0

$${}^{I}v_{P/O} = v_1\hat{e}_1 + v_2\hat{e}_2 \quad , \quad \hat{e}_r = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2 \quad , \quad \hat{e}_\theta = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$$

$$\rightarrow {}^{I}v_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

a) $\dot{r} = (v_1\hat{e}_1 + v_2\hat{e}_2)\cdot\hat{e}_r = v_1\cos\theta + v_2\sin\theta$

$r\dot{\theta} = (v_1\hat{e}_1 + v_2\hat{e}_2)\cdot\hat{e}_\theta = -v_1\sin\theta + v_2\cos\theta$

$\rightarrow \dot{r} = v_1\cos\theta + v_2\sin\theta$ , $\dot{\theta} = \dfrac{-v_1\sin\theta + v_2\cos\theta}{r}$

b) $\ddot{r} - r\dot{\theta}^2 = 0$ , $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

$\rightarrow v_1\cos\theta + v_2\sin\theta = \dot{r}$ , $-v_1\sin\theta + v_2\cos\theta = r\dot{\theta}$

$\rightarrow \dot{\theta} = \dfrac{-v_1\sin\theta + v_2\cos\theta}{r}$

$$\rightarrow \ddot{r} = r\left(\frac{-v_1\sin\theta + v_2\cos\theta}{r}\right)^2 = \frac{(-v_1\sin\theta + v_2\cos\theta)^2}{r}$$

$$\rightarrow \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} = -\frac{2(v_1\cos\theta + v_2\sin\theta)}{r}\,\frac{(-v_1\sin\theta + v_2\cos\theta)}{r} = -\frac{2(v_1\cos\theta + v_2\sin\theta)(-v_1\sin\theta + v_2\cos\theta)}{r^2}$$

c) $\dot{\theta} = \dfrac{-v_1\sin\theta + v_2\cos\theta}{r} = \dfrac{v\cdot\hat{e}_\theta}{r}$

$\rightarrow |\dot{\theta}| \le \dfrac{\|v\|}{r} \xrightarrow[t\to\infty]{} 0 \Rightarrow \dot{\theta} \to 0$

$\ddot{\theta} = -\dfrac{2\dot{r}\dot{\theta}}{r}$

$\rightarrow |\ddot{\theta}| \le \dfrac{2\|v\|(\|v\|/r)}{r} = \dfrac{2\|v\|^2}{r^2} \xrightarrow[t\to\infty]{} 0 \Rightarrow \ddot{\theta} \to 0$

$r(t) = r(0) + vt$

$\rightarrow \hat{e}_r(t) = \dfrac{r(t)}{\|r(t)\|} = \dfrac{r(0) + vt}{\|r(0) + v(t)\|} \xrightarrow[t\to\infty]{} \dfrac{v}{\|v\|}$ (unit direction of v)

2

a) $\ddot{\theta} = -\frac{g}{l}\sin\theta \approx -\frac{g}{l}\theta$, $\omega_n = \sqrt{\frac{g}{l}}$

$\rightarrow T_0 = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$

b) $g_f = g + a$

$\rightarrow \ddot{\theta} = -\frac{g_f}{l}\sin\theta \approx -\frac{g_f}{l}\theta$

$\rightarrow \ddot{\theta} + \frac{g+a}{l}\theta = 0$, $T = 2\pi\sqrt{\frac{l}{g+a}}$

c) $T_0 = 2\pi\sqrt{\frac{l}{g}}$

$g_f = g + a > g$: $T = 2\pi\sqrt{\frac{l}{g+a}} < T_0$ (swings faster)

$g_f = g - a < g$: $T = 2\pi\sqrt{\frac{l}{g-a}} > T_0$ (swings slower)

3

a) $r = x\hat{e}_1 + y\hat{e}_2$ , $x^2 + y^2 = \ell^2$

$$m\ddot{r} = mg\hat{e}_1 - T\frac{r}{\ell}$$

$$\rightarrow m\ddot{x} = mg - T\frac{x}{\ell} \; , \quad m\ddot{y} = -T\frac{y}{\ell}$$

$$x^2 + y^2 = \ell^2$$

$$x\dot{x} + y\dot{y} = 0$$

$$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0 \rightarrow x\ddot{x} + y\ddot{y} = -(\dot{x}^2 + \dot{y}^2)$$

$$m(r\cdot\ddot{r}) = mg(r\cdot\hat{e}_1) - T\frac{r\cdot r}{\ell}$$

$$\rightarrow m(x\ddot{x} + y\ddot{y}) = mgx - T\ell$$

$$\rightarrow m(-(\dot{x}^2 + \dot{y}^2)) = mgx - T\ell \rightarrow T\ell = m(gx + \dot{x}^2 + \dot{y}^2)$$

$$\frac{T}{m} = \frac{gx + \dot{x}^2 + \dot{y}^2}{\ell}$$

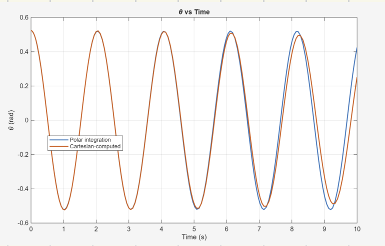
$$\rightarrow \ddot{x} = g - \left(\frac{gx + \dot{x}^2 + \dot{y}^2}{\ell}\right)\frac{x}{\ell} = g - \frac{x(gx + \dot{x}^2 + \dot{y}^2)}{\ell^2}$$

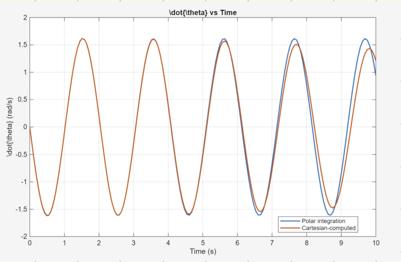
$$\rightarrow \ddot{y} = -\left(\frac{gx + \dot{x}^2 + \dot{y}^2}{\ell}\right)\frac{y}{\ell} = -\frac{y(gx + \dot{x}^2 + \dot{y}^2)}{\ell^2}$$

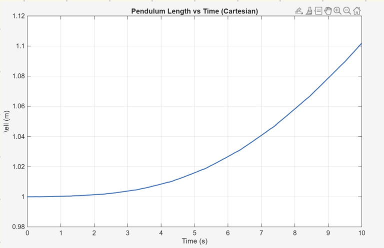
$\ell^2 = x^2 + y^2$:

$$\ddot{x} = \frac{gy^2 - x\dot{x}^2 - x\dot{y}^2}{x^2 + y^2} \; , \quad \ddot{y} = \frac{-gxy - \dot{x}^2 y - y\dot{y}^2}{x^2 + y^2}$$

b)




c) The angle and angular velocity from the Cartesian and polar simulations match very closely, showing that both approaches give the same pendulum motion. The pendulum length stays essentially constant, which confirms the Cartesian model is enforcing the rigid rod correctly.

(1)

a) DOF = 1

b) $x$ = horizontal displacement

$\rightarrow$ $s(x)$ = spring length = $\sqrt{x^2 + l^2}$

$\Delta s = s - l = \sqrt{x^2 + l^2} - l$

$f_x = -k(s-l)\frac{x}{s}$

$\rightarrow$ $m\ddot{x} = -k(s-l)\frac{x}{s}$

$\rightarrow$ $m\ddot{x} + kx\left(1 - \frac{l}{\sqrt{x^2+l^2}}\right) = 0$

5

a) Effective mass: $m = 0.330 + 0.026/3 = 0.3387$ kg

Log decrement (1 cycle): $\delta \approx 0.311$

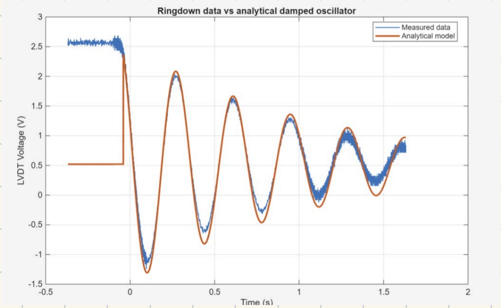
Damping ratio: $\zeta \approx 0.0494$

Damped period: $T_d \approx 0.338$ s $\rightarrow \omega_d \approx 18.59$ rad/s

Natural frequency: $\omega_n \approx 18.61$ rad/s

$\rightarrow k = m\omega_n^2 \approx 117$ N/m , $b = 2m\zeta\omega_n \approx 0.623$ N·s/m

b)




c) The analytical curve follows the overall decay and oscillation frequency of the measured ringdown pretty well, especially early on. Small mismatches later in time are likely due to noise in the data, imperfect damping assumptions, and the fact that the real system isn't a perfectly linear spring-mass-damper.