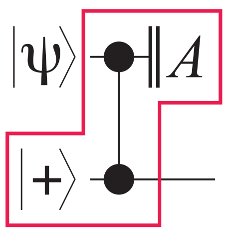
Measurement-based quantum computation • Reference solution for Tutorial 1

July 26, 2021

Question 1 [Half-teleporatation]. The picture below shows the half-teleportation circuit explaining the effect of one single measurement on the cluster state; you saw in class earlier today.

For which observables A is the resulting logical operation T_A acting on the input state $|\Psi\rangle$ (a) a unitary, and (b) a projection?



Solution. A general a one-qubit measurement is in the basis

$$\mathcal{B} = \{|m_{+1}\rangle = \alpha|0\rangle + \beta|1\rangle,\ |m_{-1}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle\},$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Now assuming an input state $|\text{in}\rangle = c_0|0\rangle + c_1|1\rangle$, we can work out the corresponding output state of the above 2-qubit circuit.

Outcome +1. (the first basis state is the post-measurement state), we obtain

$$|\text{out}_{+1}\rangle = c_0\alpha^*|+\rangle + c_1\beta^*|-\rangle,$$

where $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$. This means that the transformation executed on the input state $|\text{in}\rangle \cong (c_0, c_1)^T$ is

$$T_{+1} \cong H \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta^* \end{pmatrix} \tag{1}$$

(a) We now figure out when T_{+1} is a unitary, up to scalar λ. (The scalar accounts for the non-unit probability of obtaining the outcome +1.) From the definition of unitarity ($UU^\dagger = U^\dagger U = I$) we find the condition

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} HH \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta^* \end{pmatrix} = |\lambda|^2 I,$$

and hence

$$|\alpha| = |\beta|. \tag{2}$$

We may thus parametrize $\alpha = e^{i\varphi}/\sqrt{2}$, $\beta = e^{i\psi}/\sqrt{2}$. Then, the measured observable $A = |m_{+1}\rangle\langle m_{+1}| - |m_{-1}\rangle\langle m_{-1}|$ is

$$\boxed{A = \cos(\varphi - \psi)\sigma_x - \sin(\varphi - \psi)\sigma_y} \tag{3}$$

Remarks: (i) The observables of Eq. (3) are precisely those encountered in the lecture. Denote $\Delta := \varphi - \psi$. Then,

$$\begin{array}{rcl} e^{i\frac{\Delta}{2}Z} X e^{-i\frac{\Delta}{2}Z} &=& e^{i\Delta Z} X \\ &=& (\cos\Delta\, I + i \sin\Delta\, Z)\, X \\ &=& \cos\Delta\, X - \sin\Delta\, Y. \end{array}$$

(ii) Consistency check: If the states $|m_{\pm 1}\rangle$ are modified by a *global* phase, $|m_{\pm 1}\rangle \longrightarrow e^{i\Lambda}|m_{\pm 1}\rangle$, there is no physical effect. And indeed, under this transformation $\varphi \longrightarrow \varphi + \Lambda$, $\psi \longrightarrow \psi + \Lambda$. Thus, the angle difference Δ remains unaffected. Δ is the only angle combination that the observable A of Eq. (3) depends upon.

(b) We now figure out when T_+ is a logical measurement. With Eq. (1), this happens when either $\alpha = 0$ or $\beta = 0$. That is, the measured observable is

$$\boxed{A = \sigma_z}$$

Outcome -1. In this case, we obtain for the post-measurement state of the second qubit

$$|\text{out}_{-1}\rangle = c_0\beta|+\rangle - c_1\alpha|-\rangle.$$

The transformation thus is

$$T_{-1} \cong H \left(\begin{array}{cc} \beta & 0 \\ 0 & -\alpha \end{array} \right). \tag{4}$$

(a) unitaries. The conclusion is the same as above: T_{-1} is unitary if and only if $|\alpha| = |\beta|$. Hence, the observables A of Eq. (3) lead to unitary evolution for both outcomes.

(b) logical projection. The conclusion is the same as above. The transformation T_{-1} amounts to a logical measurement if and only if $\alpha = 0$ or $\beta = 0$. Hence, the measurement of the observable $A = \sigma_z$ leads to a logical measurement for both outcomes.

Relation between T_{+1} and T_{-1}. (a) Unitary case. Wlog. we may choose $\varphi = -\psi = \Delta/2$. Then,

$$T_{+1} \cong H \left(\begin{array}{cc} e^{-i\Delta/2} & 0 \\ 0 & e^{i\Delta/2} \end{array} \right),\; T_{-1} \cong H \left(\begin{array}{cc} e^{-i\Delta/2} & 0 \\ 0 & -e^{i\Delta/2} \end{array} \right).$$

Hence,

$$T_{-1} = T_{+1} Z.$$

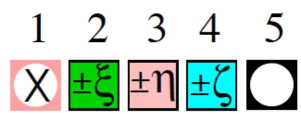
(b) Logical measurement. Wlog. assume that $\beta = 0$. Then,

$$T_{+1} \cong H \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),\; T_{-1} \cong H \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$

Thus physical measurement leads to logical measurement. The logical post-measurement states $|\text{out}_{\pm 1}\rangle$ are orthogonal for the two measurement outcomes.

Question 2 [Temporal order]. The picture below shows a five-qubit cluster state for the implementation of a general logical 1-qubit rotation. Cluster qubit #5 is the output qubit.

Which measurement outcomes does the choice of measurement basis at cluster qubit #4 depend upon?

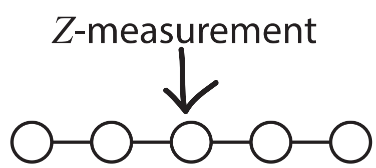


Solution. As we discussed in class, the measurement outcome of qubit 1 influences the measurement basis at qubits 2 and 4; the measurement outcome of qubit 2 influences the measurement bases at qubits 3 and 5; the measurement outcome of qubit 3 influences the measurement basis at qubit 4; the measurement outcome of qubit 4 influences the measurement basis of qubit 5; and the measurement outcome of qubit 5 influences no measurement bases.

Thus, the measurement basis at qubit 4 depends on the outcomes of qubit 1 and 3.

Question 3 [Cissors]. Think of a 1D cluster state as shown below. What happens if a qubit in the middle is measured in the Z-eigenbasis? What is the influence of the measurement outcome on the resulting state? Move on to 2D cluster states—what is the effect of Z-measurement now?

Hint: If you know the stabilizer formalism, it is helpful for this problem. If you don't, recall the creation procedure for cluster states using conditional phase gates, and note that such gates commute with Z-measurements (they are diagonal operations in the same basis).



Solution 1 (using the stabilizer formalism). Assume we have a 1D cluster chain of length n, and qubit k, $1 < k < n$ is measured in the Z-basis.

The stabilizer of the cluster state before measurement has the generators

$$K_l = Z_{l-1} X_l Z_{l+1},\ l = 2, .., n-1,$$

and $K_1 = X_1 Z_2$, $K_n = Z_{n-1} X_n$.

Now suppose that qubit k is measured in the Z-basis, and the eigenvalue $(-1)^{s_k}$ of Z, $s_k \in \mathbb{Z}_2$, is obtained. In the stabilizer of the resulting post-measurement state, K_k is removed, and replaced by $(-1)^{s_k} Z_k$. Furthermore,

$$\begin{array}{rcl} K_{k-1} = Z_{k-2} X_{k-1} Z_k & \longrightarrow & (-1)^{s_k} Z_{k-2} X_{k-1}, \\ K_{k+1} = Z_k X_{k+1} Z_{k+2} & \longrightarrow & (-1)^{s_k} X_{k+1} Z_{k+2}. \end{array}$$

These are the only changes of the stabilizer. The resulting state is thus a tensor product state of a cluster state on qubits 1 to $k-1$, a Z-eigenstate on qubit k, and another cluster state on qubits $k+1$ to n. Actually, since the new stabilizers centered on qubits $k-1$ and $k+1$ have a sign that depends on the measurement outcome s_k, the respective states to the left and right of qubit k are cluster states only up to a possible Pauli correction Z on qubits $k-1$ and $k+1$.

But the main effect is that the Z-measurement on qubit k has applies the scissors. We started with a long cluster chain and cut it into two smaller ones.