CV problem set 1

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1 Continuous-variable wavefunction

(a) Consider the wavefunction of a general single-mode pure state is given by

$$|\psi\rangle = \int \psi(q)|q\rangle dq = \int \tilde{\psi}(p)|p\rangle dp \ . \tag{1}$$

Show that $\tilde{\psi}(p)$ is the Fourier transform of $\psi(q)$. You may need to use the identity operator in the position and momentum basis, $\hat{\mathbb{I}} \equiv \int |q\rangle\langle q|dq = \int |p\rangle\langle p|dp$, and the overlap of the position and momentum eigenstates $\langle p|q\rangle = \frac{1}{\sqrt{2\pi}}e^{-iqp}$.

(b) Find, numerically, the probability of getting outcome $q \geq 2$ when a ground state is measured by homodyne detection in the $\hat{q}$ basis.

2 Coherent state

(a) Find the mean position ($\langle q\rangle$), momentum ($\langle p\rangle$), and their variances ($\langle \Delta q^2\rangle, \langle \Delta p^2\rangle, \langle\frac{\Delta q\Delta p+\Delta p\Delta q}{2}\rangle$) of a coherent state $|\alpha\rangle$, where $\alpha \equiv \frac{q_\alpha+ip_\alpha}{\sqrt{2}}$.

(b) Write down the joint wavefunction, $\Psi(q_1, q_2)$, for the two-mode state $|\alpha\rangle|\beta\rangle$.

(c) Show that no entanglement can be generated by sending two coherent state through a Beam Splitter, i.e. $|\Psi'\rangle \equiv \hat{B}(\theta)|\alpha\rangle|\beta\rangle$ is separable. You can do it either by showing $|\Psi'\rangle = |\phi_1\rangle|\phi_2\rangle$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are pure states, or showing $\Psi'(q_1, q_2) = \phi_1(q_1)\phi_2(q_2)$, or formally showing $\mathrm{Tr}_1\{|\Psi'\rangle\langle\Psi'|\}$ is pure.

3 Gaussian transformation of mode operators

By using Baker-Campbell-Hausdorff formula, or otherwise, show the following identities:

(a) $\hat{D}^\dagger(\beta)\hat{a}\hat{D}(\beta) = \hat{a} + \beta$

(b) $\hat{R}^\dagger(\phi)\hat{a}\hat{R}(\phi) = e^{-i\phi}\hat{a}$

(c) $\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \cosh r\hat{a} - \sinh r\hat{a}^\dagger$

(d) $\hat{B}^\dagger(\theta)\hat{a}_1\hat{B}(\theta) = \cos\theta\hat{a}_1 + \sin\theta\hat{a}_2$ and $\hat{B}^\dagger(\theta)\hat{a}_2\hat{B}(\theta) = -\sin\theta\hat{a}_1 + \cos\theta\hat{a}_2$

4 Two-mode squeezed state

(a) Consider two modes that are prepared in oppositely squeezed vacuum state, $|\Psi\rangle = \hat{S}_1(r)\hat{S}_2(-r)|0\rangle|0\rangle$, where $\hat{S}_i$ is the single-mode squeezing operator for mode i. By considering the wavefunction, or otherwise, show that $|\Psi'\rangle \equiv \hat{B}(\frac{\pi}{4})|\Psi\rangle$ is a two-mode squeezed state.

(b) Show that two-mode squeezed state $|TMS\rangle$ is entangled. You can either use the wavefunction we introduced in lecture, or use the Fock state expression

$$|TMS\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|n\rangle|n\rangle \ . \tag{2}$$

(c) If mode 1 is measured by homodyne detection, show that the state of mode 2 is Gaussian, i.e. find the wavefunction of the (unnormalized) state $\langle q_1|TMS\rangle$, and verify it is a Gaussian function.

(d) If mode 1 is measured by photon number detection, show that the state of mode 2 is Gaussian if the measurement outcome is $|0\rangle$.

Notation

Baker-Campbell-Hausdorff formula:

$$e^X Y e^{-X} = Y + \frac{1}{1!}[X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots \tag{3}$$