The Brush-Tastic Voyage:
A System Dynamics Model of Electric Toothbrush

MAE 3260 Final Group Work

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Outline/List of MAE 3260 Concepts Used:

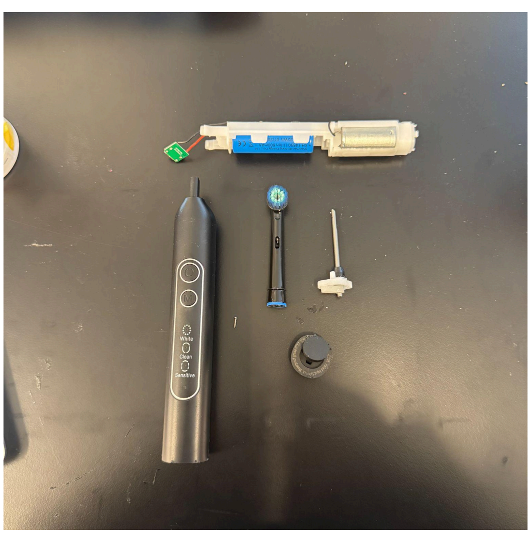
Abstract: We dissected an electric toothbrush to gain a better understanding of the dynamic behavior of a familiar electromechanical device. The brush head is driven by a DC motor that forces it into an oscillatory steady state at constant angular speed. We used mathematical tools such as ordinary differential equations and state space modeling to analyze the transient and steady state response of the toothbrush, with a particular focus on predicting its steady state angular speed. Another focus for our analysis is to understand how the system parameters influence performance. Through our model, we aim to gain insight into the device's overall performance and the impact of design choices on dynamic behavior.

Student	Role/Section	Portfolio Link
Kayoko Thornton	ODEs - Synthesized a set of simplified ODEs, which informed key parameters to note in our dissection. TF -Utilized ODEs to create the transfer function. Bode Plot - Used MATLAB to analyze amplitude and phase response of the system.	https://cornell-mae-ug.github.io/fa25-portfolio-KayokoT/projects/2025-electric-toothbrush/
Rebecca Gerola	Parameter Estimation - Performed parameter estimation to generate numerical estimates to support the final ODE and transfer-function model.	https://cornell-mae-ug.github.io/fa25-portfolio-rebeccagerola-png/projects/Systems-Electric-Toothbrush/

Michael Wywrocki	Block Diagram - Used ODEs, transfer functions, and state-space equations to understand and derive a block diagram model. Matlab Simulation - Used the dissection, parameter estimation, and ODEs to create a matlab model of our open loop system and how it behaves with disturbance.	https://cornell-mae-ug.github.io/fa25-portfolio-Mwywrocki14/projects/Electric%20Toothbrush%20Dynamic%20Model/
Nagamitesh Nagamuralee	State Space - Created a state space model for the system to better understand the outputs of the system.	https://cornell-mae-ug.github.io/spring-2025-portfolio-nagamitesh/projects/electric-toothbrush-dissection/

System Description & Steady State Behavior

Our goal of our system model is to capture the dynamic behavior of our electric toothbrush as it drives the brush head into a steady oscillating motion that uses an open loop DC motor system. This is an important design consideration because the angular speed of the brush head depends on the electrical capabilities of the motor and battery, and the mechanical properties of the tooth brush head like inertia and damping. Understanding these relationships will help us predict how the toothbrush will respond to disturbances like friction against teeth and toothpaste, and what parameters of the system influence the performance of the toothbrush. We model the toothbrush by using coupled electrical and mechanical ordinary differential equations, which describe how the motor current and angular velocity evolve over time. These ODEs help us evaluate steady-state behavior, transfer functions, and system stability without feedback control.



The following data was recorded from the toothbrush:

Battery: 600 mAh
Radius from motor to shaft: 11 mm
Mass of head: 5 grams

Ordinary Differential Equations

To model the electric toothbrush, we used two basic ODEs determined by referencing Group Work 7W, problem 3 [1]. The first equation describes the rotational motion of the head, modeled as a spring-mass-damper system:

$$J\dot{\omega} + b\omega = K_T i + T_d \quad \textbf{(1)}$$

J is the rotational inertia of the head, b is the damping coefficient due to drag and/or friction, i is the motor current, and $K_T i$ is the torque generated by the motor. T_d describes the disturbance torque; for example, from pressing the toothbrush against your teeth. ω is the angular speed, and $\dot{\omega}$ is the change of angular speed over time. Thus, this equation describes how angular speed changes based on electrical input and mechanical load.

The second ODE describes the electrical dynamics of the DC motor:

$$L\frac{di}{dt} + Ri = V_s - K_b\omega \quad \textbf{(2)}$$

L is the inductance of the motor windings and R is the resistance. The current i and its time rate of change $\frac{di}{dt}$ depends on the supply voltage V_s and the back emf $K_b\omega$. The back emf opposes the supply, keeping the current from increasing infinitely as the motor accelerates. The battery capacity of the toothbrush is 600 mAh, but this equation shows that this value defines the limit and cannot be treated as the device's constant current.

Together, these equations represent a coupled electromechanical system.

Parameter Estimation

We estimated the parameters of our system using the measurements above

$$J = mr^2 = 0.005(0.011)^2 = 6.05 \times 10^{-7}\, kg \cdot m^2$$

From the ODEs, we can solve for b at steady state by setting $\dot{\omega} = 0$

$$b\,\omega = K_T i_{ss}$$

$$b = \frac{K_T i_{ss}}{\omega}$$

Find i_{ss} by setting $\frac{di}{dt} = 0$:

$$i_{ss} = \frac{V_{ss} - K_b \omega_{ss}}{R}$$

$$b = \frac{K_T}{\omega} \cdot \frac{V_{ss} - K_b \omega_{ss}}{R}$$

The following parameters were not able to be measured directly (V_s , K_T , K_b L) and were estimated using typical values for small DC motors of comparable size and voltage. Toothbrush motors commonly operate from a single Li-ion cell (≈3.7 V) . Based on the given data from the toothbrush manufacturer, we used

$$V_s = 3.7\,V,\ K_T = K_b = 2 \times 10^{-3} N \cdot m/A, R = 5\Omega,\ L = 1.0 \times 10^{-3} \frac{V \cdot s}{A},\ f = 200\, Hz$$

$$\omega = 2\pi f = 2\pi(200) = 1256\, rad/s$$

$$b = \frac{2 \times 10^{-3} N \cdot m/A}{1256 rad/s} \cdot \frac{3.7\,V - (2 \times 10^{-3} N \cdot m/A \cdot 1256\, rad/s)}{5\,\Omega} = 3.78 \times 10^{-7} N \cdot m \cdot s/rad$$

Mechanical ODE:

$$6.05 \times 10^{-7}\, \dot{\omega} + 3.78 \times 10^{-7} \omega = 0.002 i + T_d$$

Electrical ODE:

$$1.0 \times 10^{-3} \frac{di}{dt} + 5i = 3.7 - 0.002\omega$$

Transfer Function

The transfer function of interest relates the motor voltage V_s and angular speed ω.
First, taking the LaPlace transform of Equation 1, assuming no disturbance, and solving for $\Omega(s)$:

$$\mathcal{L}\{J\dot{\omega} + b\omega = K_T i\}$$

$$Js\Omega(s) + b\Omega(s) = K_T I(s) \rightarrow \Omega(s) = \frac{K_T I(s)}{Js+b}$$

Taking the LaPlace transform of Equation 2, and rearranging to get an equation for I(s):

$$\mathcal{L}\{L\frac{di}{dt} + Ri = V_s - K_b\omega\}$$

$$LsI(s) + RI(s) = V_s(s) - K_b\Omega(s) \rightarrow I(s) = \frac{V_s(s) - K_b\Omega(s)}{Ls+R}$$

Substituting I(s) into the $\Omega(s)$ equation:

$$\Omega(s) = \frac{K_T}{Js+b} \cdot \frac{V_s(s) - K_b\Omega(s)}{Ls+R}$$

Solving for the transfer function:

$$G(s) = \frac{\Omega(s)}{V_s(s)} = \frac{K_T}{(Js+b)(Ls+R)+K_T K_b} = \frac{K_T}{JLs^2+(JR+bL)s+(bR+K_T K_b)}$$

Putting it in the standard second-order form of $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$:

$$G(s) = \frac{\Omega(s)}{V_s(s)} = \frac{K_T/JL}{s^2+(\frac{R}{L}+\frac{b}{J})s+(\frac{bR+K_T K_b}{JL})} = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

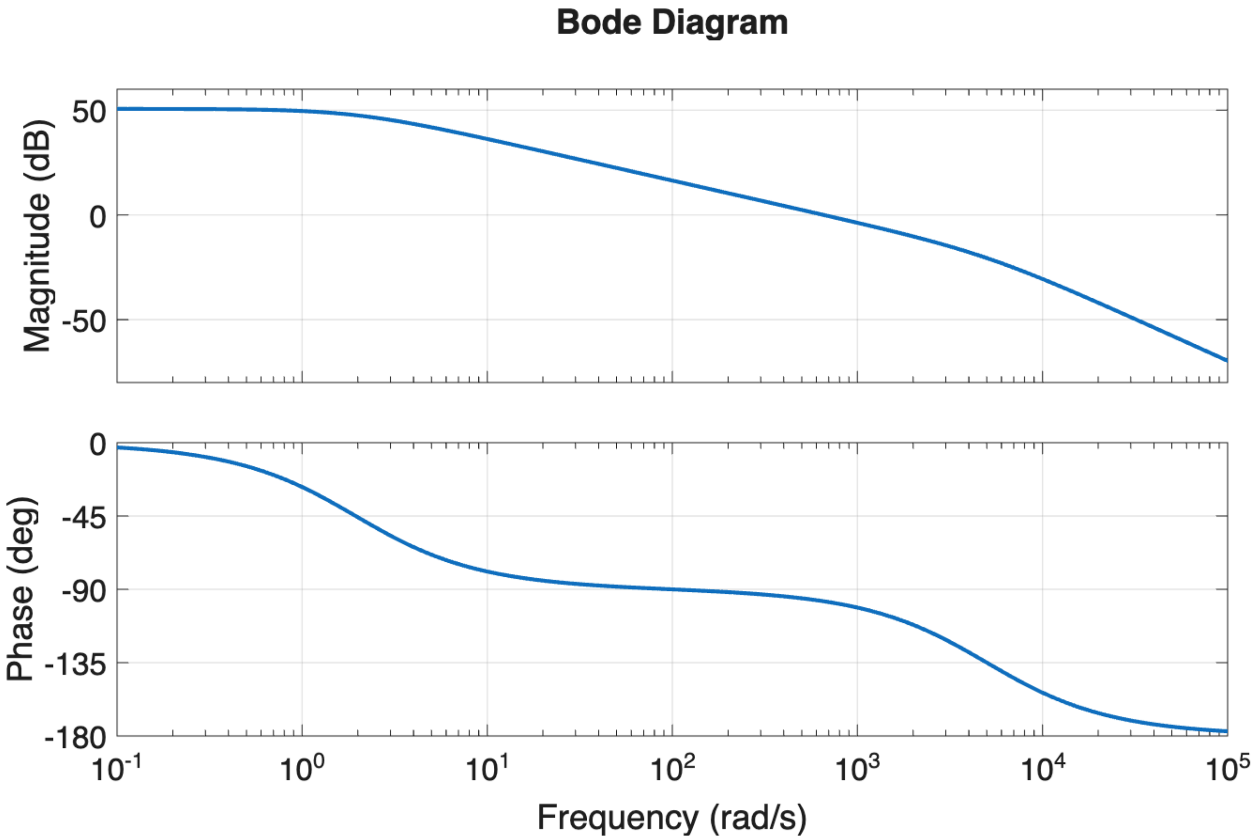
where K is the normalizing constant that equals $\frac{K_T}{bR+K_T K_b}$, $\omega_n^2 = \frac{bR+K_T K_b}{JL}$, and $\zeta = \frac{RJ+bL}{2\sqrt{JL(bR+K_T K_b)}}$

The natural frequency of the head is inversely proportional to the rotational inertia and inductance. In contrast, increasing the damping, torque, back emf constant, or motor resistance increases the natural frequency. For this system, we do not want it to vibrate at the natural frequency, as resonance would intensify vibrations, wear down the components, and hurt the user. Thus, parameters must be carefully selected to keep damping ratio high and the natural frequency away from the motor frequency.

With our estimated values, we find $\zeta =$ 25.34, indicating the system is heavily overdamped. The natural frequency of the head is around 15.6 Hz, way below the operating frequency of about 200 Hz. The brush has no risk of hitting resonance and harming the user.

Bode Plots

Using MATLAB, we graphed the Bode plots for this system:



The magnitude plot shows that at low frequencies, the magnitude is high, allowing the toothbrush to begin spinning quickly even when the applied voltage is low and ramping up. At higher voltage input frequencies, the head lags the voltage input, and eventually barely moves due to its inertia and damping. As expected, there are no resonant peaks as the system is heavily damped. For this system, making a peak appear requires inputting a damping coefficient of essentially zero, as well as electrical resistance over two orders of magnitude lower. Such drastic reductions are extremely unlikely to occur, which ensures the toothbrush is safe from resonance. This allows the toothbrush to spin smoothly and predictably.

State Space

Here we are creating a state space that focuses on creating an output of angular speed. We are starting off with the ODEs created in an earlier part of this report, namely $J\dot{\omega} + b\omega = K_T i + T_d$ and $L\frac{di}{dt} + Ri = V_s - K_b\omega$. We are not including current as one of outputs because we did not have a way to properly characterize and measure the current given by our motor so we decided to only focus on the angular speed of the motor's output shaft.

Based on these ODEs we can find the that our state vector should have two states, one is i to represent the current and this is differentiated into $\frac{di}{dt}$, and the other state variable is the angular velocity ω which is differentiated into $\dot{\omega}$. Thus based on these choices we can find that our state vector $\vec{x}$ will be $\vec{x} = \begin{bmatrix} i \\ \omega \end{bmatrix}$.

Next we need to find the inputs for this system and the inputs here will be the supply voltage which is denoted by V_s as well as the disturbance torque which is denoted as T_d. These variables then can be combined to create the input vector $\vec{u}$, and that is equal to $\vec{u} = \begin{bmatrix} V_S \\ T_d \end{bmatrix}$.

Then we can isolate the derivatives of the state variables to help create our first equation with will involve our A and B matrices to create an equation in the form of $\dot{\vec{x}} = A\vec{x} + B\vec{u}$.

$$J\dot{\omega} + b\omega = K_T i + T_d \Rightarrow J\dot{\omega} = -b\omega + K_T i + T_d \Rightarrow \dot{\omega} = \frac{1}{j}(-b\omega + K_T i + T_d)$$

$$L\frac{di}{dt} + Ri = V_s - K_b\omega \Rightarrow L\frac{di}{dt} = -Ri + V_s - K_b\omega \Rightarrow \frac{di}{dt} = \frac{1}{L}(-Ri + V_s - K_b\omega)$$

Now with these variables isolated, we can then substitute them into the A and B matrices of the form above. Since all the vectors here have a size of 2x1 the matrices A and B will have size of 2x2. We can find that with substitution the equation is as follows.

$$\begin{bmatrix} di/dt \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R/L & -K_b/L \\ K_T/J & -b/J \end{bmatrix}\begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L & 0 \\ 0 & 1/J \end{bmatrix}\begin{bmatrix} V_S \\ T_d \end{bmatrix}$$

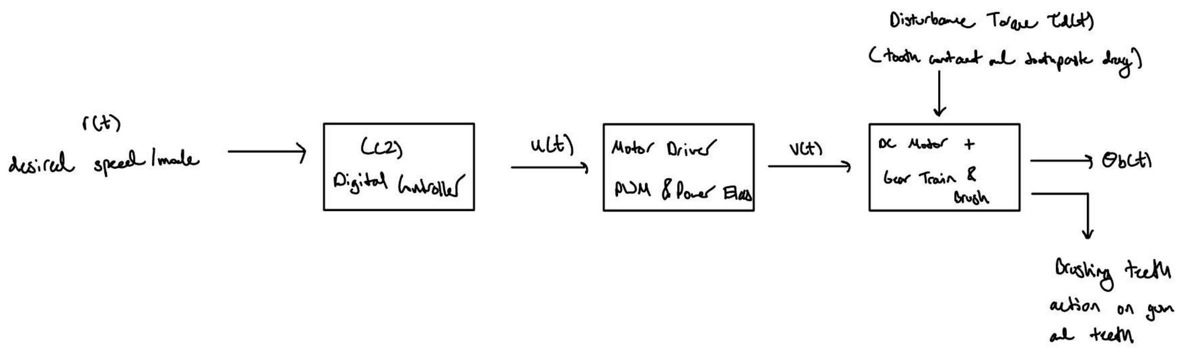
Next we can construct the output equation to relate our output ω to the inputs and the state variables. Our output vector is $\vec{y} = [\omega]$. The output will be in the form $\vec{y} = C\vec{x} + D\vec{u}$. Here since the input and state variable vectors have a size of 2x1 and the output vector has a dimension of 1x1, the C and D matrices will have dimensions of 1x2. Since $\omega = \omega$, we can substitute and find the following.

$$[\omega] = [0\ 1]\begin{bmatrix} i \\ \omega \end{bmatrix} + [0\ 0]\begin{bmatrix} V_S \\ T_d \end{bmatrix}.$$

Thus to define our state space we can define it using the following A, B, C, and D matrices similar to the methods we learned in class.

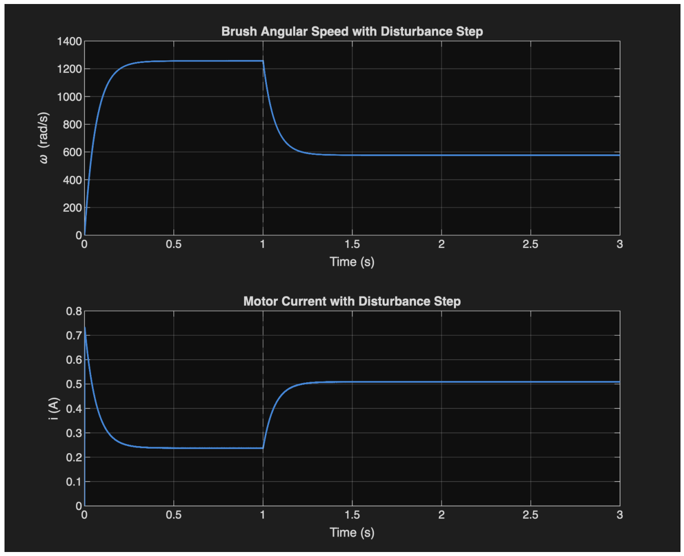
$$A = \begin{bmatrix} -R/L & -K_b/L \\ K_T/J & -b/J \end{bmatrix},\ B = \begin{bmatrix} 1/L & 0 \\ 0 & 1/J \end{bmatrix},\ C = [0\ 1],\ D = [0\ 0]$$

Block Diagram



This block diagram represents our open-loop electric toothbrush system modeling the input user command of what brushing mode is selected to the output of the brushing action of the head of the tooth brush. The first box represents the digital controller that is a microcontroller implementing a PI controller that compares r(t) with measured speed and outputs controller signal u(t). The next box is the motor driver and power electronics which converts lower-power logic signal u(t) to a high-current voltage v(t) for the motor. The DC motor and great train for the brush head is the next box and it is the physical item that converts the voltage v(t) to the brush head angular motion of θb(t). The disturbance of the teeth and toothpaste on the electric toothbrush is modeled at torque_d(t) going into the DC motor and gear train box as it loads torque as an input for the motor block.

MATLAB Simulation



The MATLAB model solves the coupled electrical and mechanical differential equations of the eclectic tooth brush, as it captures how the brush speed and motor current change over time in response to the applied voltage when the input settings is clicked and the disturbance of the teeth and toothpaste are applied to the head of the toothbrush. Using these two ODEs, the matlab code simulates with the graphs the angular velocity and current as the system spins up and encounters the disturbance at an arbitrary time at t=1 seconds. The top plot shows the speed quickly rising once the toothbrush is turned on to what would be a steady state value without disturbance, but then disturbance is applied and the angular velocity is decreased rapidly to a lower steady state value. The bottom plot shows the corresponding motor current as the current is initially spiked to overcome inertia and settles down when there is no disturbance, but when disturbance is applied then the current increases and the motor demands more torque to main rotation against the disturbance. Therefore these two plots help model the electric toothbrush system based on the two ODEs.

References

M. Campbell, “MAE 3260 Group Work 7W: State Space,” Cornell University, course handout, Fall 2025.

ChatGPT. (GPT-5). OpenAI. Accessed: Dec. 8, 2025. [Online]. Available: https://chat.openai.com/chat