Transport Equation: $^N\frac{d}{dt}\vec{v} = {}^B\frac{d}{dt}\vec{v} + {}^N\vec{\omega}^B \times \vec{v}$

1. Use the Transport Theorem to show that, in general, $\frac{^B d}{dt}\boldsymbol{\omega}^{B/N} = \frac{^N d}{dt}\boldsymbol{\omega}^{B/N}$. In our more compact notation, that equation is also written as $\overset{B}{\boldsymbol{\omega}}{}^{B/N} = \overset{N}{\boldsymbol{\omega}}{}^{B/N}$

$$^N\frac{d}{dt}\,{}^N\vec{\omega}^B = {}^B\frac{d}{dt}\,{}^N\vec{\omega}^B + \cancelto{0}{{}^N\vec{\omega}^B \times {}^N\vec{\omega}^B}, \text{ so } {}^N\frac{d}{dt}\,{}^N\vec{\omega}^B = {}^B\frac{d}{dt}\,{}^N\vec{\omega}^B$$

For problems 2-5, Let $\boldsymbol{z} = 3t\boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 1t\boldsymbol{b}_3$ and $\boldsymbol{\omega}^{B/N} = 4t\boldsymbol{b}_2$.

2. Find $\overset{B}{\boldsymbol{z}}$.
3. Find $\overset{N}{\boldsymbol{z}}$.
4. Find $\overset{BB}{\boldsymbol{z}}$.
5. Find $\overset{NN}{\boldsymbol{z}}$.

2. $\boxed{{}^B\frac{d}{dt}\vec{z} = 3\hat{b}_1 + \hat{b}_3}$

3. ${}^N\frac{d}{dt}\vec{z} = {}^B\frac{d}{dt}\vec{z} + {}^N\vec{\omega}^B \times \vec{z}$

$= 3\hat{b}_1 + \hat{b}_3 + \left[4t\hat{b}_2 \times (3t\hat{b}_1 + 2\hat{b}_2 + t\hat{b}_3)\right]$

$= 3\hat{b}_1 + \hat{b}_3 - 12t^2\hat{b}_3 + 4t^2\hat{b}_1$, $\boxed{{}^N\frac{d}{dt}\vec{z} = (3+4t^2)\hat{b}_1 + (1-12t^2)\hat{b}_3}$

4. ${}^B\frac{d^2}{dt^2}\vec{z} = {}^B\frac{d}{dt}\left({}^B\frac{d}{dt}\vec{z}\right) = {}^B\frac{d}{dt}(3\hat{b}_1 + \hat{b}_3) = \boxed{0 = {}^B\frac{d^2}{dt^2}}$

5. ${}^N\frac{d^2}{dt^2}\vec{z} = {}^N\frac{d}{dt}\left({}^N\frac{d}{dt}\vec{z}\right) = {}^B\frac{d}{dt}\left({}^N\frac{d}{dt}\vec{z}\right) + {}^N\vec{\omega}^B \times \left({}^N\frac{d}{dt}\vec{z}\right)$

$= 8t\hat{b}_1 - 24t\hat{b}_3 + (12t + 16t^3)(-\hat{b}_3) + (4t - 48t^3)\hat{b}_1 = \boxed{(12t - 48t^3)\hat{b}_1 - (36t + 16t^3)\hat{b}_3 = {}^N\frac{d^2}{dt^2}\vec{z}}$

7. OPTIONAL A camera is mounted on a spacecraft that is spinning with an angular velocity of $\boldsymbol{\omega}^{B/N} = 3\boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 1\boldsymbol{b}_3$ rad/sec. The camera's boresight $\boldsymbol{c}$ is directed along $\boldsymbol{b}_1$. What is the inertial rate of change of this boresight? I.e., compute $\frac{^N d}{dt}\boldsymbol{c}$. Write the answer in terms of the $\boldsymbol{b}_i$ basis vectors.
8. Give an example of a nonzero angular-velocity vector that allows the camera of the previous problem to remain pointed along an axis fixed in N.
9. Compute the inertial acceleration of this boresight if it is mounted on a spacecraft with an angular velocity $\boldsymbol{\omega}^{B/N} = 3t\boldsymbol{b}_1 + 2t^2\boldsymbol{b}_2 + 1\boldsymbol{b}_3$. I.e., compute $\frac{^N d^2}{dt^2}\boldsymbol{p}$. Write the answer in terms of the $\boldsymbol{b}_i$ basis vectors.

7. ${}^N\vec{\omega}^B = 3\hat{b}_1 + 2\hat{b}_2 + \hat{b}_3$

$\vec{c} = \hat{b}_1$

${}^N\frac{d}{dt}\vec{c} = \cancelto{0}{{}^B\frac{d}{dt}\vec{c}} + {}^N\vec{\omega}^B \times \vec{c}$

${}^N\frac{d}{dt}\vec{c} = (3\hat{b}_1 + 2\hat{b}_2 + \hat{b}_3) \times \hat{b}_1 = \boxed{-2\hat{b}_3 + \hat{b}_2 = {}^N\frac{d}{dt}\vec{c}}$

8. If the camera is fixed in N, ${}^N\frac{d}{dt}\vec{c} = 0 = \cancelto{0}{{}^B\frac{d}{dt}\vec{c}} + {}^N\vec{\omega}^B \times \vec{c} = \underline{{}^N\vec{\omega}^B \times \vec{c} = 0}$ To be 0, ${}^N\vec{\omega}^B = a\vec{c}$, where a is any scalar.

9. Compute the inertial acceleration of this boresight if it is mounted on a spacecraft with an angular velocity $\boldsymbol{\omega}^{BN} = 3t\boldsymbol{b}_1 + 2t^2\boldsymbol{b}_2 + 1\boldsymbol{b}_3$. I.e., compute $\frac{^N d^2}{dt^2}\boldsymbol{p}$. Write the answer in terms of the $\boldsymbol{b}_i$ basis vectors.

$$\frac{^N d}{dt}\vec{C} = \cancel{\frac{^B d}{dt}\vec{C}}^{\,0} + {}^N\vec{\omega}^B \times \vec{C} = (3t\hat{b}_1 + 2t^2\hat{b}_2 + \hat{b}_3)\times\hat{b}_1 = -2t^2\hat{b}_3 + \hat{b}_2 = \frac{^N d}{dt}\vec{C}$$

$$\frac{^N d^2}{dt^2}\vec{C} = \frac{^N d}{dt}\left(\frac{^N d}{dt}\vec{C}\right) = \frac{^B d}{dt}\left(\frac{^N d}{dt}\vec{C}\right) + {}^N\vec{\omega}^B\times\left(\frac{^N d}{dt}\vec{C}\right) = -4t\hat{b}_3 + (3t\hat{b}_1 + 2t^2\hat{b}_2 + \hat{b}_3)\times(-2t^2\hat{b}_3 + \hat{b}_2)$$

$${}^N\vec{\omega}^B \times \frac{^N d}{dt}\vec{C} = \begin{vmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\ 3t & 2t^2 & 1 \\ 0 & 1 & -2t^2 \end{vmatrix} = \hat{b}_1(-4t^4 - 1) - \hat{b}_2(-6t^3) + \hat{b}_3(3t - 0)$$

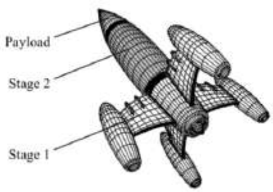
$$\frac{^N d^2}{dt^2}\vec{C} = -(4t^2+1)\hat{b}_1 + 6t^3\hat{b}_2 - t\hat{b}_3$$

10. **(10 points)** What is the total ΔV available from a two-stage launch vehicle with the following characteristics?

Mass of Payload: 1000 kg

Stage 1: Mass of Propellant = 100,000 kg
Mass of Stage 1 Structure = 5,000 kg
Isp = 300 s

Stage 2: Mass of Propellant = 15,000 kg
Mass of Stage 2 Structure = 4,000 kg
Isp = 450 s



11. Which is true of the James Webb Space Telescope?

I. Its orbit will cause it to enter Earth eclipse once per day during two months out of the year.

II. It is located at one of the Lagrange points for the Earth-Sun system.

III. Its orbit puts it at risk of collisions with "Jupiter Trojan" asteroids.

IV. The industrialist S. R. Hadden devised the initial concept for the spacecraft when it was known as the "Next Generation Space Telescope" and later won the valuable Systems Integration role for his company, Hadden Industries.

① dry mass = 5000 kg
propellant mass = 100000 kg
$I_{sp} = 300$ s
$m_0 = 125000$
$m_f = 25000$

$$\Delta V_1 = g_0 I_{sp} \ln\left(\frac{m_0}{m_f}\right)$$

$\Delta V_1 = 4736.575776$ m/s

② dry mass = 4000 kg
propellant mass = 15000 kg
$I_{sp} = 450$ s
$m_0 = 20000$ kg
$m_f = 5000$ kg

$$\Delta V_2 = g_0 I_{sp} \ln\left(\frac{m_0}{m_f}\right)$$

$\Delta V_2 = 6119.796$ m/s

$$\Delta V_{tot} = \Delta V_1 + \Delta V_2 \Longrightarrow \boxed{\Delta V_{tot} = 10856.37223 \text{ m/s}}$$

For the following problems, use the direction-cosine matrix ${}^{N}Q^{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$.

1. Consistent with Euler's Theorem, the B axes are related to the N axes by a rotation about some single axis by some angle.
   a. What is that axis?
   b. What is that angle?

2. Given ${}^{B}v = \begin{bmatrix} 0 \\ -1/2 \\ \sqrt{3}/2 \end{bmatrix}$, what is ${}^{N}v$?

3. Given ${}^{B}v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, what is ${}^{N}v$?

4. What is ${}^{B}Q^{N}$?

5. Given ${}^{N}v = \begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix}$, what is ${}^{B}v$?

6. Given ${}^{N}v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, what is ${}^{B}v$?

7. Let ${}^{A}Q^{N} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is ${}^{A}Q^{B}$?

8. Let ${}^{A}Q^{N} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and ${}^{B}v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. What is ${}^{A}v$?

NEXT PAGE

1.

a) By inspection, ${}^{B}\vec{a} = {}^{N}\vec{a}$ when

$$\boxed{{}^{B}\vec{a} = {}^{N}\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

b) We can write ${}^{N}Q^{B}$ as

${}^{N}Q^{B} = e^{-\phi a^{\times}}$, so

$$\log({}^{N}Q^{B}) = -\phi a^{\times}$$

$$\operatorname{logm}({}^{N}Q^{B}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.5236 \\ 0 & 0.5236 & 0 \end{bmatrix}$$

Matlab logm

$$-\phi a^{\times} = \begin{bmatrix} 0 & \phi a_3 & -\phi a_2 \\ -\phi a_3 & 0 & \phi a_1 \\ \phi a_2 & -\phi a_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.5236 \\ 0 & 0.5236 & 0 \end{bmatrix}$$

So, $\phi a_1 = -0.5236$, and $\therefore$ $\phi = -0.5236 \text{ rad} = \boxed{-\frac{\pi}{6} \text{ rad} = -30 \text{ deg}}$

2. $${}^{N}\vec{v} = {}^{N}Q^{B}\,{}^{B}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ -1/2 \\ \sqrt{3}/2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} = {}^{N}\vec{v}}$$

3. $${}^{N}\vec{v} = {}^{N}Q^{B}\,{}^{B}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix} = {}^{N}\vec{v}}$$

4. Columns form an ONB

$${}^{B}Q^{N} = ({}^{N}Q^{B})^{-1} = ({}^{N}Q^{B})^{T}$$

$$\boxed{{}^{B}Q^{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}}$$

5. $${}^{N}\vec{v} = {}^{N}Q^{B}\,{}^{B}\vec{v} \Rightarrow {}^{B}\vec{v} = ({}^{N}Q^{B})^{-1}\,{}^{N}\vec{v} \Rightarrow {}^{B}\vec{v} = {}^{B}Q^{N}\,{}^{N}\vec{v}$$

$${}^{B}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix} = {}^{B}\vec{v}}$$

6.

$$ {}^{B}\vec{v} = {}^{B}_{\ }Q^{NN}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3}/2 \\ -1/2 \end{bmatrix} = {}^{B}\vec{v} $$

7.

$$ {}^{A}Q^{B} = {}^{A}Q^{NN}Q^{B} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/4 & -\sqrt{2}/4 \\ -\sqrt{2}/2 & \sqrt{6}/4 & -\sqrt{2}/4 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} = {}^{A}Q^{B} $$

8.

$$ {}^{A}\vec{v} = {}^{A}Q^{BB}\vec{v} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/4 & -\sqrt{2}/4 \\ -\sqrt{2}/2 & \sqrt{6}/4 & -\sqrt{2}/4 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/4 \\ \sqrt{6}/4 \\ 1/2 \end{bmatrix} = {}^{A}\vec{v} $$

In the following problems, the basis vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ are unit vectors fixed in the spacecraft (or, equivalently, fixed in a reference frame B that rotates with the spacecraft). The basis vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ are unit vectors fixed in an inertial reference frame N. The angular-velocity vector $\boldsymbol{\omega}^{B/N}$ represents the rotation of frame B with respect to an inertial reference frame N.

9. Use the direction-cosine matrix provided below to represent the vector $\boldsymbol{c} = 3\boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 1\boldsymbol{b}_3$ in terms of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$.

$$ {}^{N}Q^{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right) \\ 0 & -\sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} $$

10. Consider a vector $\boldsymbol{c}$ that is constant in a spacecraft-fixed reference frame B. In terms of the $\boldsymbol{b}_i$ basis vectors, $\boldsymbol{c} = 2\boldsymbol{b}_1 + 1\boldsymbol{b}_2$. In this problem, $\boldsymbol{\omega}^{B/N} = 3\boldsymbol{b}_3$.

a. Find $\overset{B}{\dot{\boldsymbol{c}}}$

b. Find $\overset{N}{\dot{\boldsymbol{c}}}$

10b

c. Represent the answer for problem [redacted] in terms of the $\boldsymbol{b}_i$ basis vectors.

d. Represent the answer for problem [redacted] in terms of the $\boldsymbol{n}_i$ basis vectors at an instant when

10b

$$ {}^{N}Q^{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}. $$

9.

$$ {}^{B}\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} $$

$$ {}^{N}\vec{c} = {}^{N}Q^{B}\,{}^{B}\vec{c} $$

$$ {}^{N}\vec{c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right) \\ 0 & -\sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} $$

$$ {}^{B}\vec{c} = \begin{bmatrix} 3 \\ 2\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right) \\ -2\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) \end{bmatrix} $$

10.

$$ {}^{B}\vec{c} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix};\quad {}^{B}\vec{\omega}^{B/N} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} $$

a) $\overset{B}{\dot{\vec{c}}} = \frac{{}^{B}d}{dt}\vec{c} = 0$ ($\vec{c}$ is constant in the spacecraft body frame).

b) $\overset{N}{\dot{\vec{c}}} = \frac{{}^{N}d}{dt}\vec{c} = \frac{{}^{B}d}{dt}\vec{c} + \vec{\omega}^{B/N} \times \vec{c} = 0 + {}^{B}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \times {}^{B}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = {}^{B}\begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} = \frac{{}^{N}d}{dt}\vec{c}$

transport thm

c) $\frac{{}^{N}d}{dt}\vec{c} = {}^{B}\begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}$, already represented using B frame coordinates.

d) $${}^{N}\dot{\vec{c}} = {}^{N}Q^{B}\,{}^{B}\dot{\vec{c}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} = {}^{N}\begin{bmatrix} -3 \\ 3\sqrt{3} \\ 3 \end{bmatrix} = {}^{N}\dot{\vec{c}}$$

11. Which one of the following matrices can be a direction-cosine matrix (i.e. it meets all the requirements for being a member of the algebraic group SO(3), the right-handed, orthonormal 3x3 matrices)?

~~I.~~ $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det(Q) = -1 \neq 1 \therefore \underline{\text{NOT}}\ SO(3)$

~~II.~~ $Q = \begin{bmatrix} 1/3 & -1/3 & -8/9 \\ 2/3 & 2/3 & 0 \\ 2/3 & -2/3 & 4/9 \end{bmatrix}$ $\det(Q) = 0.9877 \neq 1 \therefore \underline{\text{NOT}}\ SO(3)$

III. $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det(Q) = 0$, and columns form an orthonormal basis $\therefore SO(3)$

~~IV.~~ $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$ $\det(Q) = 0.5 \neq 1 \therefore \underline{\text{NOT}}\ SO(3)$

Multiple Choice. Select only one answer for each question and indicate your choice by circling the Roman numeral next to the chosen answer.

1. $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ are mutually orthogonal unit vectors. Let $\boldsymbol{d} = 2\boldsymbol{n}_1 + 4\boldsymbol{n}_2$ and $\boldsymbol{\omega} = \boldsymbol{n}_1 + 2\boldsymbol{n}_2$. What is $\boldsymbol{\omega} \times \boldsymbol{d}$?

   i. $\boldsymbol{\omega} \times \boldsymbol{d} = 0$
   ii. $\boldsymbol{\omega} \times \boldsymbol{d} = 3\boldsymbol{n}_1 + 6\boldsymbol{n}_2$
   iii. $\boldsymbol{\omega} \times \boldsymbol{d} = 2\boldsymbol{n}_1 + 8\boldsymbol{n}_2$
   iv. $\boldsymbol{\omega} \times \boldsymbol{d} = 8\boldsymbol{n}_3$

   $\vec{d} = 2\vec{\omega}$, so $\vec{d} \parallel \vec{\omega}$
   and $\therefore \vec{\omega} \times \vec{d} = 0$

2. Consider two vectors, $\boldsymbol{a}$ and $\boldsymbol{c}$. Let $\boldsymbol{a} \times \boldsymbol{c} = 3\boldsymbol{n}_1$. What is $\boldsymbol{c} \times \boldsymbol{a}$?

   i. $-3\boldsymbol{n}_1$
   ii. $3\boldsymbol{n}_1$
   iii. Cannot be determined from the information given

   $\vec{a} \times \vec{c} = -(\vec{c} \times \vec{a})$

3. Which of the following is a statement of the Transport Theorem?

   i. $\frac{{}^{B}d}{dt}\boldsymbol{v} \equiv {}^{B}\boldsymbol{v}$
   ii. ${}^{B}\boldsymbol{c} = \boldsymbol{0}$
   iii. $\Delta V_i = g I_{sp,i} \ln \frac{m_{0,i}}{m_{f,i}}$
   iv. ${}^{N}\boldsymbol{c} = {}^{B}\boldsymbol{c} + \boldsymbol{\omega}^{B/N} \times \boldsymbol{c}$

   $\frac{{}^{N}d}{dt}\vec{c} = \frac{{}^{B}d}{dt}\vec{c} + \vec{\omega}^{B/N} \times \vec{c}$

Write a MATLAB script to solve the following:

1. Find the principal axes and principal moments of inertia of the inertia matrix

$$^{B}I = {}^{B}\begin{bmatrix} 110 & 0 & -20 \\ 0 & 130 & 20 \\ -20 & 20 & 120 \end{bmatrix} \text{kg m}^2$$

Report the principal axes as a DCM that transforms from the B coordinates to the principal (P) coordinates, i.e. ${}^{P}Q^{B}$. Make this script general, capable of performing this calculation for any inertia matrix. You can assume that the input is a symmetric, real, 3x3 matrix, although those three conditions would also be nice to check if you want to use this script for other purposes in the future.

Include in this script a verification that the inertia matrix is physically realizable, i.e. that it conforms to the so-called triangle inequality.

Outline:

Principal moments of inertia are the eigenvalues of ${}^{B}I$

Principal axes are the eigenvectors of ${}^{B}I$, which are ${}^{B}\vec{p}_1$, ${}^{B}\vec{p}_2$, ${}^{B}\vec{p}_3$

$${}^{B}Q^{P} = \begin{bmatrix} {}^{B}\vec{p}_1 & {}^{B}\vec{p}_2 & {}^{B}\vec{p}_3 \end{bmatrix}, \quad {}^{P}Q^{B} = \left({}^{B}Q^{P}\right)^T$$

$${}^{P}I = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}; \quad \begin{matrix} I_{11} + I_{22} \geq I_{33} \\ I_{11} + I_{33} \geq I_{22} \\ I_{22} + I_{33} \geq I_{11} \end{matrix}$$

CODE:

```
function[P_I, PQB] = getPI(B_I)

[BQP, P_I] = eig(B_I)

%check that result is physically realizable with triangle inequality:

if P_I(1,1)+P_I(2,2)>=P_I(3,3) &...
   P_I(1,1)+P_I(3,3)>=P_I(2,2) &...
   P_I(2,2)+P_I(3,3)>=P_I(1,1)

    PQB = BQP.' %take transpose of eigenvector matrix
else
    error('Input is not physically realizable according to the triangle inequality.')
end
```

```
B_I = [110,0,-20; 0,130,20; -20,20,120]
```

```
B_I = 3x3
   110     0   -20
     0   130    20
   -20    20   120
```

```
[P_I, PQB] = getPI_MS(B_I)
```

```
BQP = 3x3
   -0.6667    0.6667   -0.3333
    0.3333    0.6667    0.6667
   -0.6667   -0.3333    0.6667
P_I = 3x3
   90.0000         0         0
         0  120.0000         0
         0         0  150.0000
PQB = 3x3
   -0.6667    0.3333   -0.6667
    0.6667    0.6667   -0.3333
   -0.3333    0.6667    0.6667
P_I = 3x3
   90.0000         0         0
         0  120.0000         0
         0         0  150.0000
PQB = 3x3
   -0.6667    0.3333   -0.6667
    0.6667    0.6667   -0.3333
   -0.3333    0.6667    0.6667
```

Principal moments of inertia

${}^{P}Q^{B}$

Write a MATLAB/Simulink function to calculate each of the following in general. Or write MATLAB scripts if you don't want to learn Simulink, but you really should learn Simulink.

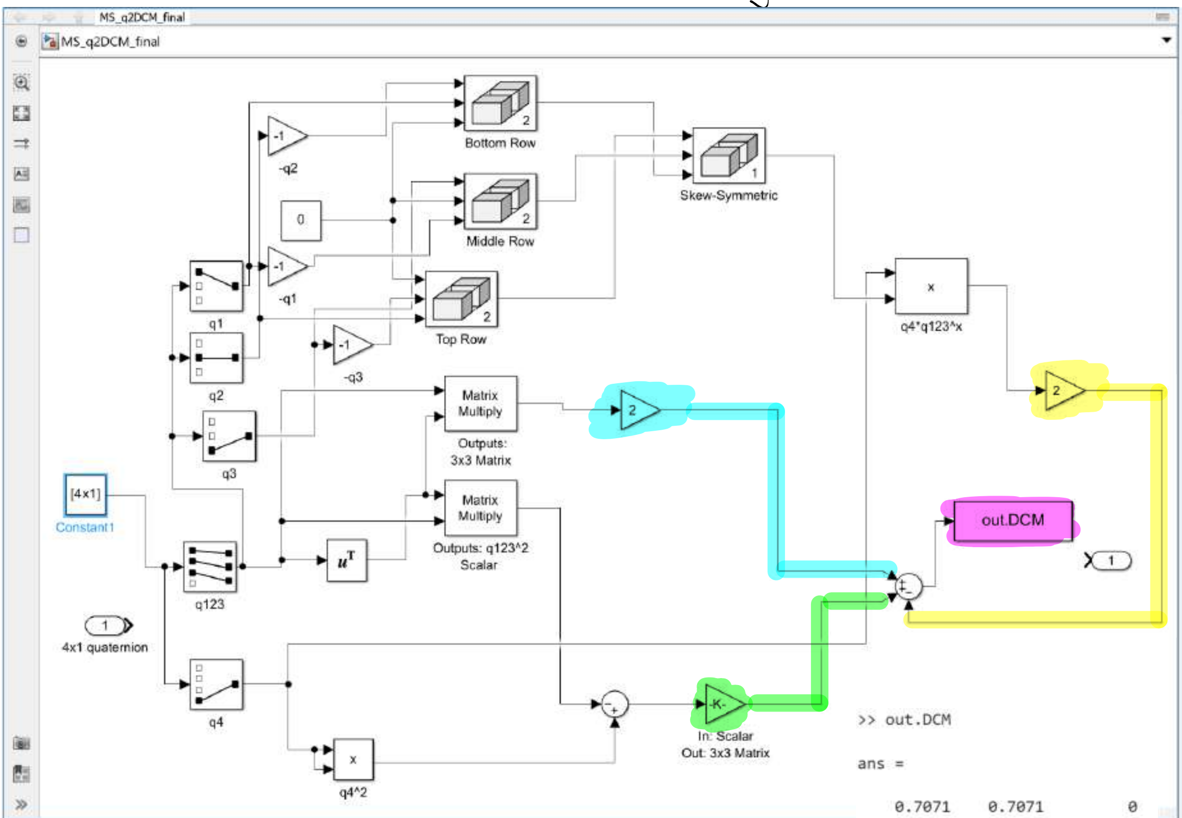
2. Transform a quaternion into a DCM. The input is a quaternion. The output is a DCM.

Outline:

$q_{123}^T q_{123} = |q_{123}|^2$

Use $\;{}^N Q^B = eye(3,3)*(q_4^2 - q_{123}^2) + 2q_{123}q_{123}^T - 2q_4 q_{123}^\times$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_{123} \\ q_4 \end{bmatrix}, \quad q_{123}^\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$




```
>> out.DCM

ans =

    0.7071    0.7071         0
   -0.7071    0.7071         0
         0         0    1.0000
```

My Simulink Model

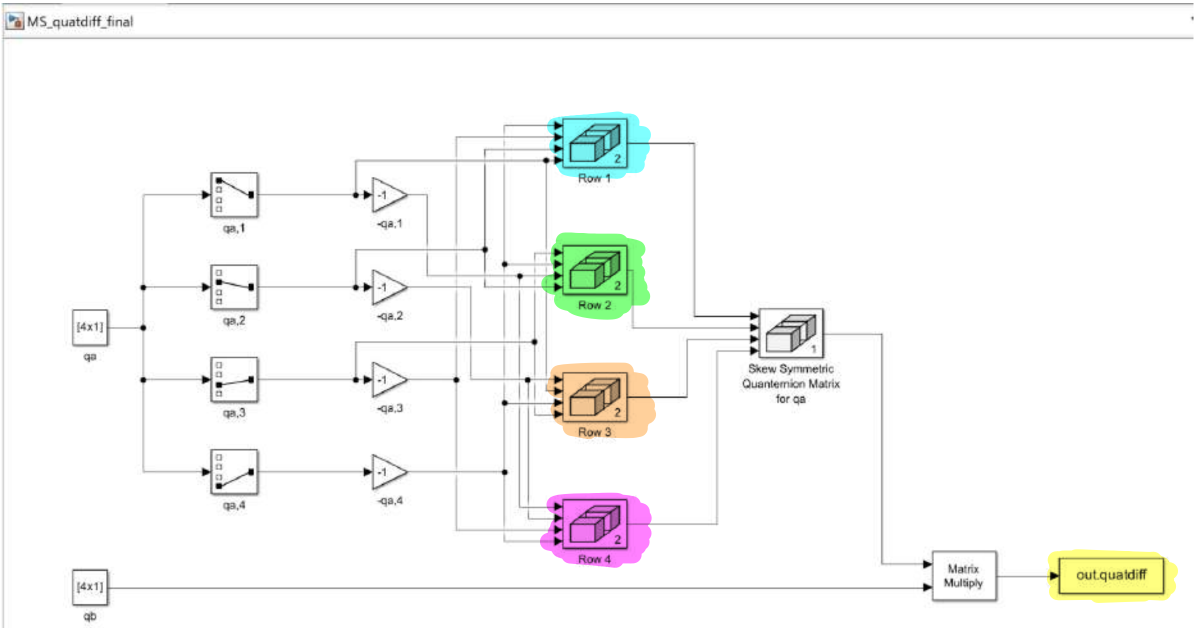
I tested it worked by inputting $q = \begin{bmatrix} 0 \\ 0 \\ 0.382683 \\ 0.92388 \end{bmatrix}$, and I got an output of

$$\begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 45° rot abt the 3-axis ✓

3. Calculate a quaternion difference. The input consists of two quaternions. The output is their difference.

Use

$$q_{diff} = q_a^{-1} \otimes q_b = \begin{bmatrix} -q_{a,4} & -q_{a,3} & q_{a,2} & q_{a,1} \\ q_{a,3} & -q_{a,4} & -q_{a,1} & q_{a,2} \\ -q_{a,2} & q_{a,1} & -q_{a,4} & q_{a,3} \\ -q_{a,1} & -q_{a,2} & -q_{a,3} & -q_{a,4} \end{bmatrix} \begin{bmatrix} q_{b,1} \\ q_{b,2} \\ q_{b,3} \\ q_{b,4} \end{bmatrix} ; \quad q_a^{-1} = \begin{bmatrix} q_{a,1} \\ q_{a,2} \\ q_{a,3} \\ -q_{a,4} \end{bmatrix}$$



My Simulink Model

I used the example from lecture to check:

$q_a = \begin{bmatrix} 0 \\ 0 \\ 0.059964 \\ 0.998201 \end{bmatrix}$, and $q_b = \begin{bmatrix} 0 \\ 0 \\ 0.079915 \\ 0.996802 \end{bmatrix}$, expected output is $\begin{bmatrix} 0 \\ 0 \\ -0.019999 \\ -0.998000 \end{bmatrix}$

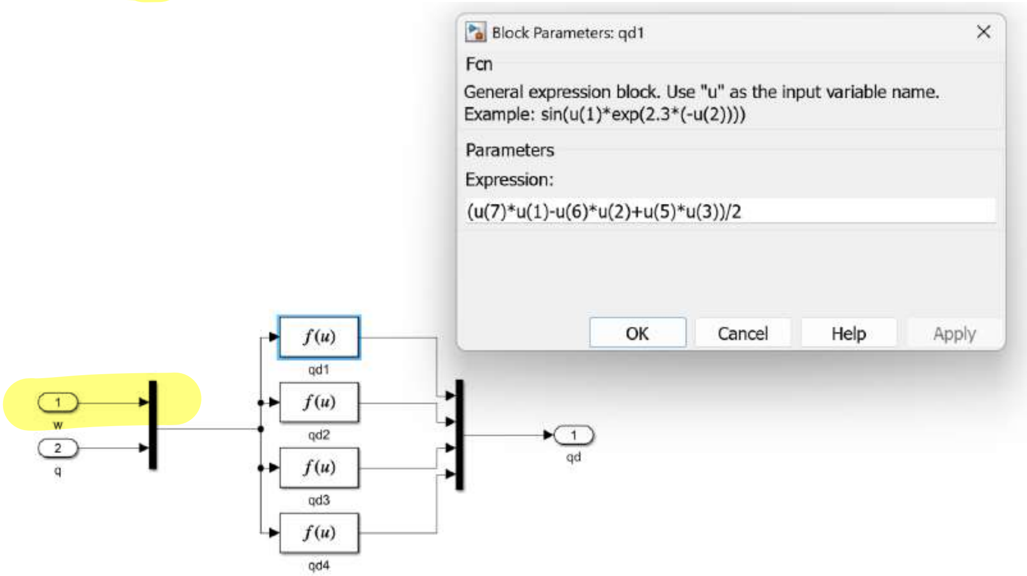
```
>> out.quatdiff

ans =

         0         0   -0.0200   -0.9998
```

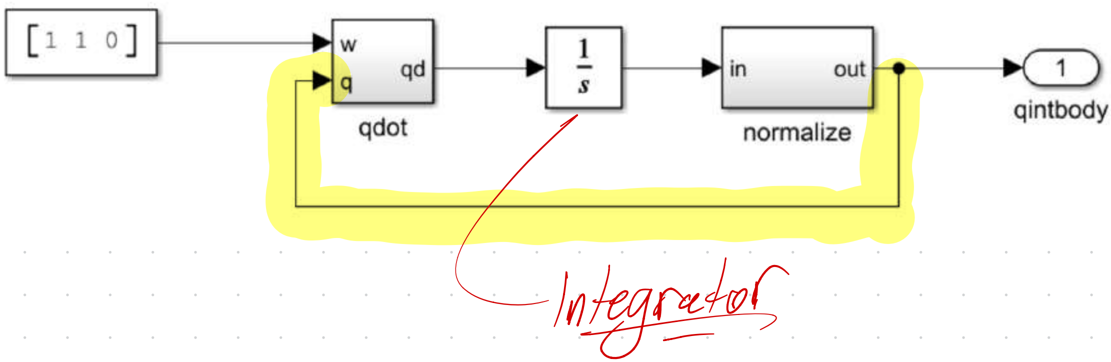
It works + passed the test ✓

4. Calculate a quaternion time derivative. The input consists of the angular velocity in body axes (${}^{B}\omega^{B/N}$) and a quaternion. For the sake of testing this algorithm, let ${}^{B}\omega^{B/N} = {}^{B}[1 \quad 1 \quad 0]$ rad/sec. The output is the derivative of that quaternion.



Simulink Model w/ first "f(u)" shown to get the first component of qdot.

5. Build upon the result of problem 4 to include an integrator that integrates the quaternion starting from the initial condition $q(0)=[0\ 0\ 0\ 1]^T$ from $t=0$ to $t=10$ sec.

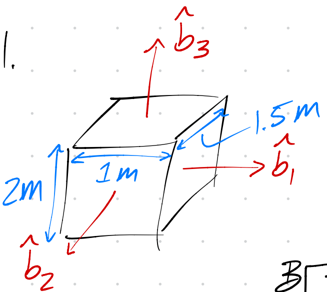


In the following problems, the basis vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ are unit vectors fixed in the spacecraft (or, equivalently, fixed in a frame B that rotates with the spacecraft). You may use an engineering reference of some sort to look up the moments of inertia of the rigid bodies described rather than carrying out the volume integrals yourself. Unless stated otherwise, each body is assumed to be rotating freely (i.e. about its center of mass) with no externally applied torque.

<u>Rigid-Body Dynamics</u>

1. Consider a box-shaped spacecraft of uniform density spinning with an angular velocity of $\boldsymbol{\omega} = 0.1\boldsymbol{b}_1 + 0.2\boldsymbol{b}_2 + 0.3\boldsymbol{b}_3$ rad/sec. This rectangular prism's length, measured along $\boldsymbol{b}_1$, is 1 m. Its width, measured along $\boldsymbol{b}_2$, is 1.5 m. It is 2 m high. Its mass is 1500 kg. What is the angular momentum vector of this spacecraft in terms of the $\boldsymbol{b}_i$ basis vectors?

1.



$M = 1500$ kg

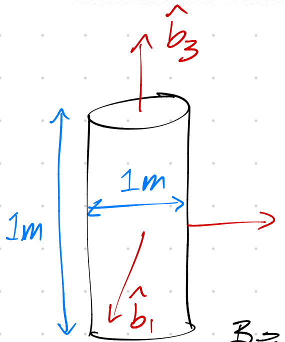
$${}^{B}I = \frac{M}{12}\,{}^{B}\begin{bmatrix} (w^2+h^2) & 0 & 0 \\ 0 & (l^2+h^2) & 0 \\ 0 & 0 & (w^2+l^2) \end{bmatrix}, \quad {}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

$${}^{B}I = {}^{B}\begin{bmatrix} 781.25 & 0 & 0 \\ 0 & 625 & 0 \\ 0 & 0 & 406.25 \end{bmatrix}$$

$${}^{B}\vec{H} = {}^{B}I\,{}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 781.25 & 0 & 0 \\ 0 & 625 & 0 \\ 0 & 0 & 406.25 \end{bmatrix} {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

$${}^{B}\vec{H} = {}^{B}\begin{bmatrix} 78.125 \\ 125 \\ 121.875 \end{bmatrix} \text{Nms}$$

2. Consider a cylindrical spacecraft of uniform density with an angular velocity of $\boldsymbol{\omega} = 0.1\boldsymbol{b}_1 + 0.2\boldsymbol{b}_2 + 0.3\boldsymbol{b}_3$ rad/sec. The cylinder's axis of symmetry is parallel to the $\boldsymbol{b}_3$ axis. Its height is 1 m; its diameter is 1 m, and its mass is 200 kg. What is the angular momentum vector of this body in terms of the $\boldsymbol{b}_i$ basis vectors?



$m = 200$ kg

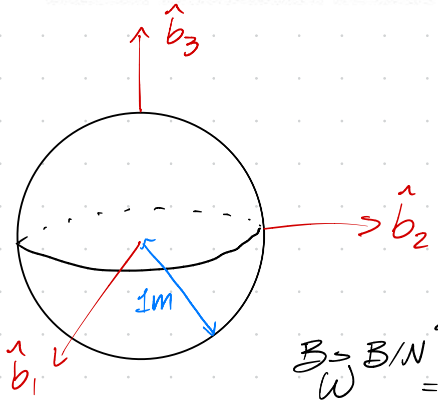
$$ {}^{B}I = \frac{m}{12}\,{}^{B}\begin{bmatrix} (3R^2+h^2) & 0 & 0 \\ 0 & (3R^2+h^2) & 0 \\ 0 & 0 & 6R^2 \end{bmatrix} $$

$$ {}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \text{rad/s} $$

$$ {}^{B}\vec{H} = {}^{B}I\,{}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 29.167 & 0 & 0 \\ 0 & 29.167 & 0 \\ 0 & 0 & 25 \end{bmatrix} {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} $$

$$ {}^{B}\vec{H} = {}^{B}\begin{bmatrix} 2.917 \\ 5.833 \\ 7.5 \end{bmatrix} \text{Nms} $$

3. Consider a 1 m radius sphere of uniform density ρ=1000 kg/m^3 with an angular velocity of $\boldsymbol{\omega} = 0.1\boldsymbol{b}_1 + 0.2\boldsymbol{b}_2 + 0.3\boldsymbol{b}_3$ rad/sec. What is the angular momentum vector of this body in terms of the $\boldsymbol{b}_i$ basis vectors?



$${}^{B}I = \frac{2}{5} m R^2 \, {}^{B}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \text{rad/s} \qquad V = \frac{4}{3}\pi R^3, \quad m = \rho V$$

$$m = 4188.8 \text{ kg}$$

$${}^{B}\vec{H} = {}^{B}I \, {}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 1675.516 & 0 & 0 \\ 0 & 1675.516 & 0 \\ 0 & 0 & 1675.516 \end{bmatrix} {}^{B}\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

$${}^{B}\vec{H} = {}^{B}\begin{bmatrix} 167.56 \\ 335.10 \\ 503.65 \end{bmatrix} \text{Nms}$$

4. The magnitude of a uniformly dense sphere's angular velocity vector is 1 rad/sec. The magnitude of its angular momentum vector is 2 rad/sec. What is the inertia matrix of this sphere?

For a sphere, the inertia tensor is the same in any direction due to symmetry.

We know $|\vec{\omega}^{B/N}| = 1$ rad/s and $|\vec{H}| = 2$ rad/s

$$|\vec{H}| = I\,|\vec{\omega}^{B/N}| \implies \underline{\underline{2 = I}}, \text{ so,} \quad {}^{B}I = {}^{B}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5. What is the nutation frequency of the cylinder in problem 2 if its angular velocity vector at a certain instant is $\boldsymbol{\omega} = 0.1\boldsymbol{b}_1 - 0.1\boldsymbol{b}_2 + 0.5\boldsymbol{b}_3$ rad/s?

$\lambda = \Omega(\sigma - 1)$; $\sigma = \frac{I_s}{I_t}$; and $\Omega = \|{}^{B}\vec{\omega}^{B/N}\| = \sqrt{0.1^2 + 0.1^2 + 0.5^2} = 0.51$

$\sigma = \frac{25}{29.167}$ from solution to question #2

Putting it all together, $\boxed{\lambda = -0.07143 \text{ rad/s}}$

7. What is the nutation frequency of a sphere?

For a sphere, the moment of inertia about any axis is equal ($I_1 = I_2 = I_3$), so $\sigma = 1$, and using $\lambda = \Omega(\sigma - 1)$, we clearly see $\lambda = 0$ for a sphere.

8. Consider a cylindrical spacecraft whose axis of symmetry is aligned with $\boldsymbol{b}_3$. Telemetry indicates that its spin rate is $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{b}_3 = 0.75$ rad/sec. Its nutation frequency is measured to be 0.1 rad/sec. Its spin moment of inertia is known to be 1200 kg-m^2. What is its transverse moment of inertia?

$\lambda = \Omega(\sigma - 1)$; $\lambda = 0.1$ rad/s ; $\Omega = 0.75$ rad/s ; $\sigma = \frac{I_3}{I_t} = \frac{1200}{I_t}$

$I_t = 1059$ kg·m^2

9. What is the angular acceleration vector of the sphere in problem 3?

Euler's Equations:
$$\left.\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2 \end{aligned}\right\}$$
For a sphere, $I_1 = I_2 = I_3$, so

$\dot{\omega} = \vec{0}$

10. Compute the angular acceleration vector of a spacecraft whose angular velocity is $\boldsymbol{\omega} = 0.2\boldsymbol{p}_1 - 0.2\boldsymbol{p}_2 - 0.2\boldsymbol{p}_3$ rad/s and whose total angular-momentum vector is $\boldsymbol{h} = 9\boldsymbol{p}_1 - 12\boldsymbol{p}_2 - 15\boldsymbol{p}_3$ Nms, where the $\boldsymbol{p}_i$ represent the principal axes of the inertia matrix.

$${}^{P}\vec{\omega}^{B/N} = {}^{P}\begin{bmatrix} 0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \text{rad/s} \; ; \quad {}^{P}\vec{h} = \begin{bmatrix} 9 \\ -12 \\ -15 \end{bmatrix} \text{Nms} \qquad {}^{P}\vec{h} = {}^{P}I \, {}^{P}\vec{\omega}^{B/N}$$

$${}^{P}\begin{bmatrix} 9 \\ -12 \\ -15 \end{bmatrix} = {}^{P}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} {}^{P}\begin{bmatrix} 0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \quad \left.\begin{aligned} 9 &= 0.2 I_1 \\ -12 &= -0.2 I_2 \\ -15 &= -0.2 I_3 \end{aligned}\right\} \quad \begin{aligned} I_1 &= 45 \\ I_2 &= 60 \\ I_3 &= 75 \end{aligned}$$

Euler's: $0 = {}^{B}I \, {}^{B}\dot{\omega} + {}^{B}\vec{\omega}^{\times} \, {}^{B}I \, {}^{B}\vec{\omega} \Longrightarrow {}^{B}\dot{\omega} = -{}^{B}I^{-1} \, {}^{B}\omega^{\times} \, \underbrace{{}^{B}I \, {}^{B}\vec{\omega}}_{{}^{B}\vec{h}}$

$${}^{B}\dot{\omega} = {}^{B}\begin{bmatrix} 1/45 & 0 & 0 \\ 0 & 1/60 & 0 \\ 0 & 0 & 1/75 \end{bmatrix} \begin{bmatrix} 0 & 0.2 & -0.2 \\ -0.2 & 0 & -0.2 \\ 0.2 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ -12 \\ -15 \end{bmatrix} = {}^{B}\begin{bmatrix} -0.01\bar{3} \\ -0.02 \\ 0.008 \end{bmatrix} \text{rad/s} = {}^{B}\dot{\omega}$$

11. Compute the angular velocity vector of a spacecraft whose total angular momentum is $\boldsymbol{h} = 200\boldsymbol{b}_1 + 300\boldsymbol{b}_2 + 1000\boldsymbol{b}_3$ Nms and whose inertia matrix is ${}^{B}I = \begin{bmatrix} 3000 & -40 & 50 \\ -40 & 4000 & -60 \\ 50 & -60 & 5000 \end{bmatrix}$.

There's no need to perform this calculation by hand, but at least show the steps if you use a computer.

$${}^{B}\vec{H} = {}^{B}I\,{}^{B}\vec{\omega}^{B/N} \Longrightarrow {}^{B}\vec{\omega}^{B/N} = {}^{B}I^{-1}\,{}^{B}\vec{h} = {}^{B}\begin{bmatrix} 3000 & -40 & 50 \\ -40 & 4000 & -60 \\ 50 & -60 & 5000 \end{bmatrix}^{-1} {}^{B}\begin{bmatrix} 200 \\ 300 \\ 1000 \end{bmatrix}$$

Using MATLAB, ${}^{B}\vec{\omega}^{B/N} = \begin{bmatrix} 0.0673 \\ 0.0760 \\ 0.202 \end{bmatrix}$ rad/s

Principal Axes and Spin Stability

1. Find the principal moments of inertia (I_1, I_2, and I_3) of the following inertia matrix by hand (i.e. show the steps in the calculation).

$${}^{B}I = \begin{bmatrix} 300 & 0 & 0 \\ 0 & 450 & -50 \\ 0 & -50 & 450 \end{bmatrix} \text{kg-m}^2$$

Principal moments of inertia are the eigenvalues of the inertia matrix

$$\det({}^{B}I - \lambda \mathbb{1}_{3\times3}) = 0 \Longrightarrow \det\left(\begin{bmatrix} 300-\lambda & 0 & 0 \\ 0 & 450-\lambda & -50 \\ 0 & -50 & 450-\lambda \end{bmatrix}\right)$$

$$(300-\lambda)\left[(450-\lambda)^2 - 50^2\right] = 0 \Longrightarrow (300-\lambda)(200{,}000 - 900\lambda + \lambda^2) = 0$$

$$(300-\lambda)(\lambda-400)(\lambda-500) = 0$$

So, $\lambda_1 = 300$; $\lambda_2 = 400$; $\lambda_3 = 500$

and $\therefore$ $I_1 = 300$
$I_2 = 400$
$I_3 = 500$

2. Write down the principal axes for the inertia matrix of problem 1 (i.e. ${}^{B}p_1$, ${}^{B}p_2$, and ${}^{B}p_3$) and the principal inertia matrix, ${}^{P}I$.

↳ (previous page)

We already determined that the principal inertia Matrix is:

$$ {}^{P}I = {}^{P}\begin{bmatrix} 300 & 0 & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 500 \end{bmatrix} kg \cdot m^2 $$

The principal axes are the eigenvectors of ${}^{B}I$

Using MATLAB, the eigenvalues I solved for by hand were confirmed, and the eigenvectors (principal axes) are:

$$ {}^{B}\vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad {}^{B}\vec{p}_2 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}; \quad {}^{B}\vec{p}_3 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} $$

$\hookrightarrow I_1 = 300$ $\quad \hookrightarrow I_2 = 400$ $\quad \hookrightarrow I_3 = 500$

3. Why isn't the following inertia matrix physically realizable? Try looking at it in principal coordinates.

$$ {}^{B}I = \begin{bmatrix} 500\cos^2\frac{\pi}{30} + 4000\sin^2\frac{\pi}{30} & 0 & 3500\cos\frac{\pi}{30}\sin\frac{\pi}{30} \\ 0 & 5000 & 0 \\ 3500\cos\frac{\pi}{30}\sin\frac{\pi}{30} & 0 & 500\sin^2\frac{\pi}{30} + 4000\cos^2\frac{\pi}{30} \end{bmatrix} \text{kg-m}^2 $$

Using MATLAB, I get:

$$ {}^{P}I = \begin{bmatrix} 500 & 0 & 0 \\ 0 & 4000 & 0 \\ 0 & 0 & 5000 \end{bmatrix}. $$

This does <u>NOT</u> satisfy the triangle inequality, as $I_1 + I_2 < I_3$, so the Matrix is NOT physically realizable.

4. About which axis is the following spacecraft stable while spinning without any form of active control? Provide your answer in terms of the B basis vectors (ignore numerical issues—this matrix is meant to be physically realizable).

$$^{B}I = \begin{bmatrix} 425 & 25 & 61.23724 \\ 25 & 425 & 61.23724 \\ 61.2372 & 61.23724 & 350 \end{bmatrix} \text{kg-m}^2$$

Using MATLAB:

$$^{P}I = {}^{P}\begin{bmatrix} 503.965 & 0 & 0 \\ 0 & 404.8662 & 0 \\ 0 & 0 & 301.1688 \end{bmatrix}$$

The $\hat{p}_1$ axis has the greatest I, so that is our stable axis

$$^{B}\vec{p}_1 = {}^{B}\begin{bmatrix} -0.6468 \\ -0.5846 \\ -0.4898 \end{bmatrix}$$

5. Compute the nutation frequency of a spacecraft in stable, equilibrium spin at Ω=1 RPM and whose inertia matrix is

↳ Spinning around maximum axis

$$^{B}I = \begin{bmatrix} 55 & 0 & 5\sqrt{3} \\ 0 & 50 & 0 \\ 5\sqrt{3} & 0 & 65 \end{bmatrix} \text{kg m}^2$$

MATLAB tells us:

$$^{P}I = {}^{P}\begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 70 \end{bmatrix}$$

$$\sigma = \frac{I_s}{I_t} = \frac{70}{50}$$

$$\Omega = 1 \frac{\text{rot}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{2\pi \text{ rad}}{1 \text{ rot}} = \frac{\pi}{30} \text{ rad/s}$$

$$\lambda = \Omega(\sigma - 1) = \frac{\pi}{30}\left(\frac{7}{5} - 1\right) = 0.0419 \text{ rad/s} = \lambda$$

6. Compute the mass, center of mass (in terms of the $\boldsymbol{b}_i$ basis vectors), and centroidal inertia matrix (in terms of the $\boldsymbol{b}_i$ basis vectors) of a spacecraft consisting of the following two components:

- The spacecraft bus, represented by a uniformly dense 200 kg sphere of radius R=0.2 m whose mass center is located at $\boldsymbol{r}_s=0.01\boldsymbol{b}_1+0.02\boldsymbol{b}_2+0.2\boldsymbol{b}_3$ m from the center of the spacecraft/launch-vehicle interface plane.
- The spacecraft payload, represented by a point mass of 50 kg located at $\boldsymbol{r}_p=-0.04\boldsymbol{b}_1-0.08\boldsymbol{b}_2+0.4\boldsymbol{b}_3$ m from the center of the spacecraft/launch-vehicle interface plane.

Mass: $M_{tot} = M_{bus} + M_{payload} = 200\text{ kg} + 50\text{ kg} = 250\text{ kg} = M_{tot}$

$$\vec{r}_{CM} = \frac{200(0.01\hat{b}_1 + 0.02\hat{b}_2 + 0.2\hat{b}_3) + 50(-0.04\hat{b}_1 - 0.08\hat{b}_2 + 0.4\hat{b}_3)}{250}$$

$\vec{r}_{CM} = 0.24\hat{b}_3$ m

Sphere:

$$^{B}\vec{r}_s - {^{B}\vec{r}_{CM}} = \begin{bmatrix} 0.01 \\ 0.02 \\ -0.04 \end{bmatrix} m$$

$$^{B}I = \frac{2}{5}MR^2 \, {}^{B}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3.2 & 0 & 0 \\ 0 & 3.2 & 0 \\ 0 & 0 & 3.2 \end{bmatrix} \text{kg}\cdot m^2$$

Payload:

$$^{B}I_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kg}\cdot m^2 \qquad ^{B}\vec{r}_p - {^{B}\vec{r}_{CM}} = {}^{B}\begin{bmatrix} -0.04 \\ -0.08 \\ 0.16 \end{bmatrix}$$

Total:

$$^{B}I = {^{B}I_s} + {^{B}I_p} - M_s(^{B}\vec{r}_s - {^{B}\vec{r}_{CM}})^{\times}(^{B}\vec{r}_s - {^{B}\vec{r}_{CM}})^{\times} - M_p(^{B}\vec{r}_p - {^{B}\vec{r}_{CM}})^{\times}(^{B}\vec{r}_p - {^{B}\vec{r}_{CM}})^{\times}$$

$$^{B}I = {}^{B}\begin{bmatrix} 3.2 & 0 & 0 \\ 0 & 3.2 & 0 \\ 0 & 0 & 3.2 \end{bmatrix} + {}^{B}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 200\begin{bmatrix} 0 & 0.04 & 0.02 \\ -0.04 & 0 & -0.01 \\ -0.02 & 0.01 & 0 \end{bmatrix}^2 - 50\begin{bmatrix} 0 & -0.16 & -0.08 \\ 0.16 & 0 & 0.04 \\ 0.08 & -0.04 & 0 \end{bmatrix}^2$$

$$^{B}I = {}^{B}\begin{bmatrix} 5.2 & -0.2 & 0.4 \\ -0.2 & 4.9 & 0.8 \\ 0.4 & 0.8 & 3.7 \end{bmatrix} \text{kg}\cdot m^2$$

Attitude Sensors and Actuators

Circle the best answer:

1. Which of the following is true of a gyroscope?
   a. A gyroscope measures attitude
   b. A gyroscope measures angular velocity and/or incremental attitude change
   c. Measurements from a gyroscope and measurements from a sun sensor, used together, provide a full three-axis attitude estimate

2. Which of the following is true of a star tracker?
   a. A star tracker provides only single-axis attitude measurements
   b. A star tracker provides a full three-axis attitude measurement from multiple star observations
   c. A star tracker work well on fast-spinning spacecraft.

3. With a magnetometer and a sun sensor, a spacecraft in Earth orbit can accomplish which of the following?
   a. Estimate attitude at all points in a LEO orbit
   b. Estimate attitude at all points in a LEO orbit except where the local magnetic field aligns with the direction from the spacecraft to the sun
   c. Estimate the time since perigee passage.

4. Which attitude-determination method is not Wahba-optimal?
   a. Davenport's q method
   b. Markley's SVD method
   c. TRIAD

5. Which is true of a spacecraft with three operating reaction wheels and no other ACS actuators?
   a. It needs no attitude sensors to estimate attitude.
   b. It is capable of three-axis attitude control if the wheels' spin axes are along linearly independent directions
   c. It can achieve high-agility motions with low electrical power.

6. Which is true of a spacecraft with four operating control-moment gyroscopes (CMGs) aligned so that their gimbal axes are directed along the vertices of a regular tetrahedron?
   a. The CMG array has internal singularities
   b. The CMG array has no internal singularities
   c. The CMG array requires at least one more CMG to impart three-axis torque for attitude control.

Attitude Determination

A spacecraft beyond Earth's solar system uses two identical optical sensors to measure the direction $\hat{\boldsymbol{p}}$ to Proxima Centauri (the nearest star) and the direction to Sol (Earth's sun) $\hat{\boldsymbol{s}}$ in a body-fixed reference coordinate system B. These directions are known in a galactic coordinate system indicated with the G superscript. Determine the attitude of the spacecraft (${}^{B}Q^{G}$).

Use the TRIAD method of attitude determination.

$$
{}^{G}\hat{p} = \begin{bmatrix} 0.0 \\ 0.8829476 \\ -0.4694716 \end{bmatrix} \quad {}^{B}\hat{p} = \begin{bmatrix} 0.4268893 \\ 0.8829476 \\ 0.1953691 \end{bmatrix}
$$

$$
{}^{G}\hat{s} = \begin{bmatrix} 0.00998334 \\ 0.8772854 \\ -0.4694716 \end{bmatrix} \quad {}^{B}\hat{s} = \begin{bmatrix} 0.3853439 \\ 0.8772854 \\ -0.2861474 \end{bmatrix}
$$

Is your estimate optimal in the sense of Wahba's loss function?

$$
{}^{B}Q^{G} = \begin{bmatrix} {}^{B}\hat{s} & {}^{B}\hat{t} & {}^{B}\hat{u} \end{bmatrix} \begin{bmatrix} {}^{G}\hat{s} & {}^{G}\hat{t} & {}^{G}\hat{u} \end{bmatrix}^{T}
$$

$$
{}^{B}\vec{t} = {}^{B}\hat{s} \times {}^{B}\hat{p} \; ; \quad {}^{B}\hat{t} = \frac{{}^{B}\vec{t}}{\| {}^{B}\hat{s} \times {}^{B}\hat{p} \|} \qquad {}^{B}\hat{u} = {}^{B}\hat{s} \times {}^{B}\hat{t}
$$

$$
{}^{G}\hat{t} = \frac{{}^{G}\hat{s} \times {}^{G}\hat{p}}{\| {}^{G}\hat{s} \times {}^{G}\hat{p} \|} \; ; \quad {}^{G}\hat{u} = {}^{G}\hat{s} \times {}^{G}\hat{t}
$$

I made a MATLAB Livescript to compute these:

(Next two pages).

```
%Define vectors as per problem statement
G_p = [0;0.8829476;-0.4694716]
```

```
G_p = 3×1
         0
    0.8829
   -0.4695
```

```
G_s = [0.00998334;0.8772854;-0.4694716]
```

```
G_s = 3×1
    0.0100
    0.8773
   -0.4695
```

```
B_p = [0.4268893;0.8829476;0.1953691]
```

```
B_p = 3×1
    0.4269
    0.8829
    0.1954
```

```
B_s = [0.3853439;0.8772854;-0.2861474]
```

```
B_s = 3×1
    0.3853
    0.8773
   -0.2861
```

```
%Make skew-symmetric matrix from vector
function [cross_matrix] = get_cross(v)
    cross_matrix = [0,-v(3),v(2);...
                    v(3),0,-v(1);...
                    -v(2),v(1),0]
end

%Get B_t
B_t_i = (get_cross(B_s))*B_p
```

```
cross_matrix = 3×3
         0    0.2861    0.8773
   -0.2861         0   -0.3853
   -0.8773    0.3853         0
B_t_i = 3×1
    0.4240
   -0.1974
   -0.0343
```

```
B_t = B_t_i/norm(B_t_i)
```

```
B_t = 3×1
    0.9041
   -0.4210
   -0.0731
```

```
%Get B_u
B_u = (get_cross(B_s))*B_t
```

```
cross_matrix = 3×3
         0    0.2861    0.8773
```

```
   -0.2861         0   -0.3853
   -0.8773    0.3853         0
B_u = 3×1
   -0.1846
   -0.2306
   -0.9554
```

```
%Get N_t
G_t_i = (get_cross(G_s))*G_p
```

```
cross_matrix = 3×3
         0    0.4695    0.8773
   -0.4695         0   -0.0100
   -0.8773    0.0100         0
G_t_i = 3×1
    0.0027
    0.0047
    0.0088
```

```
G_t = G_t_i/norm(G_t_i)
```

```
G_t = 3×1
    0.2573
    0.4537
    0.8532
```

```
%Get G_u
G_u = (get_cross(G_s))*G_t
```

```
cross_matrix = 3×3
         0    0.4695    0.8773
   -0.4695         0   -0.0100
   -0.8773    0.0100         0
G_u = 3×1
    0.9615
   -0.1293
   -0.2212
```

Sanity Check

```
%Form BQG
BQG = [B_s,B_t,B_u]*[G_s,G_t,G_u].'
```

```
BQG = 3×3
    0.0590    0.7721    0.6313
   -0.3212    0.6085   -0.7200
   -0.9403   -0.1606    0.2833
```

$\det(BQG) = 0.9901 \simeq 1$ ✓

The TRIAD method is <u>NOT</u> Wahba Optimal

Dynamic Balance and Superspin

1. What wheel angular momentum ${}^{B}h$ will allow the following spacecraft to spin about the b_3 axis without nutation at 10 RPM?

$$ {}^{B}I = \begin{bmatrix} 300 & 2 & 5 \\ 2 & 450 & -50 \\ 5 & -50 & 500 \end{bmatrix} \text{ kg-m}^2 $$

This requires dynamic balance, where we cancel out the products of inertia so that ${}^{B}I$ looks like ${}^{P}I$.

We want ${}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}$; ${}^{B}\vec{H} = {}^{B}I\,{}^{B}\vec{\omega}^{B/N} + {}^{B}\vec{h} = {}^{B}\begin{bmatrix} 300 & 2 & 5 \\ 2 & 450 & -50 \\ 5 & -50 & 500 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + {}^{B}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$

So, ${}^{B}\vec{H} = {}^{B}\begin{bmatrix} 5\Omega + h_1 \\ -50\Omega + h_2 \\ 500\Omega + h_3 \end{bmatrix}$. We want ${}^{B}\vec{H} = \begin{bmatrix} 0 \\ 0 \\ 500\Omega \end{bmatrix}$, so

$h_1 = -5\Omega$

$h_2 = 50\Omega$

$h_3 = 0$

$\underline{\Omega = \frac{\pi}{3}\ \text{rad/s}}$

$${}^{B}\vec{h} = {}^{B}\begin{bmatrix} -5.236 \\ 52.3599 \\ 0 \end{bmatrix} \text{Nms}$$

2. What wheel angular momentum ${}^{B}h$ is required for the following spacecraft to spin about the b_3 axis with an effective inertia ratio of 1.2?

$$ {}^{B}I = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 210 \end{bmatrix} \text{ kg-m}^2 $$

We want $\frac{I_{33}}{\max(I_{11}, I_{22})} > 1.2$. Right now, $\frac{I_{33}}{I_{22,11}} = 1.05$

We also want ${}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix}$ ${}^{B}\vec{H} = {}^{B}I\,{}^{B}\vec{\omega}^{B/N} + {}^{B}\vec{h} = {}^{B}\begin{bmatrix} 0 \\ 0 \\ I_{33}\Omega + h_3 \end{bmatrix}$

but we want ${}^{B}\vec{H} = {}^{B}\begin{bmatrix} 0 \\ 0 \\ 1.2 I_t \Omega \end{bmatrix}$.

So, $1.2 I_t \Omega = I_{33}\Omega + h_3$

$h_3 = \Omega(\underbrace{1.2 I_t}_{200\ \text{kg-m}^2} - \overbrace{I_{33}}^{210\ \text{kg}\cdot\text{m}^2})$

$h_3 = 30\Omega$ Nms

$${}^{B}\vec{h} = {}^{B}\begin{bmatrix} 0 \\ 0 \\ 30\Omega \end{bmatrix} \text{Nms}$$

3. What wheel angular momentum ${}^{B}h$ is required for a 100 kg, 1 m radius, uniform, spherical spacecraft to spin stably with an angular-velocity vector $\boldsymbol{\omega}^{B/N} = 1\boldsymbol{b}_1 + 2\boldsymbol{b}_2$ rad/sec?

Since it's a sphere,

$$ {}^{B}I = \frac{2}{5} m R^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^{B}\begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{kg}\cdot\text{m}^2 = {}^{B}I $$

We need ${}^{B}\vec{h} \parallel {}^{B}\vec{\omega}^{B/N}$ for stable spin. We require Superspin here:

$$ {}^{B}\vec{\omega}^{B/N} = {}^{B}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad {}^{B}\hat{\omega}^{B/N} = \frac{1}{\sqrt{5}} {}^{B}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \|{}^{B}\vec{h}\| = \underbrace{\Omega}_{\sqrt{5}}(1.2 \overbrace{I_t}^{40} - \overbrace{I_{33}}^{40}) $$

$$ \|{}^{B}\vec{h}\| = 8\sqrt{5} $$

$$ {}^{B}\vec{h} = \|{}^{B}\vec{h}\| \, {}^{B}\hat{\omega}^{B/N} = 8\,{}^{B}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = {}^{B}\begin{bmatrix} 8 \\ 16 \\ 0 \end{bmatrix} \text{Nms} = {}^{B}\vec{h} $$

4. A spacecraft stably spinning about its $\boldsymbol{b}_2$ axis at Ω=2 RPM without active control does so thanks to constant wheel angular momentum $\boldsymbol{h}=3\boldsymbol{b}_1$. What are I_{12} and I_{23}?

↳ Dynamic Balance

$$ {}^{B}\vec{h} = {}^{B}\begin{bmatrix} -I_{12}\Omega \\ 0 \\ -I_{23}\Omega \end{bmatrix} = {}^{B}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} $$

$$ -I_{12}\overbrace{\Omega}^{\frac{\pi}{15}\text{ rad/s}} = 3 \Longrightarrow I_{12} = -14.323 \text{ kg}\cdot\text{m}^2 $$

$$ -I_{23}\Omega = 0 \Longrightarrow I_{23} = 0 \text{ kg}\cdot\text{m}^2 $$

5. Rigid Body Dynamics in MATLAB

Model a rigid spacecraft in MATLAB/Simulink. Take its inertia dyadic to be

$$^{\mathrm{B}}I = {}^{\mathrm{B}}I\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 4000 & 50 \\ 0 & 50 & 5000 \end{bmatrix} \text{kg m}^2.$$

Include a block that integrates the quaternion derivative (use the functions from HW 9). Use the following initial conditions:

$$^{B}\omega^{\mathrm{B/N}}(0) = \begin{bmatrix} 3 \\ 0 \\ 0.1 \end{bmatrix} \text{RPM and } q(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Output the following to the MATLAB workspace:

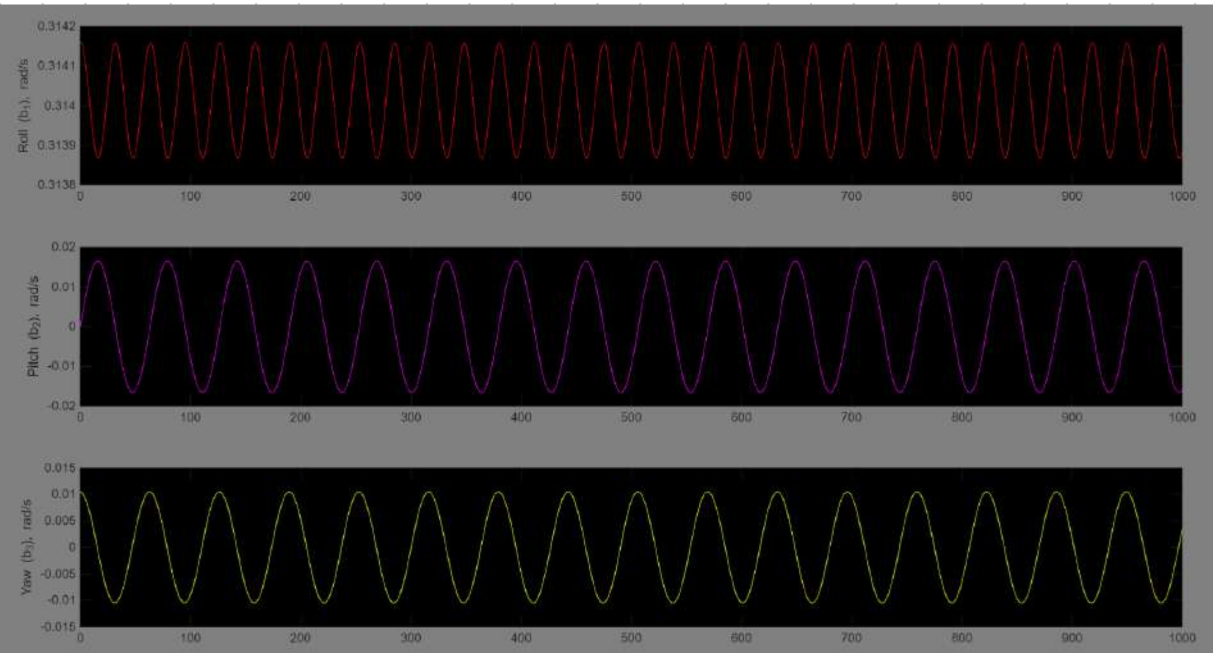
a. Time

b. The three components of $^{B}\omega^{\mathrm{B/N}}$

c. The four components of q

d. The nine components of $^{\mathrm{N}}Q^{\mathrm{B}}$

Produce a plot of each of the three values in $^{\mathrm{B}}\omega^{\mathrm{B/N}}$ vs. time for 1000 seconds. Use either the "scope" function in Simulink (preferred) or construct one from the values saved to the workspace.



HW 12:

$$^{B}I = {}^{B}\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 4000 & 0 \\ 0 & 0 & 5000 \end{bmatrix}$$

This same LEO satellite needs to perform high-agility, precise attitude maneuvers in three axes: 3 deg/sec rate, 3 deg/sec^2 acceleration, and 1e-4 deg attitude precision. Cost is not a concern. Which actuator technology is likely to be chosen to accomplish this objective? (1)

a. Reactions wheels

b. Control-moment gyroscopes

c. Thrusters

d. Magnetic torque rods

This spacecraft performs a versine slew about the $\boldsymbol{b}_3$ axis. The total angular motion about this axis is 12 deg. The maneuver lasts 10 seconds. Which of the following is an expression for the total angle vs. time for the duration of this maneuver, neglecting any errors in sensing, actuation, or mass-properties knowledge? (2)

a. $\varphi(t) = 10\left(1 - \cos\frac{\pi}{5}t\right)$

b. $\varphi(t) = 6\left(1 - \cos\frac{\pi}{10}t\right)$

c. $\varphi(t) = 12(1 - \cos\pi\, t)$

d. $\varphi(t) = 3\left(1 - \cos\frac{\pi}{12}t\right)$

4. For the maneuver described in problem 1 [redacted]., what is the maximum torque required of the momentum-actuator array? (3)

$$\ddot{\phi}(t) = \frac{s^2\phi_c}{2}\cos(s(t-t_0))$$ Max accel when cosine term is 1 @ the beginning + end of the slew

$$\frac{s^2\phi_c}{2} = \frac{\left(\frac{\pi}{10}\right)^2 \cdot 12}{2} = 0.5922 \text{ deg/s}^2$$

$$T = I_3\ddot{\phi} = 5000(0.5922)\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = \boxed{51.68 \text{ Nm} = T}$$

Shown here are the spin-axis directions for a three-RWA array. Which of them is capable of producing torque in any direction in three dimensions?

a. $\hat{a}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $\hat{a}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\hat{a}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\hat{a}_1 = -\hat{a}_2$ ×

b. $\hat{a}_1 = \frac{\sqrt{3}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\hat{a}_2 = \frac{\sqrt{3}}{3}\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ $\hat{a}_3 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $\hat{a}_1 = -\hat{a}_2$ ×

c. $\hat{a}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$ $\hat{a}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $\hat{a}_3 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\Longrightarrow$ Linearly independent Vectors

d. $\hat{a}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\hat{a}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\hat{a}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $\hat{a}_3 = 0\cdot\hat{a}_1$, $\hat{a}_3 = 0\cdot\hat{a}_2$ ×

How many CMGs, operated as scissored pairs (two CMGs per pair), are required to achieve instantaneous three-axis attitude control?

a. 1

b. 2

c. 4

d. 6

Which of the following is a sound technological reason to consider spacecraft angular jerk in the design of a CMG-driven spacecraft?

a. CMG array singularities are defined in terms of jerk

b. Spacecraft jerk drives requirements for CMG gimbal acceleration (therefore gimbal-motor torque).

c. Maximizing jerk minimizes structural-dynamics response, therefore minimizing jitter that compromises optical-payload performance.

d. Minimizing spacecraft jerk shortens the duration of a Hamilton-Jacobi-Belman optimal slew, in which the actuators are used at peak levels throughout.