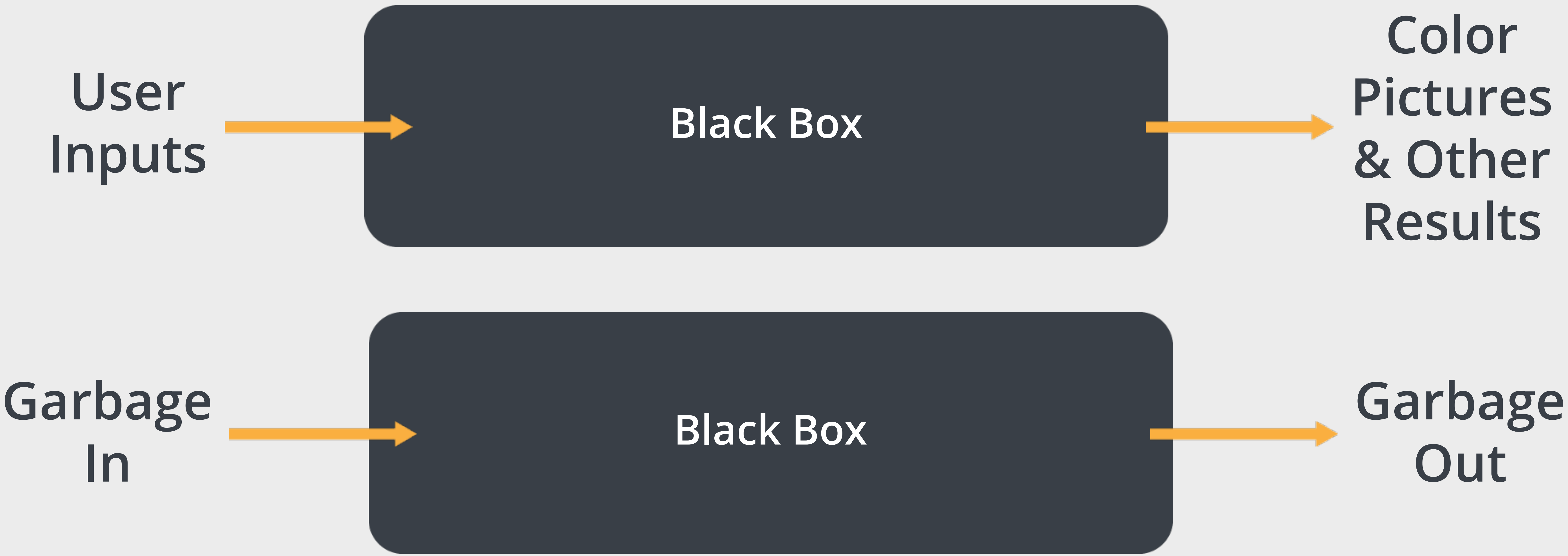
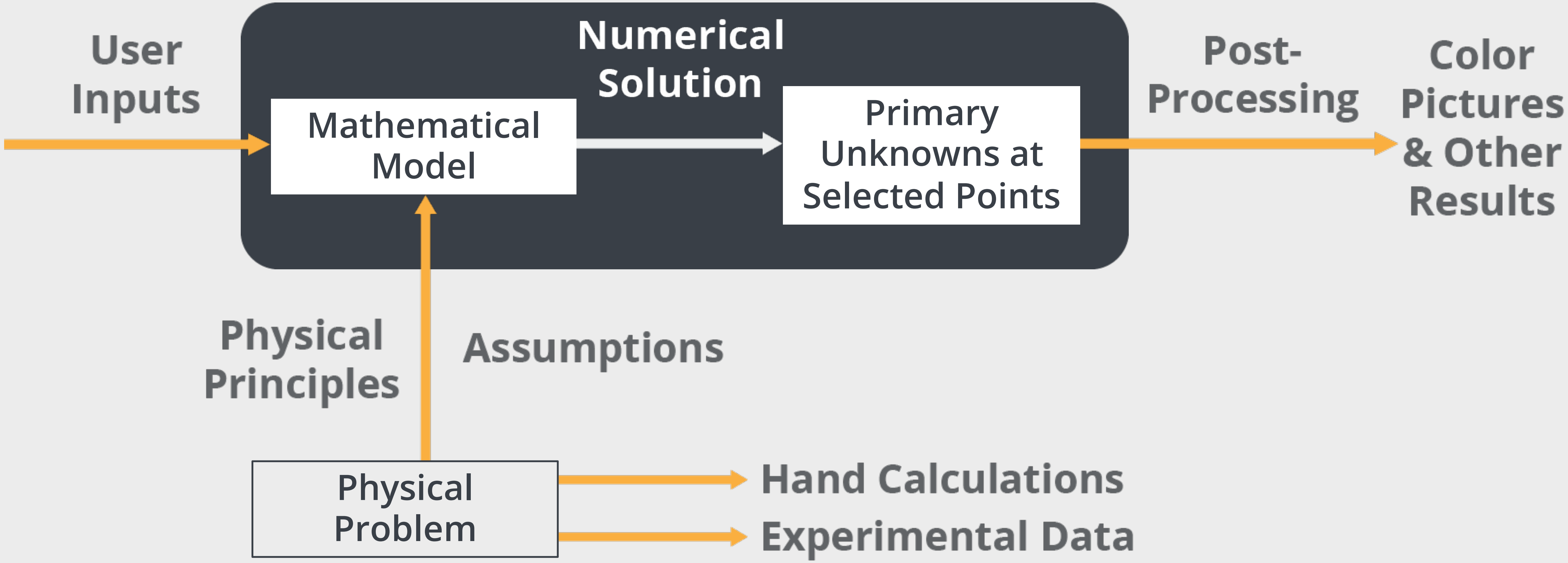
The Black Box

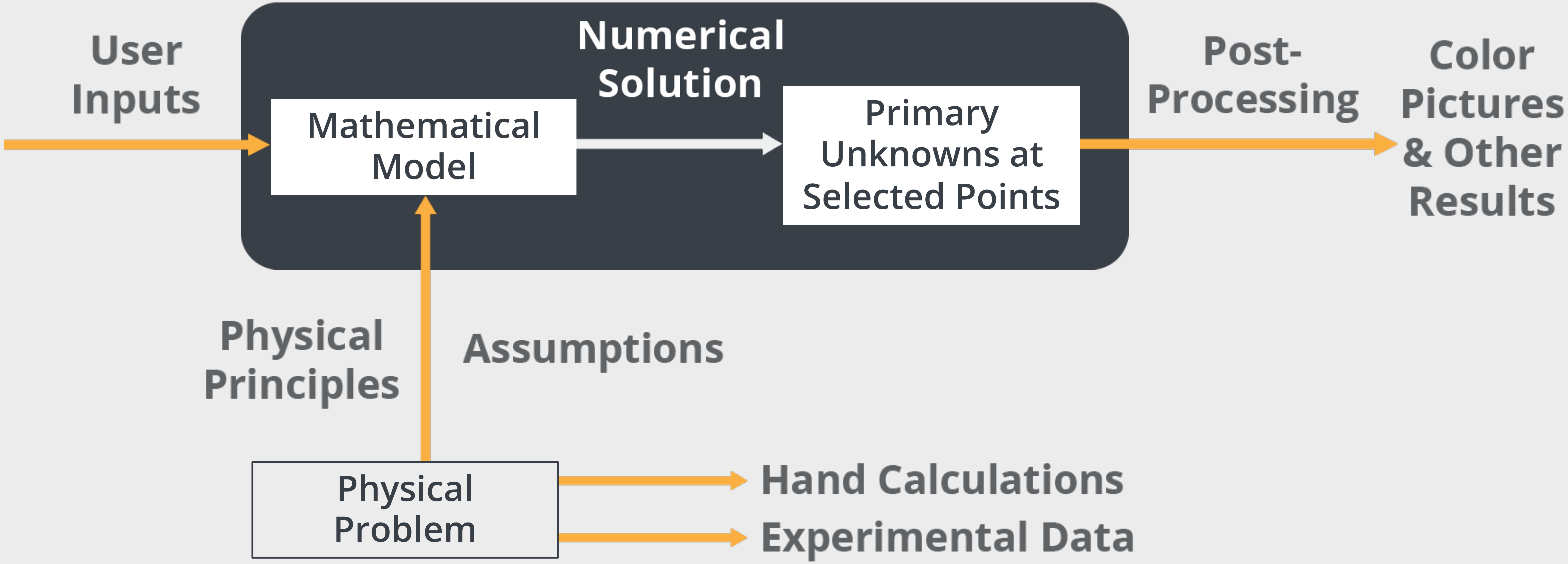


What's Inside the Black Box?



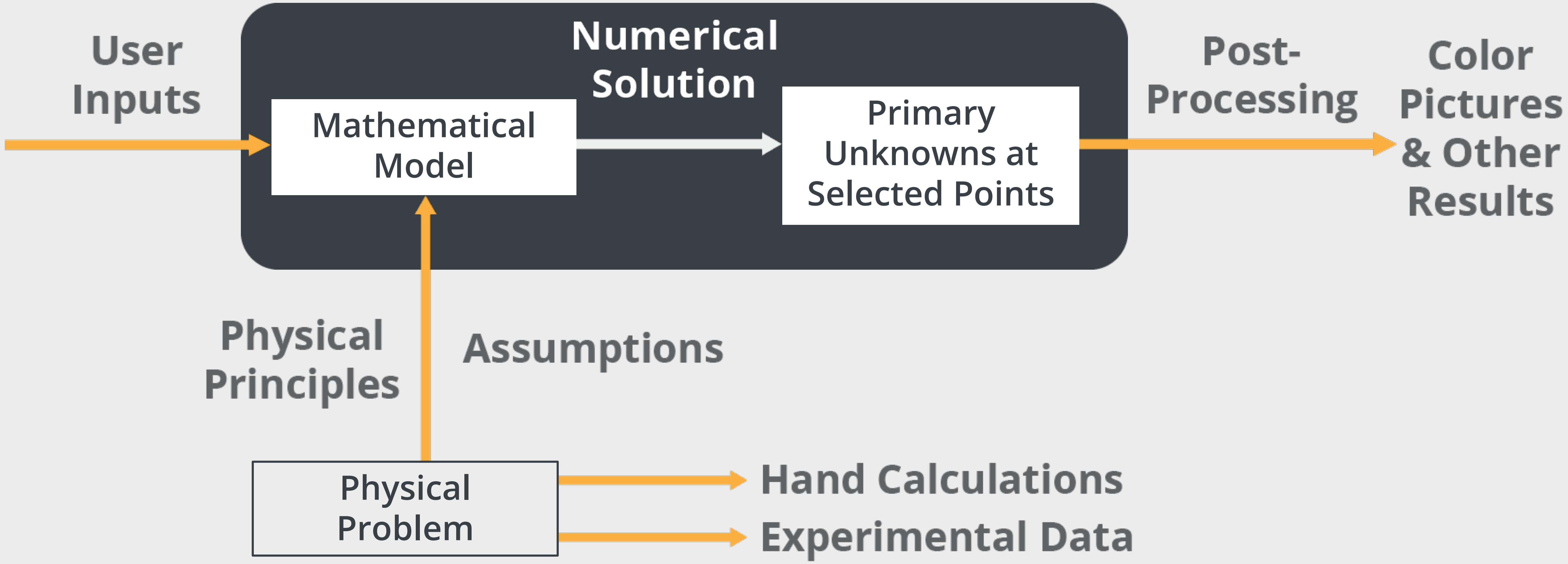
Novice → Expert

Pre-Analysis



1. Mathematical model
2. Numerical solution strategy
3. Hand-calculations to predict expected results/trends

Verification and Validation

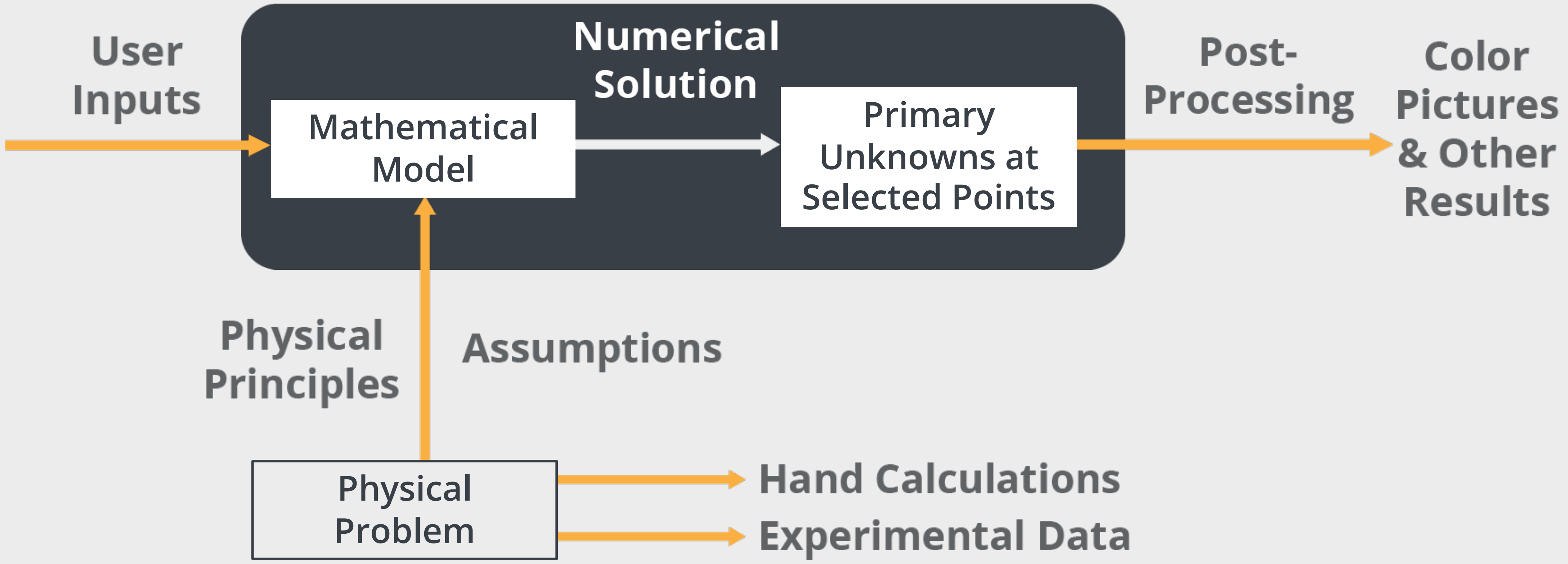


- **Verification: Did I solve the model right?**
  - Check consistency with mathematical model, level of numerical errors, comparison with hand calcs
- **Validation: Did I solve the right model?**
  - Check against experimental data

Uniform Solution Process

Problem Specification

1. Pre-analysis
2. Geometry
3. Mesh
4. Mathematical Model Setup
5. Numerical Solution
6. Post-Processing
7. Verification + Validation

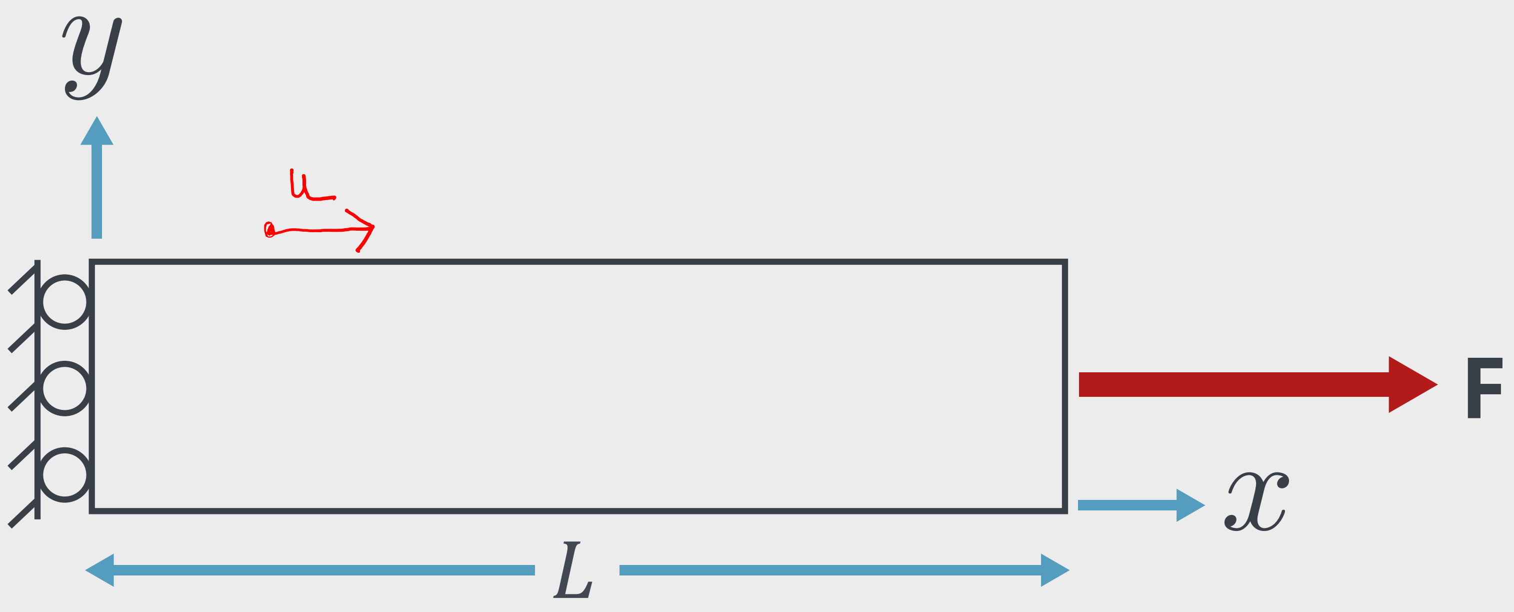


Just-in-time, problem-based learning

Conceptual Foundations of Finite Element Analysis (FEA)

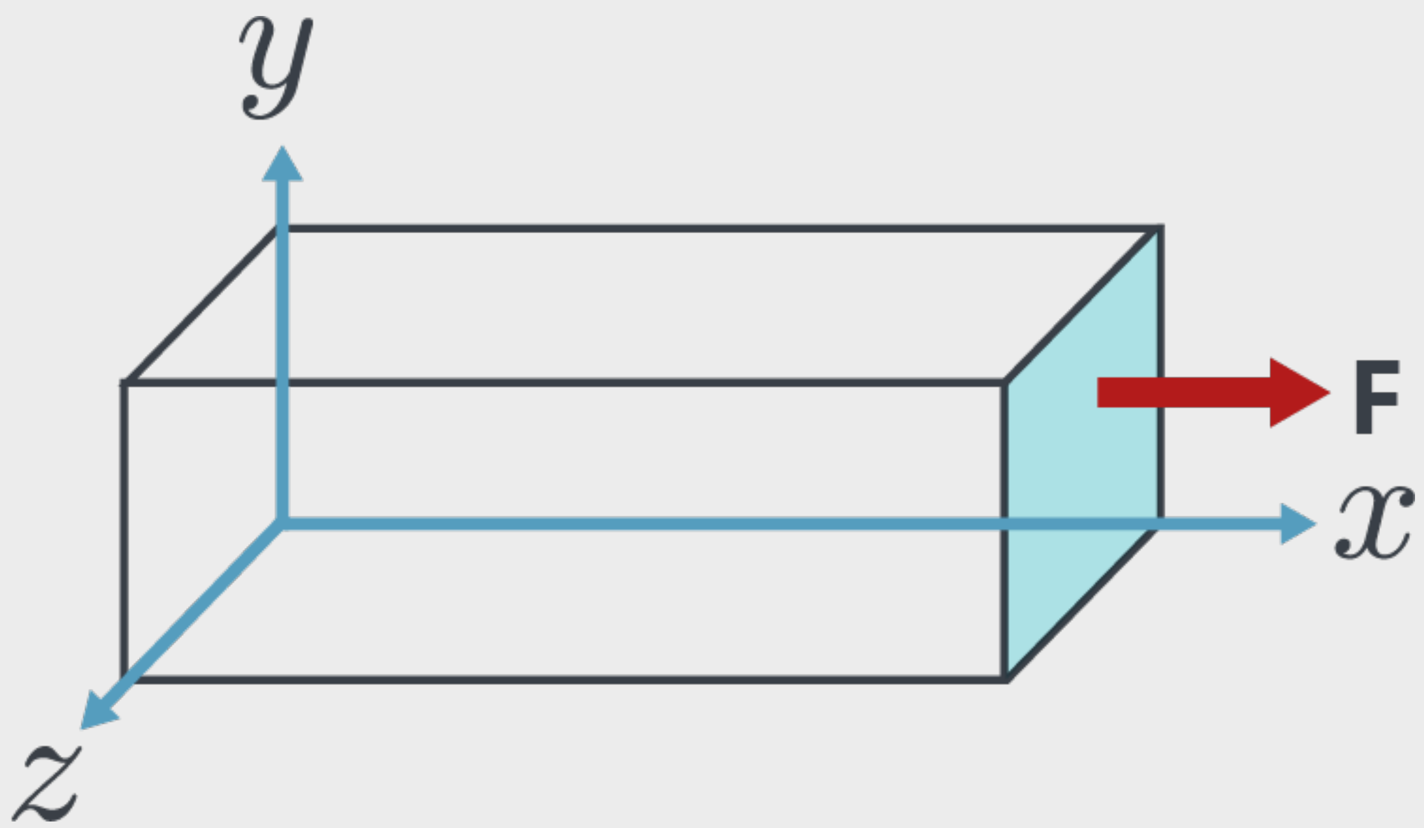
- We'll discuss the big ideas underlying FEA by considering a simple example
  - One-dimensional stretching of a bar
- Focus will be on concepts
- More complex examples are based on these concepts
- We'll come back to these concepts as we solve problems in Ansys
- They form the building blocks of FEA
- It's very important that you understand these ideas well

Example: 1D Stretching of a Bar



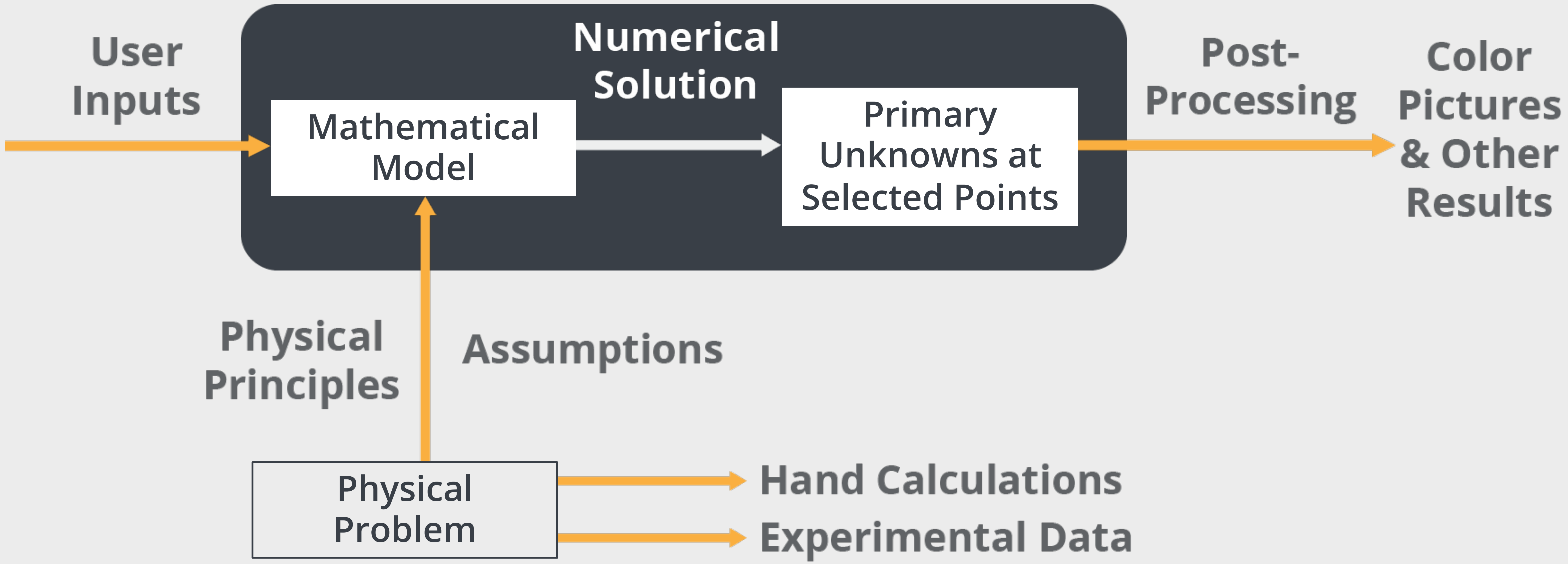
$$u = u(x, \cancel{y}, \cancel{z}) = u(x)$$

3D View



Find the displacement and stress distribution in the bar

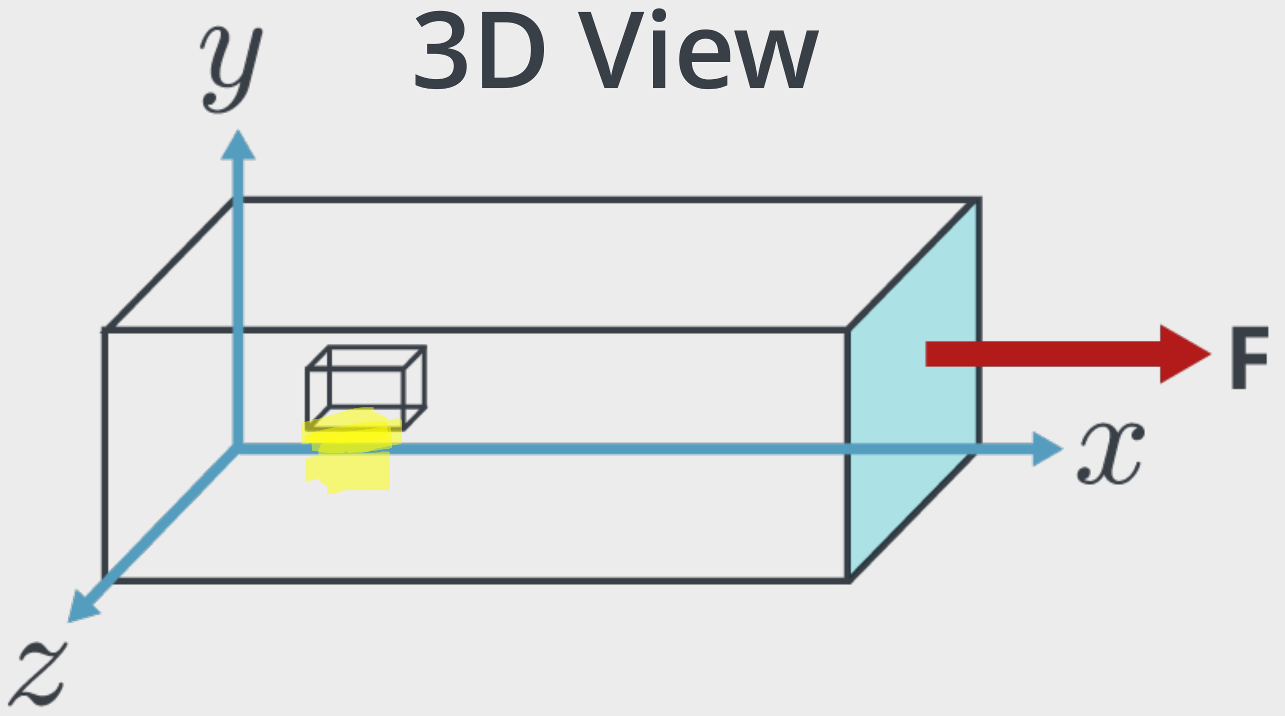
What's Inside the Blackbox?

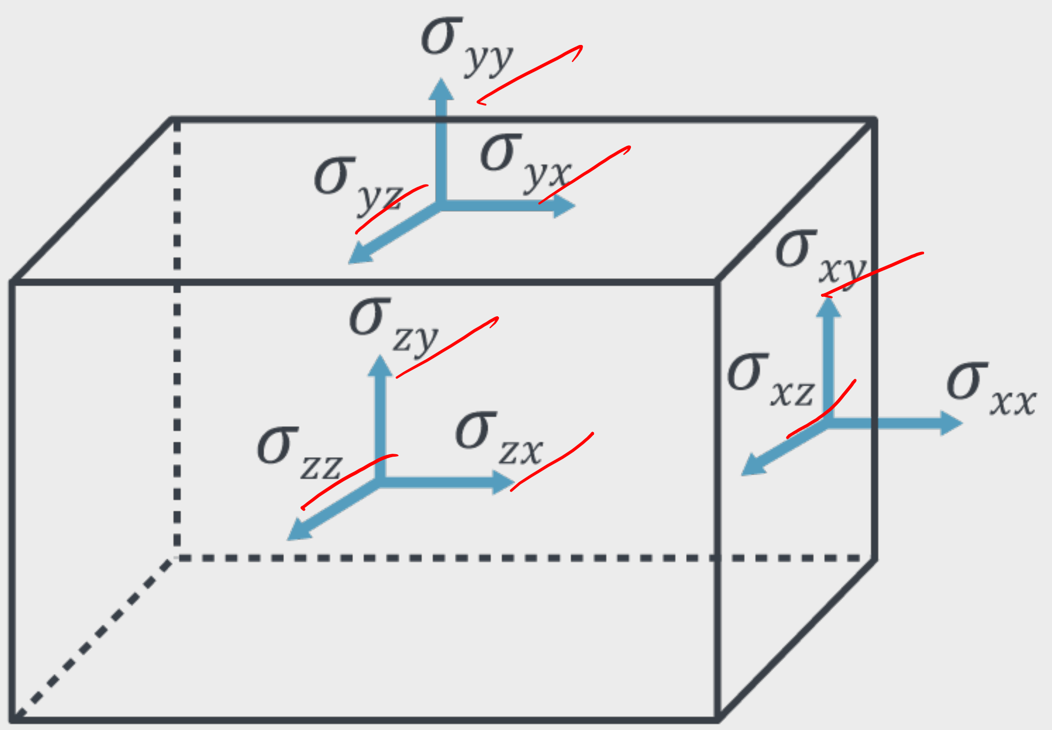


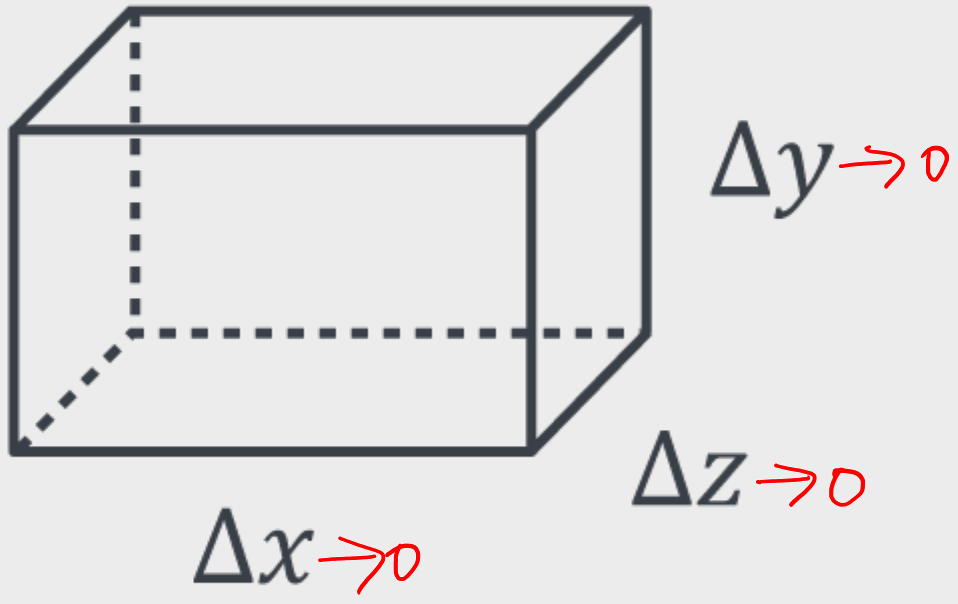
- **Mathematical model: Boundary value problem (BVP)**
  - Governing equations defined in a domain
  - Boundary conditions defined at the edges of the domain

Stresses on an Infinitesimal Element

3D View

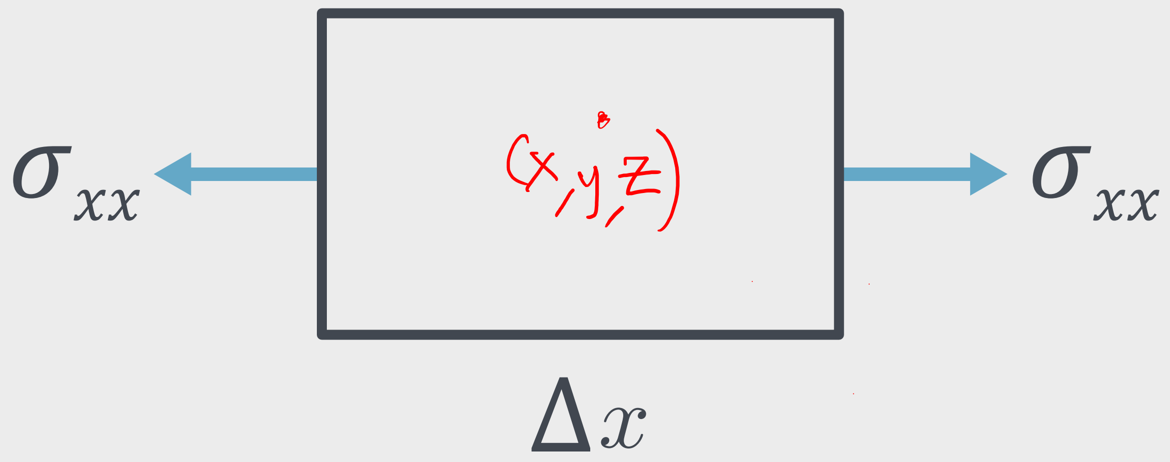






2D View

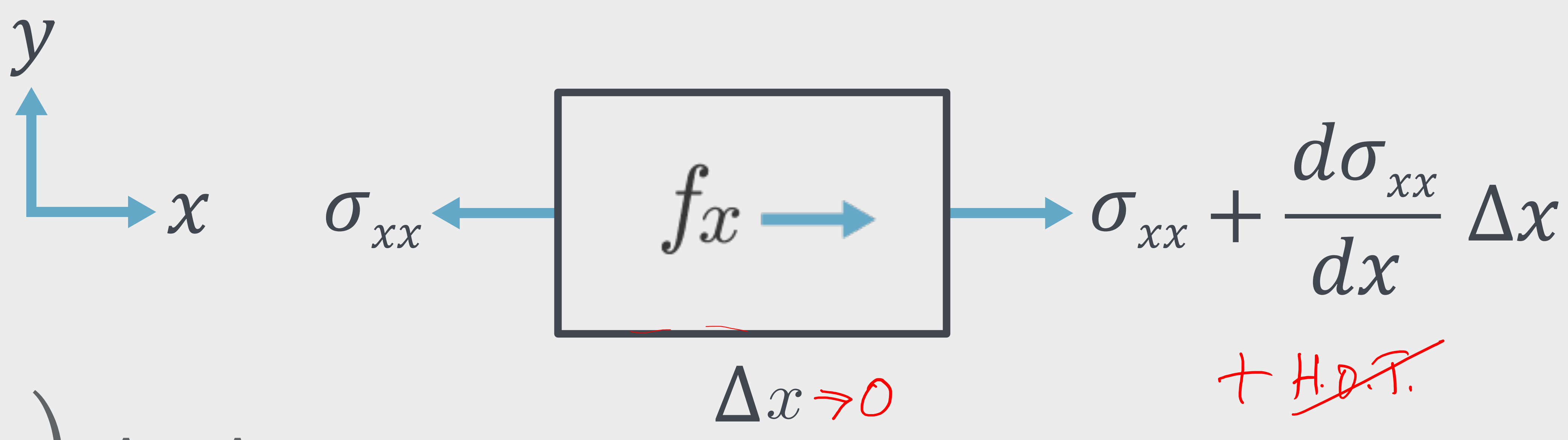
$\sigma_{xx}(x,y,z)$



Force Balance for an Infinitesimal Element

Physical principle:

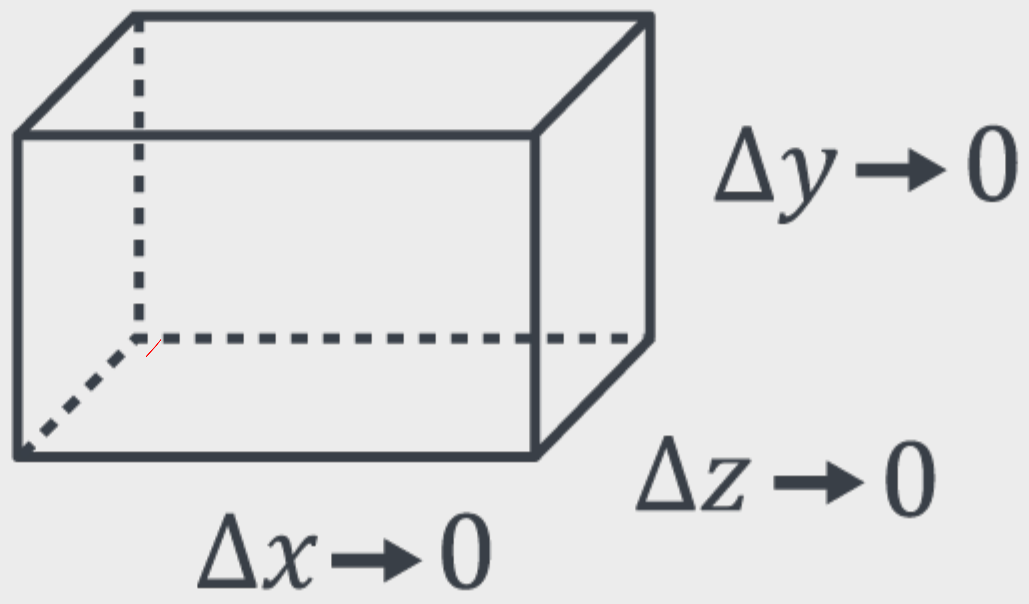
$\vec{F} = m\cancel{\vec{a}}$ or $\Sigma\vec{F_i} = 0$



$$-\cancel{\sigma_{xx}}\,\Delta y\,\Delta z + \left(\cancel{\sigma_{xx}} + \frac{d\sigma_{xx}}{dx}\,\Delta x\right)\Delta y\,\Delta z$$

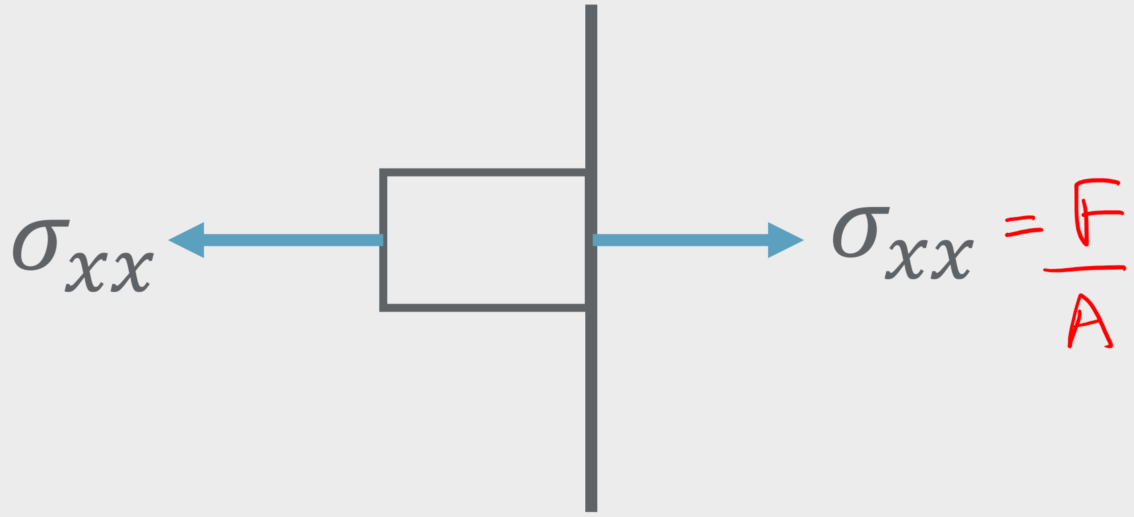
$$+ f_x\,\Delta x\,\Delta y\,\Delta z = 0$$

$$\frac{d\sigma_{xx}}{dx} + f_x = 0$$

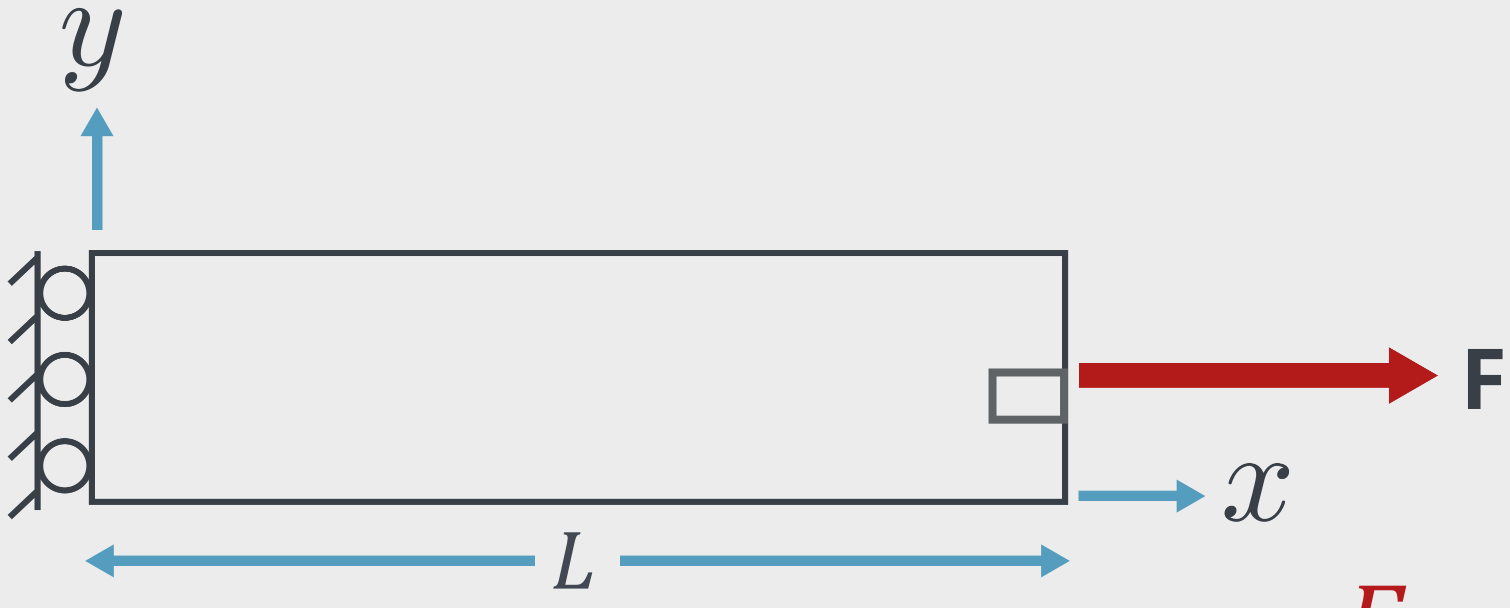


Governing Equation for 1D Bar

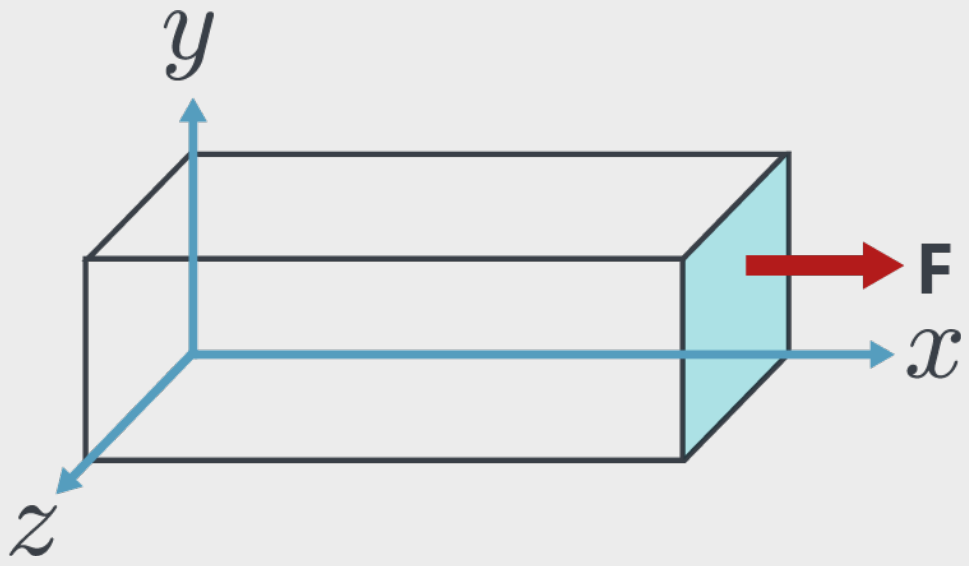
$$\frac{d\sigma_{xx}}{dx} + f_x = 0$$



Boundary conditions



$$u(0) = 0 \qquad \sigma_{xx}(L) = \frac{F}{A}$$



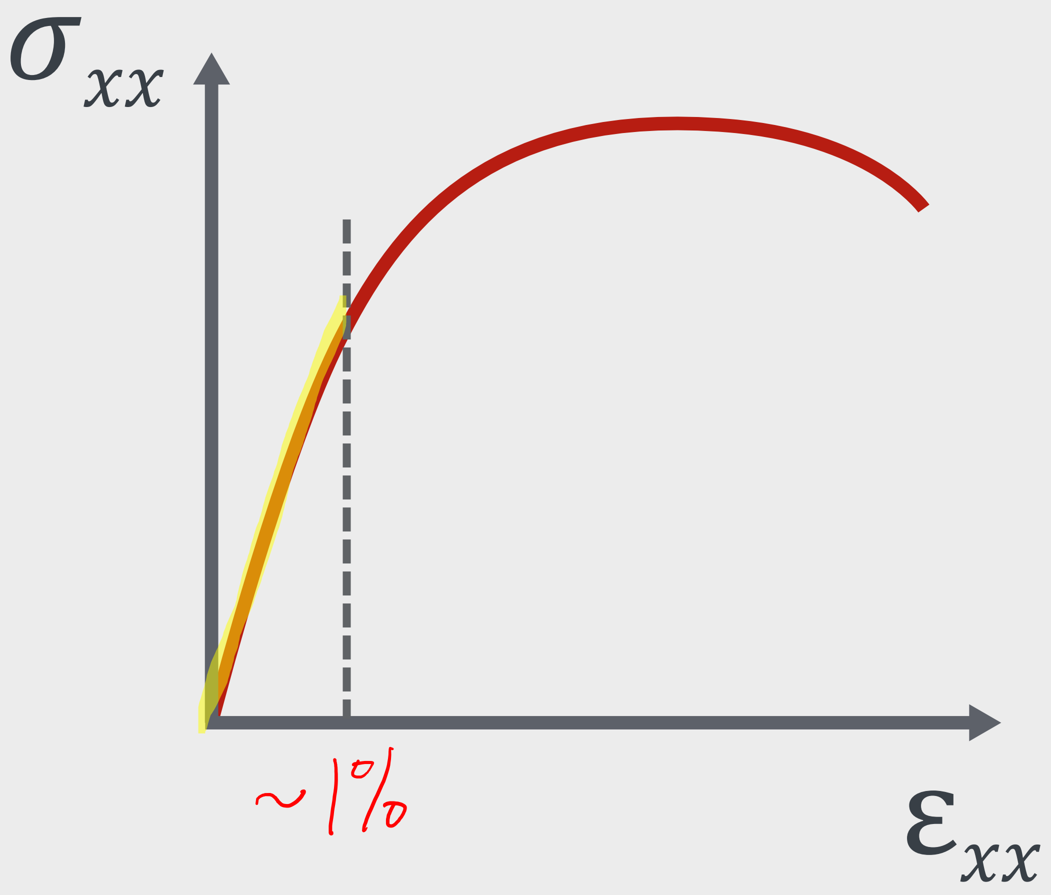
Assumption:

Small displacement

Additional Relations

$$\sigma_{xx} = E\,\epsilon_{xx}$$

1D Hooke's law
(Constitutive model)



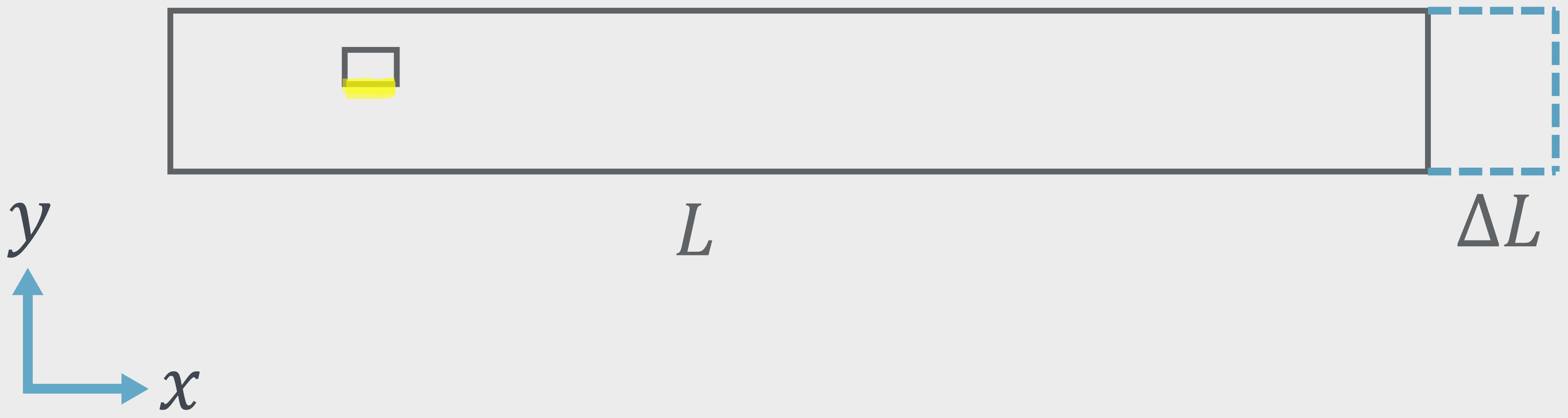
Assumption: Stress is linearly proportional to strain

$$\epsilon_{xx} = \frac{du}{dx}$$

Strain-displacement relation
(Kinematics)

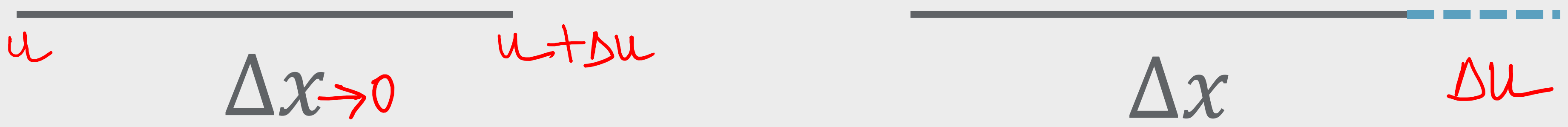
Assumption: Small strain

Strain-Displacement Relation Derivation



$$\epsilon = \frac{\Delta L}{L}$$

$$\Delta u = \frac{du}{dx}\Delta x + \cancel{H.O.T.}$$



$$\epsilon_{xx} = \frac{\Delta u}{\Delta x} = \frac{du}{dx}$$

Assumption: Small strain

Governing Equation for 1D Bar: Take 2

$$\frac{d\sigma_{xx}}{dx} + f_x = 0$$

$$\sigma_{xx} = E\,\epsilon_{xx}$$

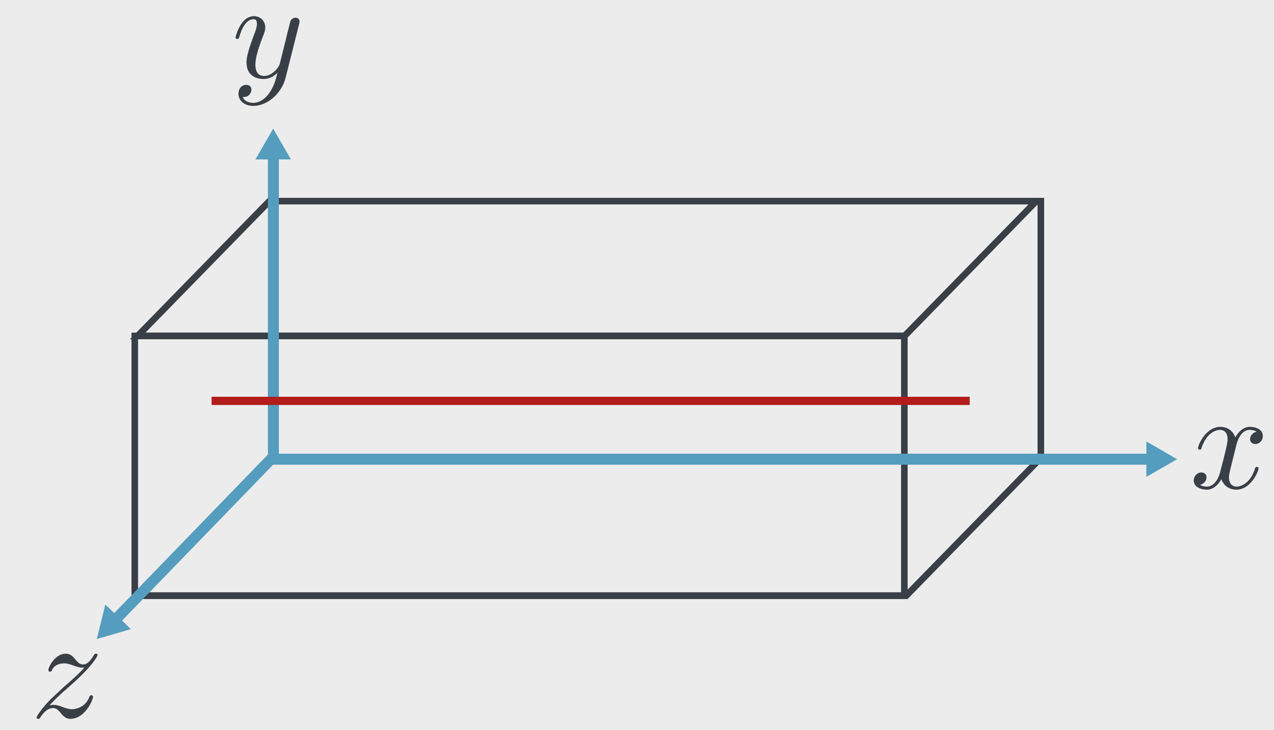
$$\frac{d}{dx}\left(\epsilon_{xx} = \frac{du}{dx}\right)$$

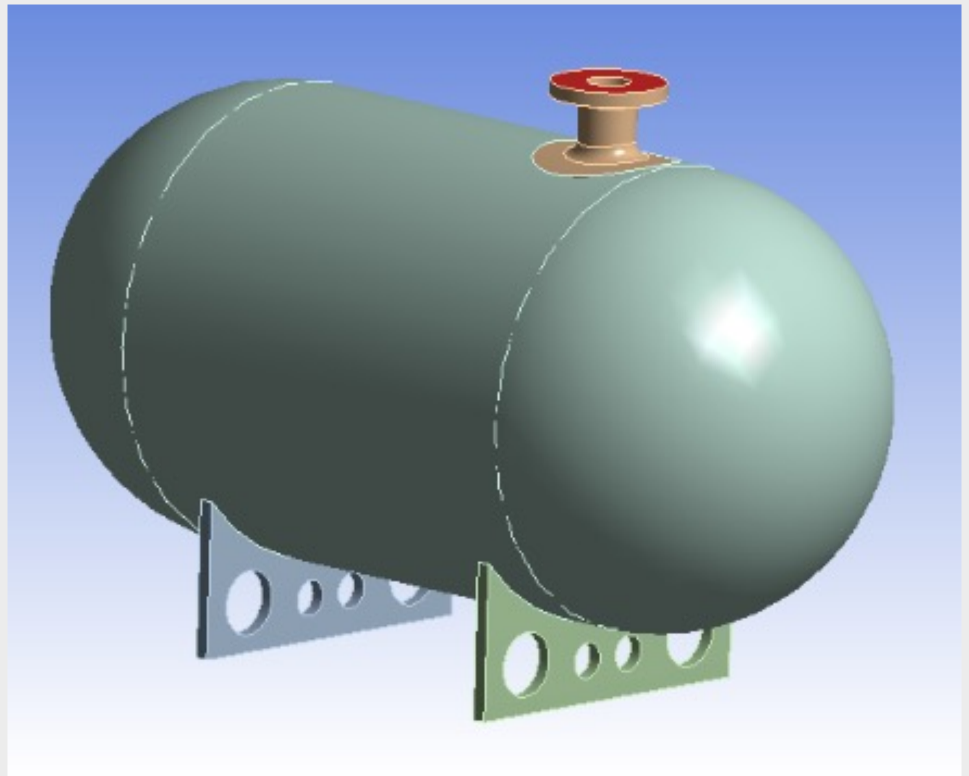
$$E\frac{d^2u}{dx^2} + f_x = 0$$

$$\frac{d\sigma_{xx}}{dx} = E\frac{d\varepsilon_{xx}}{dx} = E\frac{d^2u}{dx^2}$$

GE for 1D elasticity in terms of displacement

Domain and Boundary Conditions

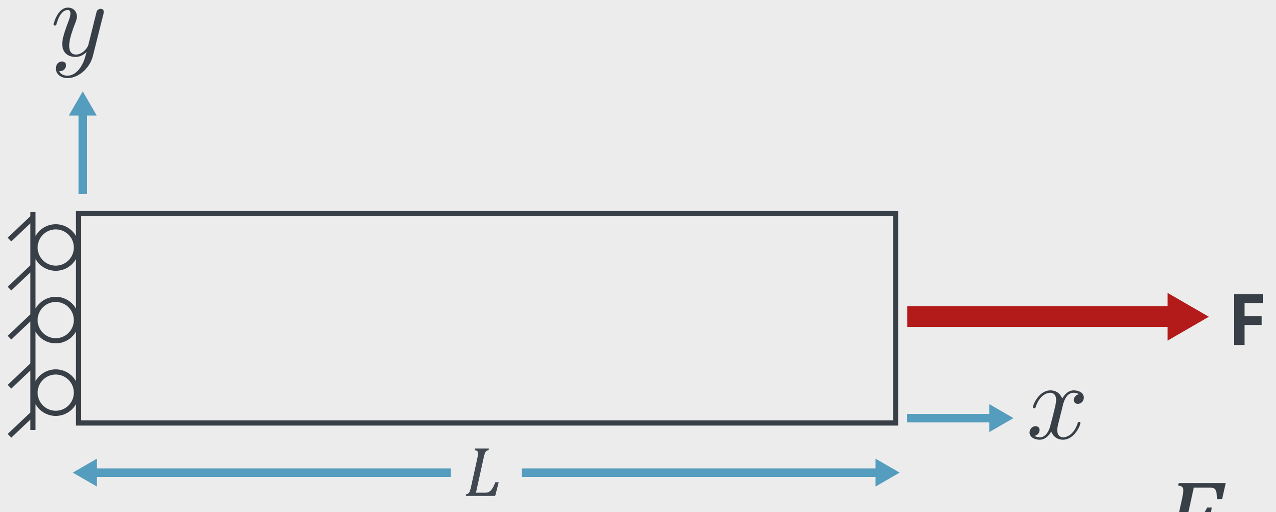




$x = 0$ $\qquad$ $x = L$

$u = 0$ $\qquad$ $\dfrac{du}{dx} = \dfrac{F}{AE}$

Assumption: Small displacement



$$E\frac{du}{dx} = \sigma_{xx}(L) = \frac{F}{A}$$

BVP for 1D Bar

$$E\frac{d^2u}{dx^2} + f_x = 0$$

$x = 0$ $\qquad$ $x = L$

$u = 0$ $\qquad$ $\dfrac{du}{dx} = \dfrac{F}{AE}$

Assumptions:

Linear material

Small strain

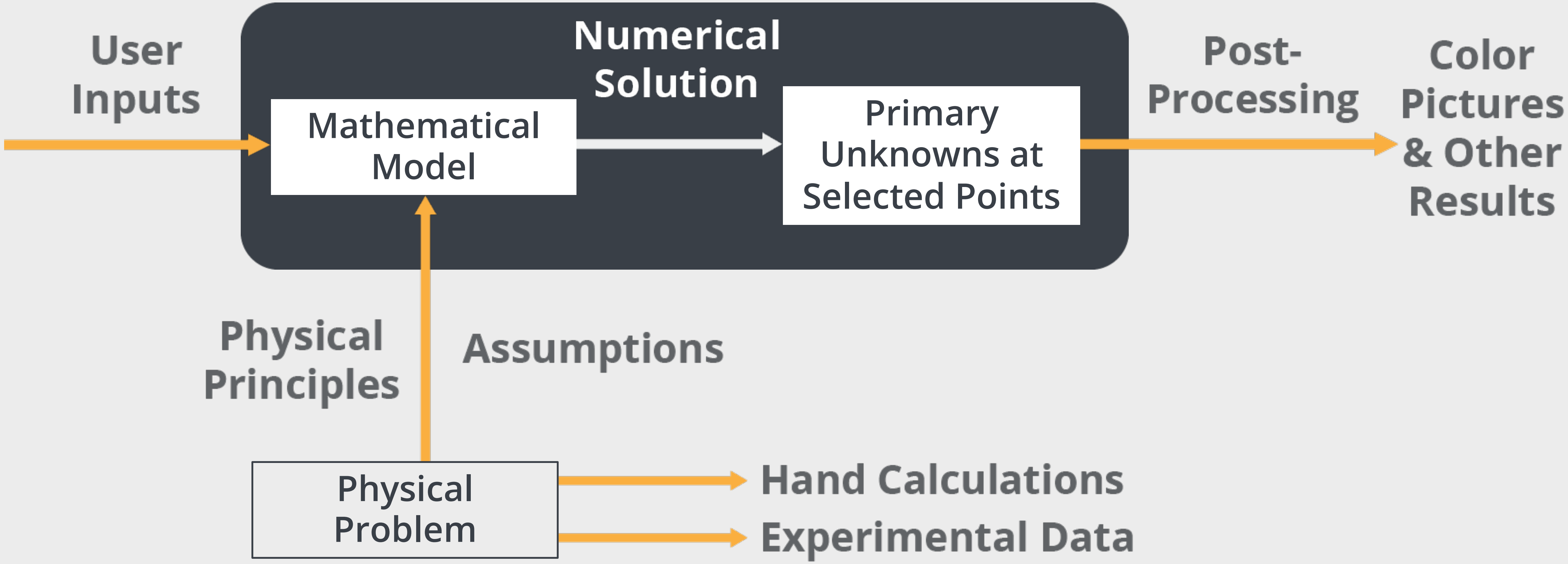
Small displacement

1D

Exact solution: $u(x) = a\,x^2 + b\,x$

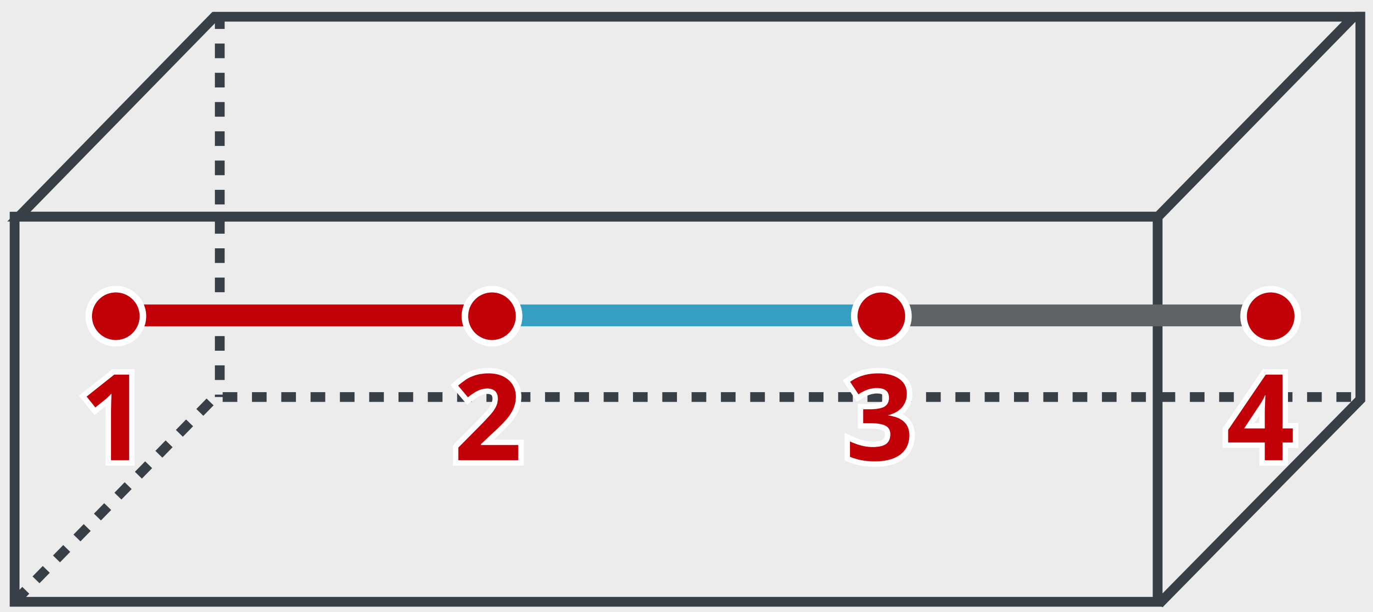
$$a = -\frac{f_x}{2E} \qquad b = \frac{F}{AE} + \frac{f_x L}{E}$$

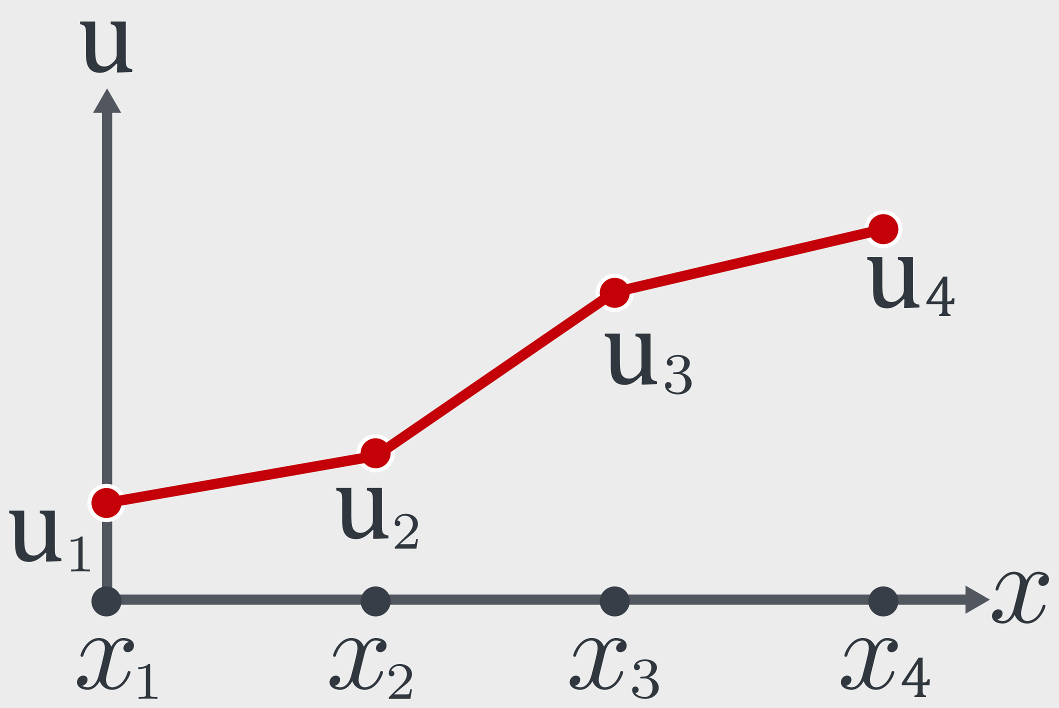
What's Inside the Blackbox?



- **Mathematical model: Boundary value problem (BVP)**
  - Governing equations defined in a domain
  - Boundary conditions defined at the edges of the domain

Discretization and Interpolation





Reduce problem to determining u values at selected points (nodes)

Use linear interpolation to determine u values between nodes

Gone from determining $u(x)$ to determining 4 $u_i's$

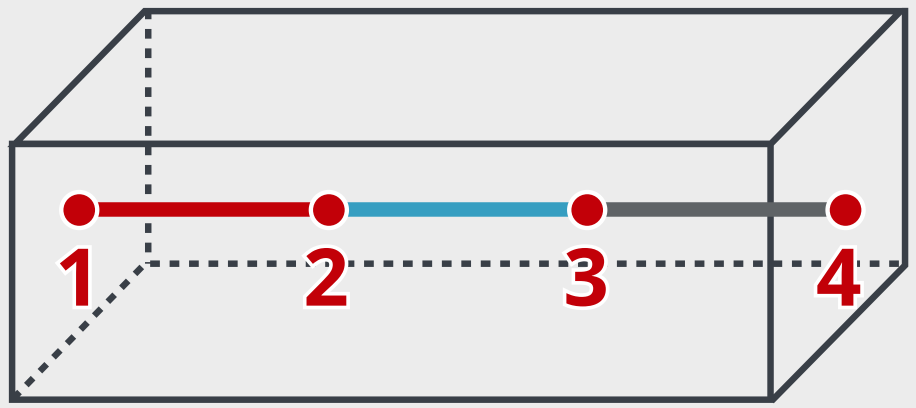
Degrees of freedom = 4

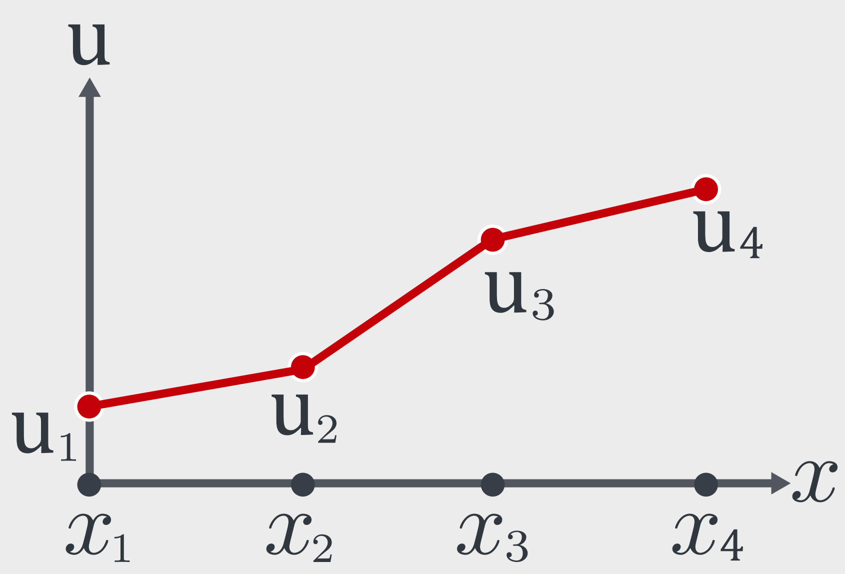
How to Find Nodal Displacement Values u_i?

Mathematical model (Boundary value problem) → System of algebraic eqs. in nodal displacements → Invert → Nodal displacements → Post-processing → $u(x), \sigma_{xx}(x)$ etc.

Element-by-element polynomial interpolation for u

$$[K]\{u\} = \{F\}$$



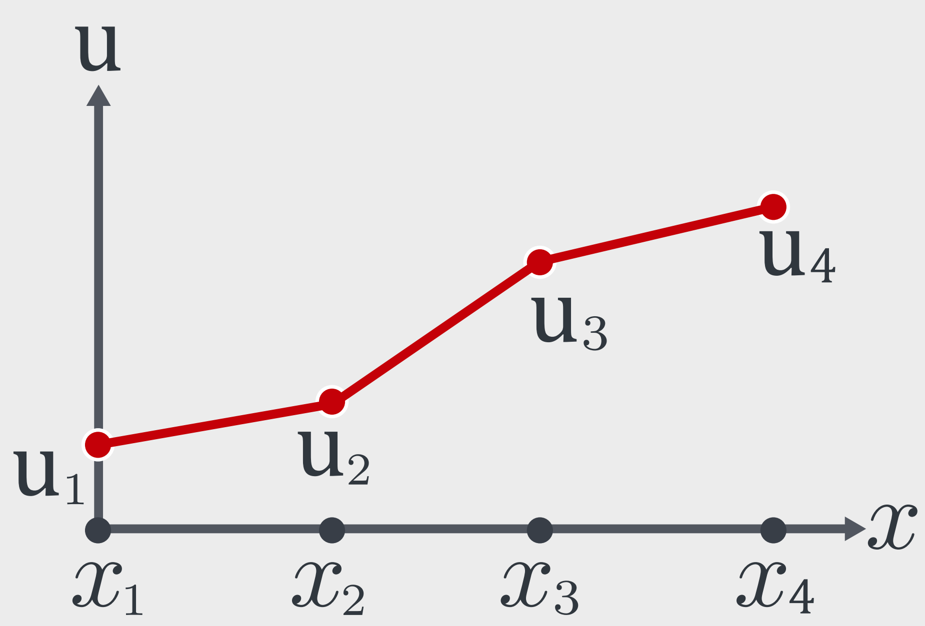


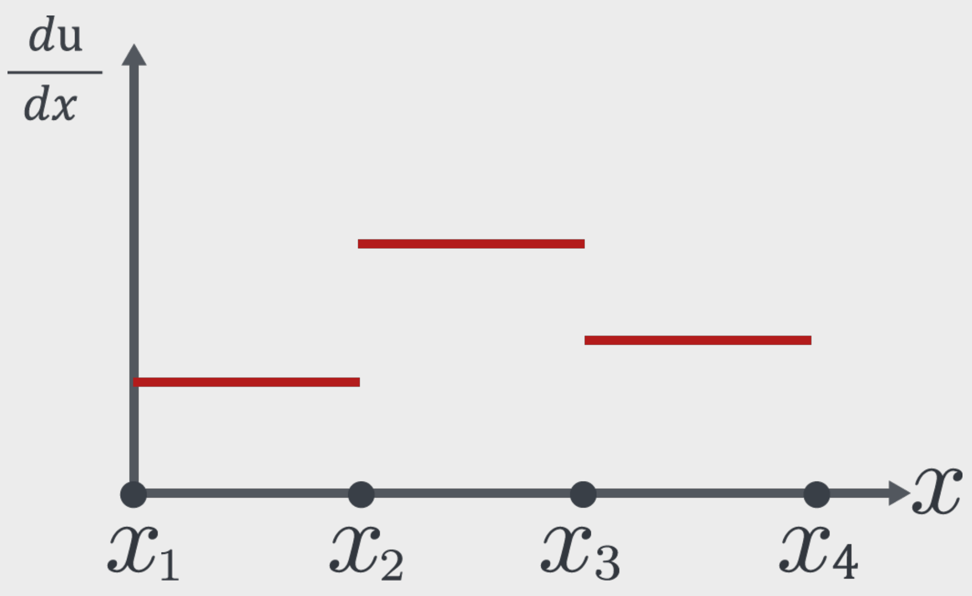
Each algebraic equation will relate neighboring u_i values

How to Derive System of Algebraic Equations? (1/3)

Element-by-element polynomial interpolation for u → $$E\frac{d^2u}{dx^2} + f_x = 0$$ → System of algebraic eqs. in nodal displacements

$0 + f_x \neq 0$





Finite element solution won't satisfy equilibrium of **infinitesimal** elements

How to Derive System of Algebraic Equations? (2/3)

Strong form

$$E\frac{d^2u}{dx^2} + f_x = 0$$

Integrate

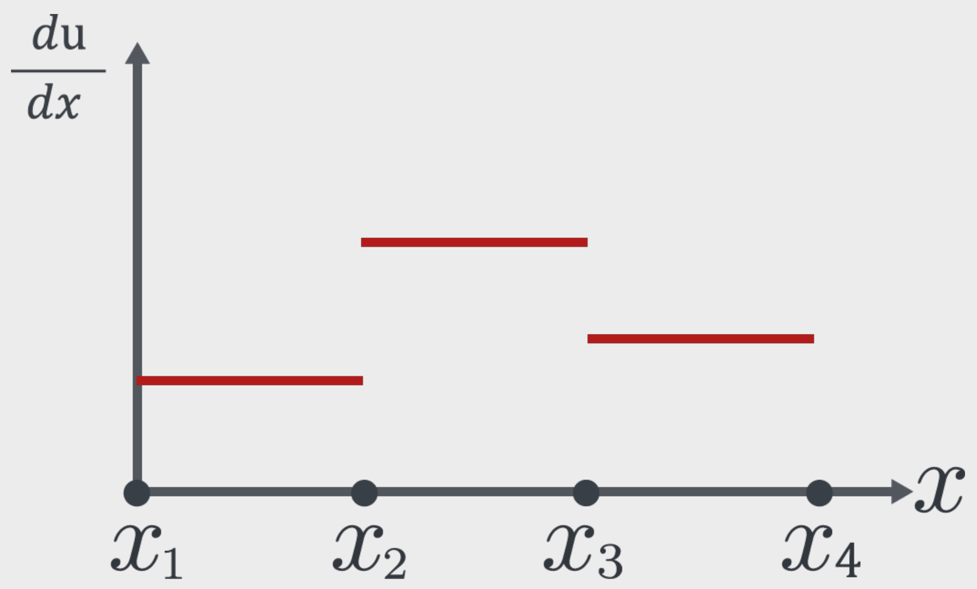
$$\int_0^L w(x)\left(E\frac{d^2u}{dx^2} + f_x\right)dx = 0$$

Integrate by parts

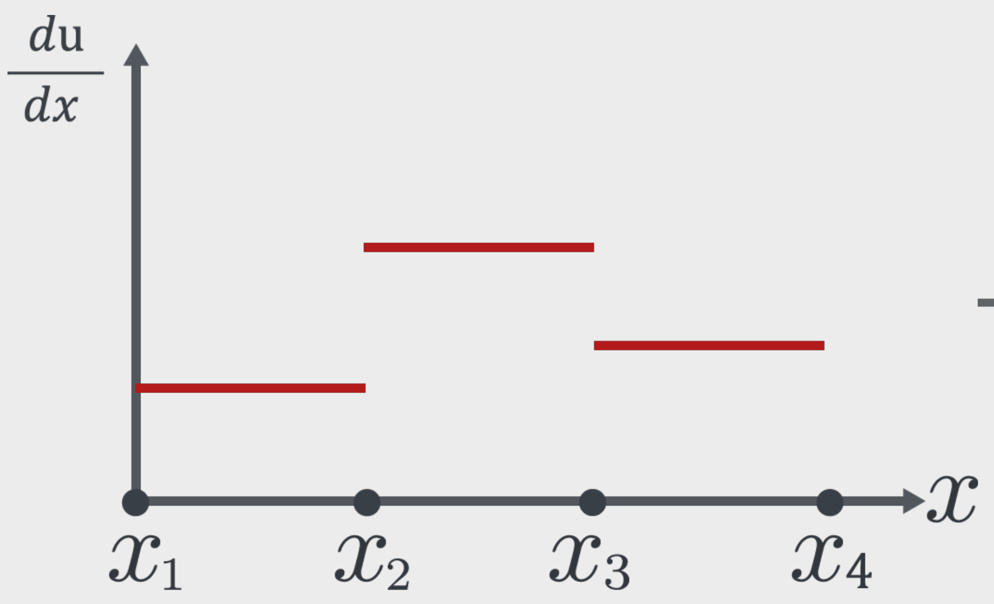
$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{du}{dx}\,dx = w\,\mathrm{E}\,\frac{du}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

Weak form

$w(x)$ is an arbitrary function

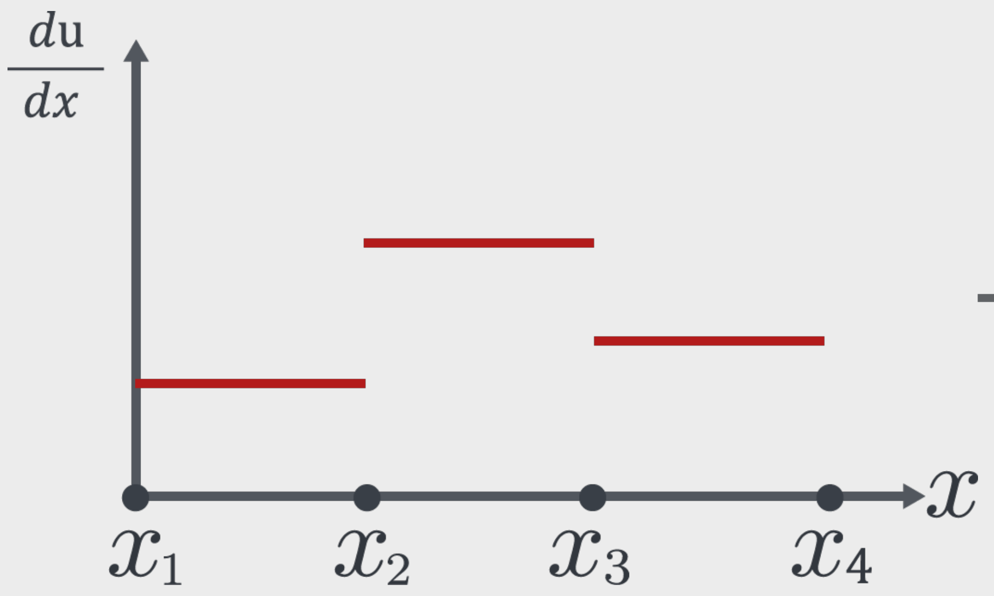


How to Derive System of Algebraic Equations? (3/3)



$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

System of algebraic eqs. in nodal displacements



$$E\frac{d^2u}{dx} + f_x = 0$$

~~System of algebraic eqs. in nodal displacements~~

Strong Form to Weak Form

$$E\frac{d^2u}{dx^2} + f_x = 0$$

Integrate

$$\int_0^L w\left(E\frac{d^2u}{dx^2} + f_x\right)dx = 0$$

$$\int_0^L wE\frac{d^2u}{dx^2}dx + \int_0^L w\, f_x\, dx = 0$$

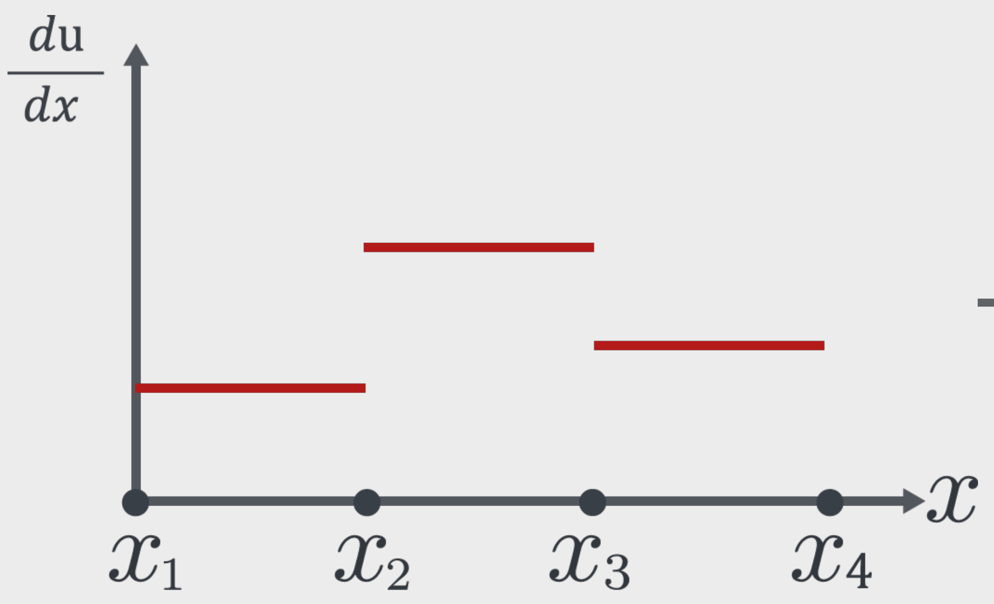
$$\int_0^L w\left(\frac{d^2u}{dx^2}\right) dx = \left[w\frac{du}{dx}\Big]_0^L - \int_0^L \frac{dw}{dx}\frac{du}{dx}dx \right] *E$$

$$w\,\mathrm{E}\frac{du}{dx}\Big]_0^L - \int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{du}{dx}dx + \int_0^L w\, f_x\, dx = 0$$

Weak form

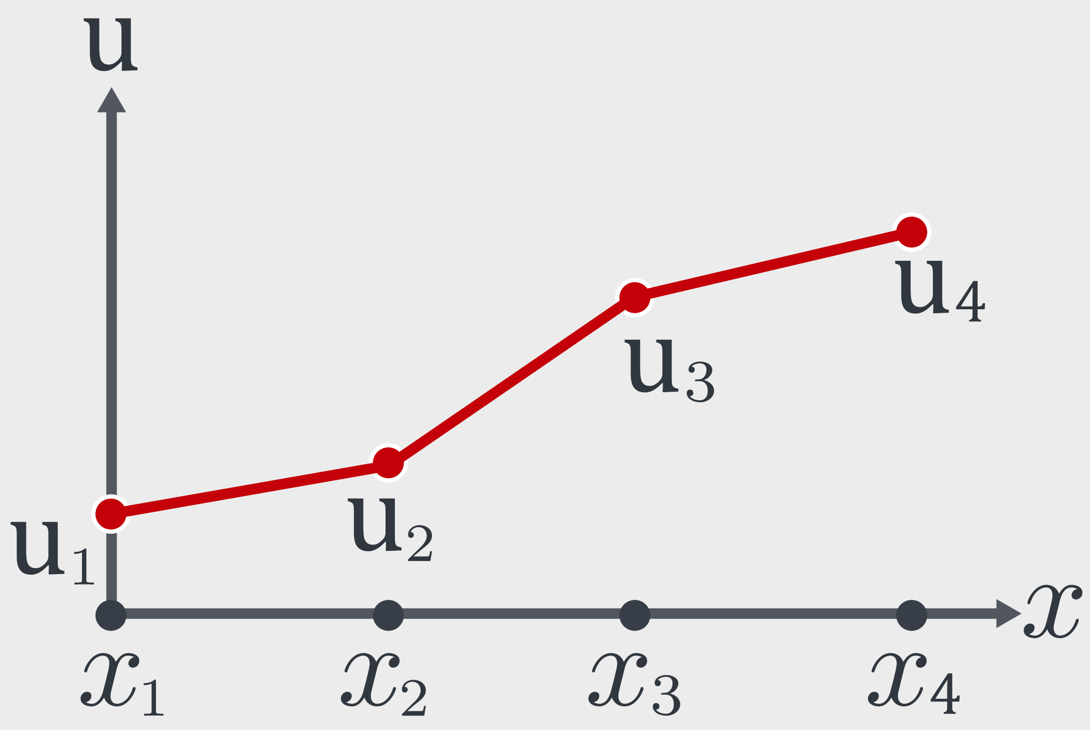
$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{du}{dx}dx = w\,\mathrm{E}\frac{du}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$

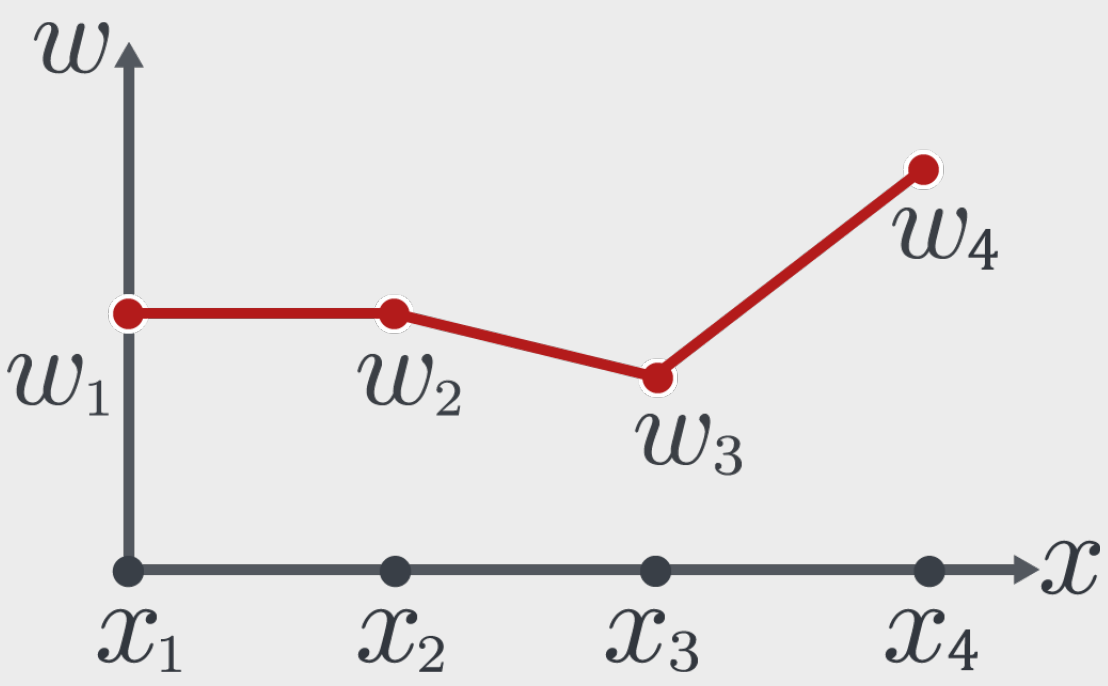
Shape of Weight Function w(x)



$$\int_0^L \frac{dw}{dx}\, \mathrm{E}\, \frac{\mathrm{du}}{dx}\, dx = w\, \mathrm{E} \frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$

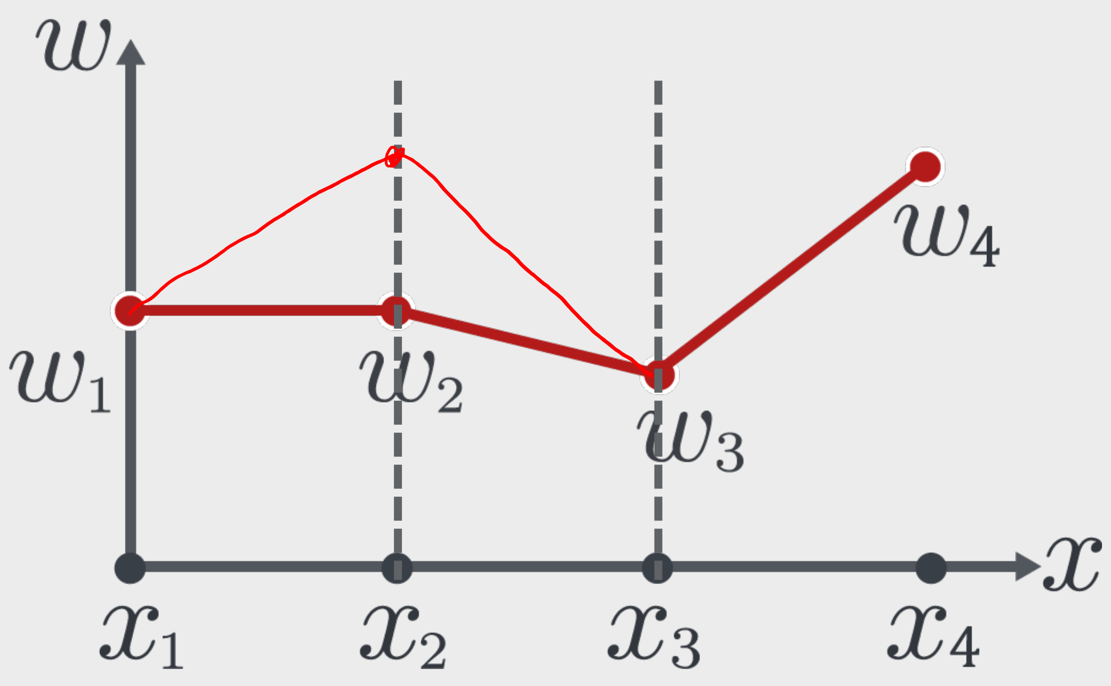
System of algebraic eqs. in nodal displacements

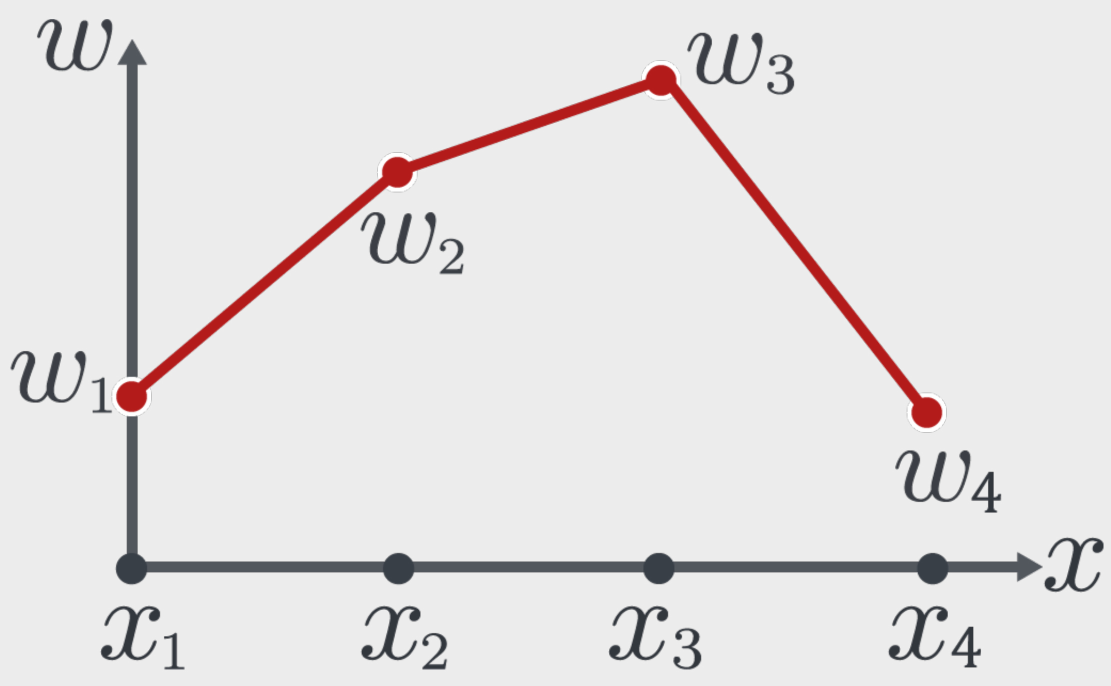




$w_i's$ are arbitrary

Examples of two admissible w variations





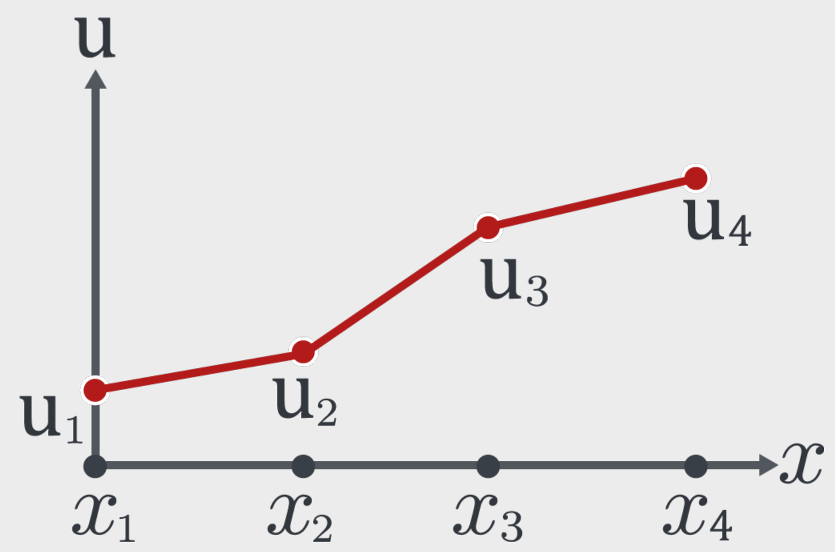
$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

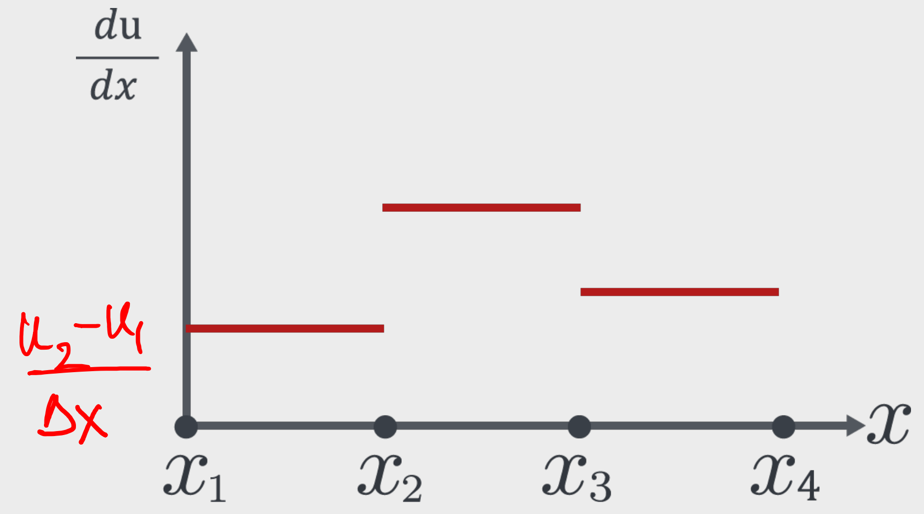
Weak form needs to be satisfied for arbitrary w_i values

Weak Form to Algebraic Equations: Process

$$\int_0^L \frac{dw}{dx}\, E\, \frac{du}{dx}\, dx = w\, E\, \frac{du}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$

Element by element integration



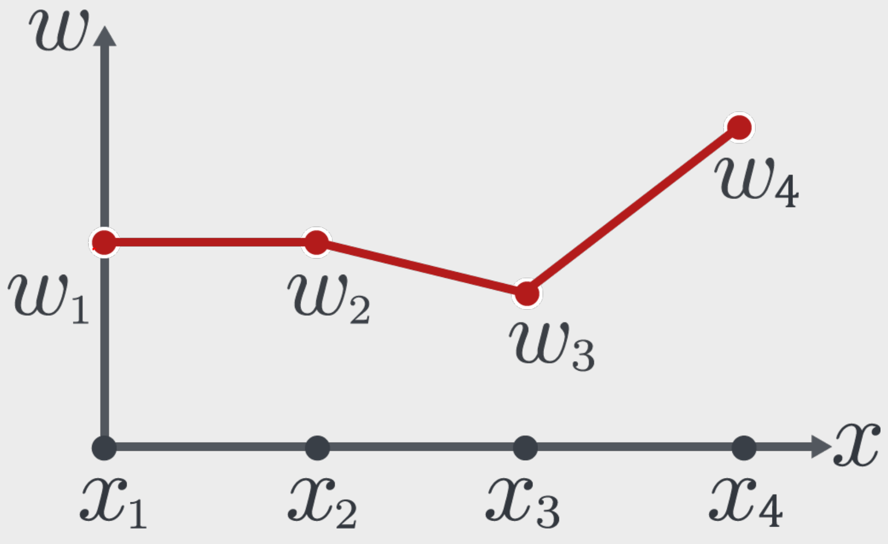


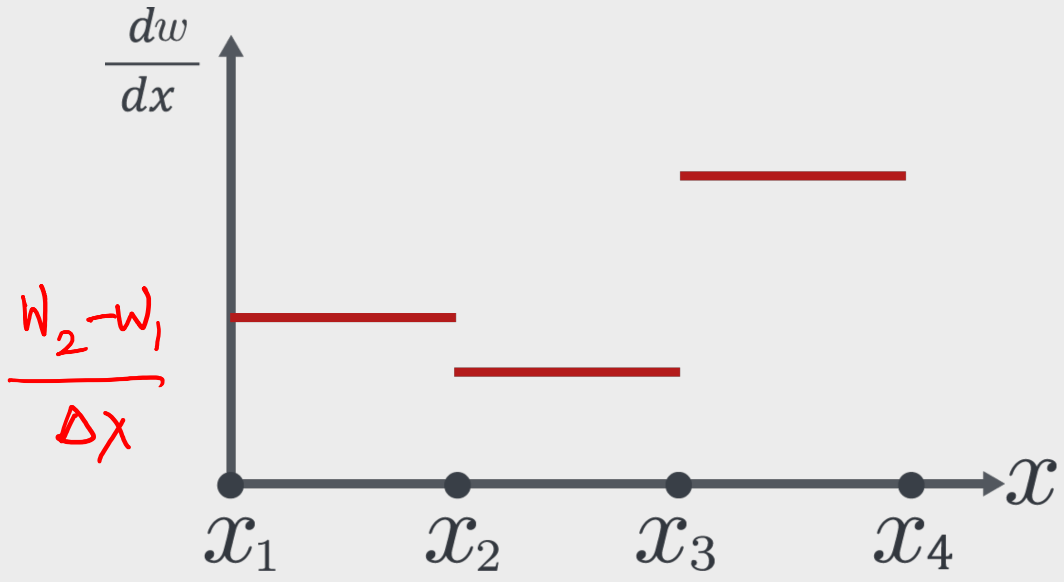
$W_2 u_2$

$-W_2 u_1$

$-W_1 u_2$

$W_1 u_1$





Weak Form to Algebraic Equations: Integration Result

$$\mathrm{w}_1 \left[\left(\frac{E}{\Delta x}\right) u_1 - \left(\frac{E}{\Delta x}\right) u_2 - 0.5 f_x \, \Delta x + \sigma_{xx}(0)\right] +$$

$$w_2 \left[-\left(\frac{E}{\Delta x}\right) u_1 + \left(\frac{2E}{\Delta x}\right) u_2 - \left(\frac{E}{\Delta x}\right) u_3 - f_x \, \Delta x\right] +$$

$$w_3 \left[-\left(\frac{E}{\Delta x}\right) u_2 + \left(\frac{2E}{\Delta x}\right) u_3 - \left(\frac{E}{\Delta x}\right) u_4 - f_x \, \Delta x\right] +$$

$$\mathrm{w}_4 \left[-\left(\frac{E}{\Delta x}\right) u_3 + \left(\frac{E}{\Delta x}\right) u_4 - 0.5 f_x \, \Delta x - \sigma_{xx}(L)\right] = 0$$

Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$

Element 1

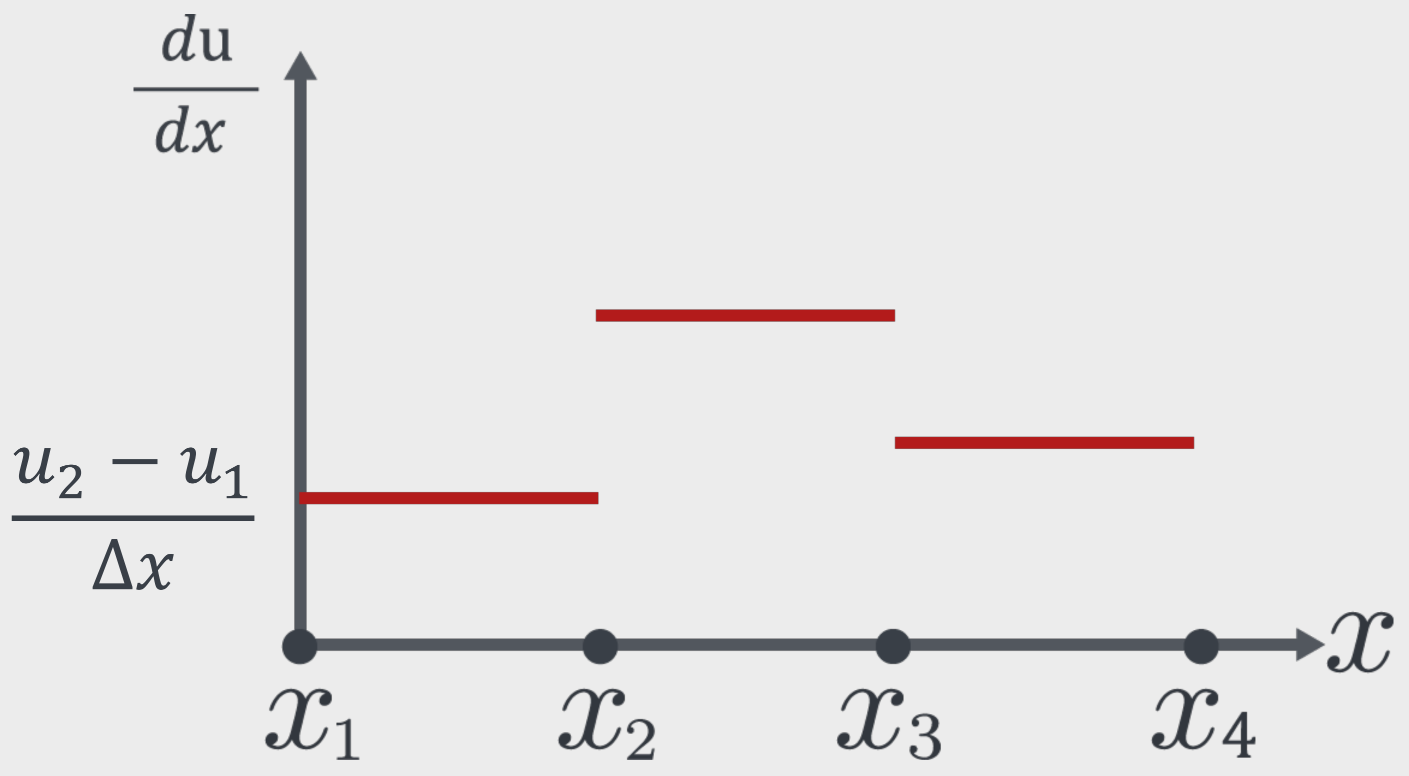
$$\int_0^{x_2} \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx$$

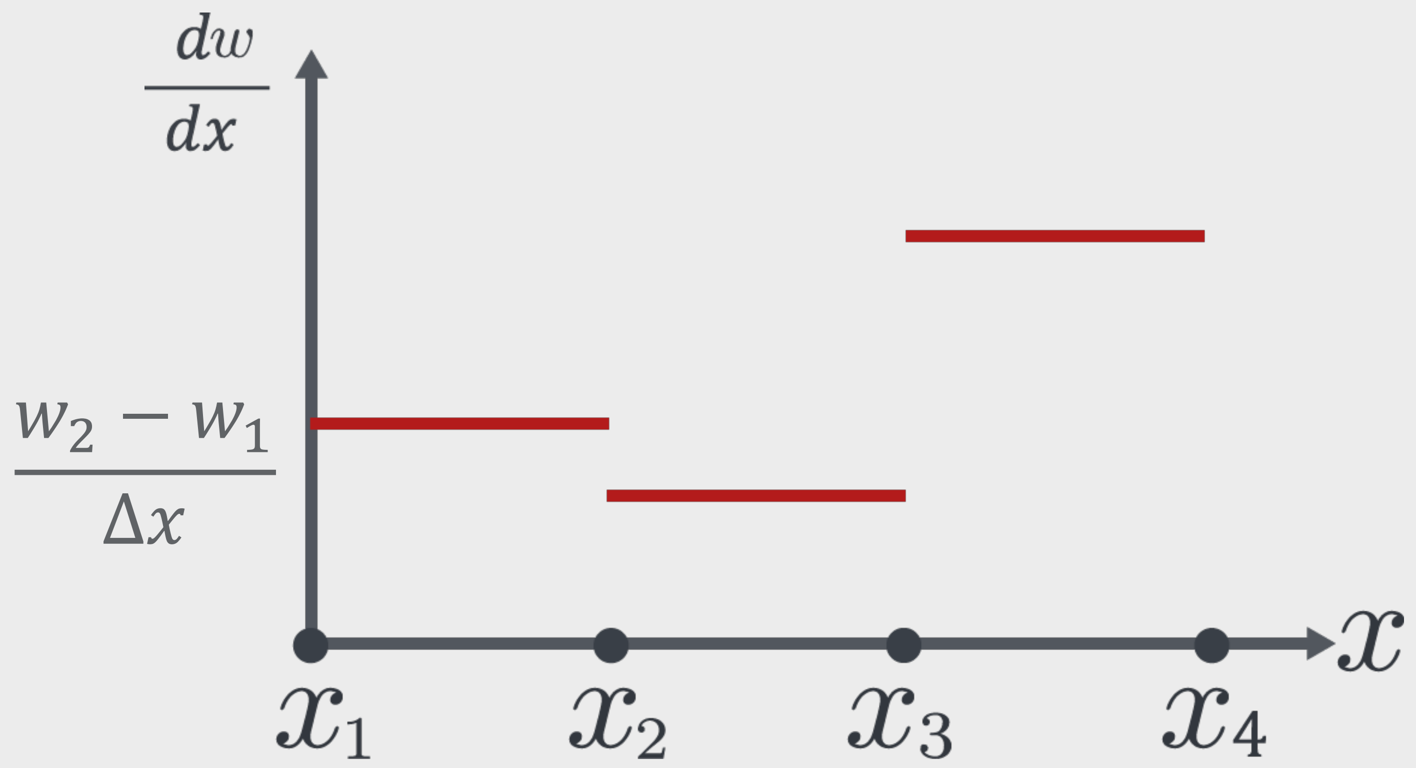
$$= \left(\frac{W_2 - W_1}{\Delta X}\right) E \left(\frac{u_2 - u_1}{\cancel{\Delta X}}\right) \cancel{\Delta X}$$

$$= \left(\frac{E}{\Delta x}\right)(w_2u_2 - w_2u_1 - w_1u_2 + w_1u_1)$$

$$= \mathrm{w}_1 \left[\left(\frac{E}{\Delta x}\right)u_1 - \left(\frac{E}{\Delta x}\right)u_2\right]$$

$$+ w_2\left[-\left(\frac{E}{\Delta x}\right)u_1 + \left(\frac{E}{\Delta x}\right)u_2\right]$$





Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

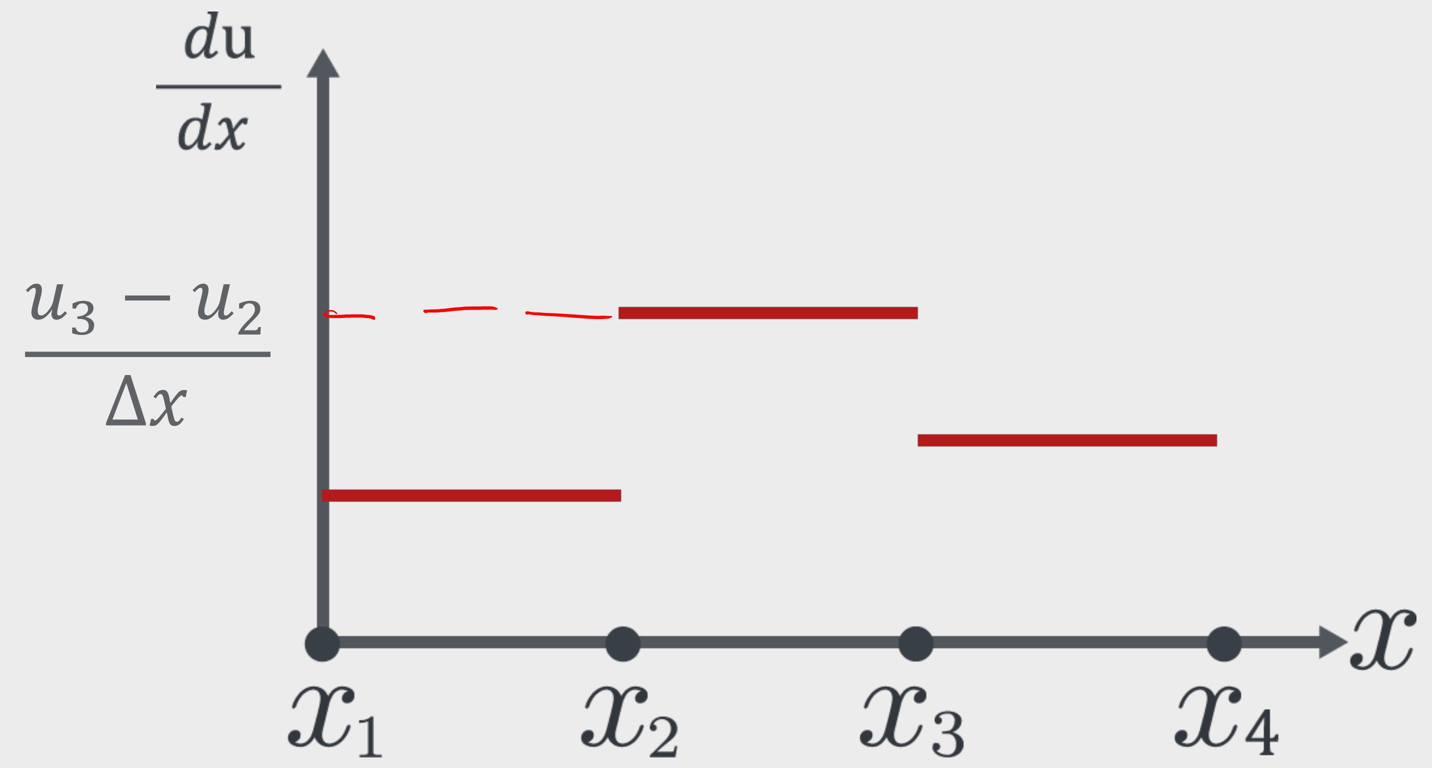
Element 2

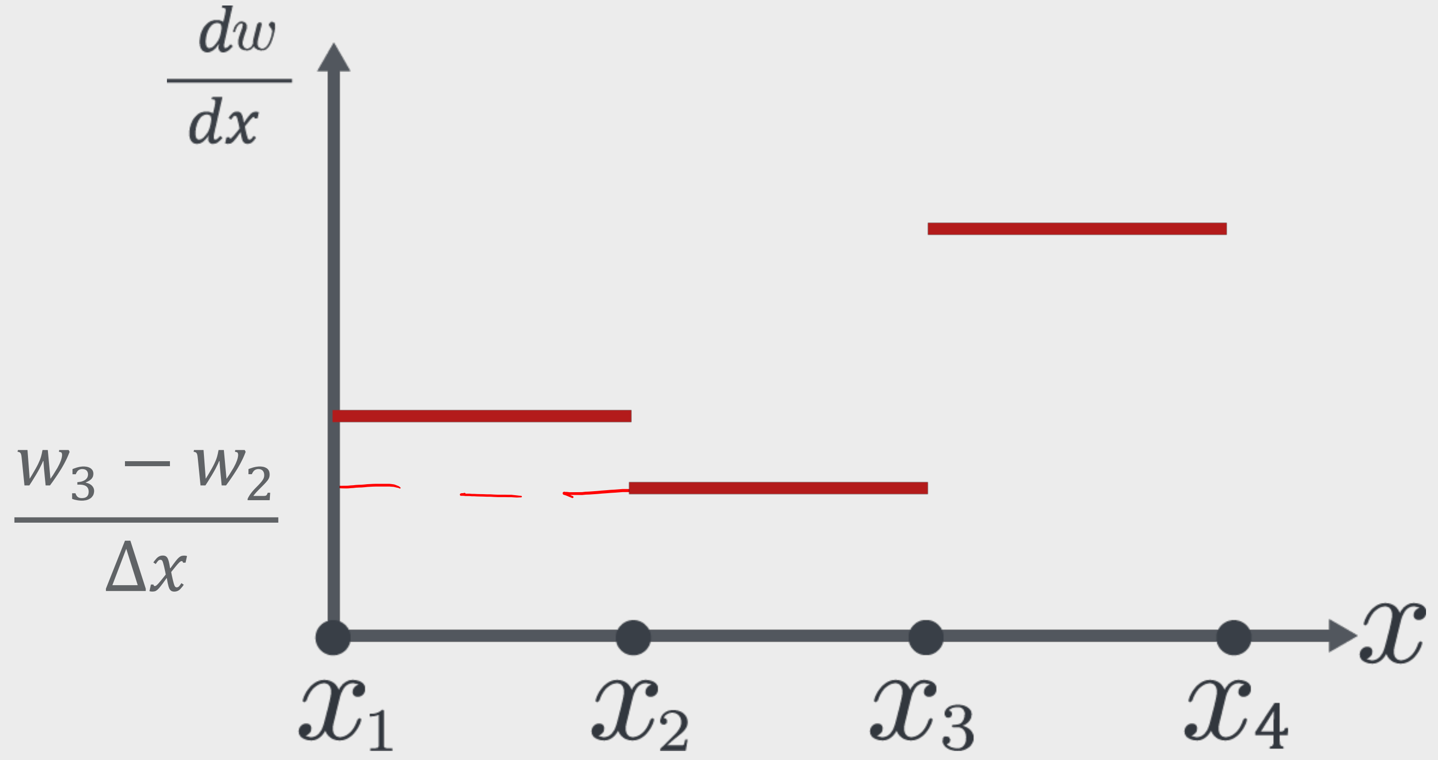
$$\int_{x_2}^{x_3} \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx$$

$$= \left(\frac{w_3 - w_2}{\Delta x}\right) E \left(\frac{u_3 - u_2}{\cancel{\Delta x}}\right) \cancel{\Delta x}$$

$$= \mathrm{w}_2 \left[\left(\frac{E}{\Delta x}\right) u_2 - \left(\frac{E}{\Delta x}\right) u_3\right]$$

$$+ w_3 \left[-\left(\frac{E}{\Delta x}\right) u_2 + \left(\frac{E}{\Delta x}\right) u_3\right]$$





Weak Form to Alg. Eqs.

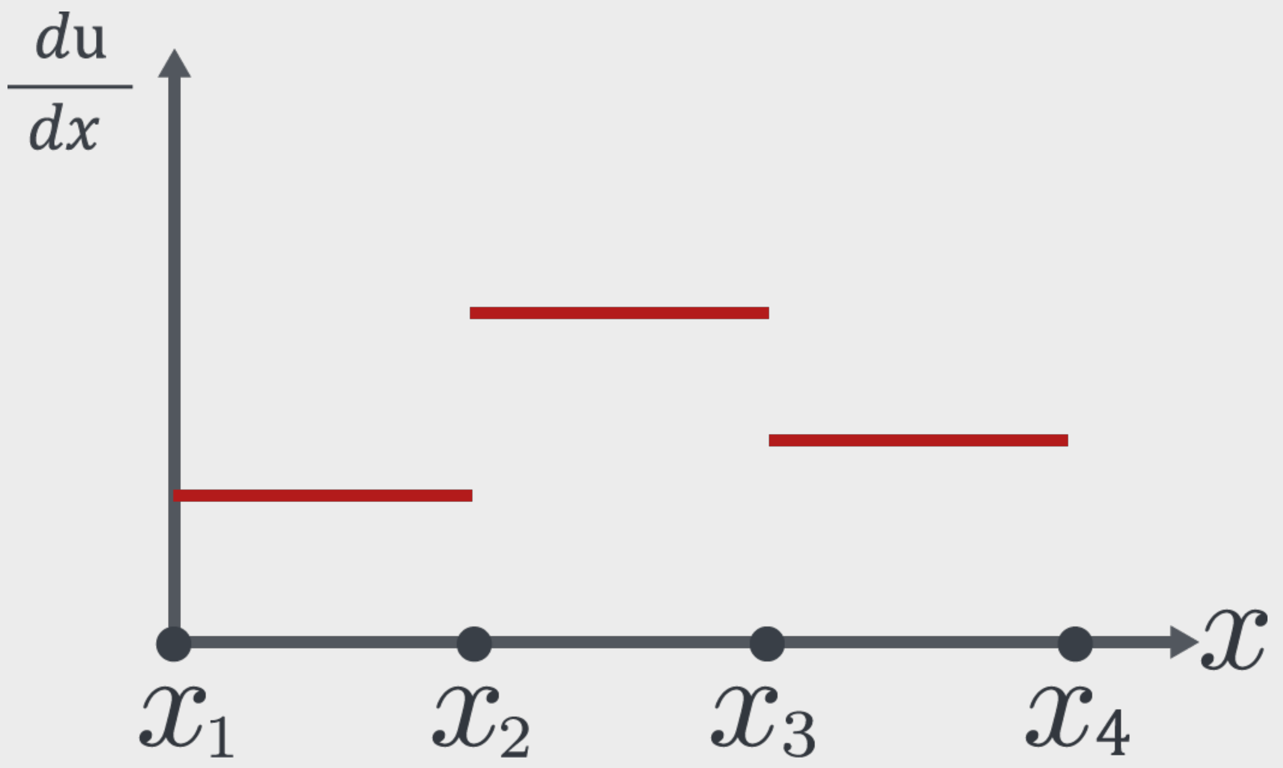
$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\bigg]_0^L + \int_0^L w\,f_x\,dx$$

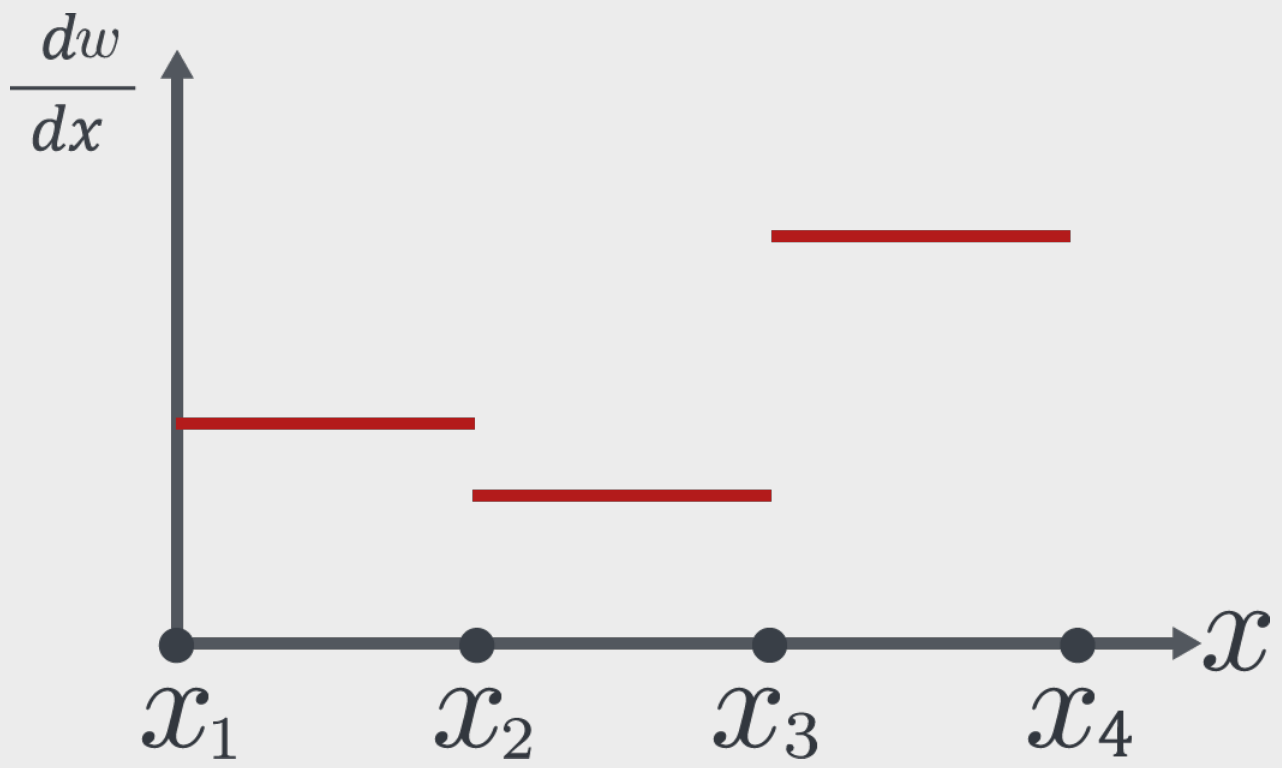
Element 3

$$\int_{x_3}^{x_4} \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx$$

$$= \mathrm{w}_3\left[\left(\frac{E}{\Delta x}\right)u_3 - \left(\frac{E}{\Delta x}\right)u_4\right]$$

$$+ w_4\left[-\left(\frac{E}{\Delta x}\right)u_3 + \left(\frac{E}{\Delta x}\right)u_4\right]$$

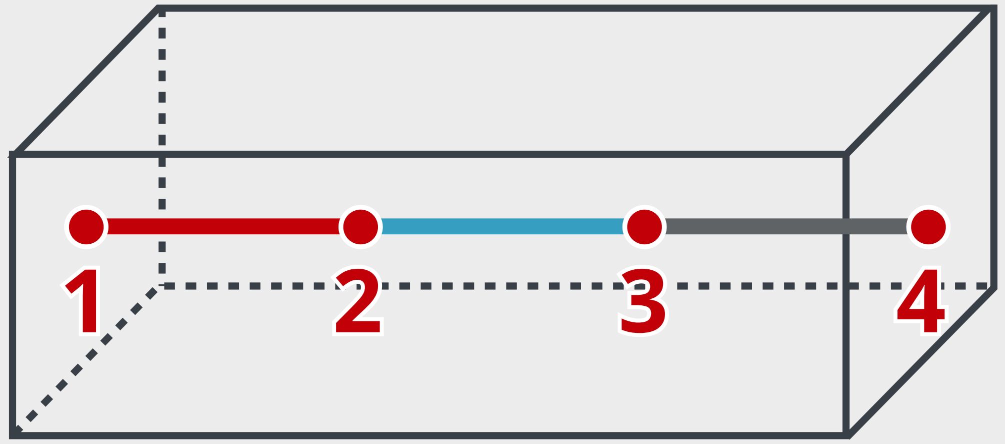




Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

Element 1+ Element 2+ Element 3



$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}dx =$$

$$\mathrm{w}_1\,(u_1 - u_2)\left(\frac{E}{\Delta x}\right) + w_2(-u_1 + 2u_2 - u_3)\left(\frac{E}{\Delta x}\right) +$$

$$w_3(-u_2 + 2u_3 - u_4)\left(\frac{E}{\Delta x}\right) + \mathrm{w}_4(-u_3 + u_4)\left(\frac{E}{\Delta x}\right)$$

Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$

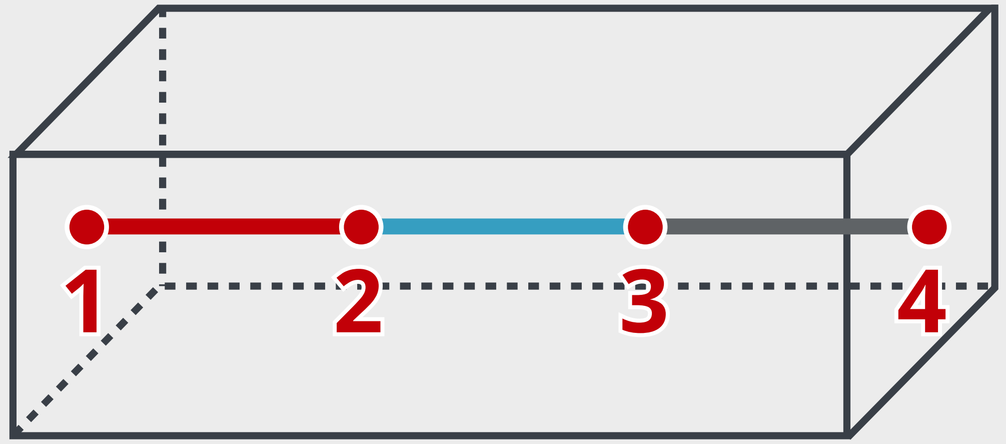
$$w\,\mathrm{E}\,\frac{\mathrm{d}u}{dx}\Big]_0^L = \mathrm{w(L)E}\frac{\mathrm{du}}{dx}(L) - \mathrm{w(0)E}\frac{\mathrm{du}}{dx}(0)$$

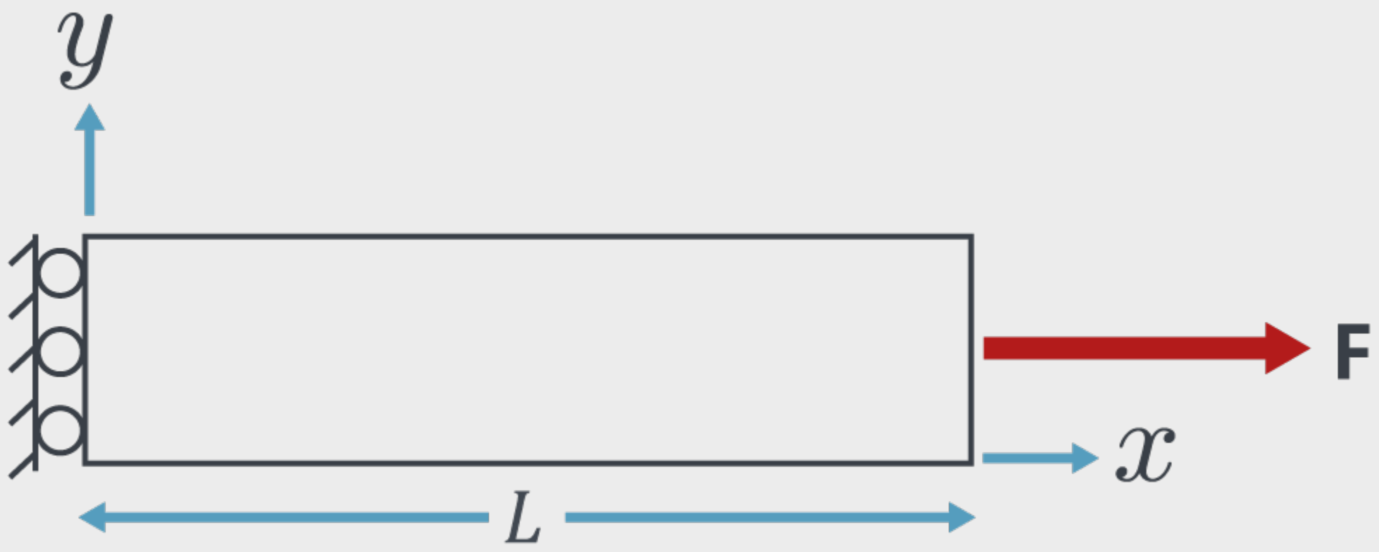
$$= W_4\,\sigma_{xx}(L) - W_1\,\sigma_{xx}(0)$$

$$= \mathrm{w}_4\sigma_{xx,4} - \mathrm{w}_1\,\sigma_{xx,1}$$

$$= \mathrm{w}_4\left(\frac{F}{A}\right) - \mathrm{w}_1\,\sigma_{xx}(0)$$

$$\sigma_{xx} = E\,\epsilon_{xx} = E\,\frac{du}{dx}$$





$$\sigma_{xx}(L) = \frac{F}{A}$$

Natural boundary condition

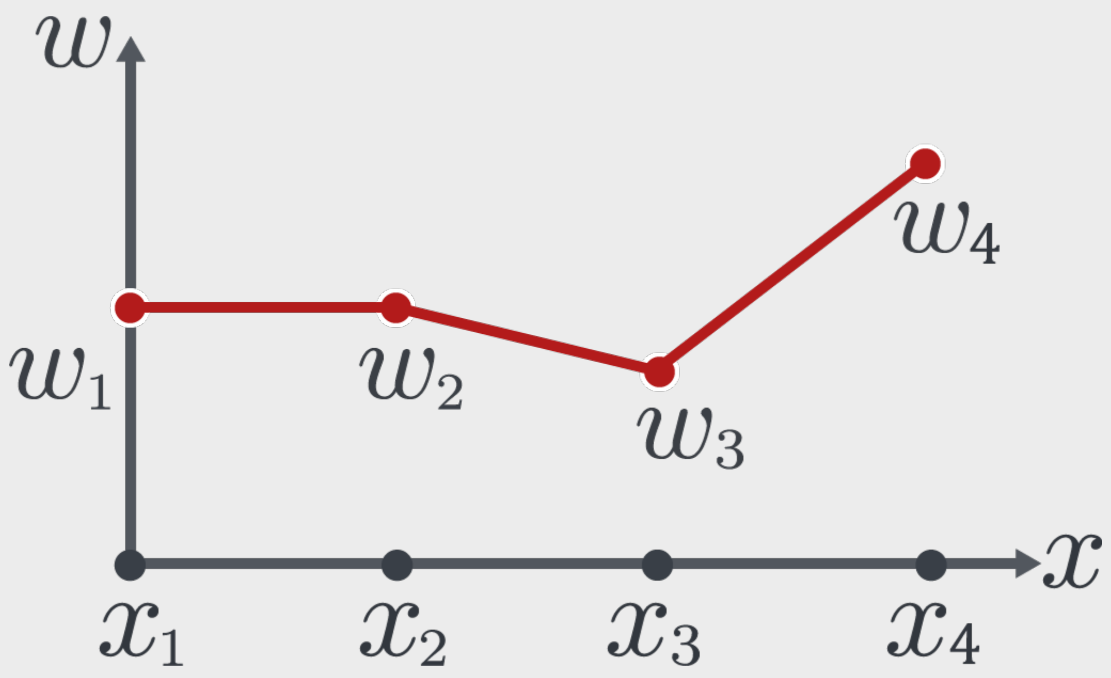
Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{du}{dx}\,dx = w\,\mathrm{E}\,\frac{du}{dx}\Bigg]_0^L + \int_0^L w\,f_x\,dx$$

Element 1

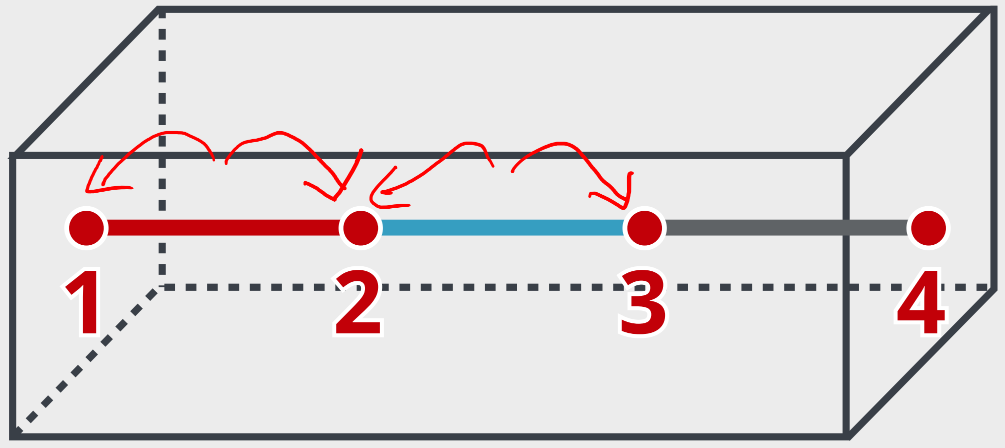
$$\int_0^{x_2} w\,f_x\,dx = f_x \int_0^{x_2} \left[w_1 + \left(\frac{w_2 - w_1}{\Delta x}\right)(x - x_1) \right] dx$$

$$= \mathrm{w}_1 \left(\frac{f_x \Delta \mathrm{x}}{2}\right) + \mathrm{w}_2 \left(\frac{f_x \Delta \mathrm{x}}{2}\right)$$



Element 2

$$\int_{x_2}^{x_3} w\,f_x\,dx = \mathrm{w}_2 \left(\frac{f_x \Delta \mathrm{x}}{2}\right) + \mathrm{w}_3 \left(\frac{f_x \Delta \mathrm{x}}{2}\right)$$

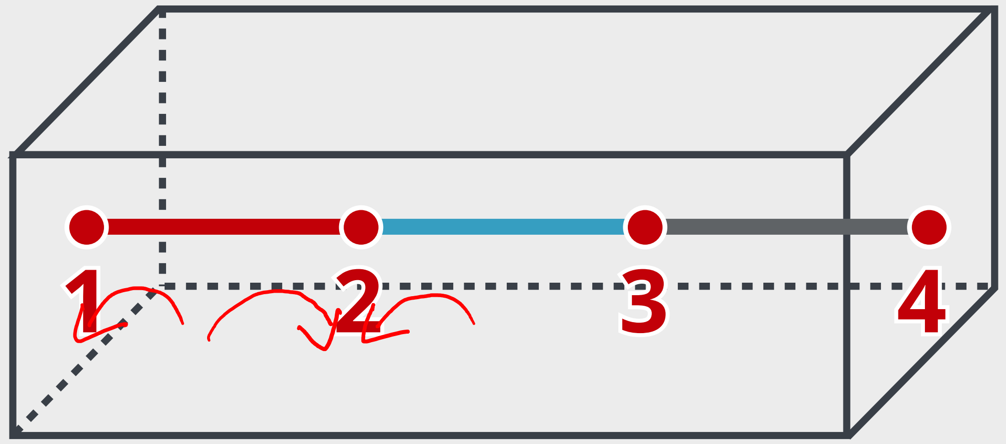


Weak Form to Alg. Eqs.

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\frac{\mathrm{du}}{dx}\Big]_0^L + \int_0^L w\,f_x\,dx$$

Element 1+ Element 2+ Element 3

$$\int_0^L w\,f_x\,dx = \mathrm{w}_1\left(\frac{f_x\Delta\mathrm{x}}{2}\right) + \mathrm{w}_2(f_x\Delta\mathrm{x}) + \mathrm{w}_3(f_x\Delta\mathrm{x}) + \mathrm{w}_4\left(\frac{f_x\Delta\mathrm{x}}{2}\right)$$



Weak Form to Algebraic Equations: Integration Result

$$w_1\left[\left(\frac{E}{\Delta x}\right)u_1-\left(\frac{E}{\Delta x}\right)u_2-0.5f_x\,\Delta x+\sigma_{xx,1}\right]+$$

$$w_2\left[-\left(\frac{E}{\Delta x}\right)u_1+\left(\frac{2E}{\Delta x}\right)u_2-\left(\frac{E}{\Delta x}\right)u_3-f_x\,\Delta x\right]+$$

$$w_3\left[-\left(\frac{E}{\Delta x}\right)u_2+\left(\frac{2E}{\Delta x}\right)u_3-\left(\frac{E}{\Delta x}\right)u_4-f_x\,\Delta x\right]+$$

$$w_4\left[-\left(\frac{E}{\Delta x}\right)u_3+\left(\frac{E}{\Delta x}\right)u_4-0.5f_x\,\Delta x-\sigma_{xx,4}\right]=0$$

$$\int_0^L \frac{dw}{dx}\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\,dx = w\,\mathrm{E}\,\frac{\mathrm{du}}{dx}\bigg]_0^L + \int_0^L w\,f_x\,dx$$

Algebraic Equations

$$\left(\frac{E}{\Delta x}\right)u_1 - \left(\frac{E}{\Delta x}\right)u_2 = 0.5 f_x\,\Delta x - \sigma_{xx,1}$$

$$-\left(\frac{E}{\Delta x}\right)u_1 + \left(\frac{2E}{\Delta x}\right)u_2 - \left(\frac{E}{\Delta x}\right)u_3 = f_x\,\Delta x$$

$$-\left(\frac{E}{\Delta x}\right)u_2 + \left(\frac{2E}{\Delta x}\right)u_3 - \left(\frac{E}{\Delta x}\right)u_4 = f_x\,\Delta x$$

$$-\left(\frac{E}{\Delta x}\right)u_3 + \left(\frac{E}{\Delta x}\right)u_4 = 0.5 f_x\,\Delta x + \sigma_{xx,4}$$

$$[K]\{u\} = \{F\}$$

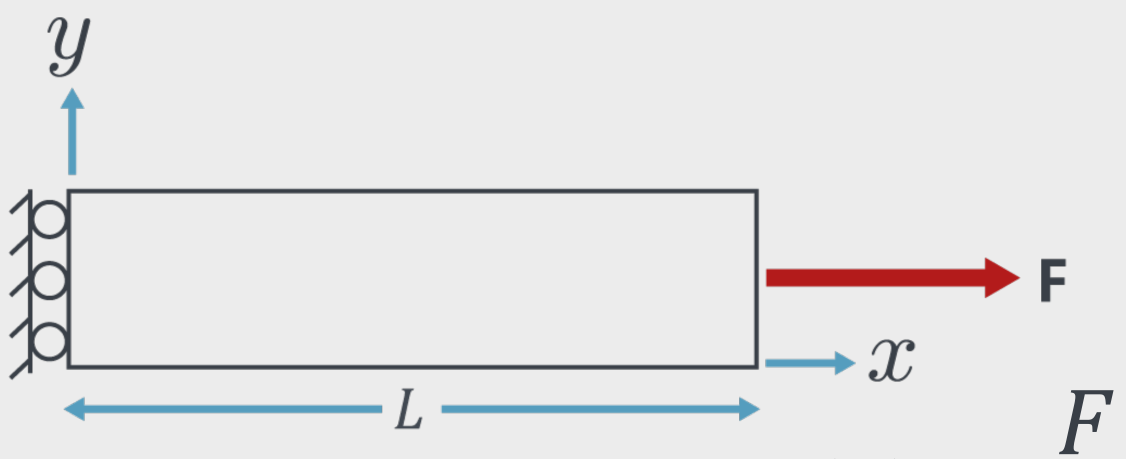
$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Essential Boundary Condition

$$\left(\frac{E}{\Delta x}\right)u_1 - \left(\frac{E}{\Delta x}\right)u_2 = 0.5 f_x\,\Delta x - \sigma_{xx,1}^{\,2} \qquad 1$$

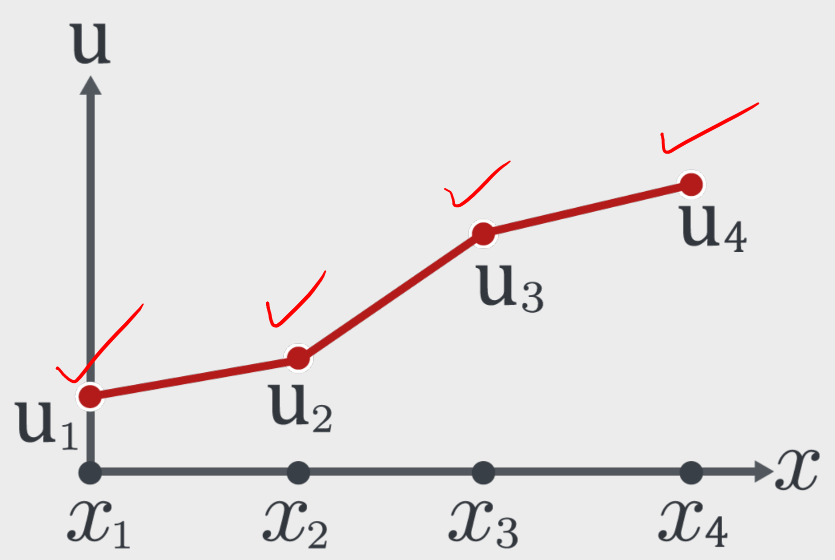
$$u_1 = 0$$

$$-\cancel{\left(\frac{E}{\Delta x}\right)u_1} + \left(\frac{2E}{\Delta x}\right)u_2 - \left(\frac{E}{\Delta x}\right)u_3 = f_x\,\Delta x \qquad 2$$



$$u(0) = 0 \qquad \sigma_{xx}(L) = \frac{F}{A}$$

$$-\left(\frac{E}{\Delta x}\right)u_2 + \left(\frac{2E}{\Delta x}\right)u_3 - \left(\frac{E}{\Delta x}\right)u_4 = f_x\,\Delta x \qquad 3$$



$$-\left(\frac{E}{\Delta x}\right)u_3 + \left(\frac{E}{\Delta x}\right)u_4 = 0.5 f_x\,\Delta x + \frac{F}{A} \qquad 4$$

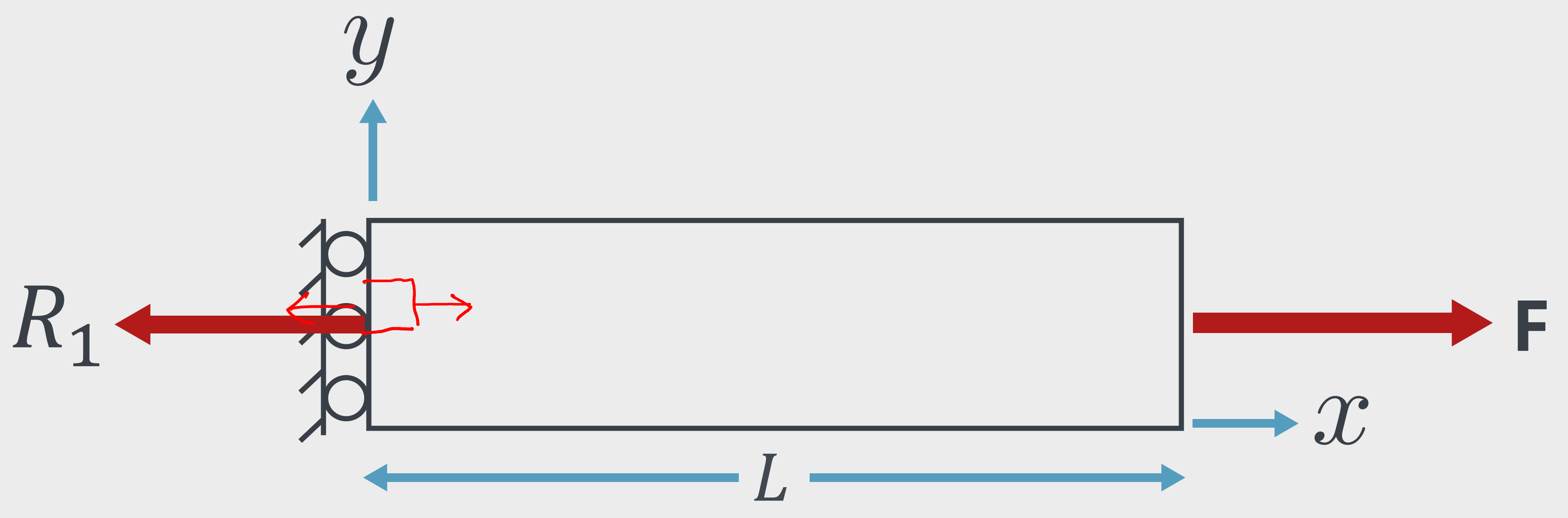
Reaction Calculation

$$\left(\frac{E}{\Delta x}\right) u_1 - \left(\frac{E}{\Delta x}\right) u_2 = 0.5 f_x\, \Delta x - \sigma_{xx,1}$$

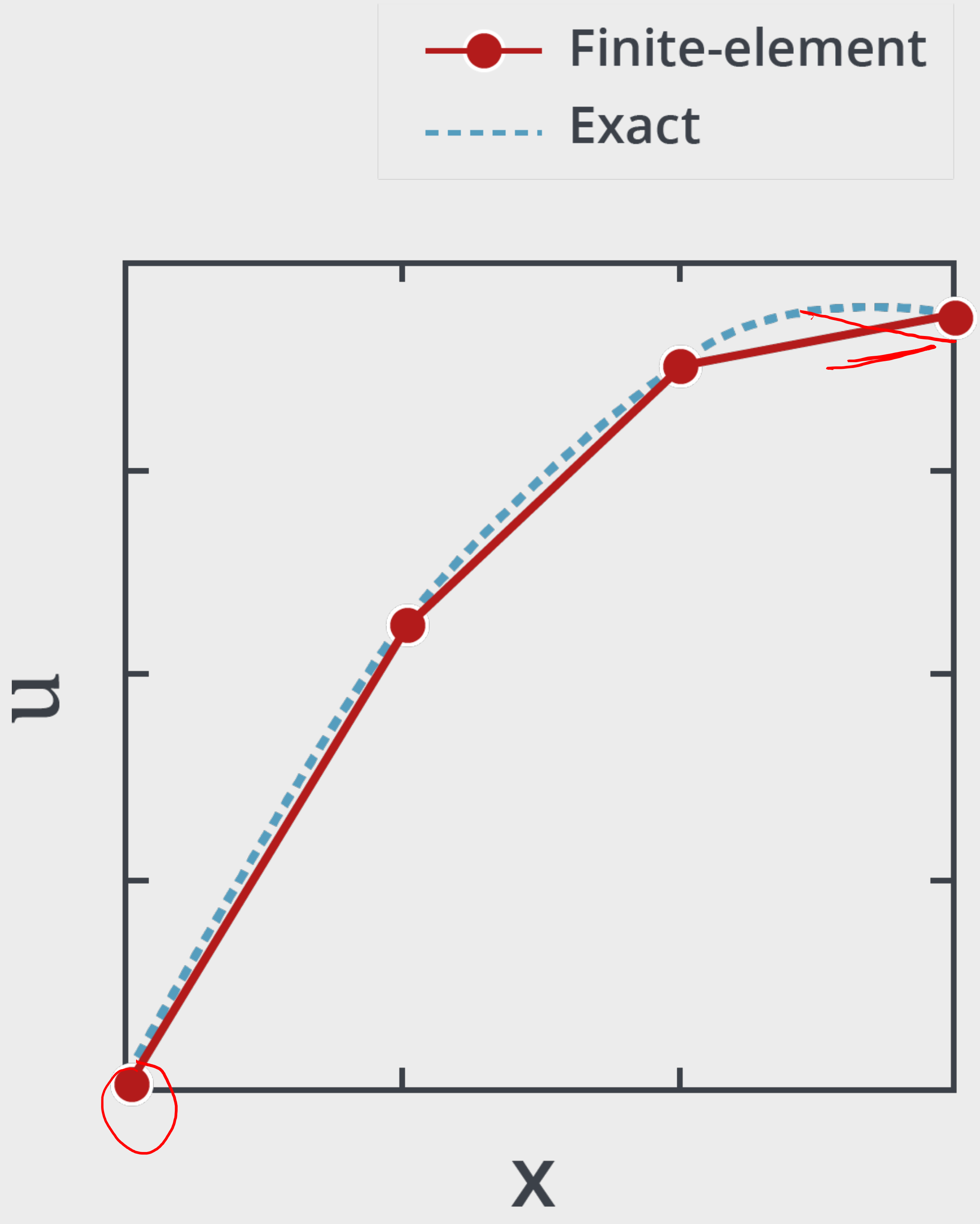
?

$$\sigma_{xx,1} = -\left(\frac{E}{\Delta x}\right) u_1 + \left(\frac{E}{\Delta x}\right) u_2 + 0.5 f_x\, \Delta x$$

$$R_1 = \sigma_{xx,1}\, A$$

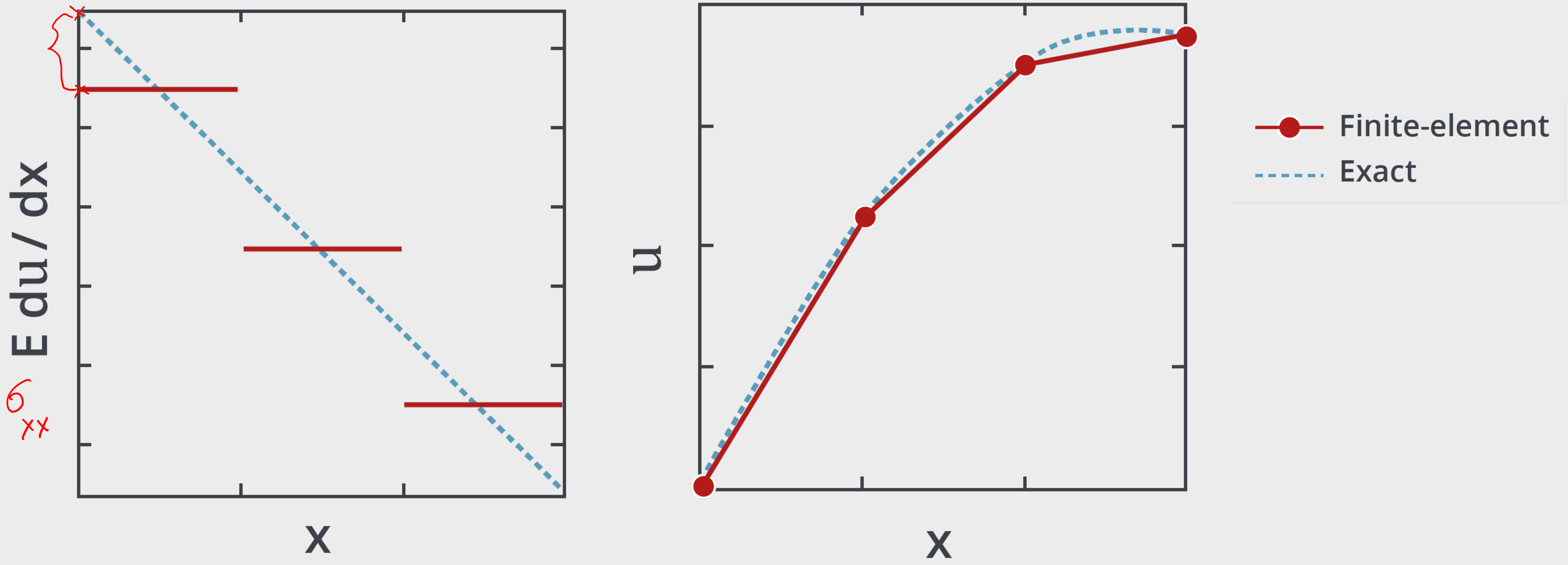


Finite-Element Solution for Displacement



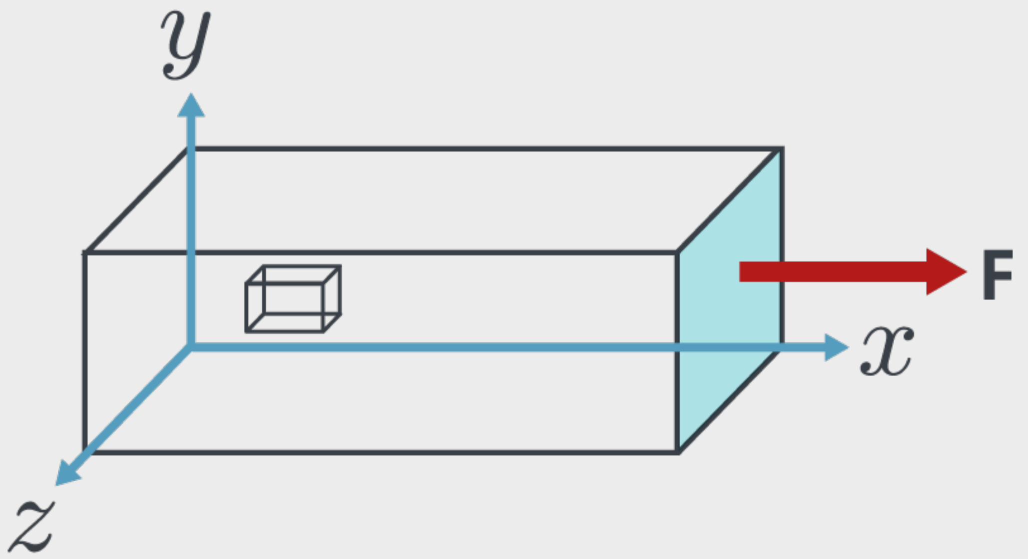
- Nodal displacement values are exact
  - Unusual property of 1D FE solution
- Essential boundary condition is satisfied exactly
- Natural boundary condition is satisfied approximately

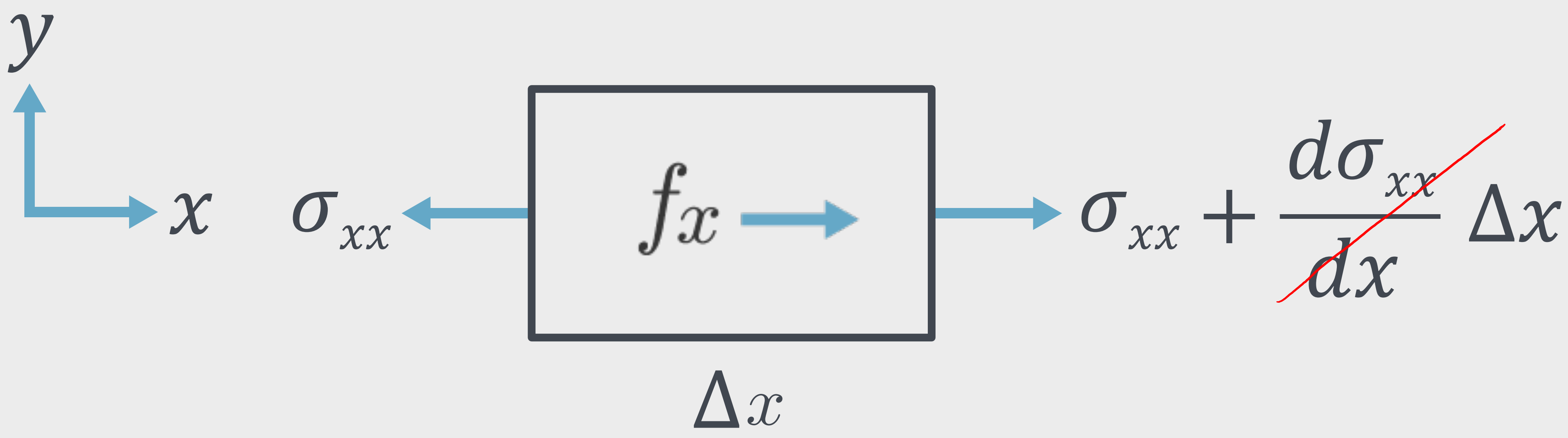
Finite-Element Solution for Stress

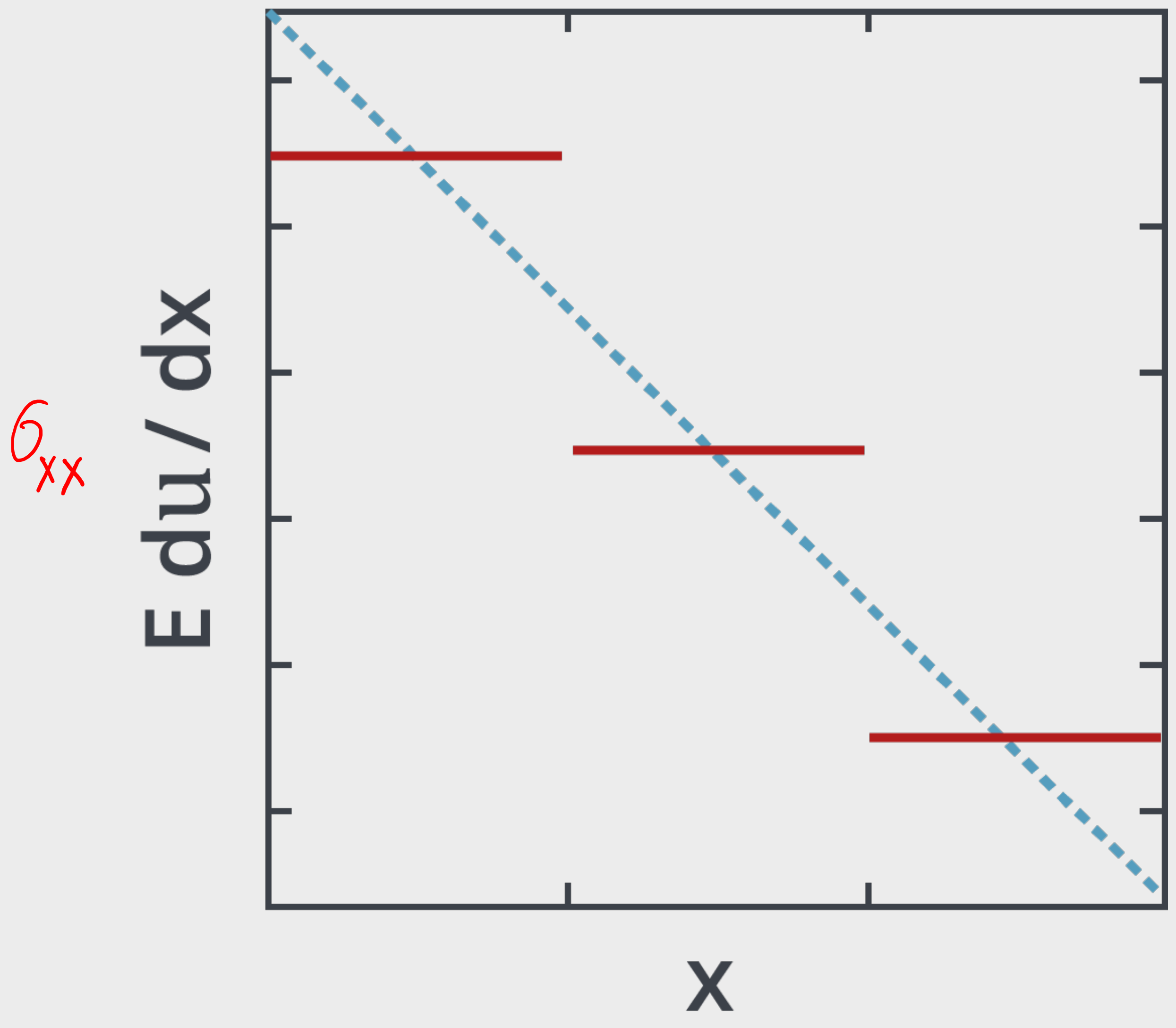


- Stress is discontinuous across elements
- Error in stress > Error in u

Check Equilibrium of Infinitesimal Elements



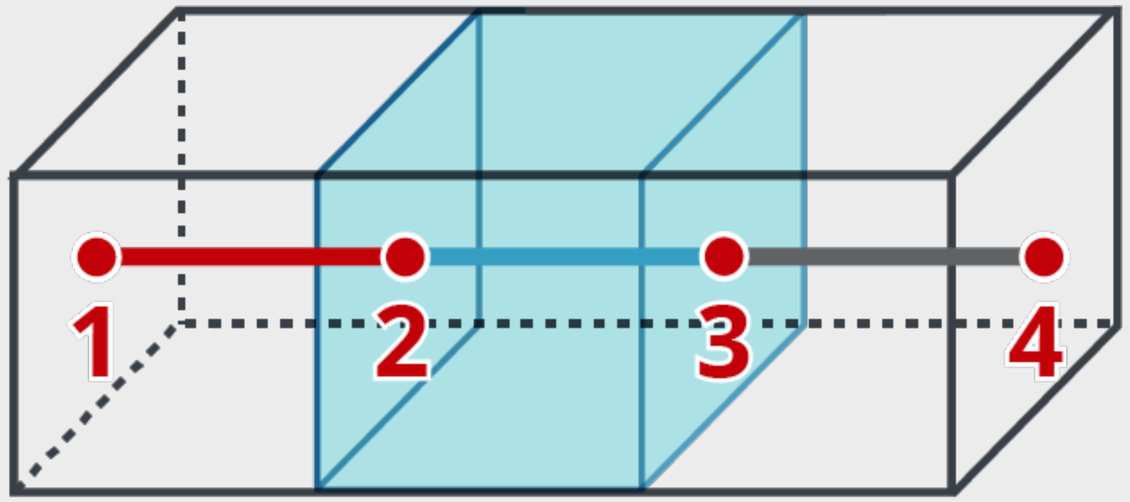




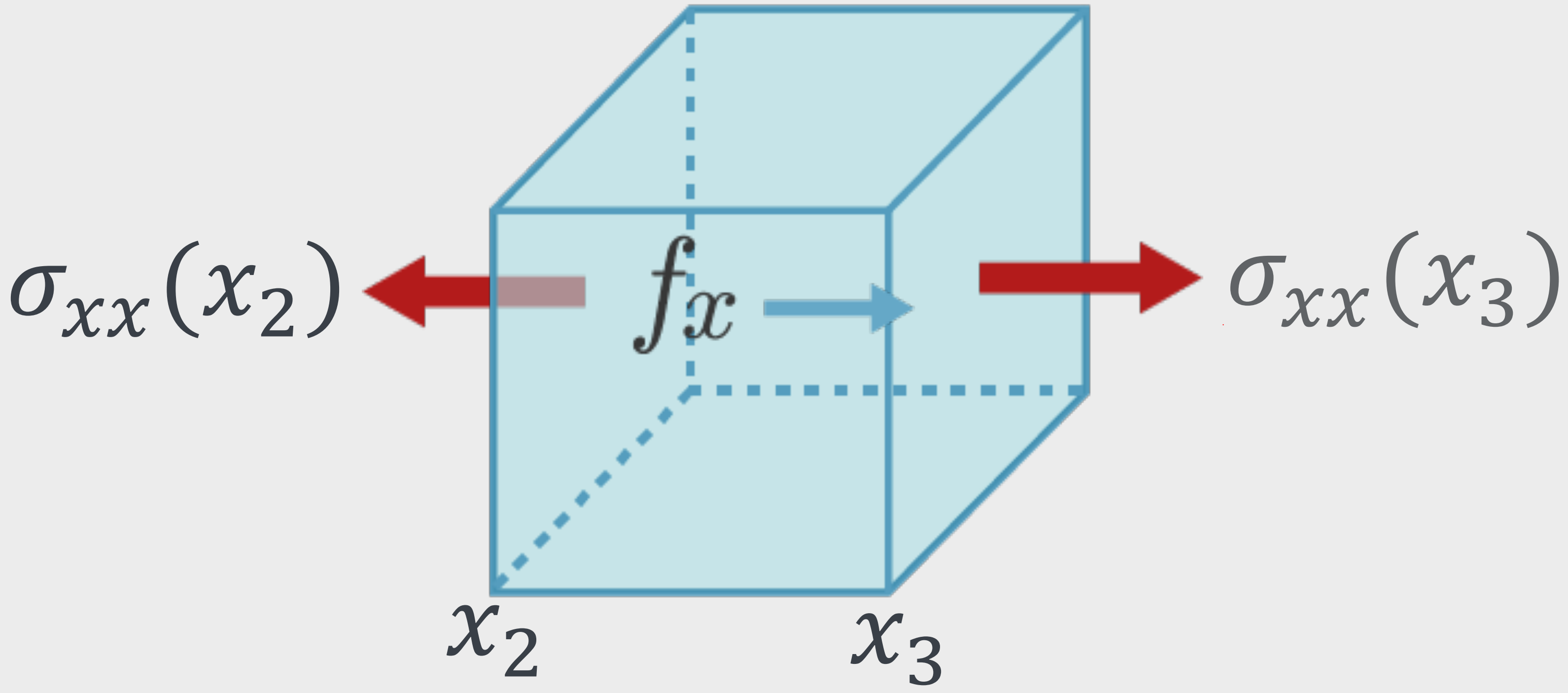
$$E\frac{d^2u}{dx} + f_x = 0$$

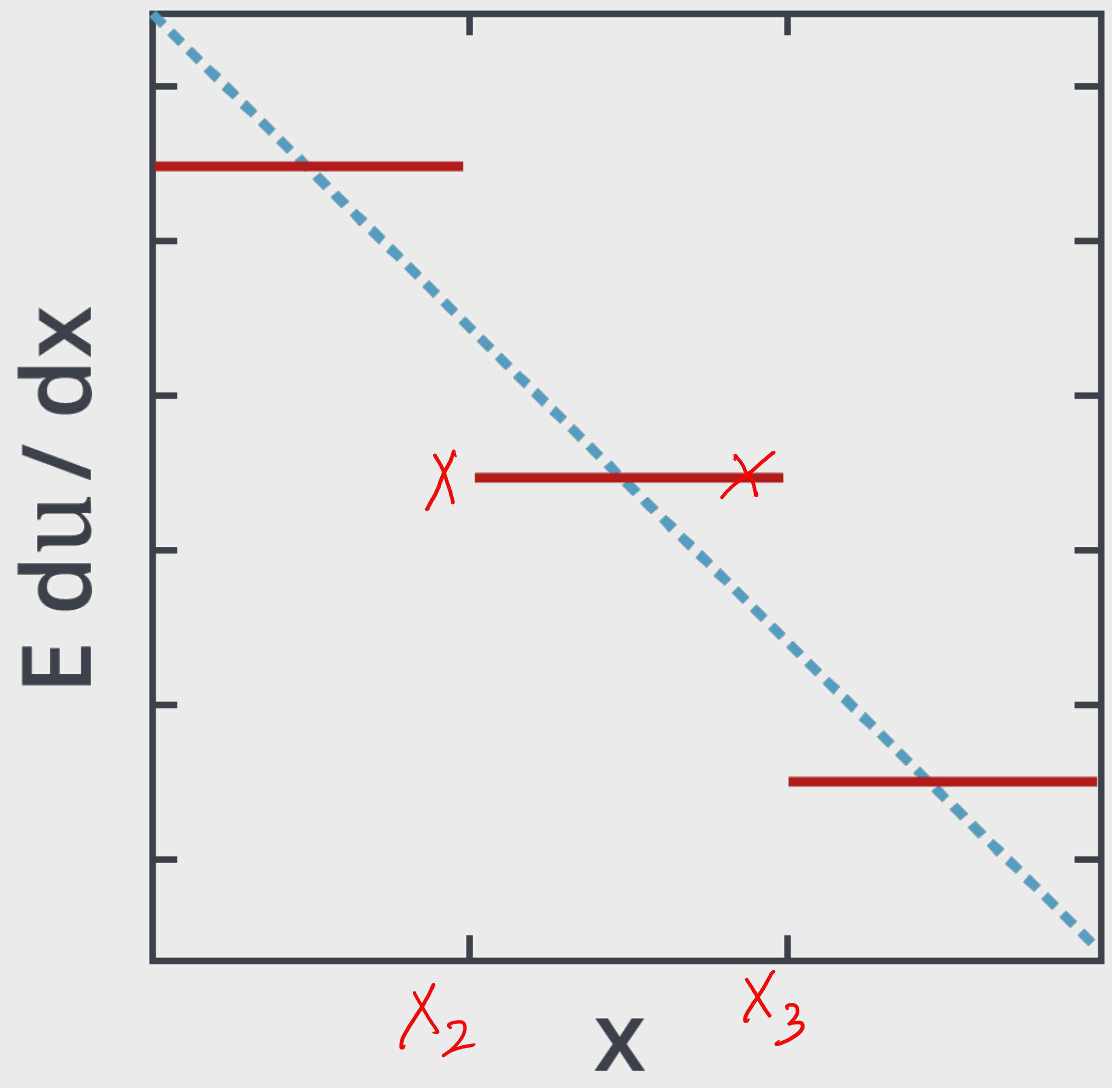
Each infinitesimal element is not in equilibrium!

Check Equilibrium of Finite Elements



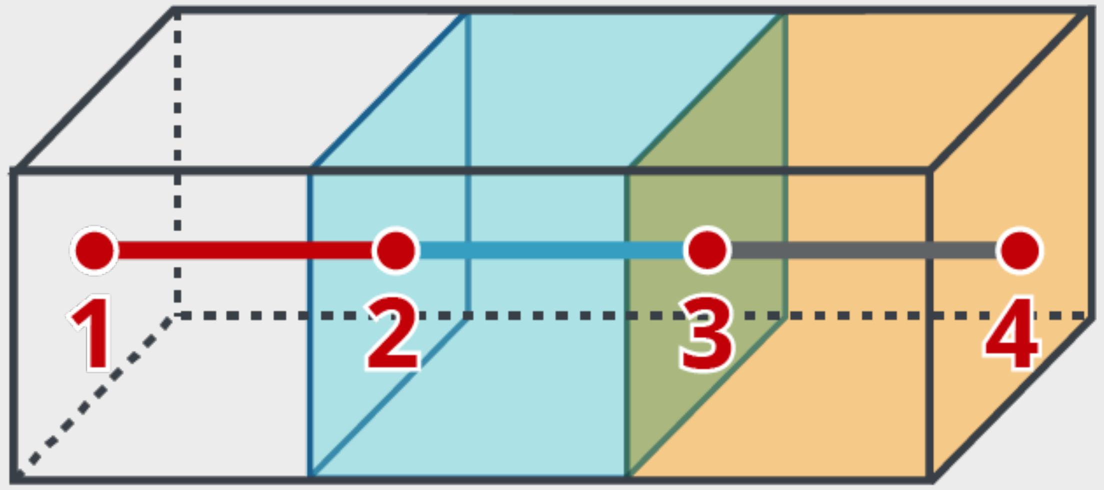
Element 2



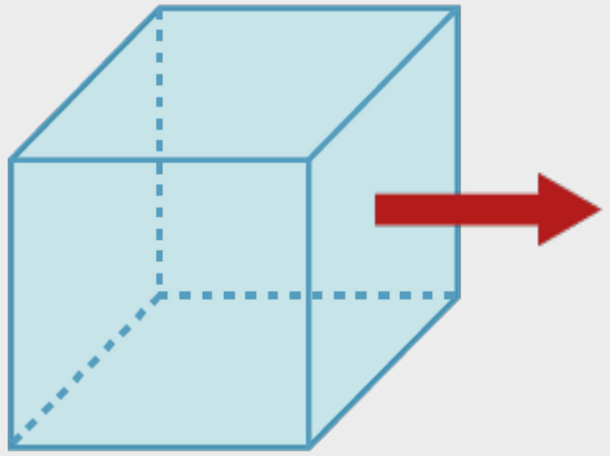


Each finite element
is not in equilibrium!

Forces at Element Interfaces

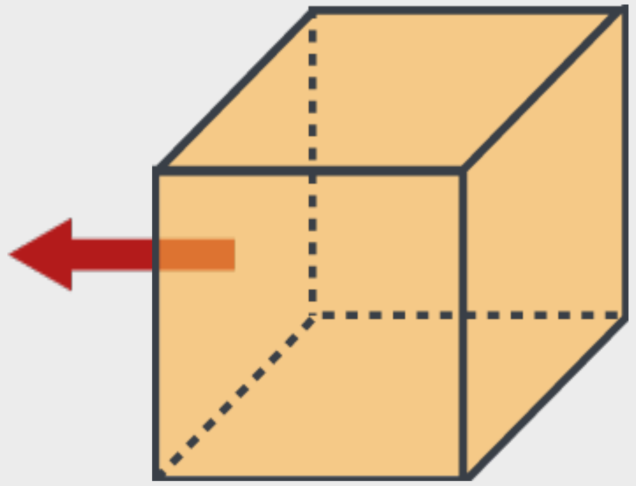


Element 2

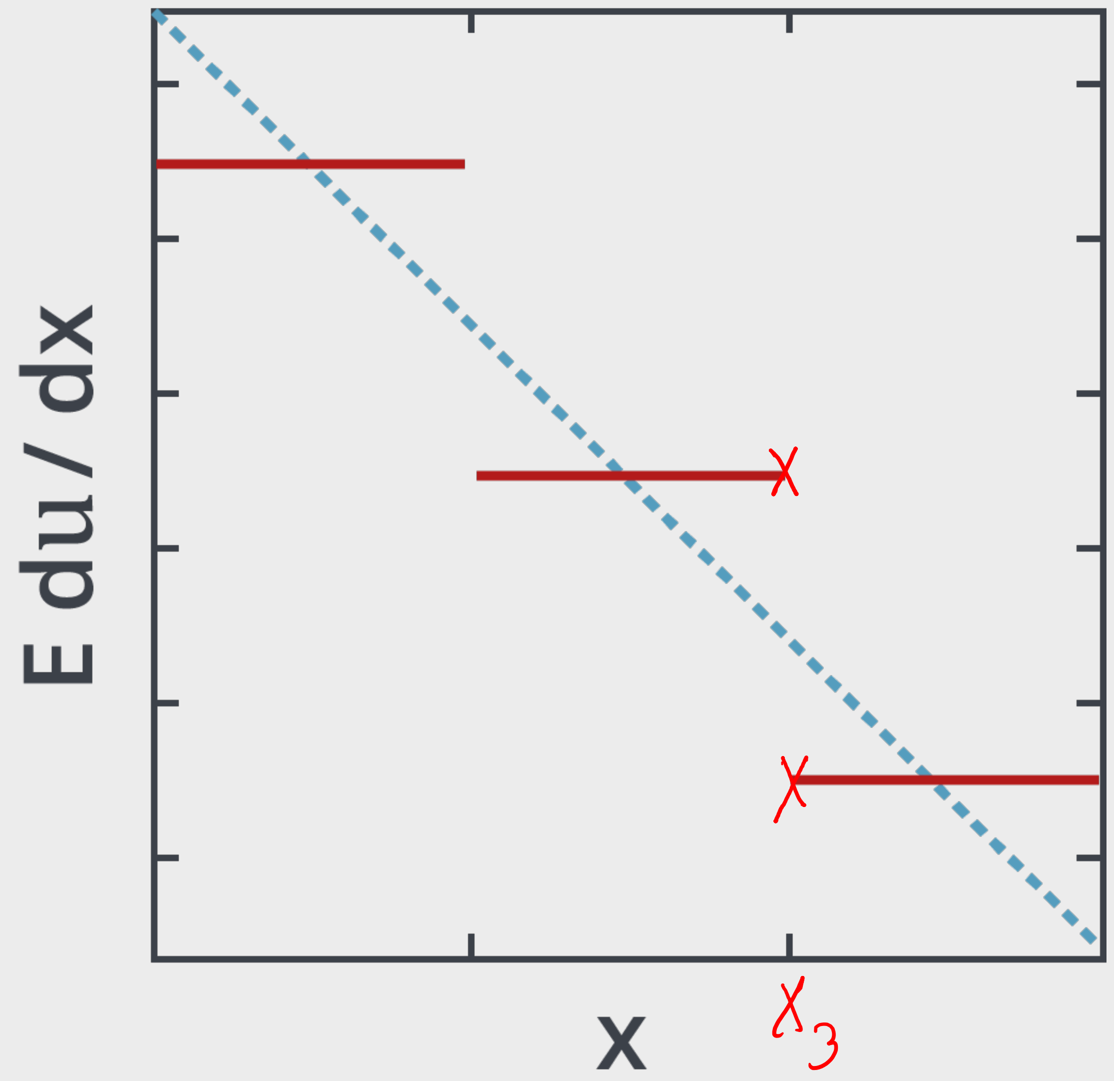


$$\sigma_{xx}(x_3) \neq \sigma_{xx}(x_3)$$

Element 3

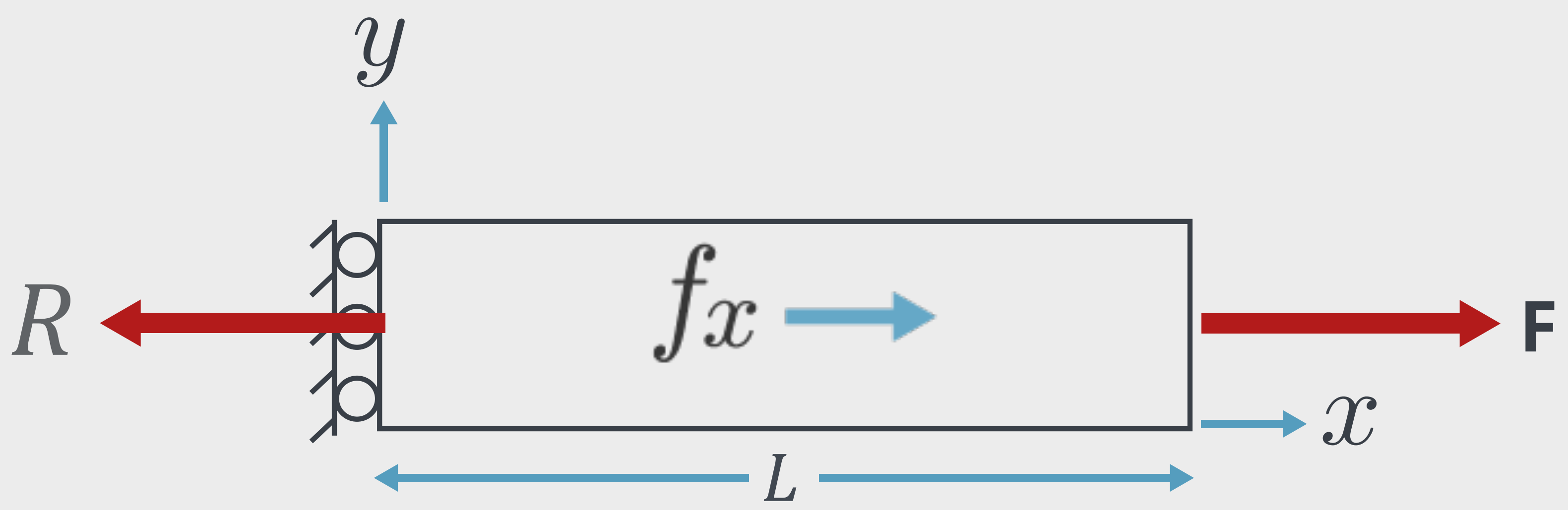


x_3 x_3



Forces at element interfaces don't match!

Check Overall Equilibrium of Bar



$u(0) = 0$ $\quad$ $\sigma_{xx}(L) = \dfrac{F}{A}$

$R = \sigma_{xx,1}\, A$

$$\left(\frac{E}{\Delta x}\right) u_1 - \left(\frac{E}{\Delta x}\right) u_2 = 0.5 f_x\, \Delta x - \overset{?}{\sigma_{xx,1}}$$

Total body force on bar, $f_{x\_tot} = f_x AL$

Overall equilibrium check: $R = F + f_x AL$

From top eq. $\quad R = 25{,}000\ N$

Example:

$F = -6250\ N$

$f_{x,tot} = 31{,}250\ N$

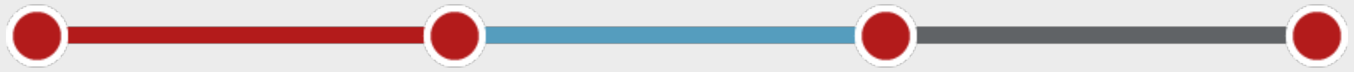
$F + f_x AL = 25{,}000\ N$

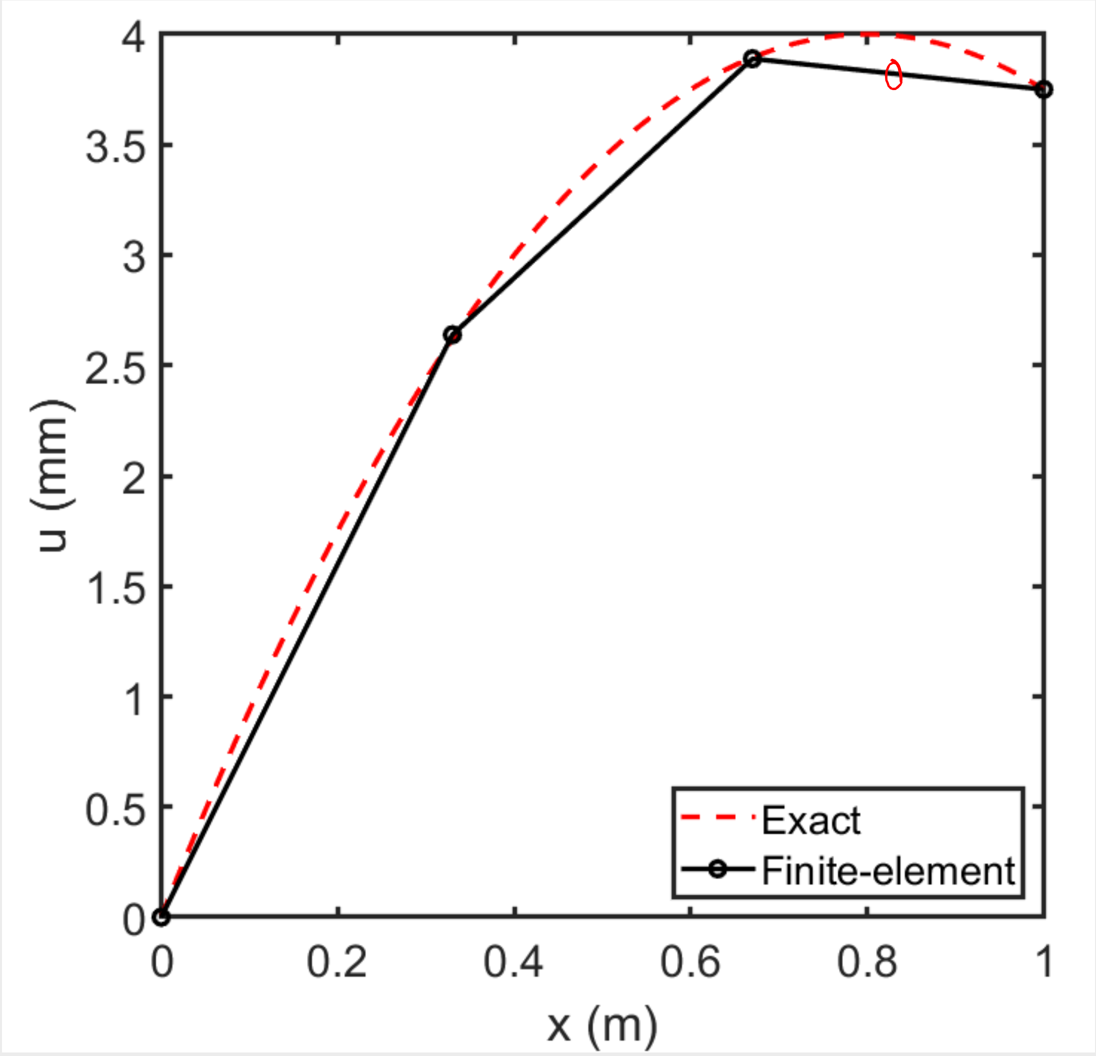
Bar is in equilibrium!

How to Reduce Numerical Error? (1/2)

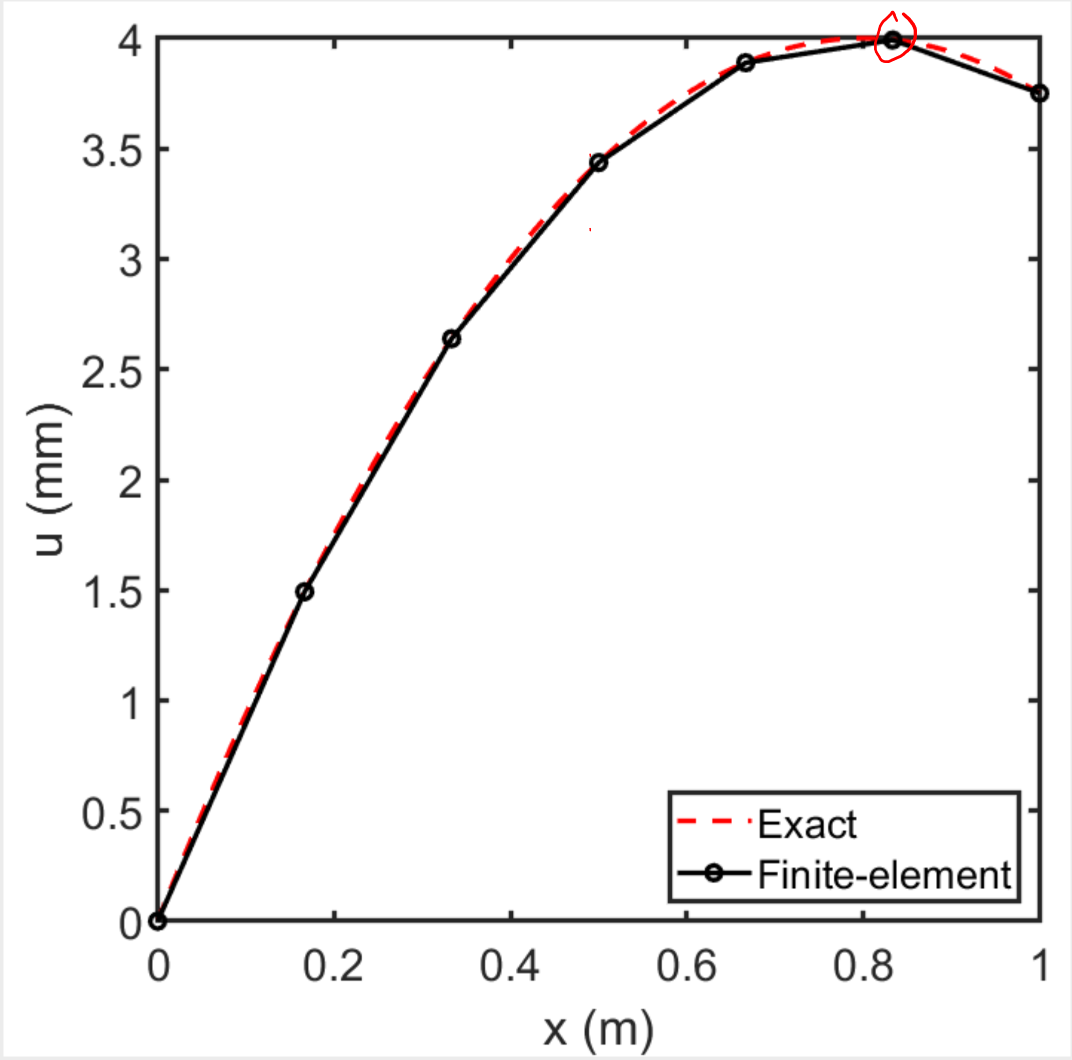
Strategy 1: Increase number of elements

Original Mesh





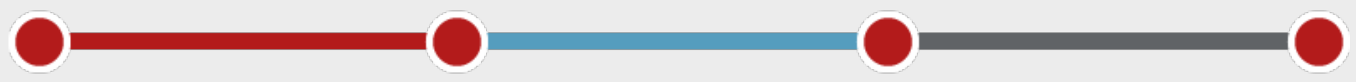
Refined Mesh

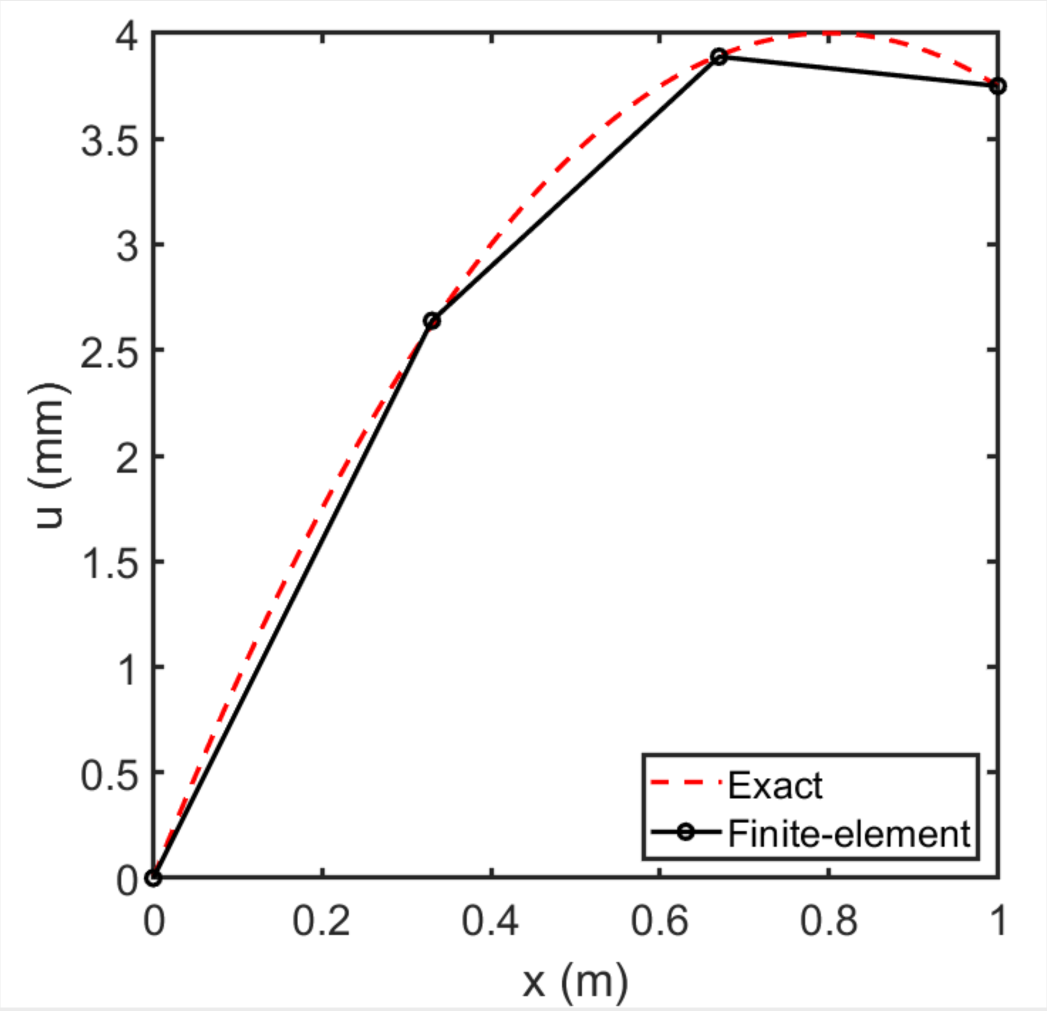


How to Reduce Numerical Error? (2/2)

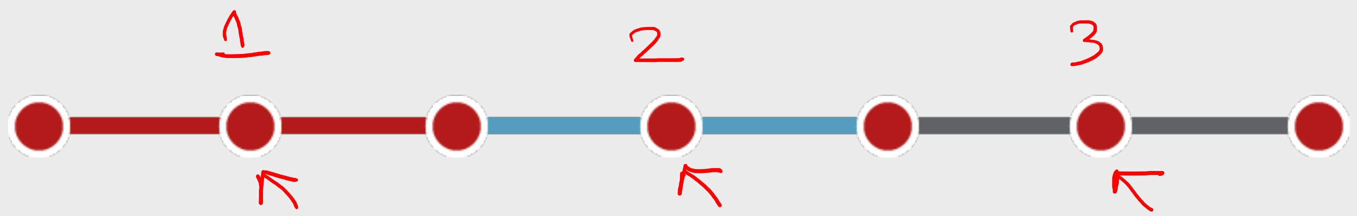
Strategy 2: Increase order of polynomial within each element

Original Mesh

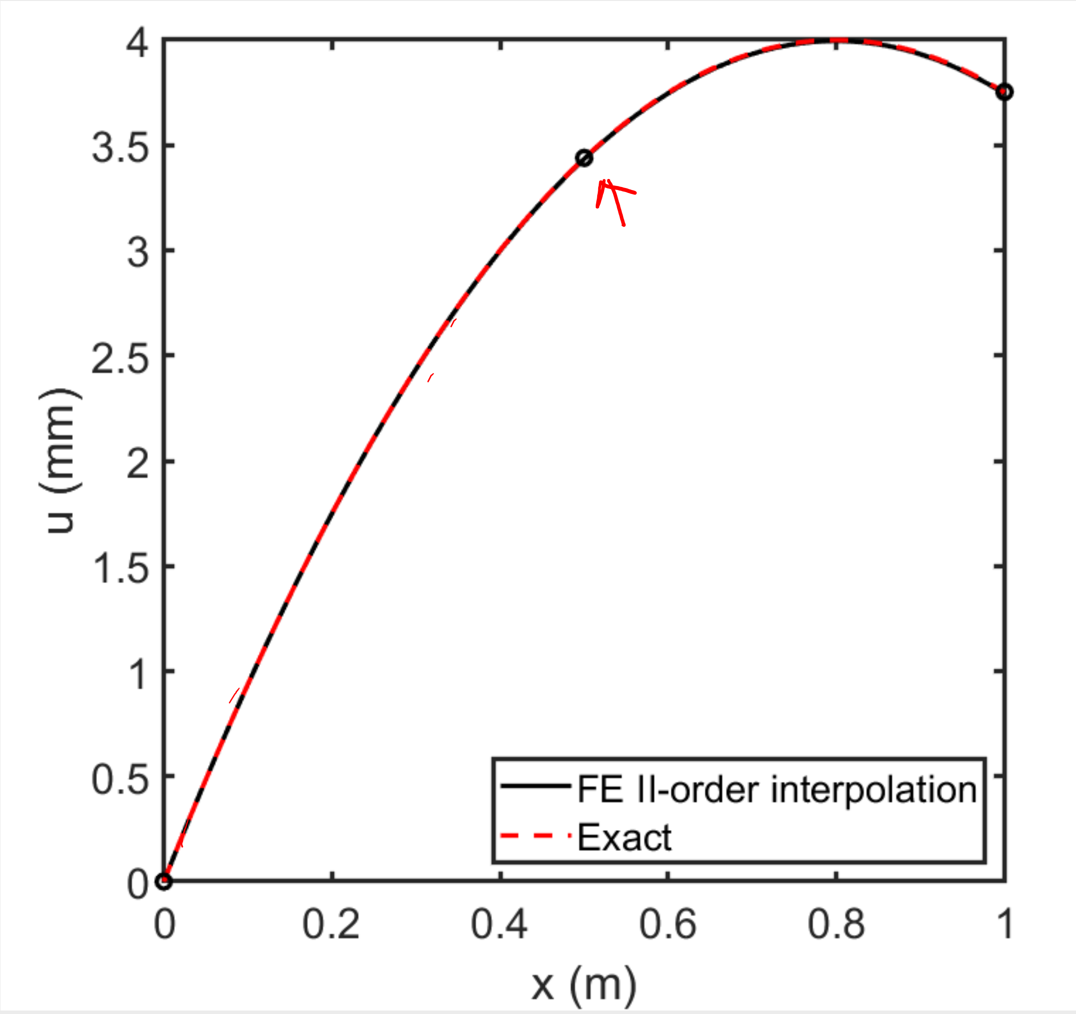




Second-Order Element

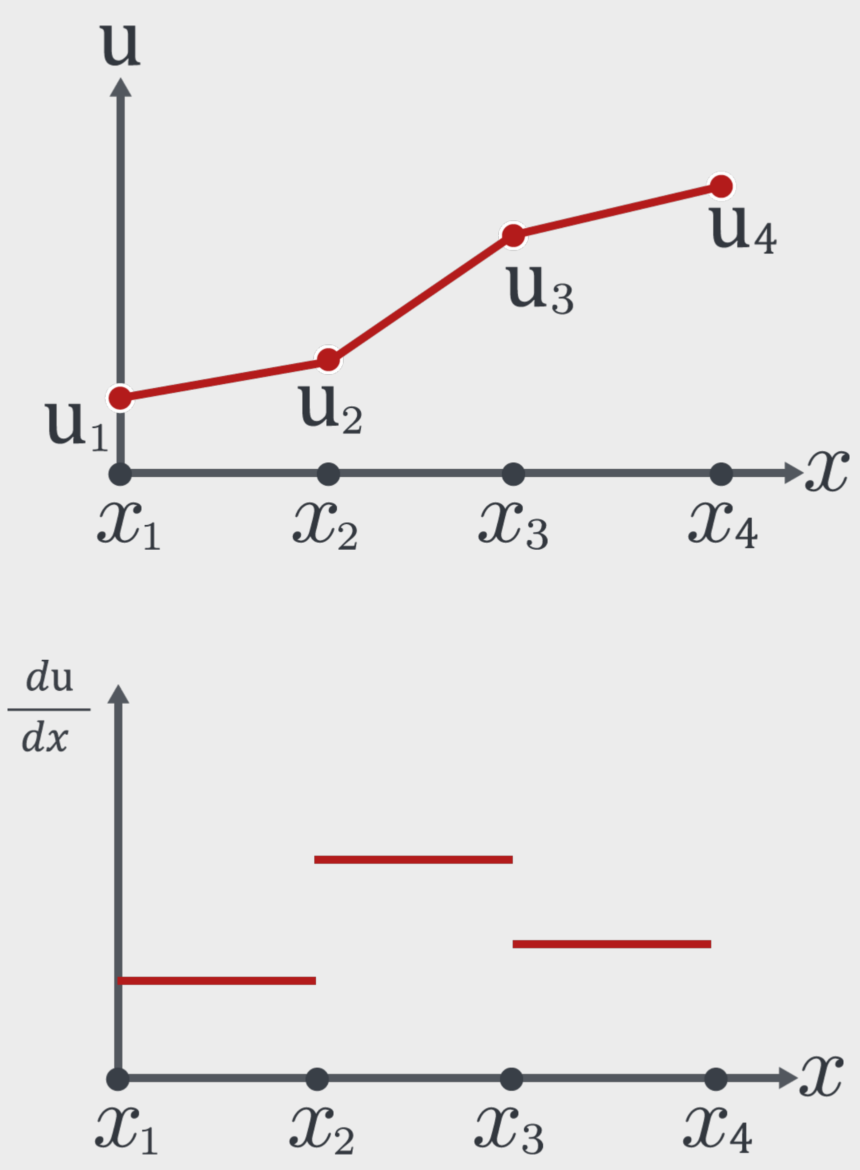


One element solution

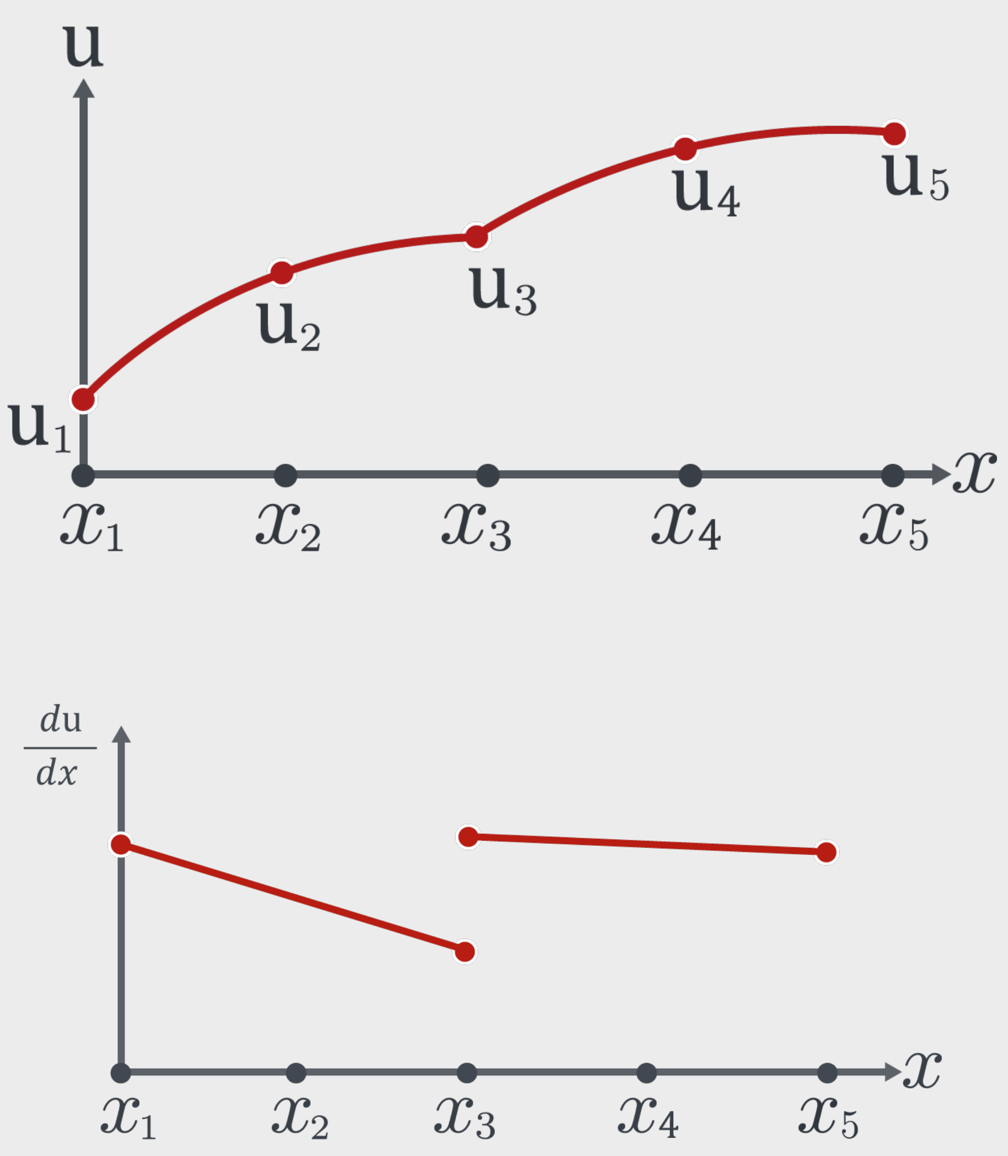


Quadratic Interpolation Implementation Strategy (1/2)

Linear interpolation



Quadratic interpolation



Quadratic Interpolation Implementation Strategy (2/2)

$$\int_0^L \frac{dw}{dx}\, \mathrm{E}\, \frac{\mathrm{d}u}{dx}\, dx = w\, \mathrm{E} \frac{\mathrm{d}u}{dx}\Big]_0^L + \int_0^L w\, f_x\, dx$$