Satellite Imaging Model and Control V1

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1 Assumptions/Setup

The satellite is stationary when it comes to translational movement in an inertial frame. Its position in the inertial frame is given by $\vec{r}_{S/O} = [x_1\, y_1\, z_1]^T$ with respect to the Earth. The position of the desired imaging location is given by $\vec{r}_{D/O} = [x_2\, y_2\, z_2]^T$. The pseudo-vector normal to the camera lens in the direction of the desired location is $\vec{r}_{D/O} - \vec{r}_{S/O} = \vec{r}_{D/S}$ which is then normalized to $\vec{L}_{D/S} = \frac{\vec{r}_{D/S}}{|\vec{r}_{D/S}|}$.

In the body frame of the spacecraft, the normal vector off the lens is given by $\vec{L}_{body} = [0\,0\,1]^T$. This is always constant within the body coordinate system. The body coordinate system is perfectly aligned with the principal axes of the spacecraft. The spacecraft contains a set of reaction wheels which provide angular momentum aligned with one of the three body/principal axes. The control input to the system is the torque on each reaction wheel direction in the form of $\bar{u} = \vec{\tau} = [\tau_x\, \tau_y\, \tau_z]$. The moment of inertia tensor for the wheels is $I_w = [I_{wx}\, I_{wy}\, I_{wy}]$ and is constant. The moment of inertia tensor for the spacecraft is $I_{tot} = I_s + I_w$ and includes the wheels and the spacecraft body.

The error in the pointing direction is given by the cross product between the desired looking vector and the actual looking vector, $\vec{e} = \vec{L}_{D/S} \times \vec{L}_{actual}$. The actual looking direction can be determined by converting the body coordinate direction to the inertial coordinate system using quaternions. The quaternion difference could also be used as the error. The state space vector for this system is given by $\bar{x} = [q\, \omega\, \Omega]$ where ω is the rotation rate of the whole spacecraft and Ω is the rotation rate of the wheels. We will use the small angle approximation and also assume low rotation rates.

2 Calculations

For a set of rotating wheels, the angular momentum is $\vec{h}_w = I_w \cdot \Omega$. The change in the angular momentum of the wheels is equal to the applied torque on the wheels. The torque (control) is given by $\frac{d\vec{h}_w}{dt} = \vec{\tau} = I_w \cdot \dot{\Omega}$ so $\dot{\Omega} = I_w^{-1}\vec{\tau} = I_w^{-1}\bar{u}$.

Being a diagonal matrix, the inverse of the wheel inertia tensor is also diagonal but with the entries inverted.

$$\dot{\Omega} = \begin{bmatrix} \frac{1}{I_{w1}} & 0 & 0 \\ 0 & \frac{1}{I_{w2}} & 0 \\ 0 & 0 & \frac{1}{I_{w3}} \end{bmatrix} \bar{u}$$

For a rotating spacecraft with reaction wheels (also called a gyrostat), the angular momentum is $\vec{h}_{tot} = I_{tot} \cdot \omega + I_w \cdot \Omega$. Using the transport theorem, the inertial derivative is given by

$$\vec{\tau}_{ext} = (I_{tot} \cdot \dot{\omega} + I_w \cdot \dot{\Omega}) + \omega \times (I_{tot} \cdot \omega + I_w \cdot \Omega)$$

The torque applied to the wheels to get them to rotate is an internal torque in the control volume including the spacecraft and the wheels. The cross product term is relatively small due to the low rotation speed and small moment of inertia tensor values for the wheels so $\dot{\omega} = -I_{tot}^{-1}\vec{\tau} = -I_{tot}^{-1}\bar{u}$.

$$\dot{\omega} = -\begin{bmatrix} \frac{1}{I_{tot1}} & 0 & 0 \\ 0 & \frac{1}{I_{tot2}} & 0 \\ 0 & 0 & \frac{1}{I_{tot3}} \end{bmatrix} \bar{u}$$

Using XYZW formatting for quaternions, the derivative of the quaternion is $\dot{q} = q \otimes [\omega\, 0]^T$. This creates a series of 4 equations with coupled terms. To linearize the system, we assume a small angle rate of change which gives $\dot{q} = \frac{1}{2}[\omega_1\, \omega_2\, \omega_3\, 0]^T$.

$$\dot{q} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \omega = \frac{1}{2} \begin{bmatrix} I_{3\times3} \\ 0_{1\times3} \end{bmatrix} \omega$$

Use linearized form of $\dot{\bar{x}} = A\bar{x} + B\bar{u}$ to discretely update through $\bar{x}_1 = \bar{x}_0 + \Delta t\dot{\bar{x}}$. Control input $\bar{u}$ is the torques on the wheels and could done with PID or some other method. Error is the cross between desired and actual so

$$\vec{e} = \vec{L}_{D/S} \times \vec{L}_{actual} = \vec{L}_{D/S} \times (q^{-1} \otimes [0\ 0\ 1\ 0] \otimes q)$$

This might not be as good as an error that looks more like $e = \bar{x}_{reference} - \bar{x}$ where $\bar{x}_{reference} = [q_{reference}\ (\omega = 0)\ (\Omega = 0)]$. The reference quaternion is the one that takes the body pointing vector to the desired inertial pointing vector.