Exercise: Maximum Cut

Session Outline

- Introduce the maximum cut example
- Exercise: develop a QUBO for the maximum cut problem
- Review the solution

Session Goals

1. Practice formulating a QUBO

Problem: Maximum Cut

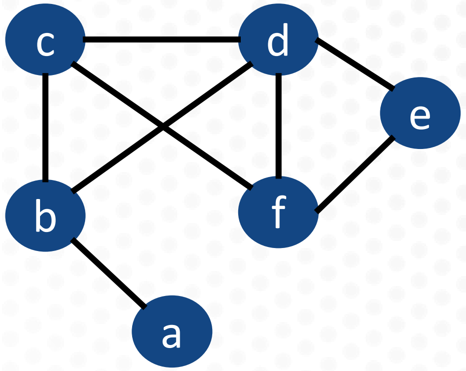
Example: Maximum Cut Problem

Problem

The maximum cut problem seeks to cut through the maximum amount of edges in a graph.

Another way of saying this is:

A maximum cut is a subset of a graph's vertices such that the number of edges between this subset and the remaining vertices is as large as possible



Example: Maximum Cut Problem

Problem

The maximum cut problem seeks to cut through the maximum amount of edges in a graph.

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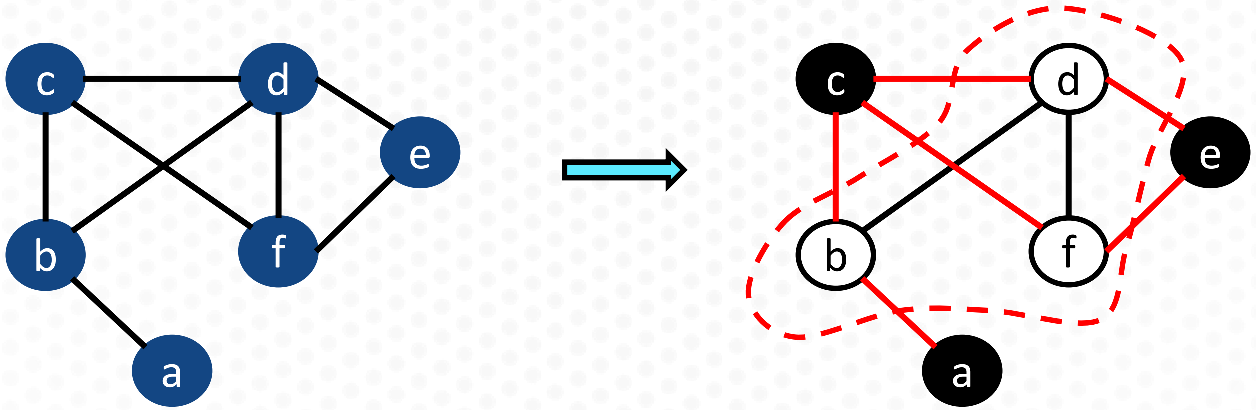


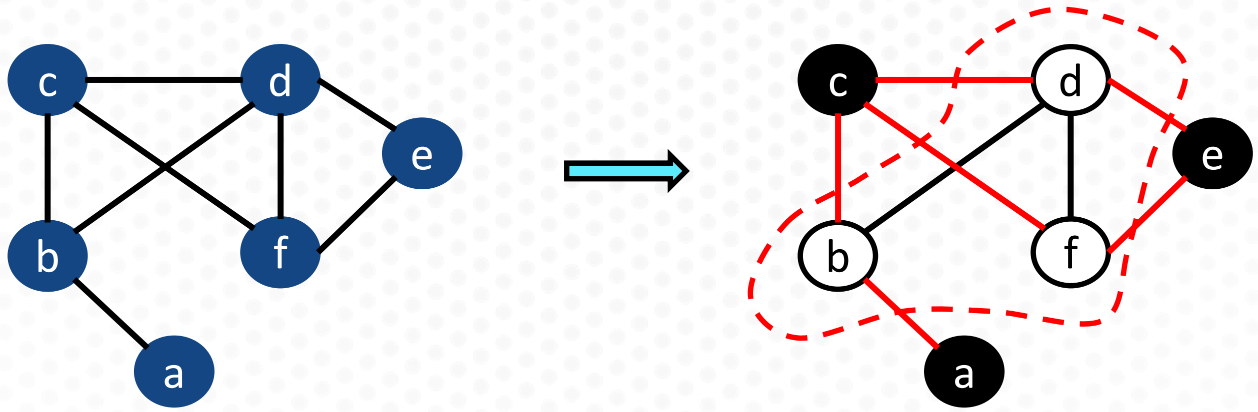
Example: Maximum Cut Problem

Problem

The maximum cut problem seeks to cut through the maximum amount of edges in a graph.

Exercise

Follow the QUBO formulation steps to write a QUBO that finds the subset of the graph below that cuts through a maximum amount of edges.



Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

QUBO Writing Process

1. Write out the objective and constraints in your problem domain
2. Define the binary variables
3. Write out objective in QUBO form
4. Write out constraints in QUBO form
5. Combine objectives and constraints
6. Solve and interpret results
7. Tune your QUBO to get better results

Example: Maximum Cut Problem

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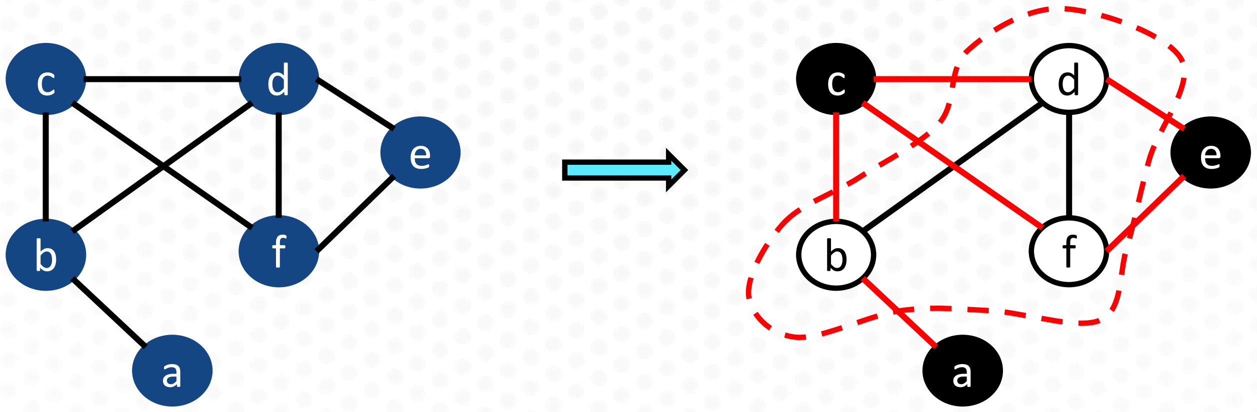
The maximum cut problem seeks to cut through the maximum amount of edges in a graph.

Exercise

Follow the QUBO formulation steps to write a QUBO that finds the subset of the graph below that cuts through a maximum amount of edges.

Hint

In this domain you're working with the graph's edges (whereas in the number partitioning problem you were thinking about the sums of numbers). You want edges in the same set to increase the QUBO's energy.



Maximum Cut Example

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 1. Write out the objective and constraints in your problem domain

Objective:
Maximize the number of cut edges

Constraints:
No constraints this time

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 2. Define the binary values

$$E_{qubo} = \sum_i a_i x_i + \sum_i b_{i,j} x_i x_j$$

We're working in QUBO so our binary variables are $x_i \in \{0, 1\}$

Let's define them as

$$x_i = \begin{cases} 1 & if\ i\ is\ in\ Set\ A \\ 0 & if\ i\ is\ in\ Set\ B \end{cases}$$

Example: Maximum Cut Problem

Problem

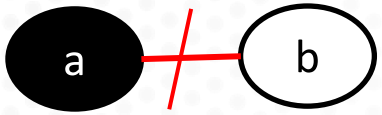
Partition the set so that the partition cuts through a maximum number of edges

Step 3. Write out the objective in QUBO form

If we cut an edge, the nodes it was connecting will be in opposite sets.

Since we want to maximize cut edges we want to favor edges between nodes with opposite values.

a	b	Cut Edge(a, b)
0	0	0
0	1	1
1	0	1
1	1	0



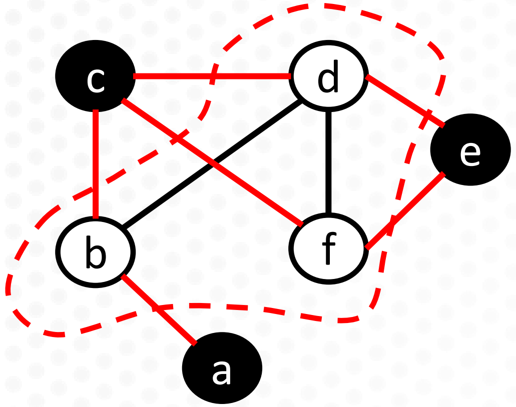
Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 3. Write out the objective in QUBO form

X_i	X_j	edge(i, j)
0	0	0
0	1	1
1	0	1
1	1	0



Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 3. Write out the objective in QUBO form

Construct a system of equations using

$$ax_i + bx_j + cx_ix_j$$

Truth table

Eqn	X_i	X_j	Cut Edge (i, j)
①	0	0	0
②	0	1	1
③	1	0	1
④	1	1	0

Example: Maximum Cut Problem

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③	1	0	1
④	1	1	0

① 0 + 0 + 0 = 0

② 0 + b + 0 = 1 → b = 1

③ a + 0 + 0 = 1 → a = 1

④ a + b + c = 0 → c = -2

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

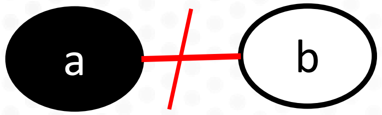
Step 3. Write out the objective in QUBO form

Construct a system of equations using

$$ax_i + bx_j + cx_ix_j$$

To describe two nodes we get

$$x_i + x_j - 2x_ix_j$$



Example: Maximum Cut Problem

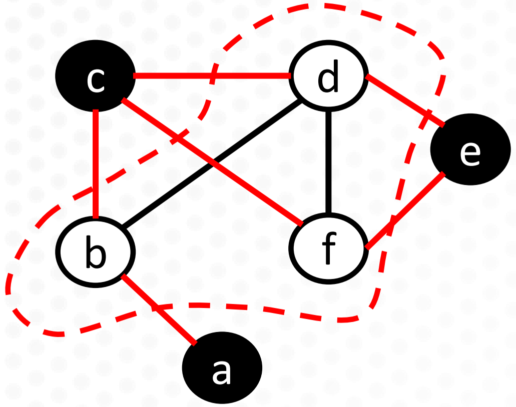
Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 3. Write out the objective in QUBO form

To describe our entire graph we can sum up our relationship over every node and edge

$$\sum_{(i,j)\in E} x_i + x_j - 2x_i x_j$$



where
E = {(a,b), (b,c), (b,d), (c,d), (c,f), (d,e), (d,f), (e,f)}

Example: Maximum Cut Problem

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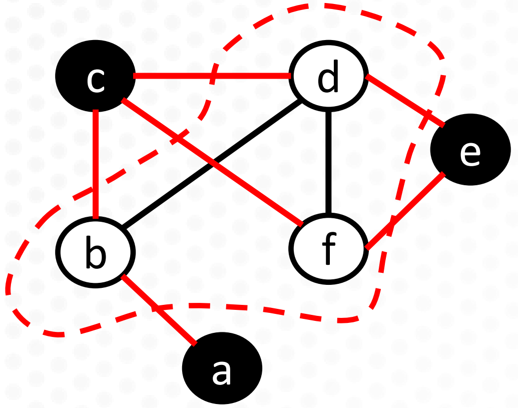
Partition the set so that the partition cuts through a maximum number of edges

Step 3. Write out the objective in QUBO form

Since the QPU naturally finds the minimum of a landscape, we change our objective function to minimize instead of maximize

$$\boxed{-\sum_{(i,j)\in E} x_i + x_j - 2x_ix_j}$$



where
E = {(a,b), (b,c), (b,d), (c,d), (c,f), (d,e), (d,f), (e,f)}

Example: Maximum Cut Problem

Problem

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Step 4. Write out the constraints in QUBO form

There aren't any constraints in this problem so we skip this step

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 5. Combine objectives and constraints

$$E_{qubo} = \min(objective) + \gamma(constraints)$$

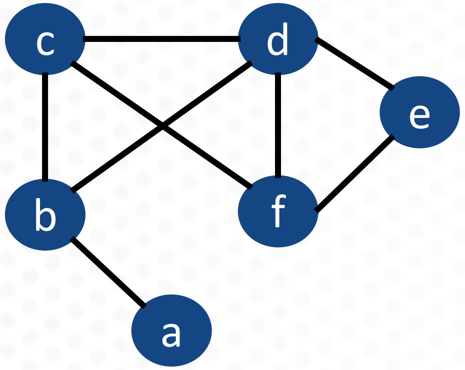
$$E_{qubo} = \sum_{(i,j)\in E} -x_i - x_j + 2x_i\, x_j$$

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 5. Combine objectives and constraints



$$E_{qubo} = \sum_{(i,j)\in E} -x_i - x_j + 2x_i\, x_j$$

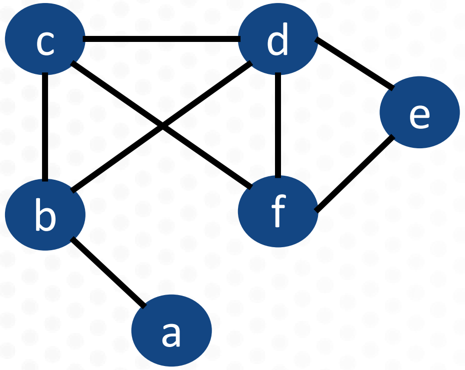
where
E = {(a,b), (b,c), (b,d), (c,d), (c,f), (d,e), (d,f), (e,f)}

Need to expand this for our problem

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges



Step 5. Combine objectives and constraints

$$E_{qubo} = \sum_{(i,j)\in E} -x_i - x_j + 2x_i\, x_j$$

Let's expand this for our problem:

$$\begin{aligned} E_{qubo} = {} & (-x_a - x_b + 2x_a\, x_b) + (-x_b - x_c + 2x_b\, x_c) + \\ & (-x_b - x_d + 2x_b\, x_d) + (-x_c - x_d + 2x_c\, x_d) + \\ & (-x_c - x_f + 2x_c\, x_f) + \ldots.. \end{aligned}$$

Example: Maximum Cut Problem

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Partition the set so that the partition cuts through a maximum number of edges

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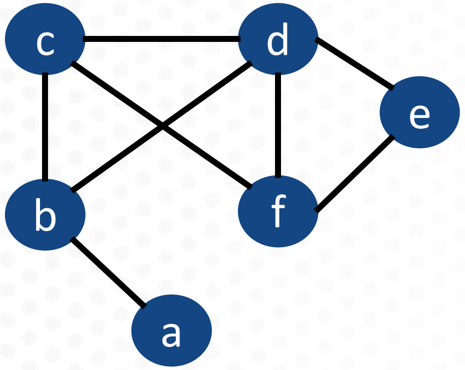


$$E_{qubo} = \sum_{(i,j)\in E} -x_i - x_j + 2x_i\, x_j$$

Let's expand this for our problem:

$$E_{qubo} = (-x_a - x_b + 2x_a\, x_b) + (-x_b - x_c + 2x_b\, x_c) + (-x_b - x_d + 2x_b\, x_d) + (-x_c - x_d + 2x_c\, x_d) + (-x_c - x_f + 2x_c\, x_f) + \ldots..$$

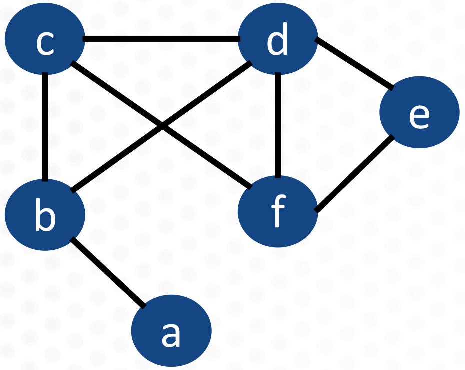
$$E_{qubo} = -x_a - 3x_b - 3x_c + \ \ldots + 2x_a\, x_b + 2x_b\, x_c + 2x_b\, x_d + \ \ldots$$

Example: Maximum Cut Problem

Problem

Partition the set so that the partition cuts through a maximum number of edges

Step 5. Combine objectives and constraints



$$E_{qubo} = \sum_{(i,j)\in E} -x_i - x_j + 2x_i\, x_j$$

Let's expand this for our problem:

$$E_{qubo} = (-x_a - x_b + 2x_a\, x_b) + (-x_b - x_c + 2x_b\, x_c) + (-x_b - x_d + 2x_b\, x_d) + (-x_c - x_d + 2x_c\, x_d) + (-x_c - x_f + 2x_c\, x_f) + \ldots..$$

The coefficients = - (the number of edges connected to the node)

$$E_{qubo} = \boxed{-x_a - 3x_b - 3x_c + \ \ldots} + 2x_a\, x_b + 2x_b\, x_c + 2x_b\, x_d + \ \ldots$$

Quadratic biases are always 2

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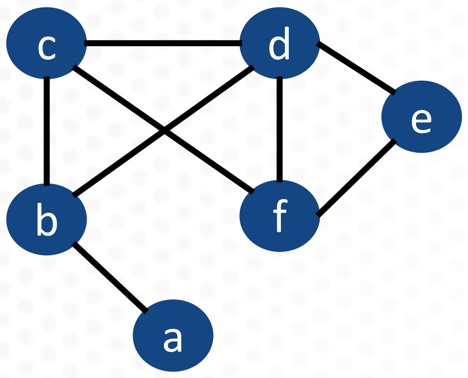

Let's expand this for our problem:

$$E_{qubo} = \begin{array}{cc} \begin{array}{cccccc} \;\;a & \;\;\;b & \;\;\;c & \;\;\;d & \;\;e & \;\;f \end{array} & \\ \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ & -3 & 2 & 2 & 0 & 0 \\ & & -3 & 2 & 0 & 2 \\ & & & -4 & 2 & 2 \\ & & & & -2 & 2 \\ & & & & & -3 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \end{array}$$

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Step 6. Solve and interpret results

You've got this (later today)!