CV 3

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Gaussian transformation

We now briefly introduce the operations, i.e. the gates, that is usually implemented in bosonic systems.
In quantum information processing, logic gates are unitary transformations applied to the logical state. Physically, unitary transformations are implemented by switching on some Hamiltonian for some time, i.e.

$$\hat{U} = \exp\{-it\hat{H}\} \quad -(1)$$

Therefore the types of gate can be implemented depend on the type of $\hat{H}$ (i.e. interaction) that can be realized.
Generally, we can always express $\hat{H}$ in terms of mode operators a and $a^\dagger$ (or equivalently $\hat{q}$ and $\hat{p}$). For example, in the single-mode case, we can formally expand the Hamiltonian in a series of mode operators:

$$\hat{H} = \underset{\text{real constant}}{h_0} + \underbrace{h_1 a + h_1^* a^\dagger}_{1^{st}\text{ order}} + \underbrace{h_2 a^2 + h_{20} a^\dagger a + h_2^* a^{\dagger 2}}_{2^{nd}\text{ order}} + \underbrace{\cdots}_{3^{rd}\text{ or higher order}} \quad -(2)$$

In practice, the terms with lower order of mode operators are easier to implement, i.e. the interaction strength of lower order terms is usually stronger than the higher order terms.
In most situation, especially in optics, we can only efficiently implement up to 2^{nd} order Hamiltonian, the transformation is thus given by

$$\hat{U}_2 = \exp\{-i\underset{\text{some degree of transformation}}{\theta}\underset{\text{Operator with up to }2^{nd}\text{ order of }a\text{ and }a^\dagger}{\hat{O}}\} \quad -(3)$$

One characteristics of $\hat{U}_2$ is that it always maps a Gaussian state to a Gaussian state. As such, this class of operation is known as Gaussian transformation.
This property has two significance:

1. Because the ground state (or generally thermal state, which is mixed so we neglect it here) is Gaussian, the states that can be prepared by applying Gaussian operation to ground state are also Gaussian. This is why Gaussian states are important: they are the type of state we can generate efficiently, if not the only type.
2. For universal CV QC, we need a $\hat{U}$ that can map Gaussian states to non-Gaussian states. Therefore, Gaussian operation alone is not universal. It is an analog of Clifford gates in qubit QC. To implement universal CVQC, we need non-Gaussian resources, either state or transformation or measurement. We will discuss it later.

In the following, we will go through some of the representative Gaussian operations: Displacement, rotation, single-mode squeezing, beam splitter. In fact, one can implement any Gaussian transformations by applying these four operations in a correct sequence [Braunstein, Phy. Rev. A 71, 055801 (2005)].

Displacement

Displacement operator is the transformation with only linear terms in the exponent:

$$\hat{D}(\beta) \equiv \exp(\beta a^\dagger - \beta^* a), \quad \text{where } \beta \equiv \frac{q_\beta + ip_\beta}{\sqrt{2}}$$

Here β is usually called displacement, which the q_β and p_β are respectively the displaced position and momentum.
To get a sense of its effect, we consider its effect on coherent state $|\alpha\rangle$.
We know $a|\alpha\rangle = \alpha|\alpha\rangle$. Now consider

$$\begin{aligned} a\hat{D}(\beta)|\alpha\rangle &= \hat{D}(\beta)\hat{D}^\dagger(\beta)a\hat{D}(\beta)|\alpha\rangle \\ &= \hat{D}(\beta)(a+\beta)|\alpha\rangle = (\alpha+\beta)\hat{D}(\beta)|\alpha\rangle \quad (\text{Hw}) \end{aligned}$$

This means $\hat{D}(\beta)|\alpha\rangle = |\alpha+\beta\rangle$, which the position and momentum of $|\alpha\rangle$ is displaced by q_β and p_β respectively. If $\alpha=0$, this is how coherent state is generated.
This effect is true for any state, as one can see

$$\langle\psi|\hat{D}^\dagger(\beta)\hat{q}\hat{D}(\beta)|\psi\rangle = \langle\psi|(\hat{q}+q_\beta)|\psi\rangle = \langle\psi|\hat{q}|\psi\rangle + q_\beta$$
$$\langle\psi|\hat{D}^\dagger(\beta)\hat{p}\hat{D}(\beta)|\psi\rangle = \langle\psi|(\hat{p}+p_\beta)|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle + p_\beta$$

Rotation/waveplate

Rotation is the single mode operation that involves the energy preserving quadratic term in the exponent:

$$\hat{R}(\theta) = \exp\{-i\theta a^\dagger a\}$$

Its effect is to mix up the position and momentum of the state.
We again get the sense of its effect by considering coherent state as an example

$$\begin{aligned} a\hat{R}(\theta)|\alpha\rangle &= \hat{R}(\theta)\hat{R}^\dagger(\theta)a\hat{R}(\theta)|\alpha\rangle \\ &= \hat{R}(\theta)\,ae^{-i\theta}|\alpha\rangle = \alpha e^{-i\theta}\hat{R}(\theta)|\alpha\rangle \quad (\text{Hw}) \end{aligned}$$

It means $\hat{R}(\theta)|\alpha\rangle = |\alpha e^{-i\theta}\rangle$, so the position and momentum are changed as

$$\frac{q_0+ip_0}{\sqrt{2}} \to e^{-i\theta}\frac{q_0+ip_0}{\sqrt{2}} = \frac{(\cos\theta\, q_0 + \sin\theta\, p_0) + i(-\sin\theta\, q_0 + \cos\theta\, p_0)}{\sqrt{2}}$$

This is a general effect for any state, as we can see

$$\langle\psi|\hat{R}^\dagger(\theta)\hat{q}\hat{R}(\theta)|\psi\rangle = \langle\psi|\cos\theta\,\hat{q} + \sin\theta\,\hat{p}|\psi\rangle = \cos\theta\langle\hat{q}\rangle + \sin\theta\langle\hat{p}\rangle$$
$$\langle\psi|\hat{R}^\dagger(\theta)\hat{p}\hat{R}(\theta)|\psi\rangle = \langle\psi|-\sin\theta\,\hat{q} + \cos\theta\,\hat{p}|\psi\rangle = -\sin\theta\langle\hat{q}\rangle + \cos\theta\langle\hat{p}\rangle$$

$\hat{R}(\theta)$ can be implemented by waveplate in linear optics.
For $\theta=\frac{\pi}{2}$, it interchanges the position and momentum, so $\hat{R}(\frac{\pi}{2})$ is sometimes called the Fourier transform gate.

Squeezing

Squeezing is the single mode operation that involves the energy non-preserving quadratic terms in the exponent

$$\hat{S}(r) \equiv \exp\{r/2\,(a^2 - a^{\dagger 2})\}$$

To understand its squeezing effect, we consider its effect on position eigenstate

$$\begin{aligned} \hat{q}\hat{S}(r)|q_0\rangle &= \hat{S}(r)\hat{S}^\dagger(r)\hat{q}\hat{S}(r)|q_0\rangle \quad (\text{Hw}) \\ &= \hat{S}(r)e^{-r}\hat{q}|q_0\rangle = e^{-r}q_0\hat{S}(r)|q_0\rangle \end{aligned}$$

As such, $\hat{S}(r)|q_0\rangle$ is proportional to $|e^{-r}q_0\rangle$.
For any state $|\psi\rangle$, we have

$$\hat{S}|\psi\rangle = \int\psi(q)\hat{S}|q\rangle dq \propto \int\psi(q)|e^{-r}q\rangle dq \propto \int\psi(e^{r}q)|q\rangle dq$$

In the last step, we re-define the integration parameter.
As we can see from the wavefunction, after the operation the argument is scaled by a factor e^r. Depending on whether r is positive or negative, the wavefunction would be squeezed or anti-squeezed.
We can also see it from the variance of general state

$$\langle\psi|\hat{S}^\dagger\hat{q}^2\hat{S}|\psi\rangle = \langle\psi|e^{-2r}\hat{q}^2|\psi\rangle = e^{-2r}\langle\hat{q}^2\rangle$$
$$\langle\psi|\hat{S}^\dagger\hat{p}^2\hat{S}|\psi\rangle = \langle\psi|e^{2r}\hat{p}^2|\psi\rangle = e^{2r}\langle\hat{p}^2\rangle$$

Note that $\hat{S}$ is energy non-preserving, so it cannot be implemented by the conventional, passive linear optics. Implementing squeezing generally needs external drive, so squeezing is sometimes called "active linear".

Beam Splitter

Finally, we need an operation to couple different modes. One typical operation is the beam splitter, which has a bi-linear term in the exponent

$$\hat{B}(\theta) \equiv \exp\{\theta(a_1^\dagger a_2 - a_2^\dagger a_1)\}$$

where a_1 and a_2 are annihilation operator of mode 1 and 2.
It is not difficult to show that

$$\hat{B}^\dagger a_1\hat{B} = \cos\theta\, a_1 + \sin\theta\, a_2 \quad (\text{hw})$$
$$\hat{B}^\dagger a_2\hat{B} = -\sin\theta\, a_1 + \cos\theta\, a_2$$

Because this $\hat{B}$ does not mix up the real and imaginary parts, we get

$$\begin{aligned} \hat{q}_1\hat{B}|q_1\rangle|q_2\rangle &= \hat{B}\hat{B}^\dagger\hat{q}_1\hat{B}|q_1\rangle|q_2\rangle = \hat{B}(\cos\theta\,\hat{q}_1 + \sin\theta\,\hat{q}_2)|q_1\rangle|q_2\rangle \\ &= (\cos\theta\, q_1 + \sin\theta\, q_2)\hat{B}|q_1\rangle|q_2\rangle \end{aligned}$$
$$\hat{q}_2\hat{B}|q_1\rangle|q_2\rangle = \hat{B}\hat{B}^\dagger\hat{q}_2\hat{B}|q_1\rangle|q_2\rangle = (-\sin\theta\, q_1 + \cos\theta\, q_2)\hat{B}|q_1\rangle|q_2\rangle$$

which tells us $\hat{B}|q_1\rangle|q_2\rangle = |\cos\theta\, q_1 + \sin\theta\, q_2\rangle|-\sin\theta\, q_1 + \cos\theta\, q_2\rangle$.
Note that the first (second) state is the state of mode 1 (2), and q_1 and q_2 are just real numbers.
For general state, we have

$$\begin{aligned} \hat{B}|\Psi\rangle = \int\Psi(q_1,q_2)\hat{B}|q_1\rangle|q_2\rangle dq_1dq_2 &= \int\Psi(q_1,q_2)\left|\tfrac{q_1+q_2}{\sqrt{2}}\right\rangle\left|\tfrac{-q_1+q_2}{\sqrt{2}}\right\rangle dq_1dq_2 \\ &= \int\Psi\left(\tfrac{q_1-q_2}{\sqrt{2}}, \tfrac{q_1+q_2}{\sqrt{2}}\right)|q_1\rangle|q_2\rangle dq_1dq_2 \end{aligned}$$

As its name suggests, Beam Splitter can be implemented by beam splitter in linear optics. Generally, Beam Splitter is not necessarily energy preserving, because mode 1 and 2 can have different energy.

Two-mode squeezing

By using the above four gates, one can implement any Gaussian Transformation. However, some other gates can also be implemented efficiently in experiment. One such example is Two-mode squeezing

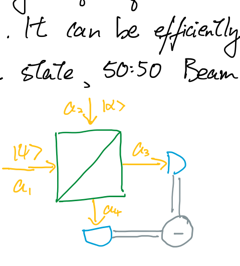
$$\hat{T}(r) \equiv \exp\{r(a_1^\dagger a_2^\dagger - a_1 a_2)\}$$

Applying this gate to vacuum will generate TMS state, i.e. $\hat{T}|0\rangle|0\rangle = |\text{TMS}\rangle$.
This is in fact the process underlying SPDC photon source.
Two-mode squeezing can be implemented by sandwiching two single-mode squeezing with two 50:50 Beam Splitter:

$$\hat{B}(-\tfrac{\pi}{4})\hat{S}_1(-r)\hat{S}_2(r)\hat{B}(\tfrac{\pi}{4}) \quad (\text{hw})$$

Gaussian measurement

After computation, we need to read out the information. Just as in DV system, we need to specify the basis of measurement, e.g. $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ in qubit.
In CV, because our information is encoded as a continuous superposition of position eigenstate, it is desirable to choose the position eigenstate, $\{|q\rangle\langle q|\}$ is our basis, i.e. $|\psi\rangle = \int\psi(q)|q\rangle dq \xrightarrow{\text{measure}} |q_0\rangle$ with probability $|\psi(q_0)|^2\delta q$.
This measurement is called homodyne detection. It can be efficiently measured in linear optics by using an auxiliary coherent state, 50:50 Beam Splitter, power measurement, and classical processing, i.e.



The measurement operator is $a_3^\dagger a_3 - a_4^\dagger a_4 = a_1^\dagger a_2 + a_2^\dagger a_1 \approx \alpha a_1^\dagger + \alpha^* a_1$, which is proportional to the position operator, $\hat{q}_1 \propto a_1 + a_1^\dagger$, when a real α is chosen.
Homodyne detection can be implemented with very high efficiency (>90%). However, as we mentioned position eigenstate is a Gaussian state, projecting a subset of multi-mode Gaussian state to Gaussian state will leave the unmeasured modes in Gaussian state. As such, homodyne measurement, as a Gaussian measurement, together with Gaussian state is not universal. Generally, we can compute the Gaussian measurement outcome of Gaussian state efficiently with classical computer.

Non-Gaussian element

To implement universal quantum computer, we thus need one non-Gaussian element, either the input state, gate, or measurement.
One common choice of non-Gaussian state is single-boson state, $|1\rangle \equiv a^\dagger|0\rangle$.
As we have seen, the wavefunction of $|1\rangle$ is non-Gaussian. Conventionally, this state is generated probabilistically from SPDC source, so scaling up will exponentially reduce the success rate. Recently, alternative photon source, e.g. quantum dot, has become mature, a deterministic single photon is very possible in near future.
For non-Gaussian gate, one could implement, e.g. $\exp\{i\chi\hat{q}^3\}$ the cubic phase gate. However, such gate relies on non-linearity of the system, which is very weak in optics. On the other hand, in superconducting microwave platform, the Josephson effect provides a strong Kerr non-linearity, i.e. $\hat{H} \propto a^{\dagger 2}a^2$. This is in fact the basis of Josephson qubit.
In optical platform, the most promising approach is perhaps using non-Gaussian measurement. One such example is photon number measurement, $\{|k\rangle\langle k|\}$, i.e.

$$|\psi\rangle = \int\psi(q)|q\rangle dq = \sum_{n=0}^{\infty} C_n|n\rangle \xrightarrow{\text{measure}} |k\rangle \text{ with prob. } |C_k|^2$$

Unless the outcome is $|0\rangle$, which is Gaussian, otherwise a multi-mode Gaussian state will be projected to non-Gaussian when a subset is measured.

Reference:

Braunstein and van Loock, Review of Modern Physics 77, 513 (2005)
Weedbrook et. al., Review of Modern Physics 84, 621 (2012)