Measurement-based quantum computation

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Reference Solution for Tutorial 2

July 26, 2021

Question 1 [probabilistic heralded gates]. In class earlier today it was shown how to compute on 1D cluster states if the probabilistic entangling gate has a success probability larger than 2/3.

Find a strategy that achieves the same, but for a success probability smaller than 2/3. How small a success probability of the heralded entangling gate can your method handle? Discuss the efficiency of your method.

Solution: The basic idea is to create 1D cluster states of a fixed length n offline, by a probabilistic process. Those states are then connected to the already existing long 1D cluster state by the probabilistic heralded conditional phase gate. The length l of the cluster chain thereby changes as follows

$$\begin{aligned} l &\longrightarrow l+n \quad \text{(success, with probability } p\text{)},\\ l &\longrightarrow l-2 \quad \text{(failure, with probability } 1-p\text{)}. \end{aligned}$$

Thus, for the expected length of the cluster chain it holds that

$$\langle l \rangle \longrightarrow \langle l \rangle + p(n+2) - 2.$$

The threshold is the success probability therefore becomes

$$\boxed{p_{\text{threshold}} = \frac{2}{n+2}}$$

Thus, in the limit of large n, the required success probability of the entangling gate tends to zero.

Now let's look at the cost of preparing 1D cluster states of length n with probabilistic heralded gates. The simplest method is the following: Attempt all $n-1$ required conditional phase gates. If they succeed, output the resulting cluster state; if one or more gate fails, then discard everything and start over. The success probability of this method of cluster state creation is p^{n-1}. Hence the expected cost $C(n)$ of creating such a cluster state is

$$C(n) = \frac{n-1}{p^{n-1}}.$$

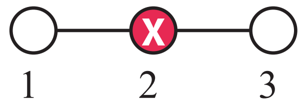
The number of length n 1D cluster states needed to create a cluster state of length $L \gg n$ is $L/(p(n+2)-2)$. Hence the total cost C_L of creating a cluster state of length $L \gg n$ is

$$C_L = L\frac{n-1}{(p(n+2)-2)p^{n-1}}, \text{ with } p > 2/(n+2).$$

Although the construction is very inefficient for small values of p, the main thing to note is that the cost is proportional to the chain length L! It is thus efficient from the scaling point of view.

There are more clever and efficient ways of creating the 1D cluster states of fixed length n, and you may explore those in the programming part of this exercise.

Question 2 [Fault-tolerance]. (a) Consider a three-qubit cluster state as shown below, with qubit number #2 being measured in the X-basis (for simplicity, you may assume the outcome of the measurement was $+1$).



Show that after this measurement, qubits 1 and 3 are in a Bell state.

Hint: Recall that all Bell states $|B_{ij}\rangle$, with $i, j \in \mathbb{Z}_2$, are described by the stabilizer relations

$$X_1 \otimes X_2 |B_{ij}\rangle = (-1)^i |B_{ij}\rangle, \quad Z_1 \otimes Z_2 |B_{ij}\rangle = (-1)^j |B_{ij}\rangle.$$

Alternatively, the explicit description of the Bell states $|B_{ij}\rangle$ in the computational basis is

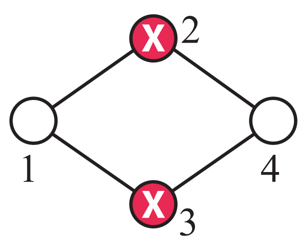
$$|B_{ij}\rangle = \frac{|0\rangle \otimes |j\rangle + (-1)^i |1\rangle \otimes |\bar{j}\rangle}{\sqrt{2}}.$$

Solution for (a): Before the measurement of qubit 2, the 3-qubit cluster state shown has stabilizer elements X_1X_3 and $Z_1X_2Z_3$. The former is unaffected by the measurement of qubit 2, and the latter turns into $\pm Z_1Z_3$, depending on the measurement outcome. The resulting stabilizer for qubits 1 and 3 is thus

$$\langle X_1X_3, \pm Z_1Z_3 \rangle,$$

which, for either sign, is the stabilizer of a Bell state.

(b) Now consider a four-qubit graph state



Show that, similar to the above, measuring qubits 2 and 3 in the X-basis results in a Bell state on qubits 1 and 4. Further, explain that from the measurement record, one can detect a single Z-error on the graph state prior to measurement, on qubit 2 or 3.

Solution for (b): The argument for the Bell state between qubits 1 and 4 is the same as in part (a). Now regarding the error detection capability, note that the Pauli operator X_2X_3 is a syndrome operator; namely it is in the graph state stabilizer,

$$X_2X_3 = (X_2Z_1Z_4)(X_3Z_1Z_4) = K_2K_3,$$

and furthermore it is of Pauli-X type. Now consider the above cluster/graph state $|\mathcal{C}_4\rangle$ with a Z-error acting on qubit 2, i.e., the state to consider is $Z_2|\mathcal{C}_4\rangle$. We then have

$$\begin{aligned}
(X_2X_3)\,(Z_2|\mathcal{C}_4\rangle) &= ((X_2X_3)\,Z_2)\,|\mathcal{C}_4\rangle \\
&= -\,(Z_2\,(X_2X_3))\,|\mathcal{C}_4\rangle \\
&= -Z_2\,((X_2X_3)\,|\mathcal{C}_4\rangle) \\
&= -Z_2|\mathcal{C}_4\rangle.
\end{aligned}$$

Therein, the first line follows by associativity of matrix multiplication, the second line by the (anti)commutation relations among Pauli operators; the third line is associativity again, and the fourth line uses the stabilizer relation for $|\mathcal{C}_4\rangle$. The upshot is that the erroneous state $Z_2|\mathcal{C}_4\rangle$ has the value -1 for the syndrome operator X_2X_3, whereas the error-free state $|\mathcal{C}_4\rangle$ has the value +1. The two states can thus be distinguished by measurement.

Doing the same calculation for an error Z_3, we find that it has exactly the same signature as Z_2. In the present case, as opposed to the graph state shown in the lecture, errors can only be detected, not identified.

Question 3 [Fault-tolerance]. In class today it was stated that X-errors on the cluster/ graph state don't matter because they are absorbed by the local X-measurements. Show this formally.

Hint: Recall the following properties of quantum measurement and Pauli operators from your QM class:

- The probability $p_A(\lambda)$ for obtaining the outcome λ in a measurement of the observable A is described by the Born rule,
$$p_A(\lambda) = \text{Tr}(\Pi_{A,\lambda}\rho),$$
where ρ is the density matrix prior to measurement, and $\Pi_{A,\lambda}$ is the projector corresponding to the eigenspace of A with eigenvalue λ. Further, the post-measurement state is given by the Dirac postulate
$$\rho \longrightarrow \frac{\Pi_{A,\lambda}\rho\Pi_{A,\lambda}}{\text{Tr}(\Pi_{A,\lambda}\rho)}.$$
- A measurement in the Pauli x-basis with outcome $\lambda = \pm 1$ is represented by the projector $\Pi_\lambda = \frac{I+\lambda\sigma_x}{2}$.
- It holds that $\sigma_x^2 = I$.

Now show that a Pauli error σ_x, affecting the density matrix as $\rho \longrightarrow \sigma_x\rho\sigma_x$, neither changes the outcome probabilities of X-measurement nor the post-measurement state.

Solution. Denote the state with X-error by τ, $\tau := \sigma_x\rho\sigma_x$. First compare the probabilities for the measurement outcome. With the Born rule we have

$$p_{X,\rho}(\lambda) = \text{Tr}\left(\frac{I+\lambda\sigma_x}{2}\rho\right).$$

For the other state we find

$$\begin{array}{rcl} p_{X,\tau}(\lambda) & = & \text{Tr}\left(\frac{I+\lambda\sigma_x}{2}\sigma_x\rho\sigma_x\right) \\ & = & \text{Tr}\left(\left(\sigma_x\frac{I+\lambda\sigma_x}{2}\sigma_x\right)\rho\right) \\ & = & \text{Tr}\left(\frac{I+\lambda\sigma_x}{2}\rho\right) \end{array}$$

Therein, in the first line we used the definition of τ, in the second line the cyclicity of trace, and in the third line cummutativity of σ_x with itself and with the identity I, and $\sigma_x^2 = I$. We thus find that the probabilities for obtaining the outcome $\lambda = \pm 1$ agree for the two states.

We now turn to the post-measurement states. Given the state ρ we obtain

$$\rho \quad \longrightarrow \quad \frac{\frac{I+\lambda\sigma_x}{2}\rho\frac{I+\lambda\sigma_x}{2}}{p_{X,\rho}(\lambda)}.$$

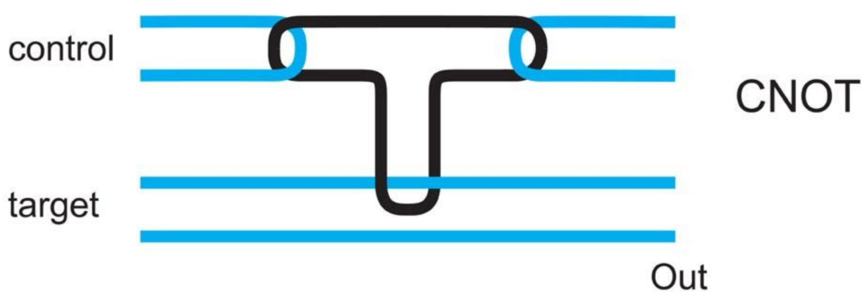
For the state τ we find

$$\begin{aligned}
\tau \quad \longrightarrow & \quad \frac{\frac{I+\lambda\sigma_x}{2}\sigma_x\rho\sigma_x\frac{I+\lambda\sigma_x}{2}}{p_{X,\tau}(\lambda)} \\
= & \quad \lambda^2\frac{\frac{I+\lambda\sigma_x}{2}\rho\frac{I+\lambda\sigma_x}{2}}{p_{X,\tau}(\lambda)} \\
= & \quad \frac{\frac{I+\lambda\sigma_x}{2}\rho\frac{I+\lambda\sigma_x}{2}}{p_{X,\rho}(\lambda)}
\end{aligned}$$

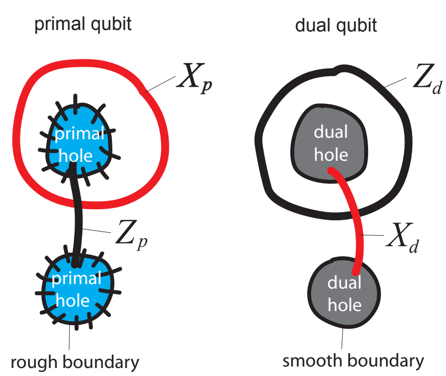
Therein, the first line is just the definition of τ, the second line used $(I \pm \sigma_x)/2\,\sigma_x = \pm(I \pm \sigma_x)/2$, and in the third line we used $(\pm 1)^2 = 1$, and the earlier result $p_{X,\tau}(\lambda) = p_{X,\rho}(\lambda)$. Comparing the post-measurement states resulting from ρ and τ, we find that they are the same.

To sum up, between the states ρ and τ there is no difference in the outcome probabilities of the X-measurement and in the post-measurement state. Hence, an X-error preceding X-measurement in the same location has no effect, as claimed.

Question 4. [take home]: Show that the topological diagram below realizes a CNOT gate. Do this by reproducing the conjugation relations for Pauli operators under a CNOT gate.



The above diagram refers to the encoding

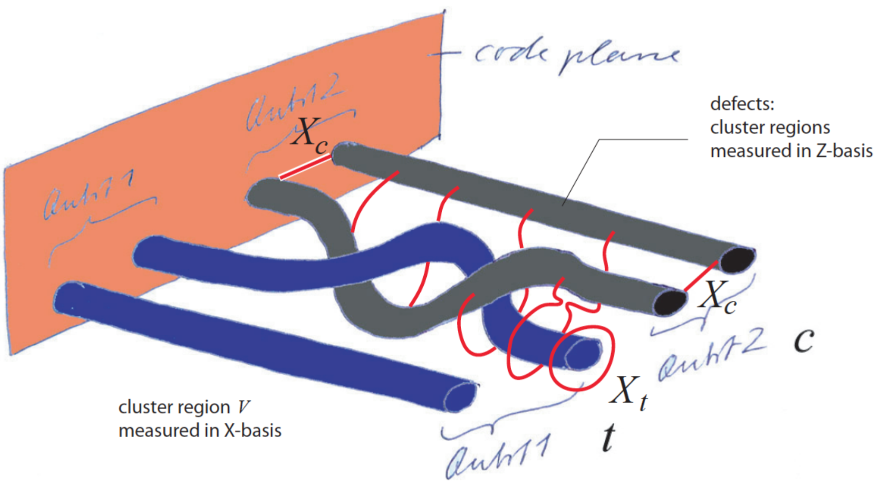


You may use the following facts without proof:

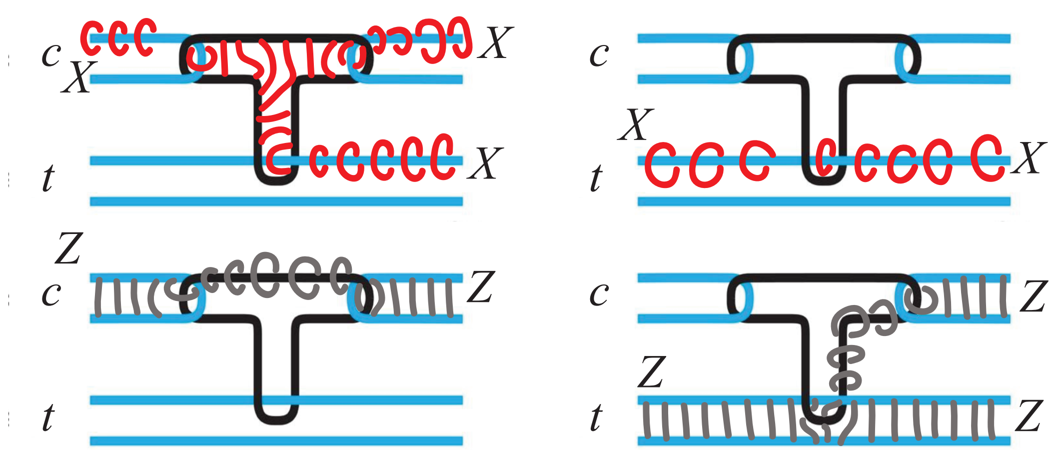
- In the same way as a 1D cluster state corresponds to the circuit model with one qubit propagating in time, the 3D cluster state corresponds to a bunch of *encoded* qubits propagating in time. The encoding is with the Kitaev surface code.
- The encoded Pauli operators for the surface code with boundary are shown above. There's two types of boundary, rough and smooth. In our example, the boundary is created by punching holes in the code surface. Two holes are required for an encoded qubit.

- As you slide along, the string operators representing the encoded Pauli observables must be deformed smoothly–Recall the example shown in class.
- To demonstrate the gate function, it suffices to show that the encoded Pauli operators transform as in conjugation under a CNOT.

Can you identify a sense in which the above CNOT gate is more general than the one shown in class (reproduced below)?



Solution. We find find



Thus we establish the following relations for the propagation of Pauli operators:

$$\begin{array}{rclcrcl} X_c & \longrightarrow & X_c X_t, & \quad & X_t & \longrightarrow & X_t \\ Z_c & \longrightarrow & Z_c, & & Z_t & \longrightarrow & Z_c Z_t. \end{array}$$

These propagation relations identify a CNOT gate with control c and target t.

The CNOT just discussed is more general than the one above, because both encoded qubits are of the same type, and the CNOT can be applied in both directions. This gives rise to a set of non-commuting gates. One the other hand, in the CNOT from the lecture, the target qubit is always primal and the control always dual. All these CNOTs pairwise commute, and are thus restricted.