Cornerstone models of QC: Measurement-based QC

Supplementary material

July 29, 2020

Contents

This set of slides will provide some extra information that will be helpful in advance of next week's lectures on measurement-based quantum computing:

- Projective measurements
- The Pauli group
- Introduction to the stabilizer formalism

Projective measurements

Projective measurements

A Hermitian matrix Π is a *projector* if $\Pi^2 = \Pi$.

A projective measurement is a set of projectors $\mathcal{B} = \{\Pi_i\}_{i=0}^{M}$ such that

$$\sum_{i=0}^{M} \Pi_i = \mathbb{1} \tag{1}$$

Exercise

Show that for a single-qubit state $|\psi\rangle$, the 2×2 matrix $|\psi\rangle\langle\psi|$ is a projector.

If we take a set of orthonormal kets $\{|\psi_i\rangle\}$ (i.e. a basis), together they form a projective measurement.

For a single qubit with basis states $|\psi_0\rangle, |\psi_1\rangle$, let

$$\Pi_0 = |\psi_0\rangle\langle\psi_0|, \quad \Pi_1 = |\psi_1\rangle\langle\psi_1|.$$

Then $\{\Pi_0, \Pi_1\}$ is a projective measurement, and

$$|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| = \mathbb{1}$$

When we make a projective measurement on a state $|\varphi\rangle$ in basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

$$\begin{aligned}\text{Pr(outcome i)} &= \text{Tr}(\Pi_i |\varphi\rangle\langle\varphi|) \\ &= |\langle\psi_i|\varphi\rangle|^2\end{aligned}$$

If we observe outcome i, following the measurement the system will be left in state $|\psi_i\rangle$[1].

[1] Actually things are a bit more subtle than this; for a good overview of projective measurements, see `https://www.people.vcu.edu/~sgharibian/courses/CMSC491/notes/Lecture%203%20-%20Measurement.pdf`

Deriving the result of a projective measurement

Let $|\varphi\rangle$ be a single-qubit state and $\Pi_i = |\psi_i\rangle\langle\psi_i|$ be one component of a projective measurement. Then

$$\begin{aligned}
\text{Pr(outcome i)} &= \text{Tr}(|\psi_i\rangle\langle\psi_i|\,|\varphi\rangle\langle\varphi|) \\
&= \sum_n \langle e_n|\psi_i\rangle\langle\psi_i|\varphi\rangle\langle\varphi|e_n\rangle \\
&= \sum_n \langle\psi_i|\varphi\rangle\langle\varphi|e_n\rangle\langle e_n|\psi_i\rangle \\
&= \langle\psi_i|\varphi\rangle\langle\varphi|\left(\sum_n |e_n\rangle\langle e_n|\right)|\psi_i\rangle \\
&= \langle\psi_i|\varphi\rangle\langle\varphi|\mathbb{1}|\psi_i\rangle \\
&= |\langle\psi_i|\varphi\rangle|^2
\end{aligned}$$

where $|e_n\rangle$ is vectors with all elements 0 except a 1 in element n.

Example: measuring in the computational basis

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

The computational basis $\{|0\rangle, |1\rangle\}$ is a projective measurement:

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1} \tag{2}$$

Then if we measure $|\psi\rangle$,

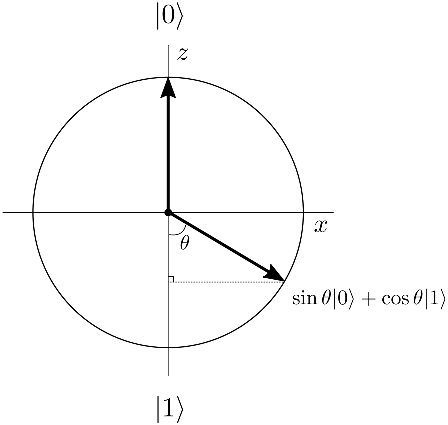
$$\begin{aligned} \Pr(0) &= |\langle 0|\psi\rangle|^2 = |\alpha|^2 \\ \Pr(1) &= |\langle 1|\psi\rangle|^2 = |\beta|^2 \end{aligned}$$

We can measure in any orthonormal basis by applying a suitable unitary transformation to the computational basis vectors.

Example: measuring in the *Hadamard basis*:

$$
\begin{aligned}
|+\rangle &= H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
|-\rangle &= H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)
\end{aligned}
$$

You can check that $|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$ is indeed a valid projective measurement.



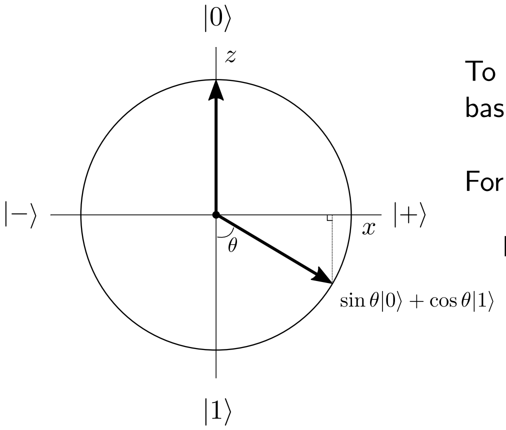
(This is just a 2D slice of the Bloch sphere, the *xz* plane.)

We can visualize measurements as projections onto the axes of the Bloch sphere:

For example,

$$\begin{aligned} \langle 1|\psi\rangle &= \sin\theta\langle 1|0\rangle + \cos\theta\langle 1|1\rangle \\ &= \cos\theta \end{aligned}$$

$$\begin{aligned} \Pr(1) &= |\langle 1|\psi\rangle|^2 \\ &= \cos^2\theta \end{aligned}$$

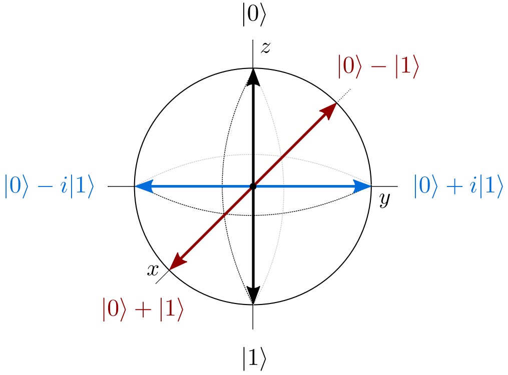


To measure in the Hadamard basis, we project onto the x axis:

For example,

$$\begin{aligned} \Pr(+) &= |\langle +|\psi\rangle|^2 \\ &= \ldots \\ &= \frac{1}{2}(1+\sin(2\theta)) \end{aligned}$$

This geometric interpretation makes the following fact very intuitive: to fully characterize the state of a single-qubit ket, we can make measurements in 3 different bases along x, y and z.



We say that these 3 bases form a *tomographically complete* set of measurements.

Measuring multi-qubit systems

Projective measurements are similar for multi-qubit systems.

Exercise

Verify that the computational basis for two qubits is a valid projective measurement.

Exercise

Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|00\rangle + |10\rangle - |11\rangle \right)$$

What is the probability of the first qubit being in state $|0\rangle$? What about state $|1\rangle$? What is the state of the second qubit upon receiving each measurement outcome?

Solution:

$$\frac{1}{\sqrt{3}}\left(|00\rangle + |10\rangle - |11\rangle\right)$$

To get the probability the first qubit is in state $|0\rangle$, sum the probabilities for each measurement outcome where this is possible:

$$\begin{aligned}
\text{Pr}(0 \text{ for qubit } 1) &= \text{Pr}(00) + \text{Pr}(01) \\
&= |\langle 00|\psi\rangle|^2 + |\langle 01|\psi\rangle|^2 \\
&= \frac{1}{3}
\end{aligned}$$

The second qubit will be in state $|0\rangle$ after the measurement.

For the second part, let's factor the state:

$$\frac{1}{\sqrt{3}}\left(|00\rangle + |1\rangle\left(|0\rangle - |1\rangle\right)\right) = \frac{1}{\sqrt{3}}\left(|00\rangle + |1-\rangle\right)$$

Then

$$\begin{aligned} \text{Pr(1 for qubit 1)} &= |\langle 10|\psi\rangle|^2 + |\langle 11|\psi\rangle|^2 \\ &= \frac{1}{3} + \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

The second qubit will be in state $|-\rangle$ after the measurement.

The Pauli group

Single-qubit Pauli group

Consider our familiar single-qubit unitaries:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ$$

If we include the identity $\mathbb{1}$, we get the single-qubit Pauli group:

$$\mathcal{P}_1 = \{\mathbb{1}, X, Y, Z\}$$

The size of $\mathcal{P}_1$ is $|\mathcal{P}_1| = 4$.

Multi-qubit Pauli group

For n qubits, the Pauli group $\mathcal{P}_n$ consists of the n-fold tensor product of the single-qubit Pauli group. For example, $\mathcal{P}_2$ is: '

$$\begin{array}{cccc} \mathbb{1}\mathbb{1} & \mathbb{1}X & \mathbb{1}Y & \mathbb{1}Z \\ X\mathbb{1} & XX & XY & XZ \\ Y\mathbb{1} & YX & YY & YZ \\ Z\mathbb{1} & ZX & ZY & ZZ \end{array}$$

where the tensor product is implied, i.e. ZZ means $Z \otimes Z$.

For each qubit in a tensor product, the group element can be one of four choices ($\mathbb{1}, X, Y$, or Z). Therefore, the size of the n-qubit Pauli group is $|\mathcal{P}_n| = 4^n$.

Useful facts about Paulis

- All Paulis have two eigenvalues: $+1$ and -1. There are an equal number of eigenvectors in each eigenspace.
- Each Pauli commutes with exactly half of the other Paulis in the group, and anticommutes with the rest.
- To check if two Paulis commute, count the qubits on which they differ (and there is no $\mathbb{1}$). If it's even, they commute.

Examples

ZXX and ZYY commute because on qubits 2 and 3 the elements are different.

But $ZYZXY$ and $\mathbb{1}ZXXZ$ do not commute, because they differ on qubits 2, 3, and 5 (we ignore the identity on qubit 1 because everything commutes with it) .

(To learn an even cooler way to do this, ask me about binary symplectic representation!)

The stabilizer formalism

Stabilizers and the Pauli group

Consider an n-qubit state $|\psi\rangle$. The stabilizer group $\mathcal{S}$ of $|\psi\rangle$ is the set of Pauli elements $\mathcal{S} = \{S_i | S_i \in \mathcal{P}_n\}$ such that

$$S_i|\psi\rangle = |\psi\rangle, \quad \forall S_i \in \mathcal{S}$$

i.e. $|\psi\rangle$ is in the $+1$ eigenspace of all the Paulis in $\mathcal{S}$.

All elements in the stabilizer group commute. If S_i is in the stabilizer, so is $-S_i$. However $-\mathbb{1}$ cannot be in the group.

Exercise

Consider the 2-qubit state $|01\rangle$. Which elements of $\mathcal{P}_2$ are in its stabilizer group?

Stabilizers and quantum error-correcting codes

Stabilizer codes are used in the field of quantum error correction.

Given a set of stabilizers $\mathcal{S}$, the codespace of $\mathcal{S}$ is the set of all states $\mathcal{C} = \{|\psi_j\rangle\}$ such that

$$S_i|\psi_j\rangle = |\psi_j\rangle \quad \forall \;\; S_i \in \mathcal{S}, \;\; |\psi_j\rangle \in \mathcal{C}$$

The idea is find operations on the qubits that will always send states in $\mathcal{C}$ to other states in $\mathcal{C}$, and then we can measure the expectation value of the stabilizers. By definition of the stabilizers, it should always be $+1$. So if we ever see -1, we'll know there was an error, and we can correct it.

More resources on stabilizers

To learn more about stabilizers, check out the following notes. They go into far more detail than is necessary for our purposes, but there is some very cool stuff:

- Dave Bacon's lecture notes
  `https://courses.cs.washington.edu/courses/cse599d/06wi/lecturenotes18.pdf`
- Scott Aaronson's lectures notes
  `https://www.scottaaronson.com/qclec/28.pdf`