Qiskit Fall Fest 2023

Quantum Computing Association x IBM Quantum

Contents

About Qiskit Fall Fest

Welcome to Cornell’s very first Qiskit Fall Fest!
We hope that it’s a fun experience for everyone to learn about how to program quantum computers, using Qiskit!

Qiskit is an open-source quantum information science (QIS) Python package that simulates quantum circuits which can be run on a real quantum computer.

Three Challenge Tracks

Qubit 🧠

Beginner-friendly introduction to quantum circuits.

Schrödinger's cat 🐈

Implement important basic algorithms, such as teleportation (yes, it's real) superdense coding, and more!

Superposition 🔥

Build up to the algorithm that kickstarted interest in the field: Shor's algorithm!

Learning resources

IBM Quantum Learn portal: https://learning.quantum-computing.ibm.com

Qiskit - Introduction to Quantum Computing: https://qiskit.org/learn/course/introduction-course

Curated Syllabus on Qiskit Learn – Peter Shor: Introduction to Quantum Algorithms: http://ibm.co/shor-intro-to-quantum-algorithms

Curated Syllabus on Qiskit Learn – Jay Gambetta: Quantum computing and Superconducting Qubits**:** http://ibm.co/gambetta-quantum-hardware

Qubit Track 🧠

Lesson overview

Contents

Descriptions of quantum information

Simplified description (this unit)

- Simpler and typically learned first
- Quantum states represented by *vectors;* operations are represented by *unitary matrices*
- Sufficient for an understanding of most quantum algorithms

General description (covered in a later unit)

- More general and more broadly applicable
- Quantum states represented by *density matrices;* allows for a more general class of measurements and operations
- Includes both the simplified description and classical information (including probabilistic states) as special cases

1. Classical information

Classical information

Consider a physical system that stores information: let us call it X.

Assume X can be in one of a finite number of *classical states* at each moment. Denote this classical state set by Σ.

Examples

- If X is a bit, then its classical state set is $\Sigma = \{0, 1\}$.
- If X is a six-sided die, then $\Sigma = \{1, 2, 3, 4, 5, 6\}$.
- If X is a switch on a standard electric fan, then perhaps $\Sigma = \{\text{high}, \text{medium}, \text{low}, \text{off}\}$.

There there may be *uncertainty* about the classical state of a system, where each classical state has some *probability* associated with it.

Classical information

For example, if X is a bit, then perhaps it is in the classical state 0 with probability 3/4 and in the classical state 1 with probability 1/4. This is a *probabilistic state* of X.

$$\Pr(X = 0) = \frac{3}{4} \quad \text{and} \quad \Pr(X = 1) = \frac{1}{4}$$

A succinct way to represent this probabilistic state is by a *column vector:*

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} \begin{array}{l} \longleftarrow \text{entry corresponding to } 0 \\ \longleftarrow \text{entry corresponding to } 1 \end{array}$$

This vector is a *probability vector:*

- All entries are nonnegative real numbers.
- The sum of the entries is 1.

Dirac notation (first part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $|a\rangle$ the *column vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Example 1

If $\Sigma = \{0, 1\}$, then

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac notation (first part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $|a\rangle$ the *column vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Example 2

If $\Sigma = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, then we might choose to order these states like this: $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$. This yields

$$|\clubsuit\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\diamondsuit\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\heartsuit\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\spadesuit\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Dirac notation (first part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $|a\rangle$ the *column vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Vectors of this form are called *standard basis vectors.* Every vector can be expressed uniquely as a linear combination of standard basis vectors.

Example

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} = \frac{3}{4}\,|0\rangle + \frac{1}{4}\,|1\rangle$$

Measuring probabilistic states

What happens if we *measure* a system X while it is in some probabilistic state?

We see a *classical state,* chosen at random according to the probabilities.

Suppose we see the classical state $a \in \Sigma$.

This changes the probabilistic state of X (from our viewpoint): having recognized that X is in the classical state a, we now have

$$\operatorname{Pr}(\mathsf{X} = a) = 1$$

This probabilistic state is represented by the vector $|a\rangle$.

Measuring probabilistic states

Example

Consider the probabilistic state of a bit X where

$$\Pr(X = 0) = \frac{3}{4} \quad \text{and} \quad \Pr(X = 1) = \frac{1}{4}$$

Measuring X selects (or reveals) a transition, chosen at random:

$$\begin{tikzcd} & \frac{3}{4}\,|0\rangle + \frac{1}{4}\,|1\rangle \arrow[dl, "\text{probability } \frac{3}{4}"'] \arrow[dr, "\text{probability } \frac{1}{4}"] & \\ |0\rangle & & |1\rangle \end{tikzcd}$$

Deterministic operations

Every function $f : \Sigma \to \Sigma$ describes a *deterministic operation* that transforms the classical state a into $f(a)$, for each $a \in \Sigma$.

Given any function $f : \Sigma \to \Sigma$, there is a (unique) matrix M satisfying

$$M\,|a\rangle = |f(a)\rangle \qquad \text{(for every } a \in \Sigma\text{)}$$

This matrix has exactly one 1 in each column, and 0 for all other entries:

$$M(b, a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$$

The action of this operation is described by *matrix-vector multiplication:*

$$v \longmapsto Mv$$

Deterministic operations

Example

For $\Sigma = \{0, 1\}$, there are four functions of the form $f : \Sigma \to \Sigma$:

a	$f_1(a)$
0	0
1	0

a	$f_2(a)$
0	0
1	1

a	$f_3(a)$
0	1
1	0

a	$f_4(a)$
0	1
1	1

Here are the matrices corresponding to these functions:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M(b, a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$$

$$M\,|a\rangle = |f(a)\rangle$$

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the *row vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Example

If $\Sigma = \{0, 1\}$, then

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the *row vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Multiplying a row vector to a column vector yields a scalar:

$$\begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} = \begin{pmatrix} * \end{pmatrix}$$

$$\langle a|b\rangle = \langle a||b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the *row vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Multiplying a row vector to a column vector yields a scalar:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

$$\langle a|b\rangle = \langle a||b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the *row vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Multiplying a row vector to a column vector yields a scalar:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\langle a|b\rangle = \langle a||b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the *row vector* having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Dirac notation (second part)

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Example

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Dirac notation (second part)

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Example

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Dirac notation (second part)

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Example

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Dirac notation (second part)

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Example

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Dirac notation (second part)

Multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * & * & * & \cdots & * \end{pmatrix} = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

In general, the matrix

$$|a\rangle\langle b|$$

has a 1 in the (a, b)-entry and 0 for all other entries.

Deterministic operations

Every function $f : \Sigma \rightarrow \Sigma$ describes a *deterministic operation* that transforms the classical state a into $f(a)$, for each $a \in \Sigma$.

Given any function $f : \Sigma \rightarrow \Sigma$, there is a (unique) matrix M satisfying

$$M\,|a\rangle = |f(a)\rangle \qquad \text{(for every } a \in \Sigma\text{)}$$

This matrix may be expressed as

$$M = \sum_{b\in\Sigma} |f(b)\rangle\langle b|$$

Its action on standard basis vectors works as required:

$$M|a\rangle = \left(\sum_{b\in\Sigma} |f(b)\rangle\langle b|\right)|a\rangle = \sum_{b\in\Sigma} |f(b)\rangle\langle b|a\rangle = |f(a)\rangle$$

Probabilistic operations

Probabilistic operations are classical operations that may introduce randomness or uncertainty.

Example

Here is a probabilistic operation on a bit:

> *If the classical state is 0, then do nothing.*
>
> *If the classical state is 1, then flip the bit with probability* $1/2$.

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Probabilistic operations are described by *stochastic matrices:*

- All entries are nonnegative real numbers
- The entries in every column sum to 1

Probabilistic operations

Probabilistic operations are classical operations that may introduce randomness or uncertainty.

Example

Here is a probabilistic operation on a bit:

If the classical state is 0, then do nothing.

If the classical state is 1, then flip the bit with probability $1/2$*.*

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Probabilistic operations are described by *stochastic matrices:*

- All entries are nonnegative real numbers
- The entries in every column sum to 1

Composing operations

Suppose X is a system and $M_1, \ldots, M_n$ are stochastic matrices representing probabilistic operations on X.

Applying the first probabilistic operation to the probability vector v, then applying the second probabilistic operation to the result yields this vector:

$$M_2(M_1 v) = (M_2 M_1) v$$

The probabilistic operation obtained by *composing* the first and second probabilistic operations is represented by the *matrix product* $M_2 M_1$.

Composing the probabilistic operations represented by the matrices $M_1, \ldots, M_n$ (in that order) is represented by this matrix product:

$$M_n \cdots M_1$$

Composing operations

Suppose X is a system and $M_1, \dots, M_n$ are stochastic matrices representing probabilistic operations on X.

Composing the probabilistic operations represented by the matrices $M_1, \dots, M_n$ (in that order) is represented by this matrix product:

$$M_n \cdots M_1$$

The order is important: matrix multiplication is *not commutative!*

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M_2 M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_1 M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

2. Quantum information

Quantum information

A *quantum state* of a system is represented by a *column vector* whose indices are placed in correspondence with the classical states of that system:

- The entries are complex numbers.
- The sum of the absolute values squared of the entries must equal 1.

Definition

The *Euclidean norm* for vectors with complex number entries is defined like this:

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \Longrightarrow \quad \|v\| = \sqrt{\sum_{k=1}^{n} |\alpha_k|^2}$$

Quantum state vectors are therefore *unit vectors* with respect to this norm.

Quantum information

A ***quantum state*** of a system is represented by a ***column vector*** whose indices are placed in correspondence with the classical states of that system:

- The entries are complex numbers.
- The sum of the absolute values squared of the entries must equal 1.

Examples of qubit states

- Standard basis states: $|0\rangle$ and $|1\rangle$
- Plus/minus states:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

- A state without a special name:

$$\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

Quantum information

A ***quantum state*** of a system is represented by a ***column vector*** whose indices are placed in correspondence with the classical states of that system:

- The entries are complex numbers.
- The sum of the absolute values squared of the entries must equal 1.

Example

A quantum state of a system with classical states ♣, ♦, ♥, and ♠:

$$\frac{1}{2}\,|♣\rangle - \frac{i}{2}\,|♦\rangle + \frac{1}{\sqrt{2}}\,|♠\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Dirac notation (third part)

The Dirac notation can be used for arbitrary vectors: any name can be used in place of a classical state. Kets are column vectors, bras are row vectors.

Example

The notation $|\psi\rangle$ is commonly used to refer to an arbitrary vector:

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

For any column vector $|\psi\rangle$, the row vector $\langle\psi|$ is the *conjugate transpose* of $|\psi\rangle$:

$$\langle\psi| = |\psi\rangle^\dagger$$

Dirac notation (third part)

The Dirac notation can be used for arbitrary vectors: any name can be used in place of a classical state. Kets are column vectors, bras are row vectors.

Example

The notation $|\psi\rangle$ is commonly used to refer to an arbitrary vector:

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

$$\langle\psi| = \frac{1-2i}{3}\langle 0| - \frac{2}{3}\langle 1|$$

For any column vector $|\psi\rangle$, the row vector $\langle\psi|$ is the *conjugate transpose* of $|\psi\rangle$:

$$\langle\psi| = |\psi\rangle^{\dagger}$$

Dirac notation (third part)

The Dirac notation can be used for arbitrary vectors: any name can be used in place of a classical state. Kets are column vectors, bras are row vectors.

Example

The notation $|\psi\rangle$ is commonly used to refer to an arbitrary vector:

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle = \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$\langle\psi| = \frac{1-2i}{3}\langle 0| - \frac{2}{3}\langle 1| = \begin{pmatrix} \frac{1-2i}{3} & -\frac{2}{3} \end{pmatrix}$$

Measuring quantum states

For this lesson will restrict our attention to *standard basis measurements:*

- The possible *outcomes* are the *classical states.*
- The probability for each classical state to be the outcome is the *absolute value squared* of the corresponding quantum state vector entry.

Example 1

Measuring the quantum state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

yields an outcome as follows:

$$\Pr(\text{outcome is } 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad \Pr(\text{outcome is } 1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Measuring quantum states

For this lesson will restrict our attention to *standard basis measurements:*

- The possible *outcomes* are the *classical states.*
- The probability for each classical state to be the outcome is the *absolute value squared* of the corresponding quantum state vector entry.

Example 2

Measuring the quantum state

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

yields an outcome as follows:

$$\Pr(\text{outcome is } 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad \Pr(\text{outcome is } 1) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Measuring quantum states

For this lesson will restrict our attention to *standard basis measurements:*

- The possible *outcomes* are the *classical states.*
- The probability for each classical state to be the outcome is the *absolute value squared* of the corresponding quantum state vector entry.

Example 3

Measuring the quantum state

$$\frac{1+2i}{3}\,|0\rangle - \frac{2}{3}\,|1\rangle$$

yields an outcome as follows:

$$\Pr(\text{outcome is } 0) = \left|\frac{1+2i}{3}\right|^2 = \frac{5}{9} \qquad \Pr(\text{outcome is } 1) = \left|-\frac{2}{3}\right|^2 = \frac{4}{9}$$

Measuring quantum states

For this lesson will restrict our attention to *standard basis measurements:*

- The possible *outcomes* are the *classical states.*
- The probability for each classical state to be the outcome is the *absolute value squared* of the corresponding quantum state vector entry.

Example 4

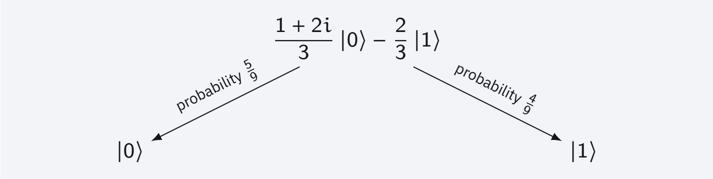
Measuring the quantum state $|0\rangle$ gives the outcome 0 with certainty, and measuring the quantum state $|1\rangle$ gives the outcome 1 with certainty.

Measuring quantum states

For this lesson will restrict our attention to *standard basis measurements:*

- The possible *outcomes* are the *classical states.*
- The probability for each classical state to be the outcome is the *absolute value squared* of the corresponding quantum state vector entry.

Measuring a system changes its quantum state: if we obtain the classical state a, the new quantum state becomes $|a\rangle$.



Unitary operations

The set of allowable *operations* that can be performed on a quantum state is different than it is for classical information.

Operations on quantum state vectors are represented by *unitary matrices.*

Definition

A square matrix U having complex number entries is *unitary* if it satisfies the equalities

$$U^\dagger U = \mathbb{1} = U U^\dagger$$

where $U^\dagger$ is the conjugate transpose of U and $\mathbb{1}$ is the identity matrix.

Both equalities are equivalent to $U^{-1} = U^\dagger$.

Unitary operations

Definition

A square matrix U having complex number entries is *unitary* if it satisfies the equalities

$$U^\dagger U = \mathbb{1} = U U^\dagger$$

where $U^\dagger$ is the conjugate transpose of U and $\mathbb{1}$ is the identity matrix.

The condition that an $n \times n$ matrix U is unitary is equivalent to

$$\|Uv\| = \|v\|$$

for every n-dimensional column vector v with complex number entries.

If v is a quantum state vector, then Uv is also a quantum state vector.

Qubit unitary operations

1. Pauli operations

Pauli operations are ones represented by the Pauli matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Common alternative notations: $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$.

The operation σ_x is also called a *bit flip* (or a NOT operation) and the σ_z operation is called a *phase flip:*

$$\begin{aligned} \sigma_x|0\rangle &= |1\rangle & \qquad \sigma_z|0\rangle &= |0\rangle \\ \sigma_x|1\rangle &= |0\rangle & \qquad \sigma_z|1\rangle &= -|1\rangle \end{aligned}$$

Qubit unitary operations

2. Hadamard operation

The Hadamard operation is represented by this matrix:

$$\mathrm{H} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Checking that H is unitary is a straightforward calculation:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{\dagger} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Qubit unitary operations

3. Phase operations

A phase operation is one described by the matrix

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

for any choice of a real number θ.

The operations

$$S = P_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad T = P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

are important examples.

Qubit unitary operations

Example 1

$$H\,|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle$$

$$H\,|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle$$

$$H\,|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$H\,|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Qubit unitary operations

Example 1

$$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$$

$$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$$

$$H\left(\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{-1+2i}{3\sqrt{2}} \\ \frac{3+2i}{3\sqrt{2}} \end{pmatrix}$$

$$= \frac{-1+2i}{3\sqrt{2}}|0\rangle + \frac{3+2i}{3\sqrt{2}}|1\rangle$$

Qubit unitary operations

Example 2

$$T|0\rangle = |0\rangle \qquad \text{and} \qquad T|1\rangle = \frac{1+i}{\sqrt{2}}|1\rangle$$

$$\begin{aligned} T\,|+\rangle &= T\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{\sqrt{2}}T\,|0\rangle + \frac{1}{\sqrt{2}}T\,|1\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle \end{aligned}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Qubit unitary operations

Example 2

$$T|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$$

$$\begin{aligned}
HT|+\rangle &= H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}H|0\rangle + \frac{1+i}{2}H|1\rangle \\
&= \frac{1}{\sqrt{2}}|+\rangle + \frac{1+i}{2}|-\rangle \\
&= \left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + \left(\frac{1+i}{2\sqrt{2}}|0\rangle - \frac{1+i}{2\sqrt{2}}|1\rangle\right) \\
&= \left(\frac{1}{2} + \frac{1+i}{2\sqrt{2}}\right)|0\rangle + \left(\frac{1}{2} - \frac{1+i}{2\sqrt{2}}\right)|1\rangle
\end{aligned}$$

$$H|0\rangle = |+\rangle$$
$$H|1\rangle = |-\rangle$$

Composing unitary operations

Compositions of unitary operations are represented by *matrix multiplication* (similar to the probabilistic setting).

Example: square root of NOT

Applying a Hadamard operation, followed by the phase operation S, followed by another Hadamard operation yields this operation:

$$HSH = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

Applying this unitary operation twice yields a NOT operation:

$$(HSH)^2 = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Lesson overview

Contents

Classical states

Suppose that we have two systems:

- X is a system having classical state set Σ.
- Y is a system having classical state set Γ.

Imagine that X and Y are placed side-by-side, with X on the left and Y on the right, and viewed together as if they form a single system.

We denote this new compound system by (X, Y) or XY.

Question

What are the classical states of (X, Y)?

Answer

The classical state set of (X, Y) is the *Cartesian product*

$$\Sigma \times \Gamma = \{(a, b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Classical states

Question

What are the classical states of (X, Y)?

Answer

The classical state set of (X, Y) is the *Cartesian product*

$$\Sigma \times \Gamma = \{(a, b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Example

If $\Sigma = \{0, 1\}$ and $\Gamma = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, then

$$\Sigma \times \Gamma = \{(0, \clubsuit), (0, \diamondsuit), (0, \heartsuit), (0, \spadesuit), (1, \clubsuit), (1, \diamondsuit), (1, \heartsuit), (1, \spadesuit)\}$$

Classical states

This description generalizes to more than two systems in a natural way.

Suppose $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are systems having classical state sets $\Sigma_1, \ldots, \Sigma_n$, respectively.

The classical state set of the n-tuple $(\mathsf{X}_1, \ldots, \mathsf{X}_n)$, viewed as a single compound system, is the Cartesian product

$$\Sigma_1 \times \cdots \times \Sigma_n = \bigl\{(a_1, \ldots, a_n) \,:\, a_1 \in \Sigma_1,\; \ldots,\; a_n \in \Sigma_n\bigr\}$$

Example

If $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{0, 1\}$, then the classical state set of $(\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3)$ is

$$\begin{aligned}\Sigma_1 \times \Sigma_2 \times \Sigma_3 = \bigl\{&(0,0,0), (0,0,1), (0,1,0), (0,1,1),\\ &(1,0,0), (1,0,1), (1,1,0), (1,1,1)\bigr\}\end{aligned}$$

Classical states

An n-tuple $(a_1, \ldots, a_n)$ may also be written as a *string* $a_1 \cdots a_n$.

Example

Suppose $\mathsf{X}_1, \ldots, \mathsf{X}_{10}$ are bits, so their classical state sets are all the same:

$$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{10} = \{0, 1\}$$

The classical state set of $(\mathsf{X}_1, \ldots, \mathsf{X}_{10})$ is the Cartesian product

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

Classical states

An n-tuple $(a_1, \ldots, a_n)$ may also be written as a *string* $a_1 \cdots a_n$.

Example

The classical state set of $(\mathsf{X}_1, \ldots, \mathsf{X}_{10})$ is the Cartesian product

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

Written as strings, these classical states look like this:

$$\begin{gathered} 0000000000 \\ 0000000001 \\ 0000000010 \\ 0000000011 \\ \vdots \\ 1111111111 \end{gathered}$$

Classical states

Convention

Cartesian products of classical state sets are ordered *lexicographically* (i.e., dictionary ordering):

- We assume the individual classical state sets are already ordered.
- Significance decreases from left to right.

Example

The Cartesian product $\{1,2,3\} \times \{0,1\}$ is ordered like this:

$$(1,0),\ (1,1),\ (2,0),\ (2,1),\ (3,0),\ (3,1)$$

When n-tuples are written as strings and ordered in this way, we observe familiar patterns, such as $\{0,1\} \times \{0,1\}$ being ordered as $00, 01, 10, 11$.

Probabilistic states

Probabilistic states of compound systems associate probabilities with the Cartesian product of the classical state sets of the individual systems.

Example

This is a probabilistic state of a pair of bits (X, Y):

$$\begin{aligned}
\Pr\big((X, Y) = (0, 0)\big) &= \tfrac{1}{2} \\
\Pr\big((X, Y) = (0, 1)\big) &= 0 \\
\Pr\big((X, Y) = (1, 0)\big) &= 0 \\
\Pr\big((X, Y) = (1, 1)\big) &= \tfrac{1}{2}
\end{aligned}$$

Probabilistic states

Probabilistic states of compound systems associate probabilities with the Cartesian product of the classical state sets of the individual systems.

Example

This is a probabilistic state of a pair of bits (X, Y):

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{array}{l} \leftarrow \text{probability associated with state } 00 \\ \leftarrow \text{probability associated with state } 01 \\ \leftarrow \text{probability associated with state } 10 \\ \leftarrow \text{probability associated with state } 11 \end{array}$$

Probabilistic states

Definition

For a given probabilistic state of (X, Y), we say that X and Y are *independent* if

$$\Pr((X, Y) = (a, b)) = \Pr(X = a)\Pr(Y = b)$$

for all $a \in \Sigma$ and $b \in \Gamma$.

Suppose that a probabilistic state of (X, Y) is expressed as a vector:

$$|\pi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle$$

The systems X and Y are independent if there exist probability vectors

$$|\phi\rangle = \sum_{a\in\Sigma} q_a|a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} r_b|b\rangle$$

such that $p_{ab} = q_a r_b$ for all $a \in \Sigma$ and $b \in \Gamma$.

Probabilistic states

Example

The probabilistic state of a pair of bits (X, Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probability vectors:

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

Probabilistic states

Example

For the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

of two bits (X, Y), we have that X and Y are not independent.

If they were, we would have numbers q_0, q_1, r_0, r_1 such that

$$\begin{aligned} q_0 r_0 &= \tfrac{1}{2} \\ q_0 r_1 &= 0 \\ q_1 r_0 &= 0 \\ q_1 r_1 &= \tfrac{1}{2} \end{aligned}$$

But if $q_0 r_1 = 0$, then either $q_0 = 0$ or $r_1 = 0$ (or both), contradicting either the first or last equality.

Tensor products of vectors

Definition

The *tensor product* of two vectors

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

is the vector

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalently, the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by this condition:

$$\langle ab|\pi\rangle = \langle a|\phi\rangle\langle b|\psi\rangle \qquad \text{(for all } a \in \Sigma \text{ and } b \in \Gamma)$$

Tensor products of vectors

Definition

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Example

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

Tensor products of vectors

Definition

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Alternative notation for tensor products:

$$|\phi\rangle|\psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

$$|\phi \otimes \psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

Tensor products of vectors

Following our convention for ordering the elements of Cartesian product sets, we obtain this specification for the tensor product of two column vectors:

$$
\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 \\ \vdots \\ \alpha_1 \beta_k \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_2 \beta_k \\ \vdots \\ \alpha_m \beta_1 \\ \vdots \\ \alpha_m \beta_k \end{pmatrix}
$$

Tensor products of vectors

Example

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_1\beta_4 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \\ \alpha_2\beta_4 \\ \alpha_3\beta_1 \\ \alpha_3\beta_2 \\ \alpha_3\beta_3 \\ \alpha_3\beta_4 \end{pmatrix}$$

Tensor products of vectors

Observe the following expression for tensor products of standard basis vectors:

$$|a\rangle \otimes |b\rangle = |a\rangle|b\rangle = |ab\rangle$$

Alternatively, writing (a, b) as an ordered pair rather than a string, we could write

$$|a\rangle \otimes |b\rangle = |(a, b)\rangle$$

but it is more common to write

$$|a\rangle \otimes |b\rangle = |a, b\rangle$$

(It is a standard convention in mathematics to eliminate parentheses when they do not serve to add clarity or remove ambiguity.)

Tensor products of vectors

Important property of tensor products

The tensor product of two vectors is *bilinear.*

1. Linearity in the first argument:

$$\big(|\phi_1\rangle + |\phi_2\rangle\big) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$

$$\big(\alpha|\phi\rangle\big) \otimes |\psi\rangle = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big)$$

2. Linearity in the second argument:

$$|\phi\rangle \otimes \big(|\psi_1\rangle + |\psi_2\rangle\big) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$

$$|\phi\rangle \otimes \big(\alpha|\psi\rangle\big) = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big)$$

Notice that scalars “float freely” within tensor products:

$$\big(\alpha|\phi\rangle\big) \otimes |\psi\rangle = |\phi\rangle \otimes \big(\alpha|\psi\rangle\big) = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big) = \alpha|\phi\rangle \otimes |\psi\rangle$$

Tensor products of vectors

Tensor products generalize to three or more systems.

If $|\phi_1\rangle, \ldots, |\phi_n\rangle$ are vectors, then the tensor product

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

is defined by the equation

$$\langle a_1 \cdots a_n | \psi \rangle = \langle a_1 | \phi_1 \rangle \cdots \langle a_n | \phi_n \rangle$$

Equivalently, the tensor product of three or more vectors can be defined recursively:

$$|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle = \big(|\phi_1\rangle \otimes \cdots \otimes |\phi_{n-1}\rangle\big) \otimes |\phi_n\rangle$$

The tensor product of three or more vectors is *multilinear.*

Measurements of probabilistic states

Measurements of compound systems work in the same way as measurements of single systems — provided that all of the systems are measured.

Example

Suppose that two bits (X, Y) are in the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

Measuring both bits yields the outcome 00 with probability 1/2 and the outcome 11 with probability 1/2.

Measurements of probabilistic states

Question

Suppose two systems (X, Y) are together in some probabilistic state.
What happens when we measure X and do nothing to Y?

Answer

1. The probability to observe a particular classical state $a \in \Sigma$ when just X is measured is

$$\Pr(\mathsf{X} = a) = \sum_{b \in \Gamma} \Pr\bigl((\mathsf{X}, \mathsf{Y}) = (a, b)\bigr)$$

2. There may still exist uncertainty about the classical state of Y, depending on the outcome of the measurement:

$$\Pr(\mathsf{Y} = b \,|\, \mathsf{X} = a) = \frac{\Pr\bigl((\mathsf{X}, \mathsf{Y}) = (a, b)\bigr)}{\Pr(\mathsf{X} = a)}$$

Measurements of probabilistic states

These formulas can be expressed using the Dirac notation as follows.

Suppose that (X, Y) is in some arbitrary probabilistic state:

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}\lvert ab\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}\lvert a\rangle \otimes \lvert b\rangle = \sum_{a\in\Sigma}\lvert a\rangle \otimes \biggl(\sum_{b\in\Gamma} p_{ab}\lvert b\rangle\biggr)$$

1. The probability that a measurement of X yields an outcome $a \in \Sigma$ is

$$\Pr(\mathsf{X} = a) = \sum_{b\in\Gamma} p_{ab}$$

2. Conditioned on the outcome $a \in \Sigma$, the probabilistic state of Y becomes

$$\frac{\sum_{b\in\Gamma} p_{ab}\lvert b\rangle}{\sum_{c\in\Gamma} p_{ac}}$$

Measurements of probabilistic states

Example

Suppose (X, Y) is in the probabilistic state

$$\frac{1}{12}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{3}|10\rangle + \frac{1}{3}|11\rangle$$

We write this vector as follows:

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Measurements of probabilistic states

Example

Suppose (X, Y) is in the probabilistic state

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Case 1: the measurement outcome is 0.

$$\Pr(\text{outcome is } 0) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle}{\frac{1}{3}} = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$$

Measurements of probabilistic states

Example

Suppose (X, Y) is in the probabilistic state

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Case 2: the measurement outcome is 1.

$$\Pr(\text{outcome is } 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle}{\frac{2}{3}} = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Measurements of probabilistic states

The same method can be used when Y is measured rather than X. Suppose that (X, Y) is in some arbitrary probabilistic state:

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|a\rangle\otimes|b\rangle = \sum_{b\in\Gamma}\left(\sum_{a\in\Sigma} p_{ab}|a\rangle\right)\otimes|b\rangle$$

1. The probability that a measurement of Y yields an outcome $a \in \Sigma$ is

$$\Pr(Y = b) = \sum_{a\in\Sigma} p_{ab}$$

2. Conditioned on the outcome $b \in \Gamma$, the probabilistic state of X becomes

$$\frac{\sum_{a\in\Sigma} p_{ab}|a\rangle}{\sum_{c\in\Sigma} p_{c,b}}$$

Operations on probabilistic states

Probabilistic operations on compound systems are represented by stochastic matrices having rows and columns that correspond to the Cartesian product of the individual systems' classical state sets.

Example

A *controlled-NOT* operation on two bits X and Y:

> *If* $\mathsf{X} = 1$*, then perform a NOT operation on* Y*, otherwise do nothing.*

X is the *control bit* that determines whether or not a NOT operation is applied to the *target bit* Y.

Operations on probabilistic states

Example

A *controlled-NOT* operation on two bits X and Y:

> *If* X = 1*, then perform a NOT operation on* Y*, otherwise do nothing.*

X is the *control bit* that determines whether or not a NOT operation is applied to the *target bit* Y.

Action on standard basis

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle \end{aligned}$$

Matrix representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operations on probabilistic states

Example

Here is a different operation on two bits (X, Y):

> *With probability 1/2, set* Y *to be equal to* X*, otherwise set* X *to be equal to* Y*.*

The matrix representation of this operation is as follows:

$$\begin{pmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Operations on probabilistic states

Question

Suppose we have two probabilistic operations, each on its own system, described by stochastic matrices:

1. M is an operation on X.
2. N is an operation on Y.

If we *simultaneously* perform the two operations, how do we describe the effect on the compound system (X, Y)?

Tensor products of matrices

Definition

The *tensor product* of two matrices

$$M = \sum_{a,b\in\Sigma} \alpha_{ab} |a\rangle\langle b| \quad \text{and} \quad N = \sum_{c,d\in\Gamma} \beta_{cd} |c\rangle\langle d|$$

is the matrix

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab}\beta_{cd} |ac\rangle\langle bd|$$

Equivalently, $M \otimes N$ is defined by this condition:

$$\langle ac|M \otimes N|bd\rangle = \langle a|M|b\rangle\langle c|N|d\rangle \quad \text{(for all } a, b \in \Sigma \text{ and } b, d \in \Gamma\text{)}$$

Tensor products of matrices

Definition

$$M = \sum_{a,b\in\Sigma} \alpha_{ab} |a\rangle\langle b| \quad \text{and} \quad N = \sum_{c,d\in\Gamma} \beta_{cd} |c\rangle\langle d|$$

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab} \beta_{cd} |ac\rangle\langle bd|$$

An alternative, but equivalent, way to define $M \otimes N$ is that it is the unique matrix that satisfies the equation

$$(M \otimes N)|\phi \otimes \psi\rangle = M|\phi\rangle \otimes N|\psi\rangle$$

for every choice of vectors $|\phi\rangle$ and $|\psi\rangle$.

Tensor products of matrices

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{pmatrix} \otimes \begin{pmatrix} \beta_{11} & \cdots & \beta_{1k} \\ \vdots & \ddots & \vdots \\ \beta_{k1} & \cdots & \beta_{kk} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{11}\beta_{11} & \cdots & \alpha_{11}\beta_{1k} & & \alpha_{1m}\beta_{11} & \cdots & \alpha_{1m}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{11}\beta_{k1} & \cdots & \alpha_{11}\beta_{kk} & & \alpha_{1m}\beta_{k1} & \cdots & \alpha_{1m}\beta_{kk} \\ & \vdots & & \ddots & & \vdots & \\ \alpha_{m1}\beta_{11} & \cdots & \alpha_{m1}\beta_{1k} & & \alpha_{mm}\beta_{11} & \cdots & \alpha_{mm}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{k1} & \cdots & \alpha_{m1}\beta_{kk} & & \alpha_{mm}\beta_{k1} & \cdots & \alpha_{mm}\beta_{kk} \end{pmatrix}$$

Tensor products of matrices

Example

$$
\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \otimes \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}
= \begin{pmatrix}
\alpha_{00}\beta_{00} & \alpha_{00}\beta_{01} & \alpha_{01}\beta_{00} & \alpha_{01}\beta_{01} \\
\alpha_{00}\beta_{10} & \alpha_{00}\beta_{11} & \alpha_{01}\beta_{10} & \alpha_{01}\beta_{11} \\
\alpha_{10}\beta_{00} & \alpha_{10}\beta_{01} & \alpha_{11}\beta_{00} & \alpha_{11}\beta_{01} \\
\alpha_{10}\beta_{10} & \alpha_{10}\beta_{11} & \alpha_{11}\beta_{10} & \alpha_{11}\beta_{11}
\end{pmatrix}
$$

Tensor products of matrices

Tensor products of three or more matrices are defined in an analogous way.

If $M_1, \ldots, M_n$ are matrices, then the tensor product $M_1 \otimes \cdots \otimes M_n$ is defined by the condition

$$\langle a_1 \cdots a_n | M_1 \otimes \cdots \otimes M_n | b_1 \cdots b_n \rangle = \langle a_1 | M_1 | b_1 \rangle \cdots \langle a_n | M_n | b_n \rangle$$

Alternatively, the tensor product of three or more matrices can be defined recursively, similar to what we observed for vectors.

The tensor product of matrices is *multiplicative:*

$$(M_1 \otimes \cdots \otimes M_n)(N_1 \otimes \cdots \otimes N_n) = (M_1 N_1) \otimes \cdots \otimes (M_n N_n)$$

Operations on probabilistic states

Question

Suppose we have two probabilistic operations, each on its own system, described by stochastic matrices:

1. M is an operation on X.
2. N is an operation on Y.

If we *simultaneously* perform the two operations, how do we describe the effect on the compound system (X, Y)?

Answer

The action is described by the tensor product $M \otimes N$.

Tensor products represent *independence* — this time between operations.

Operations on probabilistic states

Example

Recall this operation from Lesson 1:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Suppose this operation is performed on a bit X, and a NOT operation is (independently) performed on a second bit Y.

The combined operation on the compound system (X, Y) then has this matrix representation:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Operations on probabilistic states

A common situation that we encounter is one in which one operation is performed on one system and *nothing* is done to another system.

The same prescription is followed, noting that doing nothing is represented by the *identity matrix.*

Example

Resetting a bit X to the 0 state and doing nothing to a bit Y yields this operation on (X, Y):

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle|0\rangle + \frac{i}{2}|0\rangle|1\rangle - \frac{1}{2}|1\rangle|0\rangle - \frac{i}{2}|1\rangle|1\rangle$$

$$\frac{3}{5}|0\rangle|0\rangle - \frac{4}{5}|1\rangle|1\rangle$$

$$|0\rangle|1\rangle$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{i}{2}|0\rangle \otimes |1\rangle - \frac{1}{2}|1\rangle \otimes |0\rangle - \frac{i}{2}|1\rangle \otimes |1\rangle$$

$$\frac{3}{5}|0\rangle \otimes |0\rangle - \frac{4}{5}|1\rangle \otimes |1\rangle$$

$$|0\rangle \otimes |1\rangle$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X,Y) is

$$\{0,1\}\times\{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X,Y):

$$\frac{1}{2}|0\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} + \frac{i}{2}|0\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}} - \frac{1}{2}|1\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} - \frac{i}{2}|1\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

$$\frac{3}{5}|0\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} - \frac{4}{5}|1\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

$$|0\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

Quantum states

Tensor products of quantum state vectors are also quantum state vectors.

Let $|\phi\rangle$ be a quantum state vector of a system X and let $|\psi\rangle$ be a quantum state vector of a system Y. The tensor product

$$|\phi\rangle \otimes |\psi\rangle$$

is then a quantum state vector of the system (X, Y).

States of this form are called *product states.* They represent *independence* between the systems X and Y.

More generally, if $|\psi_1\rangle, \ldots, |\psi_n\rangle$ are quantum state vectors of systems $\mathsf{X}_1, \ldots, \mathsf{X}_n$, then

$$|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$$

is a quantum state vector representing a product state of the compound system $(\mathsf{X}_1, \ldots, \mathsf{X}_n)$.

Quantum states

Example

The quantum state vector

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

is an example of a product state:

$$\begin{aligned}\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle \\ &= \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)\end{aligned}$$

Quantum states

Example

The quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state.

Quantum states

Suppose it were possible to write

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\phi\rangle \otimes |\psi\rangle$$

It would then follow that

$$\langle 0|\phi\rangle\langle 1|\psi\rangle = \langle 01|\phi \otimes \psi\rangle = 0$$

implying that

$$\langle 0|\phi\rangle = 0 \text{ or } \langle 1|\psi\rangle = 0 \text{ (or both)}$$

This contradicts these equalities:

$$\langle 0|\phi\rangle\langle 0|\psi\rangle = \langle 00|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

$$\langle 1|\phi\rangle\langle 1|\psi\rangle = \langle 11|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

Example

The quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state.

Quantum states

The previous example of a quantum state vector is one of the four *Bell states,* which collectively form the *Bell basis.*

The Bell basis

$$
\begin{aligned}
|\phi^+\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
|\phi^-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
|\psi^+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|\psi^-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}
$$

Quantum states

Here are a couple of well-known examples of quantum state vectors for three-qubits.

GHZ state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

W state

$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Measurements

Measurements of compound systems work in the same way measurements of single systems — provided that all of the systems are measured.

If $|\psi\rangle$ a quantum state of a system $(X_1, \ldots, X_n)$, and every one of the systems is measured, then each n-tuple

$$(a_1, \ldots, a_n) \in \Sigma_1 \times \cdots \times \Sigma_n$$

(or string $a_1 \cdots a_n$) is obtained with probability

$$\left|\langle a_1 \cdots a_n | \psi \rangle\right|^2$$

Measurements

Measurements of compound systems work in the same way measurements of single systems — provided that all of the systems are measured.

Example

If the pair (X, Y) is in the quantum state

$$\frac{3}{5}|0\rangle|\heartsuit\rangle - \frac{4i}{5}|1\rangle|\spadesuit\rangle$$

then measuring both systems yields the outcome $(0, \heartsuit)$ with probability $9/25$ and the outcome $(1, \spadesuit)$ with probability $16/25$.

Measurements

Question

Suppose two systems (X, Y) are together in some *quantum* state.
What happens when we measure X and do nothing to Y?

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab} |ab\rangle$$

If both X and Y are measured, then each outcome $(a, b) \in \Sigma \times \Gamma$ appears with probability

$$\bigl|\langle ab|\psi\rangle\bigr|^2 = |\alpha_{ab}|^2$$

Measurements

Question

Suppose two systems (X, Y) are together in some *quantum* state.
What happens when we measure X and do nothing to Y?

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab} |ab\rangle$$

If just X is measured, the probability for each outcome $a \in \Sigma$ to appear must therefore be equal to

$$\Pr(\text{outcome is } a) = \sum_{b\in\Gamma} \big|\langle ab|\psi\rangle\big|^2 = \sum_{b\in\Gamma} |\alpha_{ab}|^2$$

Similar to the probabilistic setting, the quantum state of Y changes as a result...

Measurements

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab} |ab\rangle$$

We can express the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_a\rangle$$

where

$$|\phi_a\rangle = \sum_{b\in\Gamma} \alpha_{ab} |b\rangle$$

for each $a \in \Sigma$.

Measurements

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{a \in \Sigma} |a\rangle \otimes |\phi_a\rangle \qquad \text{where} \qquad |\phi_a\rangle = \sum_{b \in \Gamma} \alpha_{ab} |b\rangle$$

1. The probability to obtain each outcome $a \in \Sigma$ is

$$\Pr(\text{outcome is } a) = \sum_{b \in \Gamma} |\alpha_{ab}|^2 = \big\| |\phi_a\rangle \big\|^2$$

2. As a result of the standard basis measurement of X giving the outcome a, the quantum state of (X, Y) becomes

$$|a\rangle \otimes \frac{|\phi_a\rangle}{\big\| |\phi_a\rangle \big\|}$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and X is measured.

We begin by writing

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle \right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right)$$

and X is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

in which case the state of (X, Y) becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle}{\sqrt{\frac{3}{4}}} = |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \right)$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

and X is measured.

The probability for the measurement to result in the outcome 1 is

$$\left\| \frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

in which case the state of (X, Y) becomes

$$|1\rangle \otimes \frac{\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{1}{4}}} = |1\rangle \otimes \left(\frac{i}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and Y is measured.

We begin by writing

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

and Y is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{5}{8}}} = \left(\sqrt{\frac{4}{5}}|0\rangle + \frac{i}{\sqrt{5}}|1\rangle\right) \otimes |0\rangle$$

Measurements

Example

Suppose that (X,Y) is in the state

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

and Y is measured.

The probability for the measurement to result in the outcome 1 is

$$\left\| \frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

in which case the state of (X,Y) becomes

$$\frac{\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{3}{8}}} = \left(\sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle\right) \otimes |1\rangle$$

Unitary operations

Quantum operations on compound systems are represented by unitary matrices whose rows and columns correspond to the Cartesian product of the classical state sets of the individual systems.

Example

Suppose X has classical state set $\{1, 2, 3\}$ and Y has classical state set $\{0, 1\}$. This unitary matrix represents an operation on (X, Y):

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Unitary operations

The combined action of a collection of unitary operations applied independently to a collection of systems is represented by the *tensor product* of the unitary matrices.

That is, if $X_1, \ldots, X_n$ are quantum systems, $U_1, \ldots, U_n$ are unitary matrices representing operations on these systems, and the operations are performed independently on the systems, the combined action on $(X_1, \ldots, X_n)$ is represented by the matrix

$$U_1 \otimes \cdots \otimes U_n$$

In particular, if we perform a unitary operation U on a system X and do nothing to a system Y, the operation on (X, Y) we obtain is represented by the unitary matrix

$$U \otimes \mathbb{1} \quad \text{or alternatively} \quad U \otimes \mathbb{1}_Y$$

Unitary operations

The combined action of a collection of unitary operations applied independently to a collection of systems is represented by the *tensor product* of the unitary matrices.

That is, if $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are quantum systems, $U_1, \ldots, U_n$ are unitary matrices representing operations on these systems, and the operations are performed independently on the systems, the combined action on $(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ is represented by the matrix

$$U_1 \otimes \cdots \otimes U_n$$

In particular, if we perform a unitary operation V on a system Y and do nothing to a system X, the operation on (X, Y) we obtain is represented by the unitary matrix

$$\mathbb{1} \otimes V \quad \text{or alternatively} \quad \mathbb{1}_{\mathsf{X}} \otimes V$$

Unitary operations

Example

Suppose X and Y are qubits.

Performing a Hadamard operation on X and doing nothing to Y is equivalent to performing this unitary operation on (X, Y):

$$H \otimes \mathbb{1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Unitary operations

Example

Suppose X and Y are qubits.

Performing a Hadamard operation on Y and doing nothing to X is equivalent to performing this unitary operation on (X, Y):

$$\mathbb{1} \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

Suppose that X and Y are systems that share the same classical state set Σ. The *swap operation* on the pair (X, Y) exchange the contents of the two systems:

$$\mathrm{SWAP}|\phi \otimes \psi\rangle = |\psi \otimes \phi\rangle$$

It can be expressed using the Dirac notation as follows:

$$\mathrm{SWAP} = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

The swap operation can be expressed using the Dirac notation as follows:

$$\mathrm{SWAP} = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

For instance, when X and Y are qubits, we find that

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

$$\begin{aligned}
\mathrm{SWAP}|\phi^+\rangle &= |\phi^+\rangle \\
\mathrm{SWAP}|\phi^-\rangle &= |\phi^-\rangle \\
\mathrm{SWAP}|\psi^+\rangle &= |\psi^+\rangle \\
\mathrm{SWAP}|\psi^-\rangle &= -|\psi^-\rangle
\end{aligned}$$

$$\begin{aligned}
|\phi^+\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
|\phi^-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
|\psi^+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|\psi^-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-NOT operation (where the first qubit is the control):

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-NOT operation (where the second qubit is the control):

$$\mathbb{1} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-σ_z (or controlled-Z) operation:

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-σ_z (or controlled-Z) operation:

$$\mathbb{1} \otimes |0\rangle\langle 0| + \sigma_z \otimes |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Unitary operations

Example

A controlled-SWAP operation (on three qubits):

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This operation is also known as a *Fredkin operation* (or Fredkin gate).

Unitary operations

Example

A controlled-controlled-NOT operation (on three qubits):

$$
\begin{aligned}
|0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \bigl(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x\bigr) \\
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{aligned}
$$

This operation is better known as a *Toffoli operation* (or Toffoli gate).

Schrödinger's Cat Track 🐈

Superdense Coding

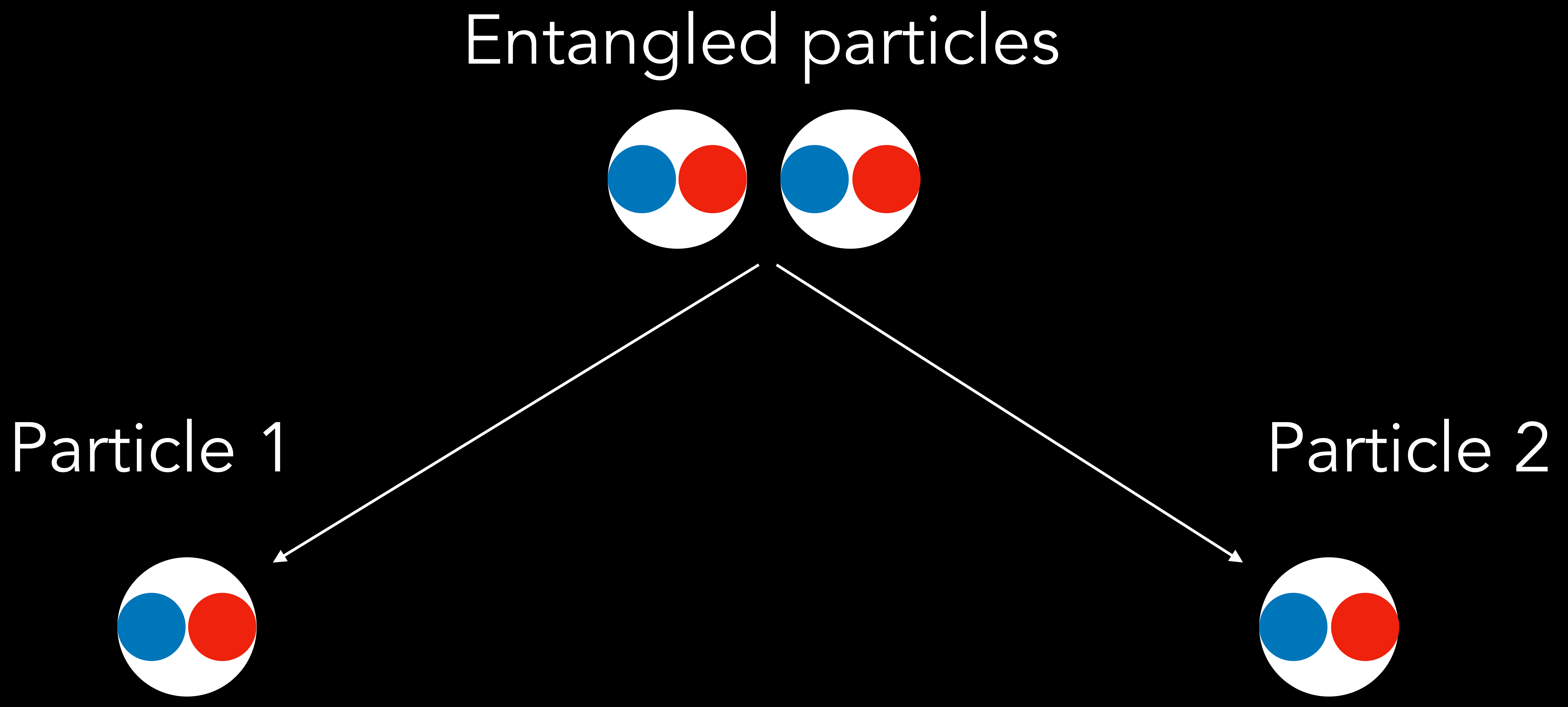
Alice is a Chimesmaster. She has climbed the Slope, and all 161 steps to get to the top of the tower.

She reaches into her pocket to find her phone when, no! She has left her phone at home, at the bottom of the hill in West Campus. 😱

But, deep inside her pocket, Alice finds an entangled particle! She knows that she can use it to communicate with her roommate Brenda on West.

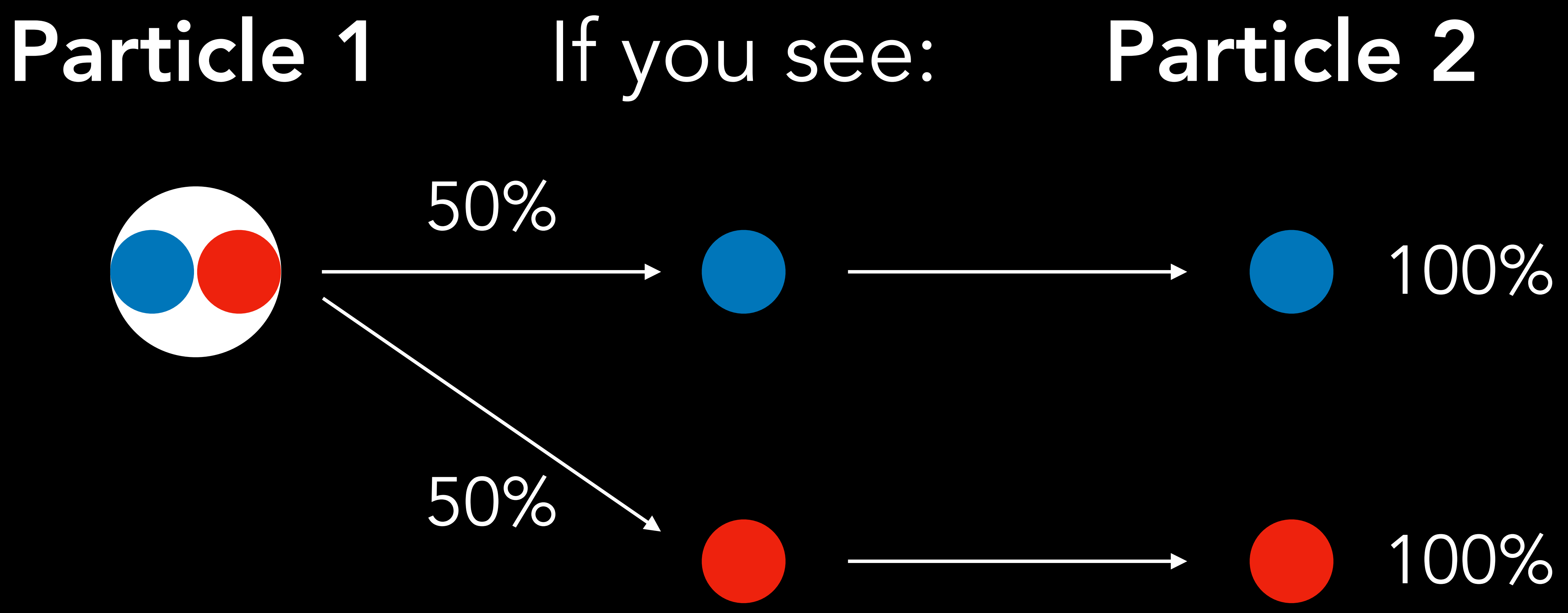
Can you help Alice use the entangled particle to send a message to Brenda?

Superdense Coding Background



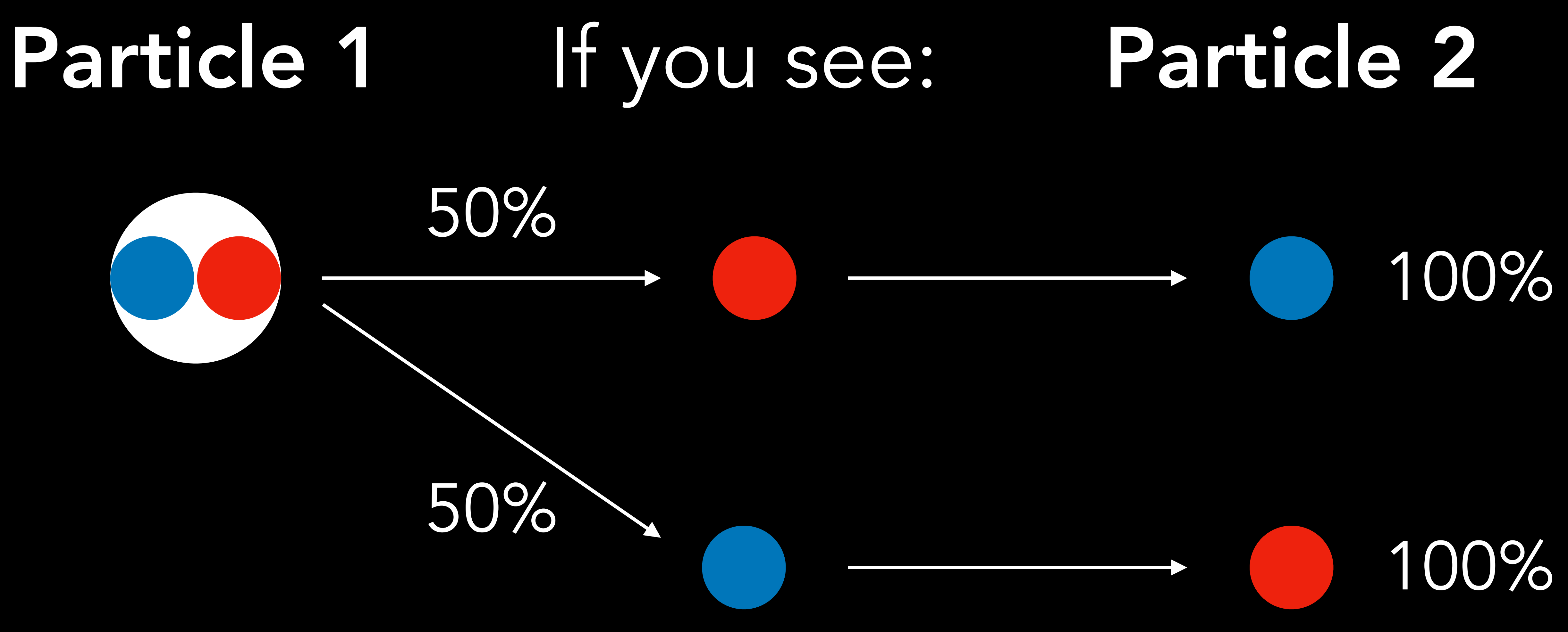
50-50 chance of finding blue or red on each particle

Entanglement Background



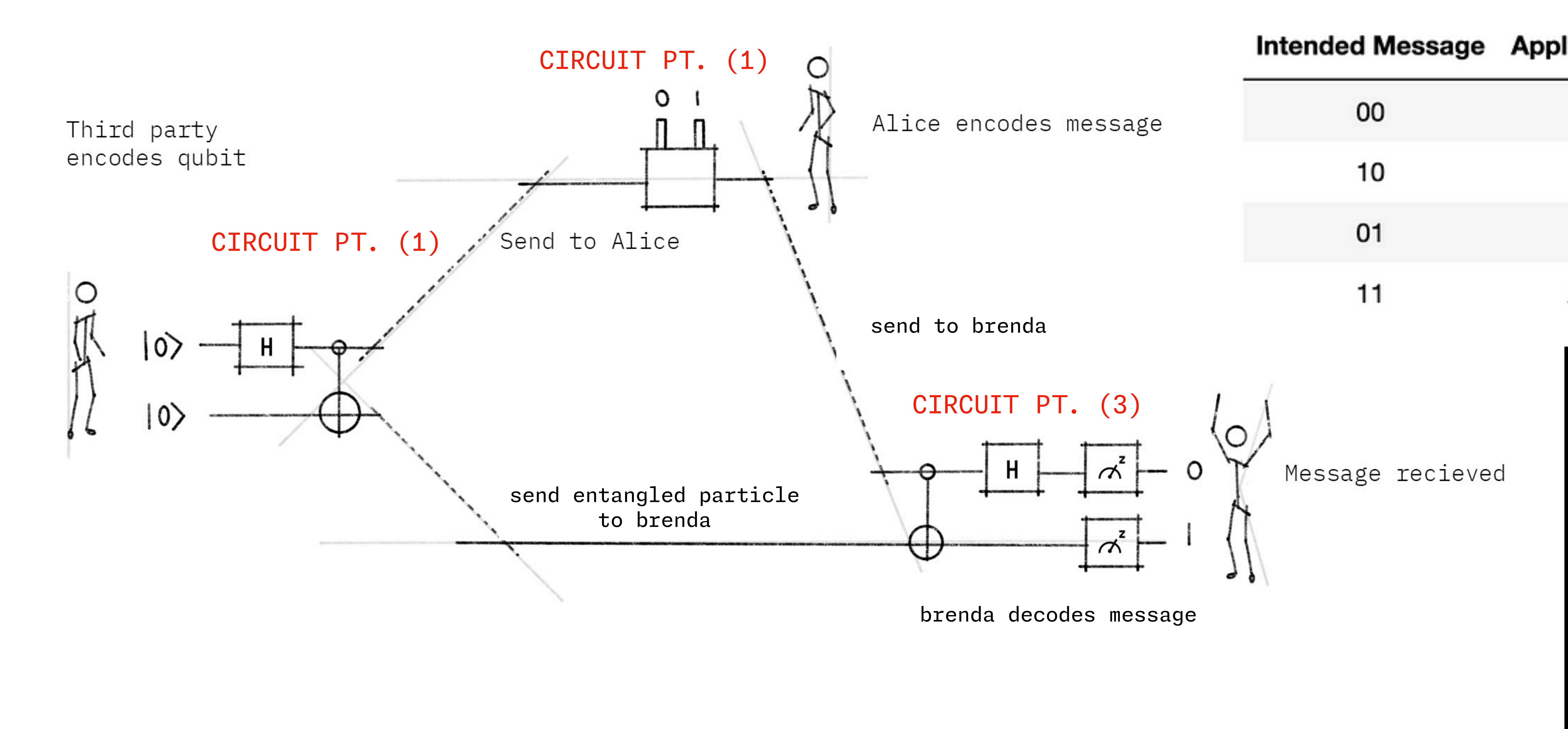
50-50 chance of finding blue or red on particle 1

Entanglement Background



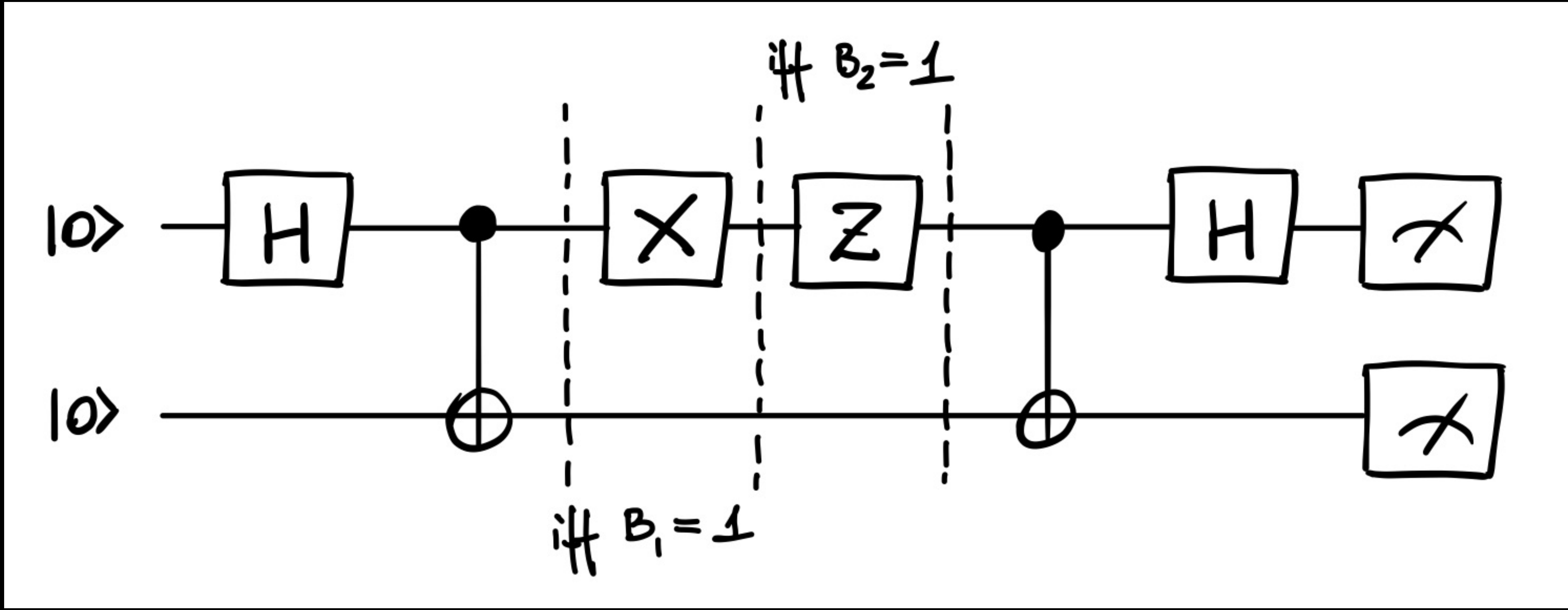
It turns out, it's possible to change the correlation between particles!

Superdense Coding Circuit



Intended Message	Applied Gate	Resulting State ($\cdot\sqrt{2}$)
00	I	$\|00\rangle + \|11\rangle$
10	X	$\|01\rangle + \|10\rangle$
01	Z	$\|00\rangle - \|11\rangle$
11	ZX	$-\|01\rangle + \|10\rangle$

Superdense Coding Circuit



Superdense Coding in 4 steps

$$|0\rangle_A |0\rangle_B$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left(Z_A^i X_A^j |0\rangle_A|0\rangle_B + Z_A^i X_A^j |1\rangle_A|1\rangle_B\right)$$

$$\downarrow$$

$$i,\ j$$

Superdense Coding

- Doing local actions on one part of an entangled state affects the overall entangled state!
- One can send **2 bits** of classical information if one has 1 entangled qubit pair and sends 1 qubit!

- How does this compare with a classical system?

Quantum Teleportation

Alice is now having lunch on North Campus and is doing a group project remotely with Brenda.

The professor has asked students to figure out how to teleport qubits to one another!

But wait, teleportation? How is that even possible?

Alice one of an entangled qubit pair, shared with Brenda.

Can you help Alice teleport her qubit to Brenda?

Quantum Teleportation Background

Not quite like this!

Start here **Want thing here**

Start.

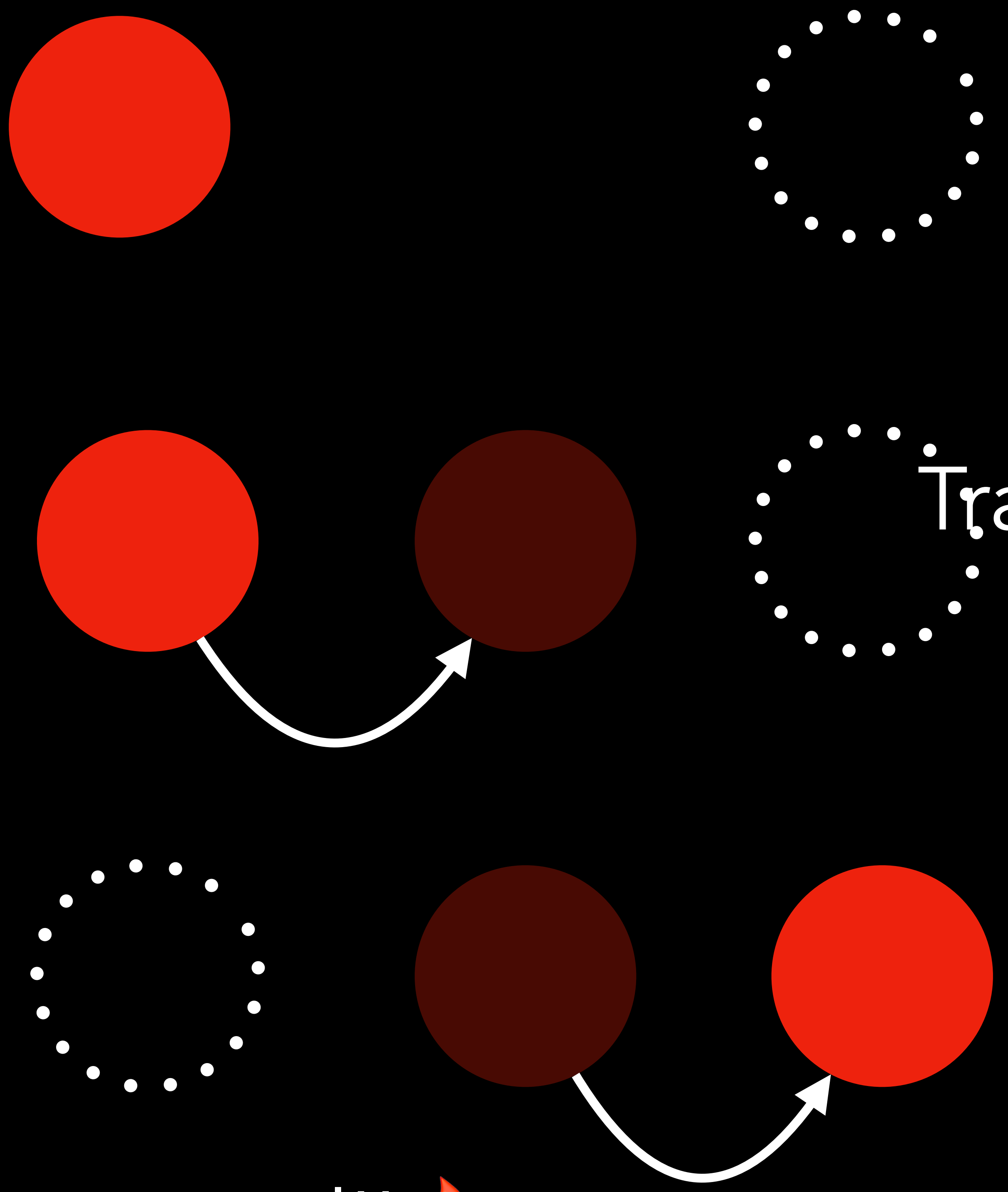
Step 1.

Transfer information to temporary holder

Step 2.

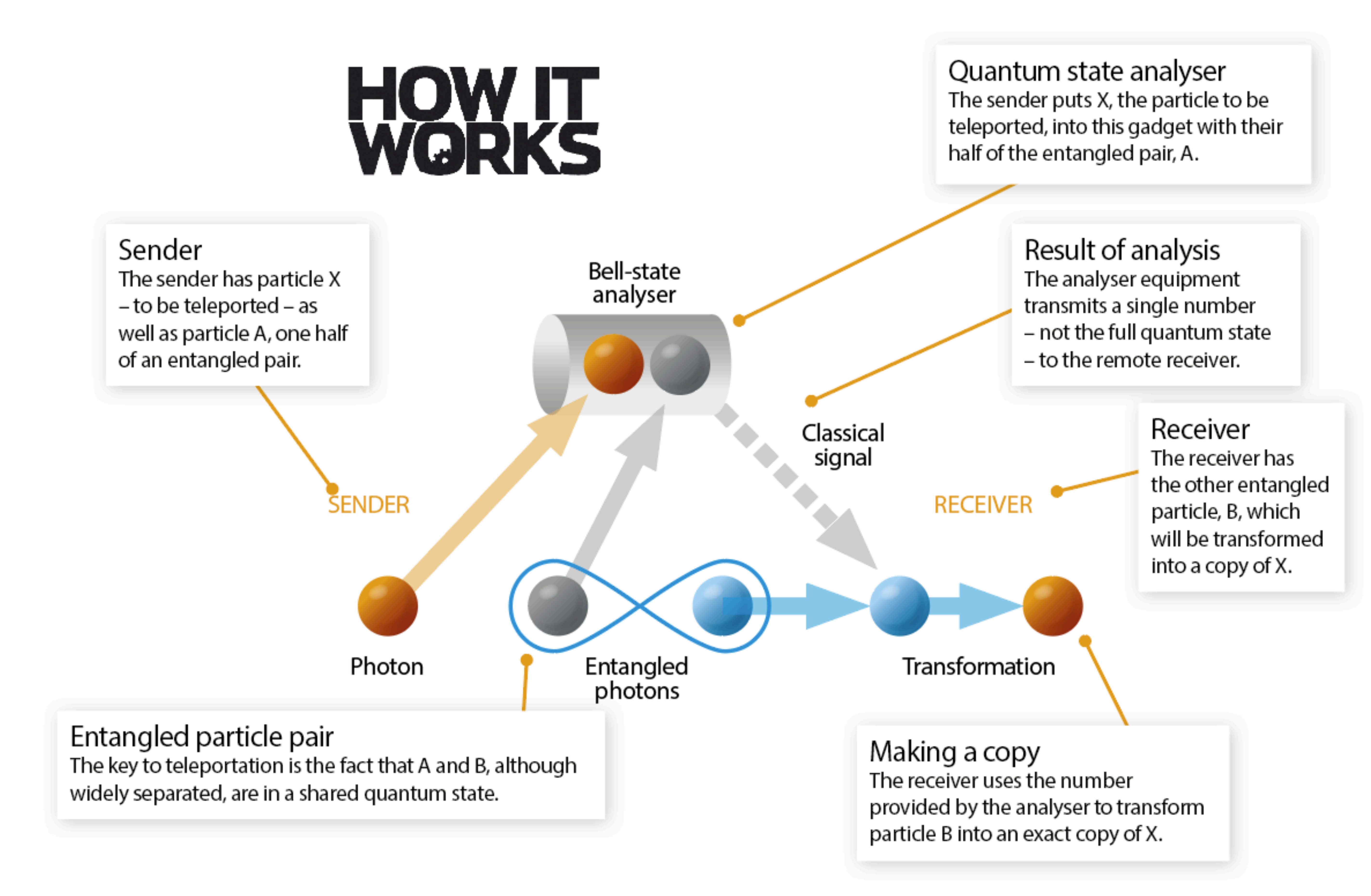
Re-born thing at destination🐣

Destroy initial!!🔥



Quantum Teleportation Background

Not quite like this!

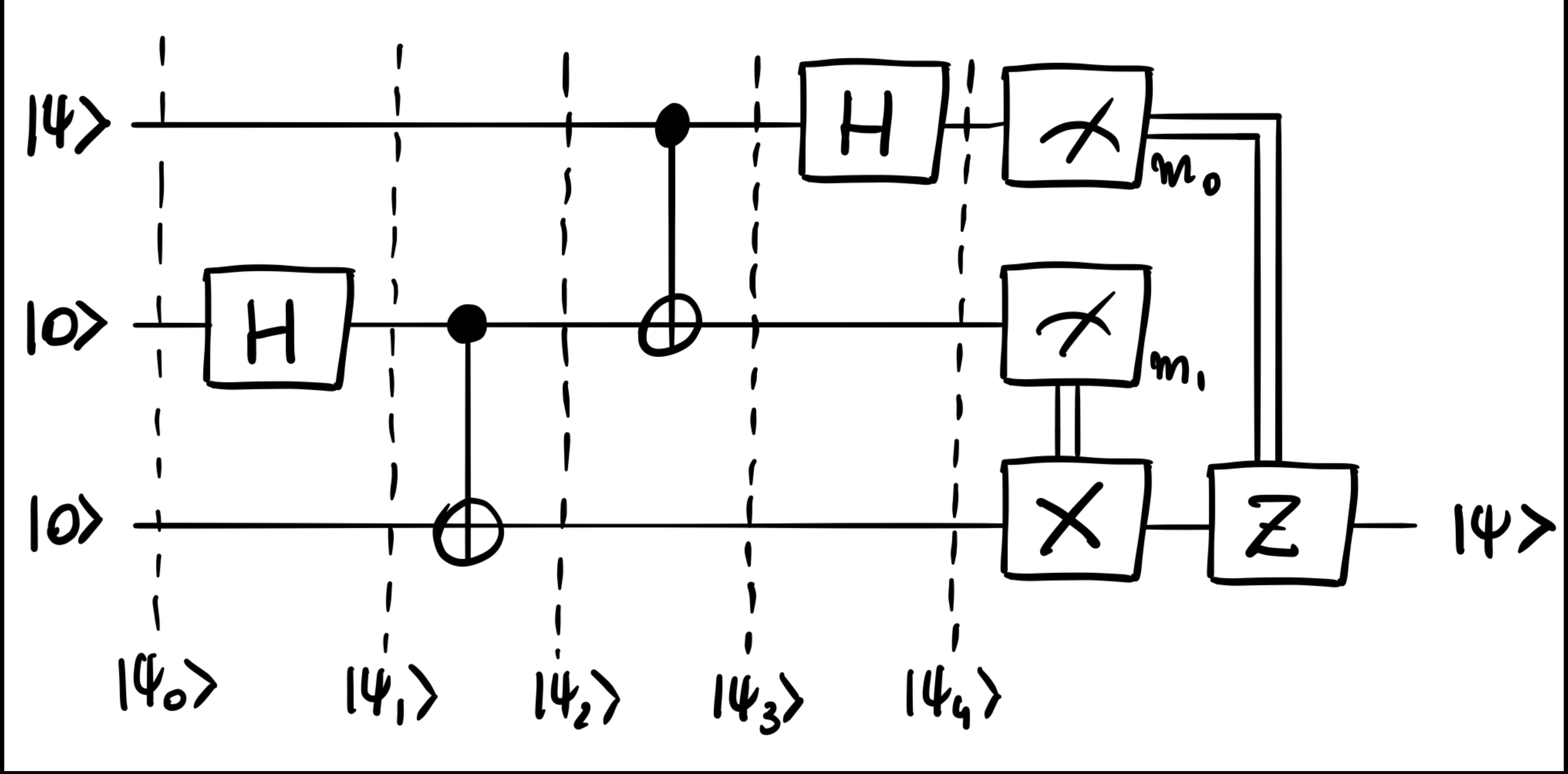


Quantum Teleportation Circuit

Alice’s data qubit

Alice’s entangled qubit

Brenda’s entangled qubit



Superposition Track 🔥