 0 & -k/J \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} k/J & -1/J \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}}$$

We already previously calculated a value for moment of inertia. In order to get a value for the controller value, Kp, we must set some system requirements. Generally, the HST aims to perform a rotation movement within a few milliarcseconds. As the telescope needs to be as precise as possible for accurate observation, we wanted the error to be as small as possible. However, due to our simplified model as well as a relatively simple controller, it may not be possible for us to get as small of an error as the real system. Hence, we took a rougher estimation in setting the steady state error to be around 2 arcseconds. From our model, we know that torque is the product of the control constant and error, so dividing the maximum torque by the error gives us a controller value of about $k = 1.65 \times 10^5$.

Unfortunately, this model was unable to be fully implemented, as our model required more than proportional control to be stabilized. With the disturbance torque and very little natural damping in space, the proportional controller can only "push" the satellite based on angle error. But we need an additional derivative controller to counteract angular velocity. Therefore, we made some modifications to the derivations of the state space model to include a new K_d.

Implementing Proportional Derivative Control

Recalculating Inertia

The simplified model remains valid for our analysis. We approximate the Hubble Space Telescope as a primary cylindrical body with two 4-ft radial masts supporting rectangular solar arrays. Although direct mass specifications for the arrays are not widely published, one source provides the power output and corresponding energy-to-weight ratio [7]. Using this relationship, we estimate the mass of each array and compute its contribution to the overall moments of inertia, along with those of the main cylindrical body.

$$m_{total} = 12200kg$$

$$m_{pan} = 2.8kW \cdot \frac{1kg}{0.0364kW} = 76.923kg$$

$$m_{cyl} = 12046.154kg$$

$$I_{cyl,z} = \frac{1}{2}m_{cyl}r^2 = \frac{1}{2}(12046.154kg)(2.15m)^2 = 27841.673kgm^2$$

$$I_{cyl,x} = I_{cyl,y} = \frac{1}{12}m_{cyl}(3r^2 + L^2) = \frac{1}{12}(12046.154kg)(3(2.15m)^2 + 13m^2) = 183570.84kgm^2$$

$$I_{p,x} = \frac{1}{12}m_{pan}h^2 = \frac{1}{12}(76.923kg)(2.5m)^2 = 40.06kgm^2$$

$$I_{p,y} = \frac{1}{12}m_{pan}L^2 = \frac{1}{12}(76.923kg)(12.1m)^2 = 938.525kgm^2$$

$$I_{p,z} = \frac{1}{12} m_{pan}(h^2 + L^2) = \frac{1}{12}(76.923kg)((2.5m)^2 + (12.1m)^2) = 978.589kgm^2$$

$$I_{pan} = I_p + m_{pan}d^2 = I_p + (76.923kg)(4ft \cdot \frac{0.3048m}{1ft} + 1.25m)^2$$

$$I_{pan,x} = 527.35kgm^2$$

$$I_{pan,y} = 1425.81kgm^2$$

$$I_{pan,z} = 1465.86kgm^2$$

$$I_{tot} = I_{cyl} + 2 \cdot I_{pan}$$

$$I_{tot,x} = 193.61 \times 10^6 kgm^2$$

$$I_{tot,y} = 523.47 \times 10^6 kgm^2$$

$$I_{tot,z} = 29.799 \times 10^3 kgm^2$$

Disturbances

For the sake of simplicity, this analysis considers only two primary environmental disturbance torques. Because the Hubble Space Telescope operates in low Earth orbit, the dominant disturbances are gravity gradient torque and aerodynamic torque.

Gravity gradient torque occurs because the part of the spacecraft closer to Earth experiences slightly stronger gravity than the part farther away. This difference creates a small twisting effect that tries to rotate the spacecraft into a preferred position. The expression for its pitch direction can be approximated as [4]:

$$G_y = \frac{3\mu}{R^3}(I_z - I_x)\theta$$

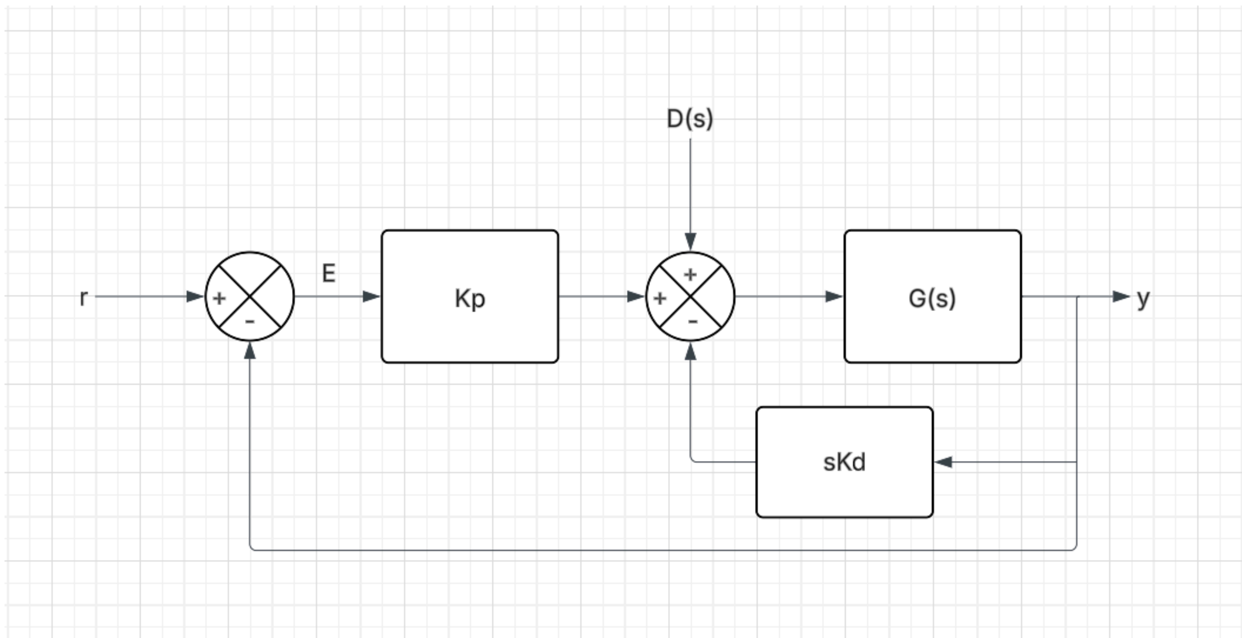
where μ is the gravitational parameter of the Earth, and R is the orbital radius for circular orbits.

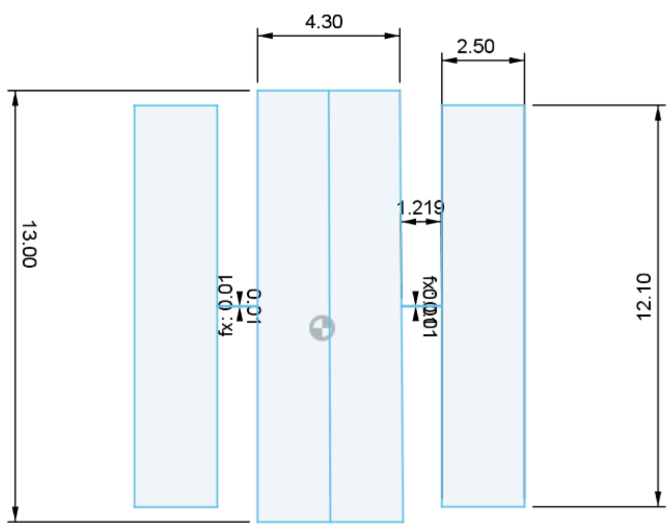
Aerodynamic torque arises from the thin traces of atmosphere still present at Hubble's altitude. As the spacecraft moves through this low-density air, drag forces act unevenly on its surfaces. If these forces do not line up with the spacecraft's center of mass, they produce a torque that can slowly change its attitude. The aerodynamic torque can be computed directly from the standard drag force equation [4]:

$$T_d = \frac{1}{2}\rho v^2 C_D A r$$

where ρ is the atmospheric density, v is the orbital velocity, C_D is the drag coefficient, A is the effective cross-sectional area, and r is the moment arm relative to the center of mass.

Dynamics





We are modeling only the pitch dynamics of the Hubble Space Telescope and implementing a PD (Proportional Derivative) controller. The rotational motion about the pitch axis is governed by the satellite's moment of inertia I_y, the control torque generated by the PD controller, and the external disturbance torques. The PD controller applies a torque proportional to the pitch angle and its rate of change, opposing deviations from the desired reference orientation. This results in a second-order closed-loop system. For the simulation and analysis, the system is represented in state space form, with the state vector comprising the pitch angle and the angular velocity, the reference angle and disturbance torque as the input, and the pitch angle as the output.

$$I_y\ddot{\theta} = T_c + G_y + T_d$$

$$T_d = \frac{1}{2}\rho v^2 C_D A r$$

$$G_y = \frac{3\mu}{R^3}(I_z - I_x)\theta$$

$$T_c = K_P(r - \theta) - K_d\dot{\theta}$$

$$I_y\ddot{\theta} = Kp(r - \theta) - K_d\dot{\theta} + \frac{3\mu(I_z - I_x)}{R^3}\theta + \frac{1}{2}\rho v^2 C_D A r$$

$$\text{Let } \alpha = \frac{3\mu(I_z - I_x)}{R^3}$$

$$\ddot{\theta} = \frac{K_P}{I_y}(r - \theta) - \frac{K_D}{I_y}\dot{\theta} + \frac{\alpha}{I_y}\theta + \frac{T_d}{I_y}$$

$$x = \begin{bmatrix}\dot{\theta}\\ \theta\end{bmatrix},\ u = \begin{bmatrix}r\\ T_d\end{bmatrix}$$

$$\dot{x} = \begin{bmatrix}\ddot{\theta}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{-K_D}{I_y} & \frac{\alpha}{I_y} - \frac{K_P}{I_y}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \theta\end{bmatrix} + \begin{bmatrix}\frac{K_P}{I_y} & \frac{1}{I_y}\\ 0 & 0\end{bmatrix}\begin{bmatrix}r\\ d\end{bmatrix}$$

$$y = [\theta] = [0 \quad 1]\begin{bmatrix}\dot{\theta}\\ \theta\end{bmatrix}$$

$$A = \begin{bmatrix}-\frac{K_D}{I_y} & \frac{\alpha}{I_y} - \frac{K_P}{I_y}\\ 1 & 0\end{bmatrix} \quad B = \begin{bmatrix}\frac{K_P}{I_y} & \frac{1}{I_y}\\ 0 & 0\end{bmatrix}$$

$$C = [0 \quad 1],\quad D = [0 \quad 0]$$

Performance Metrics

In order to calculate the proportional and derivative gains, aside from the maximum steady state error of 2 arcseconds (~0.00056 degrees) mentioned previously, we also decided that our 2% settling time should be relatively short, despite the large inertia due to the sheer size of the Hubble Space Telescope. Thus, we found a balance in $t_s = 20s$.

We will determine the damping ratio and the natural frequency will be obtained from our requirements for settling time and maximum overshoot:

$$\text{Settling time (2\%)}, \ t_S = \frac{4}{\zeta\omega_n}$$

$$\text{Maximum overshoot}, \ M_O = 100\% \times \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

We want the maximum overshoot to be as small as possible, since we want the satellite to arrive at the steady state angle as quickly as possible. Hence, the system should be as close to being critically damped as possible. Consequently, we chose the damping ratio to be approximately 1, but in order to avoid an undetermined exponent, we chose 0.99 instead. Calculating with the settling time requirement:

$$\omega_n = 0.202...$$

Now that we have the damping ratio and natural frequency, Kp and Kd are determined by the comparison between the transfer functions after the Laplace transform with the following equation:

$$\frac{C}{s^2 + (2\zeta^{\mathrm{CL}}\omega_n^{\mathrm{CL}})s + (\omega_n^{\mathrm{CL}})^2}$$

The transfer functions we have obtained are:

$K_P = \omega_n^{\ 2}(I_{yy} - \alpha)$

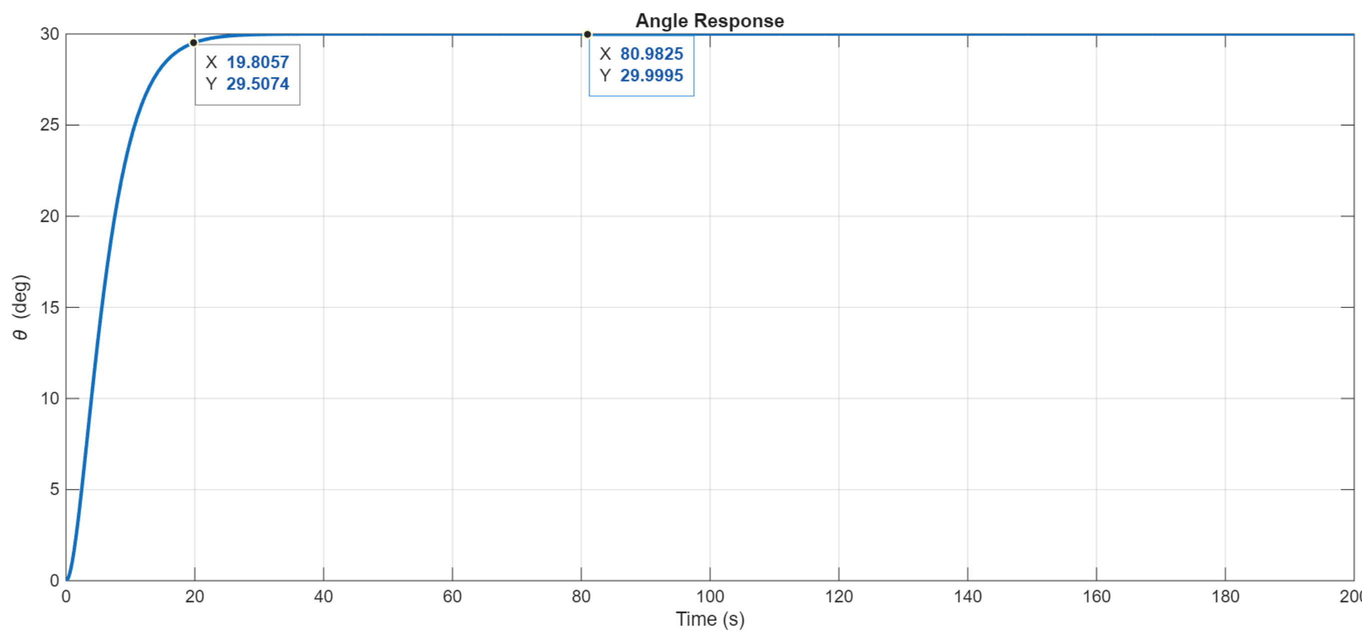
$K_d = 2\zeta\omega_n I_{yy}$

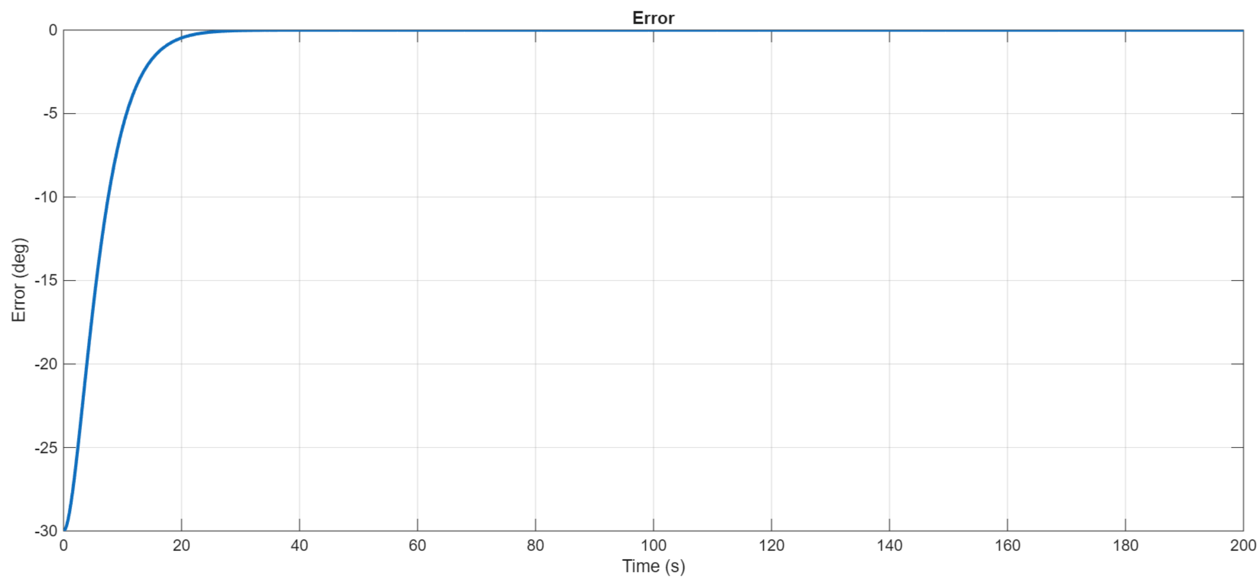
The steady state error will be controlled by a sufficiently large proportional gain, as the larger the K_p, the smaller the steady state error.

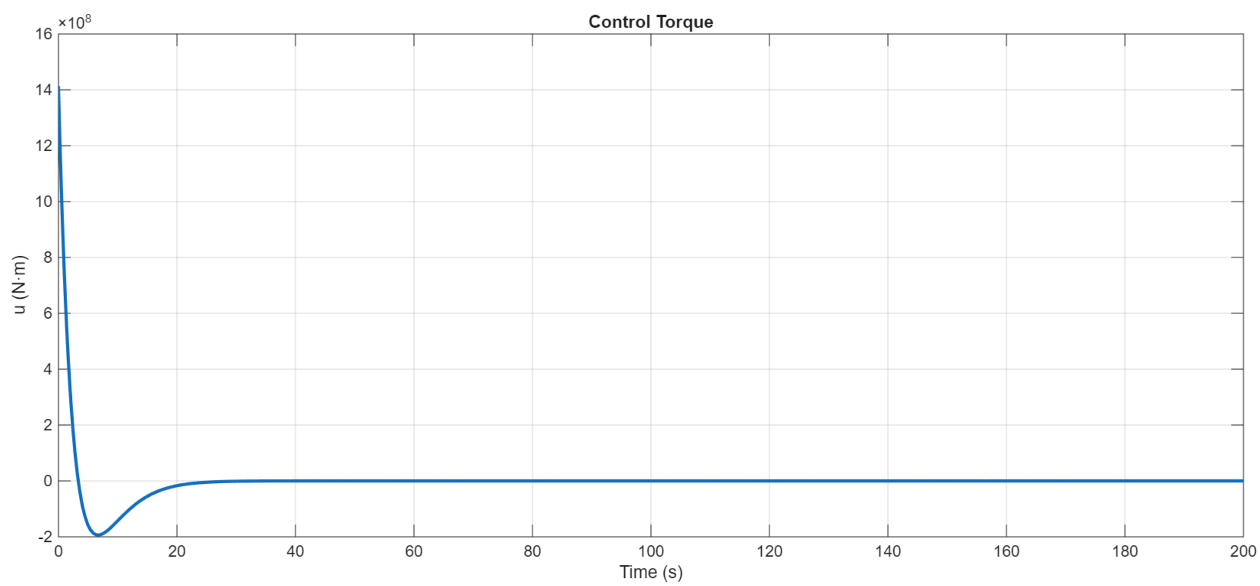
Simulation & Analysis

```
% Parameters
Ixx = 193.61*10^6;
Iyy = 523.47*10^6;
Izz = 29.799*10^3;
J = Iyy + Izz + Ixx;
mu = 3.986e14;
R = earthRadius+540e3;
rho = 2e-12;
v = 27000*1000/(60*60);
cD = 2.2;
A = 63.94;
r = 2.15;
omega_n = 0.3
zeta = 0.99
% Disturbance torque
alpha = (3*mu/R^3) * (Izz - Ixx);
```

```
d = 0.5*rho*v^2*cD*A*r;
% PD controller gains
Kp = omega_n^2*(Iyy-alpha);
Kd = 2*zeta*omega_n*Iyy;
% Reference angle
theta_ref = 30;     % degrees
% Simulation time
tspan = [0 200];
% ODE function
ode = @(t, x) [ ...
   x(2);                                                          % theta_dot
   (1/Iyy)*alpha*x(1)+(1/Iyy)*u_control(x,theta_ref,Kp,Kd) + (1/Iyy)*d ];
% theta_ddot
% Run simulation
[t, X] = ode45(ode, tspan, [0; 0]);     % initial theta=0, theta_dot=0
% Extract states
theta     = X(:,1);
theta_dot = X(:,2);
% Control torque over time
U = zeros(size(t));
for i = 1:length(t)
   U(i) = u_control(X(i,:), theta_ref, Kp, Kd);
end
```







The reference angle was set to be 30 degrees, and the initial conditions for angle and angular velocity are both 0. From the angular response graph, we can see that the system meets all our requirements in terms of settling time and steady state error. However, if we look at the maximum control torque, it is around 14×10^8Nm, which is incredibly large. Hence, this system is highly improbable in a real-life scenario. Our inertia is very large, and we ask the system to adjust to an angle in a very short amount of time. As a result, the control torque would need to be very large to compensate for speed, not to mention to offset the effects of the disturbance torques. Realistically, the system would saturate with such a large torque. To make this more realistic, we could lengthen the settling time or ramp up the reference angle in small increments. Additionally, the real Hubble Space Telescope has a much smaller steady state error than our modelled system. However, in order to achieve such finetuned accuracy, we may need to implement a PID controller instead of our current PD controller.

Conclusion

We first modeled the system similar to how a rotor system was modeled in the first laboratory of this class. While working on the model, we decided that we needed to make some modifications in order to better simulate the satellite pointing problem. We were able to develop a state space model as well as MATLAB code to run some simulations.

References:

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