Solution

1a. Given $|\psi\rangle = \int \psi(q)|q\rangle dq = \int \tilde{\psi}(p)|p\rangle dp$

Consider $\langle p|q\rangle = \frac{1}{\sqrt{2\pi}} e^{-iqp}$ and we can rewrite

$$\int \psi(q)|q\rangle dq = \int \psi(q)\ \hat{I}\ |q\rangle dq = \int \psi(q) \int |p\rangle\langle p|\, dp\, |q\rangle dq$$

$$= \iint \psi(q) \langle p|q\rangle\, |p\rangle\, dp\, dq$$

$$= \int \underbrace{\int \psi(q) \frac{1}{\sqrt{2\pi}} e^{-iqp}\, dq}\ |p\rangle dp$$

$$= \int \tilde{\psi}(p)\,|p\rangle dp$$

So $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(q) e^{-iqp} dq$ #

1.b. Consider $|\psi_0\rangle = \int \frac{1}{\sqrt[4]{\pi}} e^{-\frac{q^2}{2}} |q\rangle dq$

$$P(q\geq 2) = \int_2^\infty \langle\psi_0|q\rangle\langle q|\psi_0\rangle\, dq = \frac{1}{\sqrt{\pi}} \int_2^\infty \int e^{-\frac{q'^2}{2}} \langle q'|q\rangle dq' \int e^{-\frac{q''^2}{2}} \langle q|q''\rangle dq''\ dq$$

$$= \frac{1}{\sqrt{\pi}} \int_2^\infty \iint e^{-\frac{q'^2 - q''^2}{2}} \delta(q'-q)\, \delta(q''-q)\, dq' dq''\ dq = \frac{1}{\sqrt{\pi}} \int_2^\infty e^{-q^2} dq = 0.00233887 //$$

2.a. For the expectation value:

We have $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

Considering $\langle\alpha|\,\hat{a}\,|\alpha\rangle = \langle\alpha|\left(\frac{\hat{q}+i\hat{p}}{\sqrt{2}}\right)|\alpha\rangle = \frac{\langle\hat{q}\rangle + i\langle\hat{p}\rangle}{\sqrt{2}} = \alpha = \frac{q_\alpha + i p_\alpha}{\sqrt{2}}$

$\Rightarrow$ Real part: $\langle\hat{q}\rangle = q_\alpha$ //, imaginary part: $\langle\hat{p}\rangle = p_\alpha$ //

For the variances, we have $\Delta q = \hat{q} - \langle\hat{q}\rangle$, $\Delta p = \hat{p} - \langle\hat{p}\rangle$

our target
$$\begin{cases} \langle\Delta q^2\rangle = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 \\ \langle\Delta p^2\rangle = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 \\ \left\langle\frac{\Delta q\Delta p + \Delta p\Delta q}{2}\right\rangle = \frac{\langle\hat{q}\hat{p} + \hat{p}\hat{q}\rangle}{2} - \langle\hat{q}\rangle\langle\hat{p}\rangle \end{cases}$$

(We have $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ from previous calculation)

In order to obtain $\langle q^2\rangle$, $\langle p^2\rangle$ and $\langle\hat{q}\hat{p} + \hat{p}\hat{q}\rangle$, we can consider

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \langle\alpha|\alpha^*\alpha|\alpha\rangle = |\alpha|^2\langle\alpha|\alpha\rangle = |\alpha|^2 = \frac{q_\alpha^2 + p_\alpha^2}{2}$$

$$\Rightarrow \langle\alpha|\frac{(\hat{q}-i\hat{p})(\hat{q}+i\hat{p})}{2}|\alpha\rangle = \langle\alpha|\frac{\hat{q}^2 - i\hat{p}\hat{q} + i\hat{q}\hat{p} + \hat{p}^2}{2}|\alpha\rangle = \frac{\langle\hat{q}^2\rangle + \langle\hat{p}^2\rangle - 1}{2} = \frac{q_\alpha^2 + p_\alpha^2}{2} \quad \cdots (1)$$

Also

$$\langle\alpha|\hat{a}^2|\alpha\rangle = \alpha^2\langle\alpha|\alpha\rangle = \alpha^2 = \frac{q_\alpha^2 - p_\alpha^2 + 2i q_\alpha p_\alpha}{2}$$

$$= \langle\alpha|\frac{(\hat{q}+i\hat{p})(\hat{q}+i\hat{p})}{2}|\alpha\rangle = \frac{\langle\hat{q}^2\rangle - \langle\hat{p}^2\rangle + i\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle}{2}$$

Real part : $\dfrac{\langle\hat{q}^2\rangle - \langle\hat{p}^2\rangle}{2} = \dfrac{q_\alpha^2 - p_\alpha^2}{2}$ ... (2)

Imaginary part : $\dfrac{\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle}{2} = q_\alpha p_\alpha$

With (1) and (2), $\langle\hat{q}^2\rangle = q_\alpha^2 + \frac{1}{2}$, $\langle\hat{p}^2\rangle = p_\alpha^2 + \frac{1}{2}$

So $\langle\Delta q^2\rangle = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 = \frac{1}{2}$ //

$\langle\Delta p^2\rangle = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = \frac{1}{2}$ //

$$\left\langle\frac{\Delta q\Delta p + \Delta p\Delta q}{2}\right\rangle = \frac{\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle}{2} - \langle\hat{q}\rangle\langle\hat{p}\rangle = 0 \text{ //}$$

Or considering $\hat{q} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$ and $\hat{p} = \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$

$$\hat{q}^2 = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 1)$$

$$\therefore \langle\hat{q}^2\rangle = \frac{1}{2}(\alpha^2 + \alpha^{*2} + 2\alpha^*\alpha + 1) = \frac{1}{2}[(\overbrace{\alpha+\alpha^*}^{\sqrt{2}Re[\alpha]})^2 + 1]$$
$$= q_\alpha^2 + \frac{1}{2}$$

$$\hat{p}^2 = -\frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{a} - 1)$$

$$\therefore \langle\hat{p}^2\rangle = \frac{-1}{2}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha - 1) = -\frac{1}{2}[(\overbrace{\alpha-\alpha^*}^{i\sqrt{2}Im[\alpha]})^2 - 1]$$
$$= p_\alpha^2 + \frac{1}{2}$$

$$\hat{q}\hat{p} + \hat{p}\hat{q} = 2(\hat{a}^2 - \hat{a}^{\dagger 2})$$

$$\therefore \langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle = 2(\alpha^2 - \alpha^{*2}) = 2q_\alpha p_\alpha$$

2b. $|\alpha\rangle|\beta\rangle = \int\psi_\alpha(q_1)|q_1\rangle dq_1 \int\psi_\beta(q_2)|q_2\rangle dq_2 = \iint \psi_\alpha(q_1)\psi_\beta(q_2)\,|q_1\rangle|q_2\rangle dq_1dq_2$

So $\Psi(q_1,q_2) = \psi_\alpha(q_1)\psi_\beta(q_2)$

$$= \frac{1}{\sqrt[4]{\pi}} e^{-\frac{(q_1-q_\alpha)^2}{2}} e^{ip_\alpha q_1} \frac{1}{\sqrt[4]{\pi}} e^{-\frac{(q_2-q_\beta)^2}{2}} e^{ip_\beta q_2}$$

$$= \frac{1}{\sqrt{\pi}} \exp\left[-\frac{(q_1-q_\alpha)^2}{2} - \frac{(q_2-q_\beta)^2}{2}\right] \exp\left[ip_\alpha q_1 + ip_\beta q_2\right] \text{ //}$$

2c. We have $|\Psi'\rangle = \hat{B}(\theta)|\alpha\rangle|\beta\rangle$, $\hat{B}^\dagger(\theta)\hat{a}_1\hat{B}(\theta) = \cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2$ and $\hat{B}^\dagger(\theta)\hat{a}_2\hat{B}(\theta) = -\sin\theta\,\hat{a}_1 + \cos\theta\,\hat{a}_2$

Considering $\hat{a}_1|\Psi'\rangle = \hat{a}_1\hat{B}(\theta)|\alpha\rangle|\beta\rangle = \hat{B}(\theta)(\cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2)|\alpha\rangle|\beta\rangle = (\cos\theta\,\alpha + \sin\theta\,\beta)\underbrace{\hat{B}(\theta)|\alpha\rangle|\beta\rangle}_{|\Psi'\rangle}$

$\hat{a}_2|\Psi'\rangle = \hat{a}_2\hat{B}(\theta)|\alpha\rangle|\beta\rangle = \hat{B}(\theta)(-\sin\theta\,\hat{a}_1 + \cos\theta\,\hat{a}_2)|\alpha\rangle|\beta\rangle = (-\sin\theta\,\alpha + \cos\theta\,\beta)\overbrace{\hat{B}(\theta)|\alpha\rangle|\beta\rangle}^{|\Psi'\rangle}$

So $|\Psi'\rangle$ is eigenstate of $\hat{a}_1$ and $\hat{a}_2$ with eigenvalue $(\cos\theta\alpha + \sin\theta\beta)$ and $(-\sin\theta\alpha + \cos\theta\beta)$ respectively.

It implies $|\Psi'\rangle = |\alpha\cos\theta + \beta\sin\theta\rangle|-\alpha\sin\theta + \beta\cos\theta\rangle$ which is separable.

Alternative method : Consider $\hat{B}(\theta)|q_1\rangle|q_2\rangle = |\cos\theta q_1 + \sin\theta q_2\rangle|-\sin\theta q_1 + \cos\theta q_2\rangle$

$$|\Psi'\rangle = \hat{B}(\theta)|\alpha\rangle|\beta\rangle = \iint \Psi(q_1,q_2)\;\hat{B}(\theta)|q_1\rangle|q_2\rangle dq_1dq_2$$

$$= \iint \Psi(q_1,q_2)\,|\cos\theta q_1 + \sin\theta q_2\rangle|-\sin\theta q_1 + \cos\theta q_2\rangle dq_1dq_2$$

$$= \iint \underbrace{\left\|\begin{matrix}\frac{\partial q_1}{\partial q_+} & \frac{\partial q_1}{\partial q_-} \\ \frac{\partial q_2}{\partial q_+} & \frac{\partial q_2}{\partial q_-}\end{matrix}\right\| \Psi(q_1,q_2)}_{\Psi'(q_+,q_-)}\,|q_+\rangle|q_-\rangle dq_+dq_-$$

Here we let $q_+ = \cos\theta\, q_1 + \sin\theta\, q_2$ and $q_- = -\sin\theta\, q_1 + \cos\theta\, q_2$

$\Rightarrow$ $q_1 = \cos\theta\, q_+ - \sin\theta\, q_-$ and $q_2 = \sin\theta\, q_+ + \cos\theta\, q_-$

then $dq_1 dq_2 = \left\| \begin{vmatrix} \frac{\partial q_1}{\partial q_+} & \frac{\partial q_1}{\partial q_-} \\ \frac{\partial q_2}{\partial q_+} & \frac{\partial q_2}{\partial q_-} \end{vmatrix} \right\| dq_+ dq_- = |\cos^2\theta + \sin^2\theta|\, dq_+ dq_- = dq_+ dq_-$

And $\Psi(q_1, q_2) = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{(q_1 - q_\alpha)^2}{2} - \frac{(q_2 - q_\beta)^2}{2}\right] \exp\left[i p_\alpha q_1 + i p_\beta q_2\right]$

$$= \frac{1}{\sqrt{\pi}} \exp\left[\underbrace{-\frac{(q_+ - q_u)^2}{2}} \underbrace{-\frac{(q_- - q_v)^2}{2}}\right] \exp\left[\underbrace{i p_u q_+} + \underbrace{i p_v q_-}\right]$$

Separable in q_+ and q_-

where $q_u = q_\alpha \cos\theta + q_\beta \sin\theta$, $q_v = -q_\alpha \sin\theta + q_\beta \cos\theta$

$p_u = p_\alpha \cos\theta + p_\beta \sin\theta$, $p_v = -p_\alpha \sin\theta + p_\beta \cos\theta$

Then $\Psi'(q_+, q_-) = \left\| \begin{vmatrix} \frac{\partial q_1}{\partial q_+} & \frac{\partial q_1}{\partial q_-} \\ \frac{\partial q_2}{\partial q_+} & \frac{\partial q_2}{\partial q_-} \end{vmatrix} \right\| \Psi(q_1, q_2) = \phi_1(q_+)\, \phi_2(q_-)$ is separable.

3 a. since $\hat{D}^\dagger(\beta) \equiv \exp(\beta^* \hat{a} - \beta \hat{a}^\dagger)$, $\hat{X} = \beta^* \hat{a} - \beta \hat{a}^\dagger$

$[\hat{X}, \hat{a}] = [\beta^* \hat{a} - \beta \hat{a}^\dagger, \hat{a}] = \beta$, and then $[\hat{X}, [\hat{X}, \hat{a}]] = [\hat{X}, \beta] = 0$

By BCH formula,

$$\hat{D}^\dagger(\beta)\, \hat{a}\, \hat{D}(\beta) = \hat{a} + [\hat{X}, \hat{a}] = \hat{a} + \beta \quad \#$$

b. since $\hat{R}^\dagger(\phi) \equiv \exp(i\phi\, \hat{a}^\dagger \hat{a})$, $\hat{X} = i\phi\, \hat{a}^\dagger \hat{a}$

$[\hat{X}, \hat{a}] = i\phi [\hat{a}^\dagger \hat{a}, \hat{a}] = -i\phi\, \hat{a}$, $[\hat{X}, [\hat{X}, \hat{a}]] = -(i\phi)^2 [\hat{a}^\dagger \hat{a}, \hat{a}] = (i\phi)^2 \hat{a}$, ...

By BCH formula,

$$\hat{R}^\dagger(\phi)\, \hat{a}\, \hat{R}(\phi) = \hat{a} - i\phi\, \hat{a} + \frac{(i\phi)^2}{2!} \hat{a} - \frac{(i\phi)^3}{3!} \hat{a} + \cdots = e^{-i\phi}\, \hat{a} \quad \#$$

c. Since $\hat{S}^\dagger(r) \equiv \exp\left[\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]$, $\hat{X} = \frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})$

$[\hat{X}, \hat{a}] = \frac{r}{2}[\hat{a}^{\dagger 2}, a] = -r\, \hat{a}^\dagger$, $[\hat{X}, [\hat{X}, \hat{a}]] = \frac{r^2}{2}[\hat{a}^2, \hat{a}^\dagger] = r^2 \hat{a}$

$[\hat{X}, [\hat{X}, [\hat{X}, \hat{a}]]] = -r^3 \hat{a}^\dagger$, $[\hat{X}, [\hat{X}, [\hat{X}, [\hat{X}, \hat{a}]]]] = r^4\, \hat{a}$, ...

By BCH formula,

$$\hat{S}^{\dagger}(r)\hat{a}\hat{S}(r) = \hat{a} - r\hat{a}^{\dagger} + \frac{r^2}{2!}\hat{a} - \frac{r^3}{3!}\hat{a}^{\dagger} + \frac{r^4}{4!}\hat{a} + \cdots$$

$$= \cosh r\,\hat{a} - \sinh r\,\hat{a}^{\dagger} \quad ※$$

d. $\hat{B}^{\dagger}(\theta) \equiv \exp[-\theta(\hat{a}_1^{\dagger}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{a}_1)]$, $\hat{X} = -\theta(\hat{a}_1^{\dagger}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{a}_1)$

$[\hat{X},\hat{a}_1] = -\theta[\hat{a}_1^{\dagger}\hat{a}_2, \hat{a}_1] = \theta\hat{a}_2$, $[\hat{X},[\hat{X},\hat{a}_1]] = \theta^2[\hat{a}_2^{\dagger}\hat{a}_1, \hat{a}_2] = -\theta^2\hat{a}_1$, $[\hat{X},[\hat{X},[\hat{X},\hat{a}_1]]] = \theta^3[\hat{a}_1^{\dagger}\hat{a}_2, \hat{a}_1] = -\theta^3\hat{a}_2$

$[\hat{X},\hat{a}_2] = \theta[\hat{a}_2^{\dagger}\hat{a}_1, a_2] = -\theta\hat{a}_1$, $[\hat{X},[\hat{X},\hat{a}_2]] = \theta^2[\hat{a}_1^{\dagger}a_2, \hat{a}_1] = -\theta^2\hat{a}_2$, $[\hat{X},[\hat{X},[\hat{X},\hat{a}_2]]] = -\theta^3[\hat{a}_2^{\dagger}a_1, \hat{a}_2] = \theta^3\hat{a}_1$

∴ By BCH formula,

$$\hat{B}^{\dagger}(\theta)\hat{a}_1\hat{B}(\theta) = \hat{a}_1 + \theta\hat{a}_2 - \frac{\theta^2}{2!}\hat{a}_1 - \frac{\theta^3}{3!}\hat{a}_2 + \frac{\theta^4}{4!}\hat{a}_1 + \cdots = \cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2 \quad ※$$

$$\hat{B}(\theta)\hat{a}_2\hat{B}(\theta) = \hat{a}_2 - \theta\hat{a}_1 - \frac{\theta^2}{2!}\hat{a}_2 + \frac{\theta^3}{3!}\hat{a}_1 + \frac{\theta^4}{4!}\hat{a}_2 + \cdots = -\sin\theta\,\hat{a}_1 + \cos\theta\,\hat{a}_2 \quad ※$$

4.a. Start from

$|\Psi'\rangle = \hat{B}(\frac{\pi}{4})\hat{S}_1(r)\hat{S}_2(-r)|0\rangle_1|0\rangle_2$

$$= \hat{B}(\tfrac{\pi}{4})\int\frac{1}{\sqrt[4]{\pi e^{2r}}}e^{-\frac{q_1^2}{2e^{2r}}}|q_1\rangle dq_1 \int\frac{1}{\sqrt[4]{\pi e^{2r}}}e^{-\frac{q_2^2}{2e^{2r}}}|q_2\rangle dq_2$$

$$= \frac{1}{\sqrt{\pi}}\iint e^{-\frac{q_1^2}{2e^{2r}}}e^{-\frac{q_2^2}{2e^{2r}}}\hat{B}(\tfrac{\pi}{4})|q_1\rangle|q_2\rangle dq_1 dq_2$$

$$= \frac{1}{\sqrt{\pi}}\iint\exp\left(-\frac{q_1^2}{2e^{2r}} - \frac{q_2^2}{2e^{2r}}\right)\left|\frac{q_1+q_2}{\sqrt{2}}\right\rangle\left|\frac{-q_1+q_2}{\sqrt{2}}\right\rangle dq_1 dq_2$$

$$= \frac{1}{\sqrt{\pi}}\iint\exp\left(-\frac{(q_+-q_-)^2}{4}\underbrace{e^{2r}}_{\sigma^2} - \frac{(q_++q_-)^2}{4\underbrace{e^{2r}}_{\sigma^2}}\right)|q_+\rangle|q_-\rangle dq_+dq_-$$

is a two-mode squeezed state.

Let $q_+ = \frac{q_1+q_2}{\sqrt{2}}$, $q_- = \frac{-q_1+q_2}{\sqrt{2}}$

$q_1 = \frac{q_+-q_-}{\sqrt{2}}$, $q_2 = \frac{q_++q_-}{\sqrt{2}}$

∴ $dq_1dq_2 = dq_+dq_-$

Comparing with $\Psi_{TMS}(q_1,q_2) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{(q_1-q_2)^2}{4}\sigma^2\right)\exp\left(-\frac{(q_1+q_2)^2}{4\sigma^2}\right)$

b. $|TMS\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|n\rangle_1|n\rangle_2$ where $|\lambda|<1$

Clearly it is not separable, i.e. $|TMS\rangle \neq |\phi_1\rangle|\phi_2\rangle$

We can also check the purity of reduced density matrix $\sigma_2 = Tr_1[|TMS\rangle\langle TMS|]$

$$\sigma_2 = Tr_1[|TMS\rangle\langle TMS|] = (1-\lambda^2)\,Tr_1\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda^n\lambda^m\,|n\rangle_1\langle m|\,|n\rangle_2\langle m|\right]$$

$$= (1-\lambda^2)\sum_{n=0}^{\infty}\lambda^{2n}|n\rangle_2\langle n|$$

$$\text{Purity} = Tr_2[\sigma_2^2] = (1-\lambda^2)^2 \sum_{n=0}^{\infty} \lambda^{4n} = \frac{(1-\lambda^2)^2}{1-\lambda^4} = \underline{1 - \frac{2\lambda^2(1-\lambda^2)}{1-\lambda^4} < 1} \quad ※$$

Here we use $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

So $|TMS\rangle$ is entangled.

C. After the homodyne detection on mode 1, the state becomes $|\Psi_f\rangle \propto \langle q'|TMS\rangle$

It is more convenient to express $|TMS\rangle$ in wave function form.

$$|TMS\rangle = \iint \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{q_1-q_2}{2\sigma}\right)^2\right] \exp\left[-\left(\frac{q_1+q_2}{2}\sigma\right)^2\right] |q_1\rangle|q_2\rangle dq_1 dq_2$$

$$\therefore \langle q'|TMS\rangle = \iint \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{q_1-q_2}{2\sigma}\right)^2\right] \exp\left[-\left(\frac{q_1+q_2}{2}\sigma\right)^2\right] \langle q'|q_1\rangle |q_2\rangle dq_1 dq_2$$

(not yet normalized)

$$= \iint \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{q_1-q_2}{2\sigma}\right)^2\right] \exp\left[-\left(\frac{q_1+q_2}{2}\sigma\right)^2\right] \delta(q_1-q') |q_2\rangle dq_1 dq_2$$

(delta function reduce one integral)

$$= \iint \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{q_2-q'}{2\sigma}\right)^2 - \left(\frac{q_2+q'}{2}\sigma\right)^2\right] |q_2\rangle dq_2$$

$$= \int \frac{1}{\sqrt{\pi}} \exp\left[\frac{q'^2}{1+\sigma^2}\right] \underline{\exp\left[-\left(\frac{q_2-\bar{q}}{\sqrt{2}\bar{\sigma}}\right)^2\right]} |q_2\rangle dq_2 \quad ※$$

where $\bar{q} = \frac{1-\sigma^2}{1+\sigma^2} q'$ and $\bar{\sigma} = \sqrt{\frac{2\sigma^2}{1+\sigma^2}}$.

We can see $\langle q_1|TMS\rangle$ is still a Gausian function. ※

d. After the number detection (with $|0\rangle$ as outcome), the state becomes $|\Psi_f\rangle \propto \langle 0|TMS\rangle$

For this question, it is better to express $|TMS\rangle = \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle|n\rangle$

$$\therefore \langle 0|TMS\rangle \propto \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n \langle 0|n\rangle_1 |n\rangle_2$$

(Note that Fock states are orthonormal to each other, so $\langle n|m\rangle = \delta_{nm}$)

$$= \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n \delta_{n0} |n\rangle_2$$

$$= \sqrt{1-\lambda^2}\, \lambda^0 |0\rangle_2 = \sqrt{1-\lambda^2}\, |0\rangle_2 \quad ※$$

We know $|0\rangle$ is the vacuum state and it is also a Gaussian state. ※