MAE 3260 Final Group Work: Exploring a System of Interest

Report

Title: Spherologists

Topic of Interest: Sphero Bot

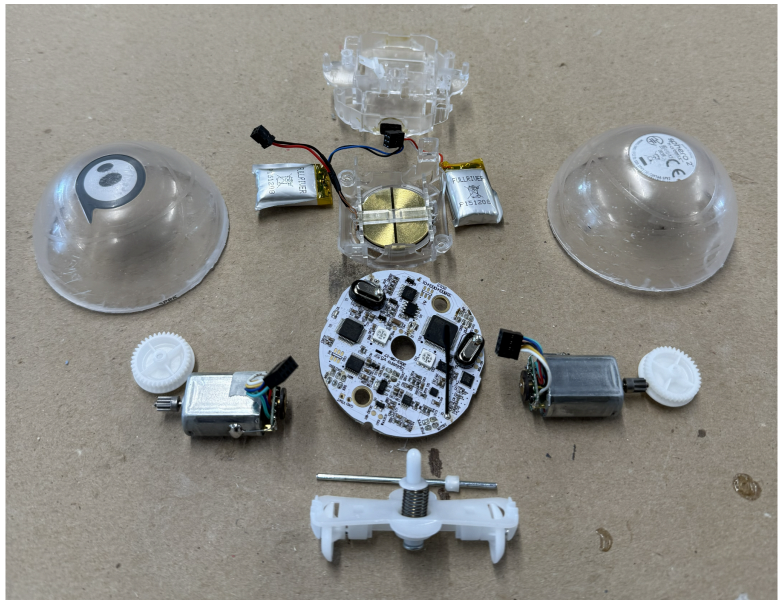
Abstract: We are doing a dissection of the sphero bot. We decided on this project because we unanimously wanted to take something apart because we feel that's one of the best ways to learn. We originally wanted to dissect a quad copter or toy helicopter, but upon learning too many groups also wanted to, we switched to the sphero bot because its way of moving intrigued us. We plan to study how it moves, how quickly it starts and stops, and its physical makeup. Unfortunately, the battery life of the Sphero Bot we received was near 0, so we could not physically study its movement. We continued with the dissection and measured what we could, researching and estimating the rest of the parameters.

Students/Roles:

Student	**Task/Role**
Name of student	1-3 sentences for what the student plans to work on, what skills in the class they will use/build upon, what they expect the final result to look like
Anish Dhawan	I plan to focus on understanding the robot's batteries and voltage, modeling that as a 2nd order system and solving for tau.
Liam Bayne	Measure, research, and estimate parameters, then create plots of input (voltage to motor) vs output (displacement of sphero bot)
Dan O'Malley	I plan to focus on learning about what decisions went into the physical design of the system to understand whether it's working well and how its components (ie. sensors) impact loops.
Matthew Heering	Derive and explain the ODE's of the system. Thus, explaining the physics of the system and how it fundamentally works.

List of MAE 3260 concepts or skills used in this group work:

- Mathematical modeling
  - ODE's
  - Transfer Functions
  - Bode Plots
  - Block Diagrams
- Battery modeling



ODE's

The Sphero rolls by shifting its center of mass using motors and gears ultimately in contact with the inside of the outer shell. This internal relative motion induces a torque on the system, resulting in the desired roll or movement. Imperative to understanding the system, deriving its Ordinary Differential Equations (ODE's) will help us understand the fundamental physics of its motion, and later derive transfer functions to clearly model and understand the relation between inputs and outputs of the system.

At its core the system works around its motor. A voltage is sent into the motor and a resulting torque is produced. This is the first ODE we will model.

If we use the standard armature equation

$$V(t) = LI(t)\frac{d}{dt} + RI(t) + K_e\omega(t) \Rightarrow I\frac{d}{dt} = \frac{-1}{L}(RI + K_e\omega - V)$$

In the Sphero, if assume in the system a constant inductance and resistance and that torque is proportional to current, then we can solve the ODE for torque (τ).

$$\text{Assume } \tau\frac{d}{dt} = K_t I\frac{d}{dt}$$

Resulting in the motor-electrical ODE

$$(1) \Rightarrow \tau\frac{d}{dt} = \frac{-K_t}{L}(RI + K_e\omega - V)$$

Secondly, we can derive an ODE to describe the relation between the angular acceleration of the internal components to the torque produced by the motor.

Assuming that the no slip condition applies we can use a torque balance to derive

$$\tau - \tau_{friction} = J_{internal}\omega\frac{d}{dt}$$

Also, assuming that the torque induced by friction is $\tau_{friction} = b\omega$ we can arrive at the final ODE for the rotation of the internal core.

$$(2) \Rightarrow J_{internal}\omega\frac{d}{dt} = \tau - b\omega$$

Thirdly, we can derive an ODE to describe the rotation of the outer shell. The torque on the shell can be described by the dynamic dynamical equation

$$\tau_{shell} = J_{shell}\Omega\frac{d}{dt}$$

Since the internal torque on the system is transferred via gears to the shell we can use a constant (k) to relate the internal torque to the external torque

$$J_{internal}\omega\frac{d}{dt} = \tau - b\omega$$

$$\tau_{shell} = b\omega \Rightarrow \tau_{shell} = k(\tau - J_{internal}\omega\frac{d}{dt})$$

$$\Rightarrow \tau_{shell} = J_{shell}\Omega\frac{d}{dt} = k(\tau - J_{internal}\omega\frac{d}{dt})$$

Thus, the ODE coupling the internal dynamics to the external dynamics is given by

$$(3)\ J_{shell}\Omega\frac{d}{dt} = k(\tau - J_{internal}\omega\frac{d}{dt})$$

Finally, using the rolling without slip assumption we can formulate the kinematic ODE.

$$\Omega = \theta\frac{d}{dt}$$

$$(4) \Rightarrow x\frac{d}{dt} = R\omega$$

By following these differential equations we can now relate a voltage to the dynamics of the system. Ultimately allowing a user input on their phone to drive a desired output of the Sphero.

Transfer Functions & Block Diagram

We can model the Sphero's motion using the same ODE set introduced earlier in the writeup, then use those ODEs to derive the transfer functions that map the input voltage to currents, angular velocities, and linear motion. This provides the foundation for later block diagram representation of the robot's dynamics.

From the previous derivation, we treat the system as four coupled subsystems: Motor dynamics, internal core rotational dynamics, shell rotational dynamics, and shell-level kinematics (rolling without slip). The following equations capture the entire system chain:

$$V(t) \rightarrow I(t) \rightarrow \omega(t) \rightarrow \Omega(t) \rightarrow \theta(t), x(t)$$

To obtain the transfer functions, the ODEs are transformed into the Laplace domain and algebraically solved to express each variable as a function of the input voltage $V(s)$. The three dynamic ODEs form a coupled system:

$$(s + \frac{R}{L})I(s) + \frac{K_e}{L}\omega(s) = \frac{1}{L}V(s)$$

$$(s + \frac{b}{J_c})\omega(s) = \frac{K_t}{J_c}I(s)$$

$$(s + \frac{kb}{J_s})\Omega(s) = \frac{kK_t}{J_s}I(s)$$

Solving this system produces a shared denominator representing the true dynamics:

$$D(s) = (s + \frac{R}{L})(s + \frac{b}{J_c}) + \frac{K_e K_t}{LJ_c}$$

The electromechanical time constant is $\frac{L}{R}$, the internal mechanical damping is $\frac{b}{J_c}$ and the motor coupling $K_e K_t$ all combine into one second-order term.

From the algebraic solution we obtain the final transfer functions that express the dynamic relationship from voltage input to rotational and translational outputs:

Motor Current: $\frac{I(s)}{V(s)} = \frac{\frac{1}{L}(s + \frac{b}{J_c})}{D(s)}$

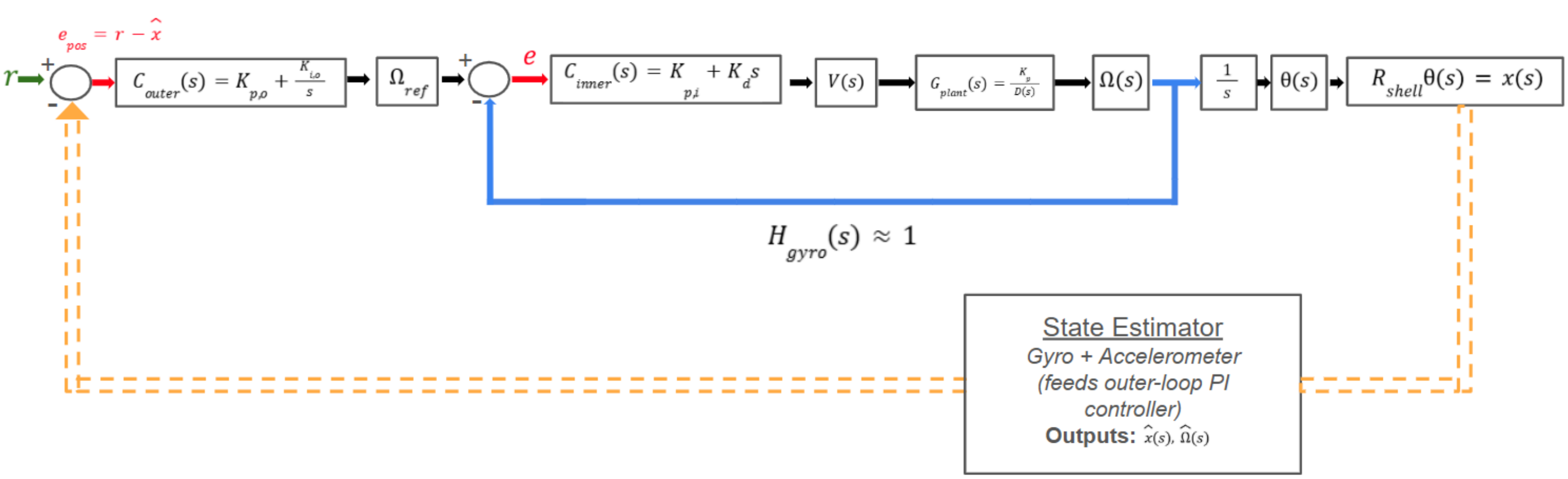
Internal Core Angular Rate: $\frac{\omega(s)}{V(s)} = \frac{\frac{K_t}{LJ_c}}{D(s)}$

Shell Angular Rate: $\frac{\Omega(s)}{V(s)} = \frac{\frac{kK_t}{LJ_s}}{D(s)}$

Shell Angle: $\frac{\theta(s)}{V(s)} = \frac{\frac{kK_t}{LJ_s}}{sD(s)}$

Linear Displacement: $\frac{x(s)}{V(s)} = \frac{R_{shell}\frac{kK_t}{LJ_s}}{sD(s)}$

Knowing the transfer functions, we can visually represent the system flow from Voltage through to the position of the Sphero bot. The overall schematic of this system contains an inner and outer loop. The inner controller uses the gyro feedback to generate a motor voltage command, which is processed by the plant that maps voltage to shell rotation according to the second-order dynamics derived from the ODEs. The outer controller receives estimated position and velocity from the state estimator, which also uses accelerometer measurements, and provides a slower corrective reference that ensures the Sphero tracks the user's commanded motion.



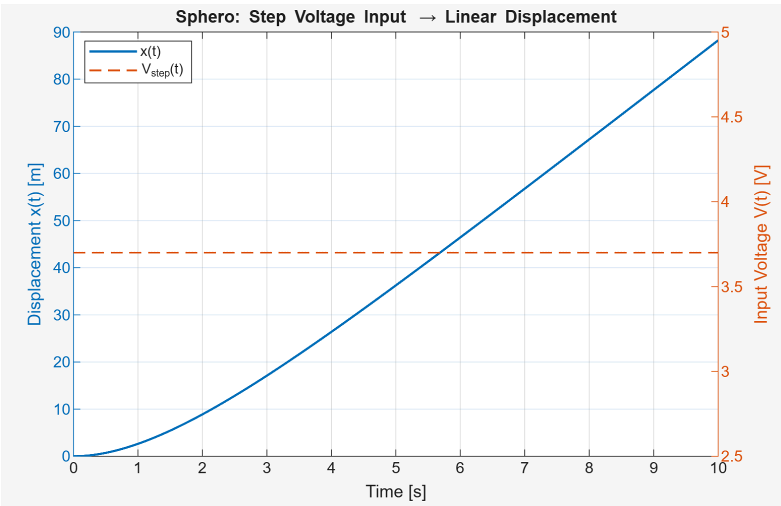
By starting from the physical ODEs that describe the dynamics of the shell, we obtained a set of Laplace-domain transfer functions modeling the behavior of the Sphero. These functions are used to give us an understanding of the control functions, feedback loops, and state estimator contribution through a robust and fully encompassing block diagram.

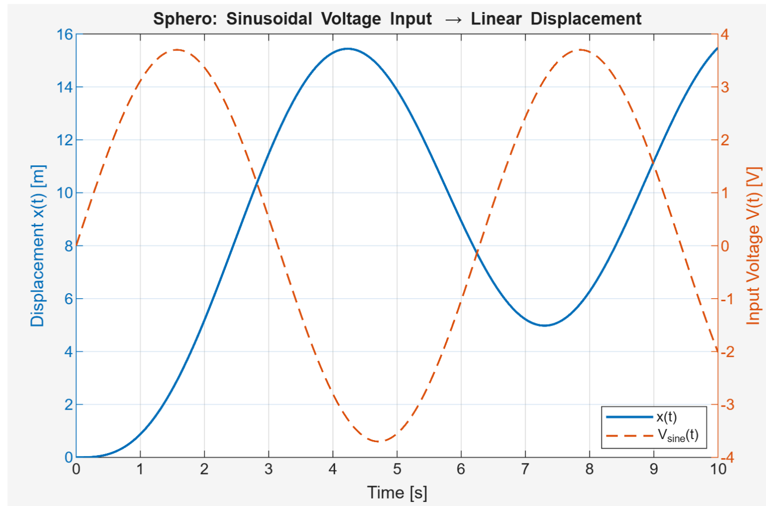
Plots

Now that we've derived ODE's and transfer functions for our system, we can create a plot of input vs output where our input is voltage and our output is linear displacement. Firstly, we must define parameter values. During our dissection, we measured the following shell mass to be 1.5 ounces, inside mass to be 4⅝ ounces, and diameter to be 74 millimeters. In the MATLAB script, we convert these to SI units and use simple rigid body formulas for our inertia values. For the motor, we were able to identify the motor as a Standard FP130-KT/10225 DC 6V-12V 13500RPM High Speed Mini 20mm Electric Motor. Unfortunately we could not find the full electrical specs through research so we estimated the remaining key parameters by comparing it to a very similar 130-size 6 V motor with full specs. Since 130-size motors are quite standardized, it's reasonable to use the following numbers as a proxy for our FP130 in our modeling context. For a brushed DC motor, stall is essentially a short through the windings so $R \approx \frac{V}{I_{stall}} \approx \frac{6\,V}{0.8\,A} \approx 7.5\ \Omega$. At no load using free-run current and free-run speed,

$K_e \approx \frac{V - I_0 R}{\omega} \approx 0.00455\ Vs/rad$. We assumed K_t to be approximately K_e for our brushed DC motor. We were unable to find inductance specs, but assuming an electrical time constant on the order of milliseconds, we get an inductance of around 1 mH.

In the step-input plot, representing a constant input voltage, displacement grows rapidly at first and then transitions into a linear increase. This is ideal for a toy that is designed to settle quickly. In the sinusoidal-input plot, the input voltage oscillates around zero, representing a desire to change directions back and forth. However, the displacement does not oscillate around zero. This represents an increasing phase lag. In terms of controls, this is a result of the integrator dominating because any imbalance in positive versus negative torque integrates in a long-term "drifting mean".





Battery Modeling

The Sphero SPRK uses a small single-cell lithium polymer (Li-Po) battery that runs at 3.7 volts and stores around 350 mAh of energy. When connected to the Sphero, it provides the energy necessary to turn on the robot's internal electronics — including the microcontroller, sensors, — and delivers the high bursts of current necessary to drive the motors. Like other batteries, its voltage changes dramatically when affected by a gradual change in speed, from internal resistance and chemical effects inside the cell. Also, when the motors pull in extra current during acceleration or turns, the battery voltage drops for a moment before slowly recovering, this influencing how stable the robot's power system is. Modeling this is important since it affects performance, the motor response, and overall runtime.

One way of doing this is to treat the battery as a 2nd order system using a dual-RC (two resistor, two capacitor) equivalent circuit. This approach captures how the battery reacts when the Sphero's motor draws more power — such as (above) the quick drop in voltage during acceleration and the slow recovery as the battery chemistry settles. In effect, the battery behaves similarly to a spring-mass damper system with two distinct response speeds, which helps visualize the fast and slow voltage changes captured by the dual-RC model. Modeling it this way predicts how the voltage behaves during operation, simulates power demands, and gives us an understanding of how the Sphero remains stable even when the motors fluctuate.

Figure 1: A Pair of Single-Cell Li-Po Batteries labeled 3.7 V and 350 mAh

Defining Parameters: R_0 = ohmic resistance, R_1 & C_1 = 1st RC transient (fast electrochemical reaction, R_2 & C_2 = 2nd RC transient (slow diffusion effect).

The terminal voltage is:

$$V_{term}(t) = V_{OC}(SOC) - I(t)R_0 - V_1(t) - V_2(t)$$

(SOC = State of Charge)

with:

$$\dot{V}_1 = \frac{-1}{R_1C_1}V_1 + \frac{1}{C_1}I \ \& \ \dot{V}_2 = \frac{-1}{R_2C_2}V_2 + \frac{1}{C_2}I$$

This is a standard 2nd-order linear ODE with poles at:

$$p_1 = \frac{-1}{\tau_1} \ \& \ p_2 = \frac{-1}{\tau_2}$$

with time constants:

$$\tau_1 = R_1C_1 \ \& \ \tau_2 = R_2C_2$$

If current I(t) steps from $0 \rightarrow I_0$, the voltage transient is:

$$V_{term}(t) = V_0 - I_0R_0 - I_0R_1(1- e^{-t/\tau1}) - I_0R_2(1- e^{-t/\tau2})$$

This is identical to a 2nd-order mechanical system, so the battery relaxes after a current step just like how a mass-spring-damper system relaxes after an impulse. ($R_0 = 0.12\ \Omega$)

$$\tau_1 = R_1C_1 = (0.25\ \Omega)(2.8\ F) = 0.7s$$

$$\tau_2 = R_2C_2 = (0.40\ \Omega)(55\ F) = 22s$$

τ_1 affects short bursts (motor load) whilst τ_2 affects long-term voltage sag during motion. Both combine to give the battery a 2nd order exponential relaxation when the robot changes load rapidly (e.g., accelerating or breaking).

If you collapse the two RC branches into a single 2nd order transfer function:

$$G(s) = \frac{\Delta V(s)}{\Delta I(s)} = -\left(R_0 + \frac{R_1}{1+s\tau_1} + \frac{R_2}{1+s\tau_2}\right)$$

Equivalent 2nd order form:

$$G(s) = -\left(R_0 + \frac{K}{s^2+(\frac{1}{\tau_1}+\frac{1}{\tau_2})s+\frac{1}{\tau_1\tau_2}}\right)$$

References

"Standard FP130-KT/10225 DC 6V–12V 13500RPM High Speed Mini 20mm Electric Motor." eBay, https://www.ebay.com/itm/265258629617. Accessed 10 Dec. 2025.

Hu, Xiaoxuan. SLAM with the Sphero Robot. Universitat Politècnica de Catalunya, Escola Tècnica Superior d'Enginyeria Industrial de Barcelona, 2018. Master's thesis.

Disclaimer: ChatGPT was used for various parts of research.