System Dynamics (MAE 3260) Final Report: LQR Control of Ball on a Ramp

Jonathan Distler (jhd94@cornell.edu), April Wang (aw763@cornell.edu), Hugo Mazzali (hlm89@cornell.edu)

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This is a final project for MAE 3260 (System Dynamics) course. The group was tasked with learning about and developing a control behavior for a system of our choice. Our group decided on a ball on a beam because of its physical nature, as well as its uniqueness compared with other systems of study throughout the semester. The system is as follows: a ball is placed on a beam at an arbitrary position with an arbitrary velocity, with the beam at an arbitrary angle moving at an arbitrary angular velocity. The goal of our control system is to apply a torque, resembling a linear actuator in practice, to balance the ball before it leaves the ramp. This work represents accumulated knowledge from coursework over the course of this semester, as well as applied work in Solid Body Dynamics MAE 2030. The key topics of study for our system include: Ordinary Differential Equations, State Space, and Feedback Control Law. Although our proposal highlighted the Proportional-Integral-Derivative Control (PID) nature of our control system, the non-linearity and high-orderedness of our system led us to delve into a Linear Quadratic Regulator (LQR) method of control. The project includes a state-space model of the system and a Matlab script simulating the physical system.

1. Modeling the System

A. Dynamics. The system was modeled with 2D rigid body dynamics. In the model, a sphere rolls without slip on a beam that can rotate about its center.

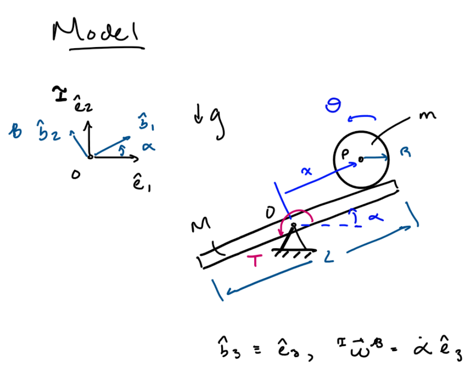

Fig. 1. Model of the ball on beam system.

As shown in figure 1, a beam of mass M and length L is attached at its center point O to a pivot, which it can freely rotate about. The angle α represents the angle of the beam relative to the horizontal. It is also the angle of the $\mathcal{B}$-frame that is fixed to the beam's axis relative to the inertial $\mathcal{I}$-frame.

A sphere with mass m and radius R sits on top of the beam, with the center point of the sphere P being a distance x from O. The sphere rotates about P by an angle of θ.

Finally, there is an applied torque T about O, which we can control in order to achieve our goal of keeping the ball at the center of the beam.

A.1. Kinematics. The kinematic equations for the position $\vec{r}_{P/O}$, velocity ${}^{\mathcal{I}}\vec{v}_{P/O}$, and acceleration ${}^{\mathcal{I}}\vec{a}_{P/O}$ of the center of the ball P relative to center of the beam O in the inertial frame can be written as a function of x and α:

$$\begin{aligned}\vec{r}_{P/O} &= x\hat{b_1} + R\hat{b_2}\\ {}^{\mathcal{I}}\vec{v}_{P/O} &= \dot{x}\hat{b_1} + x\dot{\alpha}\hat{b_2} - R\dot{\alpha}\hat{b_3}\\ {}^{\mathcal{I}}\vec{a}_{P/O} &= (\ddot{x} - x\dot{\alpha}^2 - R\ddot{\alpha})\hat{b_1} + (2\dot{x}\dot{\alpha} + x\ddot{\alpha} - R\dot{\alpha}^2)\hat{b_2}\end{aligned} \quad [1]$$

The angular momenta of the ball ${}^{\mathcal{I}}\vec{h}_P^{(ball)}$ and the beam ${}^{\mathcal{I}}\vec{h}_O^{(beam)}$ about their respective centers can be written as functions of θ and α (as well as their time derivatives):

Ball:

$${}^{\mathcal{I}}\vec{h}_P^{(ball)} = I_{sphere}\dot{\theta}\hat{e_3} \qquad {}^{\mathcal{I}}\frac{d}{dt}{}^{\mathcal{I}}\vec{h}_P^{(ball)} = I_{sphere}\ddot{\theta}\hat{e_3} \quad [2]$$

Beam:

$${}^{\mathcal{I}}\vec{h}_O^{(beam)} = I_{beam}\dot{\alpha}\hat{e_3} \qquad {}^{\mathcal{I}}\frac{d}{dt}{}^{\mathcal{I}}\vec{h}_O^{(beam)} = I_{beam}\ddot{\alpha}\hat{e_3} \quad [3]$$

I_{sphere} is the rotational moment of inertia of a sphere about its center, and I_{beam} is the rotational moment of inertia of a thin beam (rod) about an axis through its center:

$$I_{sphere} = \frac{2}{5}mR^2,\ I_{beam} = \frac{1}{12}ML^2$$

Since the ball is modeled as rolling without slip (visualized in figure 2), $x = -R\theta$, and therefore

$$\ddot{x} = -R\ddot{\theta} \quad [4]$$

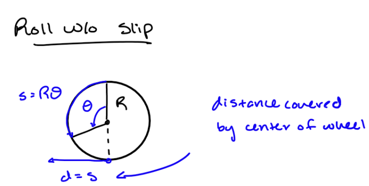

Fig. 2. Rolling without slip visual.

A.2. Free Body Diagrams. As shown in the free body diagrams of the ball and beam in figure 3, the forces acting on each object are the forces due to gravity, as well as the equal and opposite normal and friction forces between the objects. For the beam, there is also a reaction force from the pivot and the applied torque.

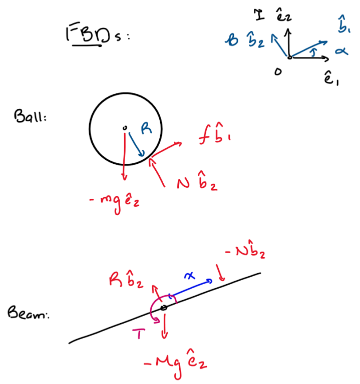

Fig. 3. Free body diagrams of the ball and beam.

A.3. Euler's 1st Law for Ball. Using the acceleration equation [1] and the forces acting on the ball (shown in figure 3), Euler's First Law for Rigid Bodies was applied to the ball.

Euler's 1st Law for Ball:

$$m^{\mathcal{I}}\vec{a}_{P/O} = \vec{F}_P \qquad [5]$$

The components of [5] in the $\mathcal{B}$-frame are:

$$\hat{b_1} : m(\ddot{x} - x\dot{\alpha}^2 - R\ddot{\alpha}) = f - mg\sin\alpha \qquad [6]$$

$$\hat{b_2} : m(2\dot{x}\dot{\alpha} + x\ddot{\alpha} - R\dot{\alpha}^2) = N - mg\cos\alpha \qquad [7]$$

The friction f and normal N forces between the ball and the beam can be found by rearranging equations [6] and [7]:

$$f = m(\ddot{x} - x\dot{\alpha}^2 - R\ddot{\alpha}) + mg\sin\alpha \qquad [8]$$

$$N = m(2\dot{x}\dot{\alpha} + x\ddot{\alpha} - R\dot{\alpha}^2) + mg\cos\alpha \qquad [9]$$

A.4. Euler's 2nd Law for Ball and Beam. To find the equations of motion for the rotation of the ball and beam, Euler's Second Law can be applied to each body using the time derivatives of the angular momenta from equations [2] and [3].

Euler's 2nd Law for Ball:

$$^{\mathcal{I}}\frac{d}{dt}{}^{\mathcal{I}}\vec{h}_P^{(ball)} = \vec{M}_P$$

$$I_{sphere}\ddot{\theta} = -fR \qquad [10]$$

Plugging [8] and the rolling without slip relation [4] into [10] and simplifying, the equation of motion for the linear acceleration of the center of the ball relative to the center of the beam is:

$$\boxed{\ddot{x} = -\frac{5}{3}(x\dot{\alpha}^2 - R\ddot{\alpha} + g\sin\alpha)} \qquad [11]$$

Euler's 2nd Law for Beam:

$$^{\mathcal{I}}\frac{d}{dt}{}^{\mathcal{I}}\vec{h}_O^{(beam)} = \vec{M}_O$$

$$I_{beam}\ddot{\theta} = -Nx + T \qquad [12]$$

Plugging [9] into [12] and simplifying, the equation of motion for the angular acceleration of the system about the center of the beam is:

$$\boxed{\ddot{\alpha} = \frac{1}{I_{beam} + mx^2}[-mx(2\dot{x}\dot{\alpha} - R\dot{\alpha}^2 + g\cos\alpha) + T]} \qquad [13]$$

B. State Space Model. A linear dynamical system can be written in state space form:

$$\dot{\vec{z}} = A\vec{z} + B\vec{u}$$

$$\vec{y} = C\vec{z} + D\vec{u}$$

where $\vec{z}$ is the state, $\vec{u}$ is the input, and $\vec{y}$ is the output.

In our system, the only input is the input torque T, and our desired output is the distance of the ball from the center of the beam x. The states are the linear and angular positions and velocities:

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ \alpha \\ \dot{x} \\ \dot{\alpha} \end{bmatrix}, \text{ so } \dot{\vec{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \ddot{x} \\ \ddot{\alpha} \end{bmatrix}$$

Thus, based on 11 and 13, the first order ODEs that represent the system are:

$$\dot{z}_1 = z_3$$

$$\dot{z}_2 = z_4$$

$$\dot{z}_3 = -\frac{5}{3}(z_1 z_4^2 - R\dot{z}_4 + g\sin z_2)$$

$$\dot{z}_4 = \frac{mz_1(2z_3z_4 - Rz_4^2 + g\cos z_2)}{I_{beam} + mz_1^2} + \frac{T}{I_{beam} + mz_1^2}$$

B.1. Linearization. It can be seen that $\dot{z}_3$ and $\dot{z}_4$ are non-linear functions, so they must be linearized in order to approximate them for a linear state space model. This can be achieved with linear Taylor Series expansions about the equilibrium $\{R\}$, which can then be simplified by evaluating them at the equilibrium condition [1]. This result can be used to model the system response at small deviations from equilibrium, as the further the state is from equilibrium, the less accurate the linear approximation is.

In our model, the equilibrium point is $\{R\} = \{z_1 = 0, z_2 = 0, z_3 = 0, z_4 = 0\}$ since the system is at steady state when neither the ball nor the beam are moving, the ball is at the center of the beam, and the beam is fully horizontal.

Taylor Expansion about $\{R\}$

Partial derivatives of $\dot{z}_4$	Partial derivatives of $\dot{z}_3$
$\frac{\partial \dot{z}_4}{\partial z_1}\Big\vert_{\{R\}} = \frac{-mg}{I_{beam}}$	$\frac{\partial \dot{z}_3}{\partial z_1}\Big\vert_{\{R\}} = (\frac{5R}{3})\frac{-mg}{I_{beam}}$
$\frac{\partial \dot{z}_4}{\partial z_2}\Big\vert_{\{R\}} = 0$	$\frac{\partial \dot{z}_3}{\partial z_2}\Big\vert_{\{R\}} = -\frac{5}{3}g$
$\frac{\partial \dot{z}_4}{\partial z_3}\Big\vert_{\{R\}} = 0$	$\frac{\partial \dot{z}_3}{\partial z_3}\Big\vert_{\{R\}} = 0$
$\frac{\partial \dot{z}_4}{\partial z_4}\Big\vert_{\{R\}} = 0$	$\frac{\partial \dot{z}_3}{\partial z_4}\Big\vert_{\{R\}} = 0$
$\frac{\partial \dot{z}_4}{\partial T}\Big\vert_{\{R\}} = \frac{1}{I_{beam}}$	$\frac{\partial \dot{z}_3}{\partial T}\Big\vert_{\{R\}} = 0$

Putting these results into state space form and exchanging the state variables for the system variables,

State Space Model of Ball on Beam System

$$\begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \ddot{x} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (\frac{5R}{3})\frac{-mg}{I_{beam}} & -\frac{5}{3}g & 0 & 0 \\ \frac{-mg}{I_{beam}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \dot{x} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_{beam}} \end{bmatrix} \begin{bmatrix} T \end{bmatrix}$$

$$\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ \dot{x} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} T \end{bmatrix}$$

where again, $I_{beam} = \frac{1}{12}ML^2$

Linear Quadratic Regulator

Drawbacks to PID Control. As seen from our group's state-space model, the system is represented by a fourth-order ODE. Our original proposal included a block-diagram with transfer functions and a PID control law [2, 3, 4]. We thought the system might be first or second order; PID control could then add a pole yielding second- or third-order dynamics, which we could manage via dominant pole placement or tuning P, I, D terms to avoid overshoot. However, because our system is fourth order, these assumptions do not hold, and applying PID would significantly complicate the controller. Moreover, our system exhibits non-linearities from coupling of the angular motion of the beam and the translational motion of the ball. Although we linearized around the midpoint of our ramp via a Taylor-series expansion, the resulting system remained too complex for effective PID control. For these reasons, we adopted a Linear Quadratic Regulator (LQR) approach.

LQR Theory. The LQR control system works well for linearized systems and is especially suited for higher-order dynamics. It is designed to minimize a cost function, represented by a weighted sum of the state and control inputs [5]. Given a linear system

$$\dot{x}(t) = A\,x(t) + B\,u(t),$$

and cost functional

$$J = \int_0^\infty \left(x^\top Q\,x + u^\top R\,u\right) dt,$$

the control law takes the form

$$u(t) = -Kx(t),$$

where the gain matrix K is chosen to minimize J. The MATLAB function `lqr` computes K given the matrices A, B, Q and scalar (or matrix) R [6].

Future Work. Future work could involve tuning the Q matrix and the R constant. In our current implementation, we chose

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R = 1.$$

This is relatively aggressive; selecting a smaller R will yield faster convergence but at the cost of higher control effort. Tuning Q and R systematically could produce different trade-offs between controller performance and effort. Additionally, because our model includes a disturbance torque, there is a non-zero steady-state error. While our immediate objective was merely to stabilize the ball, an interesting extension would be to augment the controller with integral action to eliminate steady-state error — for instance, to hold the ball at the center of the beam.

Matlab Simulation and Visualization

Following the derivation of the LQR controller, we implemented a simulation and visualizer in MATLAB to examine the system response for given initial conditions.

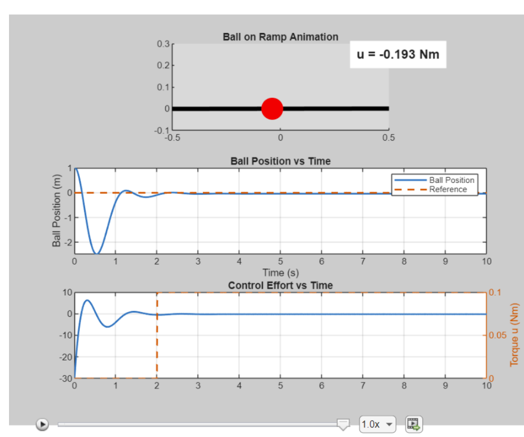

Fig. 4. Visualization of the ball-on-beam system under LQR control: the beam rotates to balance the ball, the ball moves until equilibrium is reached.

Setup. Our script asks the user for the initial conditions for the ball position [m], ramp angle [rad], ball velocity [m/s], and ramp angular velocity [rad/s]. Hard-coded parameters of the system are the ball mass (0.25 kg), ball radius (0.0125 m), acceleration due to gravity (9.8 m/s^2), ramp mass (1 kg), ramp length (1 m), moment of inertia ($I = (1/12)ML^2$), and alpha ($\alpha = -(mg)/I$). These equations for I and α are used because the ramp is being modeled as a rigid rod pivoting about its center. For animating the rotation of the ramp and the motion of the ball, the script maps from state to 2D coordinates.

Animation. The animation is comprised of 3 windows. ***Figure 1.1*** shows an animation that displays the beam reaching a steady state balance with the ball for the given initial conditions. The ball moves into the frame with its initial velocity and position, and the beam rotates about its center to balance the ball. This animation shows the back and forth motion of the beam as it overshoots, oscillates, and then reaches steady state. ***Figure 1.2*** shows a graph of the ball position over time. It is not animated, so it only shows once the simulation is complete. With position(0) being the center axis of the beam, you can quantify the balls motion as it oscillates around the center of the beam while the beam responds with a balancing torque. ***Figure 1.3*** shows the control effort over time.

References

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