Satellite Imaging Model and Control V2

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Assumptions/Setup

The satellite is stationary when it comes to translational movement in an inertial frame. Its position in the inertial frame is given by $\vec{r}_{S/O} = [x_1\, y_1\, z_1]^T$ with respect to the Earth. The position of the desired imaging location is given by $\vec{r}_{D/O} = [x_2\, y_2\, z_2]^T$. The pseudo-vector normal to the camera lens in the direction of the desired location is $\vec{r}_{D/O} - \vec{r}_{S/O} = \vec{r}_{D/S}$ which is then normalized to $\vec{L}_{desired} = \frac{\vec{r}_{D/S}}{|\vec{r}_{D/S}|}$.

In the body frame of the spacecraft, the normal vector off the lens is given by $\vec{L}_{body} = [1\,0\,0]^T$. This is always constant within the body coordinate system. The body coordinate system is perfectly aligned with the principal axes of the spacecraft. The spacecraft contains a set of reaction wheels which provide angular momentum aligned with one of the three body/principal axes. The control input to the system is the torque on each reaction wheel direction in the form of $\bar{u} = \vec{\tau} = [\tau_x\, \tau_y\, \tau_z]$. The moment of inertia tensor for the wheels is $I_w = [I_{wx}\, I_{wy}\, I_{wy}]$ and is constant. The moment of inertia tensor for the spacecraft is $I_{tot} = I_s + I_w$ and includes the wheels and the spacecraft body.

The error in the pointing direction is given by the cross product between the desired looking vector and the actual looking vector, $\vec{e} = \vec{L}_{D/S} \times \vec{L}_{actual}$. The actual looking direction can be determined by converting the body coordinate direction to the inertial coordinate system using quaternions. The quaternion difference could also be used as the error. The state space vector for this system is given by $\bar{x} = [{}^{N}Q^{B}_{3\times3\rightarrow9\times1}\, \omega\, \Omega]$ where ω is the rotation rate of the whole spacecraft and Ω is the rotation rate of the wheels. We will use the small angle approximation and also assume low rotation rates.

$$ {}^{N}Q^{B}_{3\times3} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \rightarrow {}^{N}Q^{B}_{9\times1} = \begin{bmatrix} Q_{11} \\ Q_{12} \\ \ldots \end{bmatrix} $$

Basic Planar Model

A planar satellite looks out into space with a high fidelity imaging camera. The satellite rotates to take pictures of different points in space. The rotation of the satellite is confined to a single axis with planar moment of inertia J. A torque is applied to the satellite and rotates it about the axis with known moment of inertia. This causes a change in the looking angle of the satellite described by $J\ddot{\theta} = \tau = \bar{u}$.

The state space variable for this satellite is $\bar{x} = [\theta\,\dot{\theta}]^T = [\theta\,\omega]^T$. The state space model for this system would be

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\, B = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}$$

More Complex Planar Model

The planar satellite is equipped with a reaction wheel with planar moment of inertia I. This reaction wheel has a torque applied to it via an electric motor on-board the satellite. The spin axis of the reaction wheel is the same as that defined in the previous section. The planar moment of inertia of the satellite is J and includes the mass distribution of the satellite and the reaction wheel.

The state space variable for this satellite is $\bar{x} = [\theta\,\dot{\theta}\,\Omega]^T = [\theta\,\omega\,\Omega]^T$. The state space model for this system would be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\, B = \begin{bmatrix} 0 \\ -\frac{1}{J} \\ \frac{1}{I} \end{bmatrix}$$

It is important to note that the state space variable cannot be any arbitrary triple in $\mathbb{R}^3$ since ω and Ω are linked by the condition $J\dot{\omega} + I\dot{\Omega} = 0$.

Main System Calculations

For a set of rotating wheels, the angular momentum is $\vec{h}_w = I_w \cdot \Omega$. The change in the angular momentum of the wheels is equal to the applied torque on the wheels. The torque (control) is given by $\frac{d\vec{h}_w}{dt} = \vec{\tau} = I_w \cdot \dot{\Omega}$ so $\dot{\Omega} = I_w^{-1}\vec{\tau} = I_w^{-1}\bar{u}$. Being a diagonal matrix, the inverse of the wheel inertia tensor is also diagonal but with the entries inverted.

$$\dot{\Omega} = \begin{bmatrix} \frac{1}{I_{w1}} & 0 & 0 \\ 0 & \frac{1}{I_{w2}} & 0 \\ 0 & 0 & \frac{1}{I_{w3}} \end{bmatrix} \bar{u}$$

For a rotating spacecraft with reaction wheels (also called a gyrostat), the angular momentum is $\vec{h}_{tot} = I_{tot} \cdot \omega + I_w \cdot \Omega$. Using the transport theorem, the inertial derivative is given by

$$\vec{\tau}_{ext} = (I_{tot} \cdot \dot{\omega} + I_w \cdot \dot{\Omega}) + \omega \times (I_{tot} \cdot \omega + I_w \cdot \Omega)$$

The torque applied to the wheels to get them to rotate is an internal torque in the control volume including the spacecraft and the wheels. The cross product term is relatively small due to the low rotation speed and small moment of inertia tensor values for the wheels so $\dot{\omega} = -I_{tot}^{-1}\vec{\tau} = -I_{tot}^{-1}\bar{u}$.

$$\dot{\omega} = -\begin{bmatrix} \frac{1}{I_{tot1}} & 0 & 0 \\ 0 & \frac{1}{I_{tot2}} & 0 \\ 0 & 0 & \frac{1}{I_{tot3}} \end{bmatrix} \bar{u}$$

In order to find the derivative of the DCM, the skew-symmetric matrix $[\cdot]_\times$ must be defined.

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The derivative of a direction cosine matrix is ${}^N\dot{Q}^B = {}^NQ^B\,{}^B[\omega]_\times$.

$${}^N\dot{Q}^B = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

This results in a 3×3 matrix of coupled expressions which are not easily linearized. If the small-angle approximation is applied to the Taylor series angle-axis form of a DCM, the result is greatly simplified. For small angles

$${}^NQ^B \approx \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} = \mathbf{I}_{3\times3} + [\theta]_\times \rightarrow {}^N\dot{Q}^B \approx [\dot{\theta}]_\times = [\omega]_\times$$

This simplification can also be applied to the choice of state space variable $\bar{x} = [Q\,\omega\,\Omega]^T \in \mathbb{R}^{15}$ converting it to $\bar{x} = [\theta\,\omega\,\Omega]^T \in \mathbb{R}^9$. This does a better job of encoding practically the same information while removing clutter.

The linearized, small-angle state space model for this imaging satellite is given by the block matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{0}_{3\times3} \\ -\mathbf{I}_{tot}^{-1} \\ \mathbf{I}_w \end{bmatrix}$$

Applications/Downsides

Although the small-angle approximation has been absolutely crucial in simplifying the model and allowing it to be written in a linear form, it also restricts the total movement of the satellite. When converting the actual looking vector of the satellite from the body coordinate system to the inertial system, the Taylor expansion approximate DCM is used. The matrix multiplication always isolates the first column of the DCM.

$$^{I}r_{actual} = (\mathbf{I}_{3\times 3} + [\theta]_{\times}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\psi \\ \theta \end{bmatrix}$$

After normalization of r_{actual} into L_{actual}, it is shown that the effective range of movement of the satellite is best represented by a cone. This type of linearized control would be best suited for micro-adjustments. A model capable of full attitude control would need to make use of the nonlinear dynamics of the system. The error of the pointing direction can be determined by taking the cross product of the actual pointing direction (before normalization) and the desired pointing direction, $e = r_{actual} \times L_{desired}$. Another error term should also exist which seeks to minimize the roll, pitch, and yaw of the satellite once it is actually in position.

Changes

Version 2 skips over using DCMs and makes full use of the small angle approximation. The state space becomes $\bar{x} = [\theta\, \omega\, \Omega]^T = [\theta\, \dot{\theta}\, \Omega]^T$ which has a very simple state space model. A planar model example is provided.