CV 1

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Quantum harmonic oscillator (QHO)

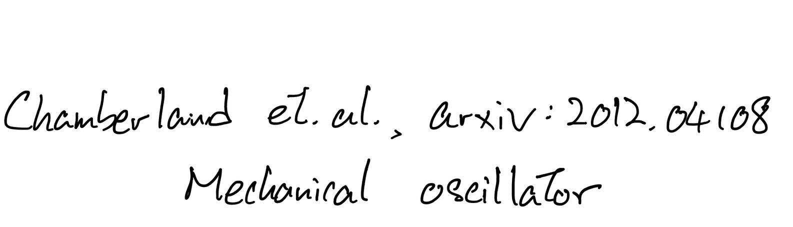
You should have learnt in quantum mechanics course that there is a class of quantum system called harmonic oscillator.
The time-independent Schrödinger equation is given by

$$E_n \Phi_n(q) = \underbrace{-\frac{1}{2}\frac{\partial^2}{\partial q^2}}_{\text{momentum}^2}\Phi_n(q) + \underbrace{\frac{1}{2}q^2}_{\text{harmonic potential}}\Phi_n(q) \qquad -(1)$$

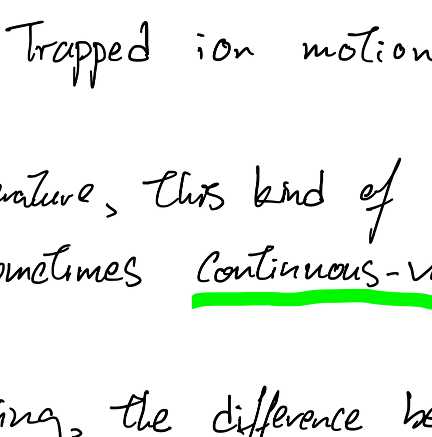
We are interested in this class of system not only because it is exactly solvable, but more importantly it is everywhere.
In fact, for most potential $V(q)$, the system behaves as QHO around equilibrium.
i.e. $$V(q) \approx \underbrace{V(q_0)}_{\text{constant}} + \underbrace{V'(q_0)}_{\substack{\text{vanish when}\\ q_0 \text{ is equilibrium}}}(q-q_0) + \underbrace{\frac{1}{2}V''(q_0)(q-q_0)^2}_{\text{harmonic potential}} + \dots$$

Indeed, several of the promising quantum computing architecture belong to this class:

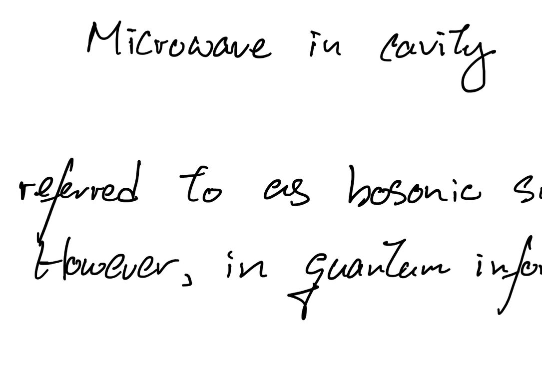
Arrazola et. al., Nature 2021
Photon



Chamberland et. al., arxiv: 2012.04108
Mechanical oscillator



Shen et al., Chem. Sci 2017
Trapped ion motion



Gao et. al., Nature 2019
Microwave in cavity

In literature, this kind of system could also be referred to as bosonic system, or sometimes continuous-variable (CV) system. However, in quantum information processing, the difference between CV and discrete-variable (DV, e.g. qubit) could be subtle. Actually, bosonic system can also be used to encode qubit, this has been a hot research topic under the name of bosonic quantum error correction code. Due to the limited time, I will just focus on the type that the information is CV. More explicitly, I consider quantum information that is represented by the amplitude of continuous-variable eigenstates.

Orthogonal basis state of QHO

To understand this idea, we consider the energy eigenstates of Eq. (1).
From standard QM textbook, we know

$$\Phi_n(q) = \frac{1}{\sqrt{2^n n!}} e^{-\frac{q^2}{2}} H_n(q)$$

H_n ← Hermite polynomial

These wavefunctions correspond to eigenstates of QHO, which are also continuous function of "position", q. Here you have a first feeling how "quantum" inhabits some continuous value.
To be more specific, let's first consider the "more advanced" way to represent QHO eigenstates, i.e. Dirac notation.
The Hamiltonian of a QHO is given by (up to unimportant constant)

$$H = \hat{a}^\dagger \hat{a}$$

where the eigenstates are $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ and eigenvalues $\{0, 1, 2, \dots\}$
a and $a^\dagger$ are the annihilation and creation operators, which satisfy

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n-1\rangle = \sqrt{n}|n\rangle$$

It is easy to check the commutation relation: $[a, a^\dagger] = 1$
The "position" and "momentum" operators can be expressed as

$$\hat{q} \equiv \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}, \quad \hat{p} \equiv \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}, \quad \text{where } [q, p] = i$$

We note that q and p represent different physical quantities for different system, e.g. they are electric field and vector potential for EM wave, but we just call them position and momentum, or q quadrature and p quadrature.

Under this formalism, we can ask a question:
How does a general QHO state, $|\psi\rangle$ looks like?
Ans: $$|\psi\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle + \dots \qquad -(2)$$
We can use $|\psi\rangle$ to encode quantum information, and in this case the information is naturally represented by the amplitudes $\{C_0, C_1, \dots\}$

Comparison with qudit

When comparing to qubit/DV quantum computer, it seems QHO is different only in the number of energy levels/orthogonal basis state. Isn't it simply a qudit with $d \to \infty$?
First of all, indeed, using a QHO does not necessarily mean the QI is CV. The representation in Eq. (2) is a DV encoding of QI.
However, before we go to the true CV QI, we emphasize two crucial difference between QHO and qudit.
First, qudit usually involves some "loop" structure that $|d-1\rangle \to |0\rangle$ is as easy as $|0\rangle \to |1\rangle$. In QHO, mapping an infinite photon state to vacuum state is not realistic.
Second, qudit computation involves generalized qubit operators.
e.g. $\hat{X} = \sum_k |k\rangle\langle k+1| + h.c.$. In QHO, physically realizable operations are usually certain power of a and $a^\dagger$, i.e.

$$\hat{a} = \sum_{k=1} \sqrt{k}\,|k-1\rangle\langle k|, \quad \hat{a}^\dagger = \sum_{k=1} \sqrt{k}\,|k\rangle\langle k-1|.$$

Therefore, the transition amplitude between basis states have strong physical relation and limitation, and it is generally difficult to engineer arbitrary transition as wish.

CV Quantum Information

We first note that there can be infinitely different choices of basis states of a Hilbert space. For QHO, Eq. (2) chose a specific basis: the number states, also called the Fock states, which have discrete eigenvalues (with respect to $\hat{a}^\dagger\hat{a}$). There are nonetheless other choice of basis that the basis states have continuous eigenvalues. One such example is the position eigenstate:

$$\hat{q}|q_0\rangle = q_0|q_0\rangle \qquad -(3)$$

Because position is a continuous quantity, q_0 is also continuous.
Just like $|n\rangle$ means the QHO has exactly n boson, $|q_0\rangle$ means the QHO is exactly located at q_0.
The position eigenstates form a complete, orthogonal set of QHO basis state:

$$\langle q|q'\rangle = \underbrace{\delta(q-q')}_{\text{Dirac delta function}} \qquad -(4)$$

Note that $\{|q\rangle\}$ are normalized by delta function, which is different from DV states that are normalized by unity, but they are nonetheless normalizable.
By using the position basis, we can then express any QHO state as:

$$|\psi\rangle = \int \psi(q)|q\rangle dq, \quad \text{where } \int|\psi(q)|^2 dq = 1. \qquad -(5)$$

$\psi(q)$ ← wave function

If we can engineer the wavefunction, we can use it to encode a continuous complex function of information into the QHO state, and that is what we call continuous variable quantum information.

Before we move on to the computational part, I add a few remarks:

1. The wavefunction in Eq. (5) is indeed the wavefunction in the differential form of Schrödinger equation, i.e. $\psi(q) = \langle q|\psi\rangle$.
2. When we conduct a (homodyne) measurement in q basis, the probability of getting an outcome between q_0 and $q_0 + \delta q$ is $|\psi(q_0)|^2 \delta q$.
   It sounds obvious, but it has important implication in quantum algorithms that involve post-selection. If your algorithm succeeds only when $q_0 = 0$, it would mean the algorithm has infinitesimally small chance to succeed.
3. We can easily generalize the picture to multiple QHOs, or what we call "modes". For example, a general two-mode state is given by
   $$|\Psi\rangle = \int \Psi(q_1, q_2)|q_1\rangle|q_2\rangle dq_1 dq_2$$
   Here $|q_1\rangle$ and $|q_2\rangle$ are respectively the position eigenstate of mode 1 and 2.
   Note that generally $\int \Psi(q_1,q_2)|q_1\rangle|q_2\rangle dq_1 dq_2 \neq \left(\int \Psi_1(q_1)|q_1\rangle dq_1\right)\left(\int \Psi_2(q_2)|q_2\rangle dq_2\right)$
   because $|\Psi\rangle$ can be entangled!
4. Position basis is not the only continuous basis set. We can also use the momentum basis $\{|p\rangle\}$, where $\hat{p}|p_0\rangle = p_0|p_0\rangle$.
   The wavefunction can be expressed as $|\psi\rangle = \int \psi(q)|q\rangle dq = \int \tilde{\psi}(p)|p\rangle dp$, where $\tilde{\psi}(p)$ is the Fourier transform of $\psi(q)$ (HW)