Measurement-based quantum computation
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Tutorial 2

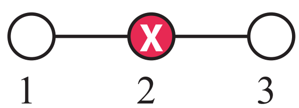
July 31, 2021

Question 1 [probabilistic heralded gates]. In class earlier today it was shown how to compute on 1D cluster states if the probabilistic entangling gate has a success probability larger than 2/3.

Find a strategy that achieves the same, but for a success probability smaller than 2/3. How small a success probability of the heralded entangling gate can your method handle? Discuss the efficiency of your method.

> **Homework suggestion:** Simulate the protocol you've come up with on a computer. Compare with your analytical predictions. Then discuss with your friends the protocols you have found.

Question 2 [Fault-tolerance]. (a) Consider a three-qubit cluster state as shown below, with qubit number #2 being measured in the X-basis (for simplicity, you may assume the outcome of the measurement was +1).



Show that after this measurement, qubits 1 and 3 are in a Bell state.

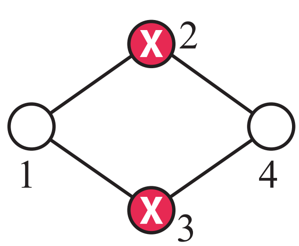
Hint: Recall that all Bell states $|B_{ij}\rangle$, with $i, j \in \mathbb{Z}_2$, are described by the stabilizer relations

$$X_1 \otimes X_2 |B_{ij}\rangle = (-1)^i |B_{ij}\rangle, \quad Z_1 \otimes Z_2 |B_{ij}\rangle = (-1)^j |B_{ij}\rangle.$$

Alternatively, the explicit description of the Bell states $|B_{ij}\rangle$ in the computational basis is

$$|B_{ij}\rangle = \frac{|0\rangle \otimes |j\rangle + (-1)^i |1\rangle \otimes |\bar{j}\rangle}{\sqrt{2}}.$$

(b) Now consider a four-qubit graph state



Show that, similar to the above, measuring qubits 2 and 3 in the X-basis results in a Bell state on qubits 1 and 4. Further, explain that from the measurement record, one can detect a single Z-error on the graph state prior to measurement, on qubit 2 or 3.

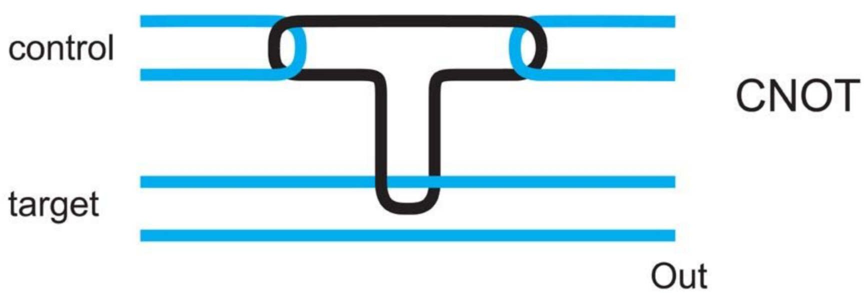
Question 3 [Fault-tolerance]. In class today it was stated that X-errors on the cluster/ graph state don't matter because they are absorbed by the local X-measurements. Show this formally.

Hint: Recall the following properties of quantum measurement and Pauli operators from your QM class:

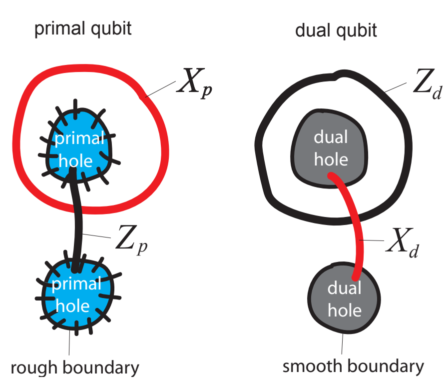
- The probability $p_A(\lambda)$ for obtaining the outcome λ in a measurement of the observable A is described by the Born rule,
$$p_A(\lambda) = \mathrm{Tr}(\Pi_{A,\lambda}\rho),$$
where ρ is the density matrix prior to measurement, and $\Pi_{A,\lambda}$ is the projector corresponding to the eigenspace of A with eigenvalue λ. Further, the post-measurement state is given by the Dirac postulate
$$\rho \longrightarrow \frac{\Pi_{A,\lambda}\rho\Pi_{A,\lambda}}{\mathrm{Tr}(\Pi_{A,\lambda}\rho)}.$$
- A measurement in the Pauli x-basis with outcome $\lambda = \pm 1$ is represented by the projector $\Pi_\lambda = \frac{I+\lambda\sigma_x}{2}$.
- It holds that $\sigma_x^2 = I$.

Now show that a Pauli error σ_x, affecting the density matrix as $\rho \longrightarrow \sigma_x\rho\sigma_x$, neither changes the outcome probabilities of X-measurement nor the post-measurement state.

Question 4*. [take home]: Show that the topological diagram below realizes a CNOT gate. Do this by reproducing the conjugation relations for Pauli operators under a CNOT gate.



The above diagram refers to the encoding



You may use the following facts without proof:

- In the same way as a 1D cluster state corresponds to the circuit model with one qubit propagating in time, the 3D cluster state corresponds to a bunch of *encoded* qubits propagating in time. The encoding is with the Kitaev surface code.
- The encoded Pauli operators for the surface code with boundary are shown above. There's two types of boundary, rough and smooth. In our example, the boundary is created by punching holes in the code surface. Two holes are required for an encoded qubit.
- As you slide along, the string operators representing the encoded Pauli observables must be deformed smoothly–Recall the example shown in class.
- To demonstrate the gate function, it suffices to show that the encoded Pauli operators transform as in conjugation under a CNOT.

Can you identify a sense in which the above CNOT gate is more general than the one shown in class (reproduced below)?