Detailed solutions for day 2 problems 2 and 3

Stabilizer formalism review

- Consider an n-qubit state $|\psi\rangle$. The stabilizer group $\mathcal{S}$ of $|\psi\rangle$ is the set of Pauli operators such that

$$S|\psi\rangle = |\psi\rangle, \quad \forall S \in \mathcal{S}.$$

- If $|\mathcal{S}| = 2^n$ then there is a unique state fixed by every element of $\mathcal{S}$. Such states are called stabilizer states.
- E.g. the Bell state $|B_{00}\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ is a stabilizer state with stabilizer group $\langle X_1X_2, Z_1Z_2\rangle$

Stabilizer formalism review

Consider a n-qubit stabilizer state $|\sigma\rangle$ with stabilizer group $\mathcal{S} = \langle g_1, g_2, \ldots, g_n \rangle$ and suppose we perform a Pauli measurement A. We have two cases:

Case 1) A commutes with every element of $\mathcal{S}$.

1. For each stabilizer generator g_j, we have

$$g_j(A|\sigma\rangle) = Ag_j|\sigma\rangle = A|\sigma\rangle \Rightarrow A|\sigma\rangle \propto |\sigma\rangle .$$

   Then since $A^2 = I$, we have $A|\sigma\rangle = \pm|\sigma\rangle$. Therefore, $\lambda A \in \mathcal{S}$ for some $\lambda \in \{1, -1\}$.

2. Then the probability of getting measurement outcome λ is

$$P(\lambda) = \mathrm{Tr}\left(\frac{I + \lambda A}{2}|\sigma\rangle\langle\sigma|\right) = 1$$

   I.e. outcome λ occurs with probability 1 and the state is unaffected by the measurement.

Stabilizer formalism review

Case 2) A anticommutes with at least one generator of $\mathcal{S}$.

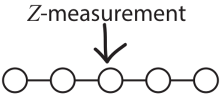
1. We can always rewrite the set of generators so that A anticommutes with exactly one generator (e.g. if A anticommutes with g_1 and g_2 then we can use the generators $\mathcal{S} = \langle g_1, g_2', g_3, \dots \rangle$ where $g_2' = g_1 g_2$.) Therefore, WLOG A anticommutes with g_1 and commutes with $g_2, \dots, g_n$.
2. The measurement outcome is $+1$ or -1 with equal probability since

$$P(+1) = \mathrm{Tr}\left(\frac{I+A}{2} |\sigma\rangle \langle\sigma|\right) = \mathrm{Tr}\left(\frac{I+A}{2} g_1 |\sigma\rangle \langle\sigma|\right) = \mathrm{Tr}\left(g_1 \frac{I-A}{2} |\sigma\rangle \langle\sigma|\right)$$
$$= \mathrm{Tr}\left(\frac{I-A}{2} |\sigma\rangle \langle\sigma| g_1\right) = \mathrm{Tr}\left(\frac{I-A}{2} |\sigma\rangle \langle\sigma|\right) = P(-1)$$

3. The postmeasurement state is $\frac{I+\lambda A}{2} |\psi\rangle$ which is a stabilizer state with stabilizer group $\langle \lambda A, g_2, g_3, \dots, g_n \rangle$

Day 1 problem 3

Think of a 1D cluster state as shown below. What happens if a qubit in the middle is measured in the Z-eigenbasis? What is the influence of the measurement outcome on the resulting state?



Day 1 problem 3, solution

The generators of the stabilizer group of the initial cluster state are

$$
\begin{aligned}
&X_1 Z_2 \\
&Z_1 X_2 Z_3 \\
&\quad Z_2 X_3 Z_4 \\
&\qquad Z_3 X_4 Z_5 \\
&\qquad \vdots \\
&\qquad Z_{n-2} X_{n-1} Z_n \\
&\qquad\qquad Z_{n-1} X_n
\end{aligned}
$$

Suppose we perform a Pauli-Z measurement on the k^{th} qubit and get measurement outcome λ.

Day 1 problem 3, solution

Suppose we perform a Pauli-Z measurement on the k^{th} qubit and get measurement outcome λ. The effect on the stabilizer group generators is

- $Z_{k-1}X_kZ_{k+1} \longrightarrow \lambda Z_k$

The rest of the generators are unaffected.

Day 1 problem 3, solution

The resulting state has stabilizer group generators

$$
\begin{array}{l}
X_1 Z_2 \\
Z_1 X_2 Z_3 \\
Z_2 X_3 Z_4 \\
Z_3 X_4 Z_5 \\
\vdots \\
Z_{k-2} X_{k-1} Z_k \\
\lambda Z_k \\
Z_k X_{K+1} Z_{k+2} \\
\vdots \\
Z_{n-2} X_{n-1} Z_n \\
Z_{n-1} X_n
\end{array}
$$

Day 1 problem 3, solution

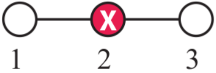
Equivalently, the resulting state has stabilizer group generators

$$
\begin{aligned}
&X_1 Z_2 \\
&Z_1 X_2 Z_3 \\
&\quad Z_2 X_3 Z_4 \\
&\qquad Z_3 X_4 Z_5 \\
&\qquad \vdots \\
&\qquad \lambda Z_{k-2} X_{k-1} \\
&\qquad \lambda Z_k \\
&\qquad \lambda X_{K+1} Z_{k+2} \\
&\qquad \vdots \\
&\qquad Z_{n-2} X_{n-1} Z_n \\
&\qquad\qquad\quad Z_{n-1} X_n
\end{aligned}
$$

The first $k-1$ generators are the generators of a cluster state on the first $k-1$ qubits, the last $n-k$ generators are the generators of a cluster state on qubits $k+1$ to n, and the generator λZ_k says that qubit k is in a Z eigenstate, as expected.

Question 2 (a) [Fault-tolerance].

(a) Consider a three-qubit cluster state as shown below, with qubit number #2 being measured in the X-basis (for simplicity you may assume the outcome of the measurement was $+1$).



Show that after this measurement, qubits 1 and 3 are in a Bell state.

Question 2, solution

- The stabilizer of the initial three qubit cluster state is

$$S = \langle X_1 Z_2, Z_1 X_2 Z_3, Z_2 X_3 \rangle = \langle X_1 Z_2, Z_1 X_2 Z_3, X_1 X_3 \rangle$$

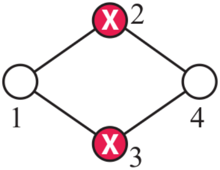
- We perform a Pauli-X measurement on qubit #2.
- The measurement outcome is $\lambda = +1$ or $\lambda = -1$ with equal probability.
- The stabilizer group of the postmeasurement state is

$$\mathcal{S} = \langle \lambda X_2, \lambda Z_1 Z_3, X_1 X_3 \rangle$$

- $\langle \lambda Z_1 Z_3, X_1 X_3 \rangle$ is the stabilizer group of a Bell state on qubits 1 and 3.

Question 2 (b)

Now consider the four-qubit graph state



Show that, similar to the above, measuring qubits 2 and 3 in the X-basis results in a Bell state on qubits 1 and 4. Further, explain that from the measurement record, one can detect a single Z-error on the graph state prior to measurement on qubits 2 and 3.

Question 2, solution

The argument is the same as before.

- The stabilizer of the initial four qubit state is

$$\begin{aligned} S &= \langle X_1 Z_2 Z_3, Z_1 X_2 Z_4, Z_1 X_3 Z_4, Z_2 Z_3 X_4 \rangle \\ &= \langle X_1 Z_2 Z_3, Z_1 X_2 Z_4, Z_1 X_3 Z_4, X_1 X_4 \rangle \end{aligned}$$

- The measurements yield outcomes $\lambda_2 = \lambda_3 = +1$ or $\lambda_2 = \lambda_3 = -1$ with equal probabilities.
- The postmeasurement state has stabilizer group

$$S = \langle \lambda_2 X_2, \lambda_2 X_3, \lambda_2 Z_1 Z_4, X_1 X_4 \rangle$$

Question 2, solution

- Recall the stabilizer group of the initial state is

$$S = \langle X_1 Z_2 Z_3, Z_1 X_2 Z_4, Z_1 X_3 Z_4, Z_2 Z_3 X_4 \rangle$$

- The Pauli operator $X_1 X_2 = (X_1 Z_2 Z_3)(Z_2 Z_3 X_4)$ is a syndrome operator.
- Consider the four qubit cluster state $|C_4\rangle$ with a Z-error on qubit #2: $Z_2 |C_4\rangle$. We have

$$\begin{aligned}(X_2 X_3)(Z_2 |C_4\rangle) &= X_2 X_3 Z_2 |C_4\rangle = -Z_2 X_2 X_3 |C_4\rangle \\ &= -Z_2 (X_2 X_3 |C_4\rangle) = -Z_2 |C_4\rangle\end{aligned}$$

- The erroneous state has value -1 for the measurement $X_2 X_3$ where as $|C_4\rangle$ has value $+1$. The same holds for a Z measurement on qubit #3.
- $\Rightarrow$ a Pauli Z error on qubits 2 and 3 can be detected.

Question 3 [Fault-tolerance].

In class it was stated that X-errors on the cluster/graph state don't matter because they are absorbed by the local X-measurements. Show this formally.

Specifically, show that a Pauli error σ_x, affecting the density matrix as $\rho \to \sigma_x \rho \sigma_x$, neither changes the outcome probabilities of the X-measurement nor the post-measurement state.

Question 3, solution

The probability of obtaining outcome λ for a Pauli X measurement on state ρ is

$$P_{X,\rho}(\lambda) = \mathrm{Tr}\left(\frac{I+\lambda\sigma_x}{2}\rho\right).$$

After a Pauli X error, the state is $\tau := \sigma_x\rho\sigma_x$. The probability of obtaining outcome λ now is

$$\begin{aligned} P_{X,\tau}(\lambda) &= \mathrm{Tr}\left(\frac{I+\lambda\sigma_x}{2}\sigma_x\rho\sigma_x\right) = \mathrm{Tr}\left(\left(\sigma_x\frac{I+\lambda\sigma_x}{2}\sigma_x\right)\rho\right) \\ &= \mathrm{Tr}\left(\frac{I+\lambda\sigma_x}{2}\rho\right) = P_{X,\rho}(\lambda) \end{aligned}$$

The probabilities for measurement outcomes are the same!

Question 3, solution

The postmeasurement state after a Pauli-X measurement is performed on state ρ yielding outcome λ is

$$\rho \longrightarrow \rho' := \frac{\frac{I+\lambda\sigma_x}{2}\rho\frac{I+\lambda\sigma_x}{2}}{P_{X,\rho(\lambda)}}$$

If the measurement is performed on the state τ, the result is

$$\tau \longrightarrow \tau' = \frac{\frac{I+\lambda\sigma_x}{2}\sigma_x\rho\sigma_x\frac{I+\lambda\sigma_x}{2}}{P_{X,\tau(\lambda)}}$$

For the denominator we proved already $P_{X,\tau}(\lambda) = P_{X,\rho}(\lambda)$. The numerator is

$$\frac{I+\lambda\sigma_x}{2}\sigma_x\rho\sigma_x\frac{I+\lambda\sigma_x}{2} = \frac{\sigma_x+\lambda I}{2}\rho\frac{\sigma_x+\lambda I}{2} = \frac{I+\lambda\sigma_x}{2}\rho\frac{I+\lambda\sigma_x}{2}$$

Therefore, $\tau' = \rho'$!