Math 376 Project: Delay Differential Equations

Rob Li

December 4, 2024

Abstract

This document presents my Math 376 project on Delay Differential Equations (DDEs). The project explores stability analysis, linearization, and solutions of DDEs, focusing on the discrete delay cases. With numerical simulations, we will further examine the effects of delays on applied models. We discover that the addition of delays can cause oscillations, limit cycles, and even chaos in simple dynamical systems.

Contents

1 Introduction

The typical class of differential equation that we have studied is of the form:

$$\dot{x} = x'(t) = f(x(t), t)$$

We can see that the dynamics of x at any give time t is a function x at that time. Among their many applications, we have the classic example of the logistic model of population growth:

$$\dot{N} = N'(t) = N(t)[r - \frac{r}{K}N(t)]$$

Here, population change $N'(t)$ is contingent on a growth rate $r > 0$, a carrying capacity $K > 0$, and the current population $N(t)$.

Now suppose we were to use this model to describe a wild animal population. The animal species may have maturation and developmental periods which would prevent the changes in population from being instantaneously affected by its current size. Due to these qualities, we can expect the effect of the current population on the growth rate to be delayed by some period $\tau \geq 0$. We now update our model to:

$$\dot{N} = N'(t) = N(t)[r - \frac{r}{K}N(t - \tau)]$$

This is an example of a delay differential equation. Delay differential equations (DDEs) are a extension of ordinary differential equations that accounts for a time delay $\tau > 0$. They can be of the form:

$$\dot{x} = x'(t) = f(x(t - \tau))$$

We may also have systems of multiple different delays, or the delay could be distributed over a continuous interval. In this project, we will provide an introduction to delay differential equations. We will aim to see how the variance of the delay parameter τ affects the dynamics of a system, and we will ground our theory with applied examples, qualitative analysis, and numerical simulation.

2 Background

This section will provide an overview of the scope of the class of delay differential equations its various subclasses. Additionally, we will ground our theories by analyzing a simple DDE. Sections 1, 2, 3, and 4 are primarily sourced from Hal Smith's An Introduction to Delay Differential Equations with Applications to the Life Sciences [1].

2.1 Equation form

2.1.1 Delay Differential Equation (DDE): A delay differential equation has a general form:

$$\dot{x} = x'(t) = f(t, x(t, \tau))$$

Where $x(t,\tau)$ refers to the position of x at a previous time from t determined by τ.

2.1.2 History segment: We will call $x(t,\tau)$ a history segment because it gives x at a historical time value.

2.2 Types of delays

There are number of different ways a delay can be represented in the history segment.

2.2.1 Discrete Delays delay time in the history segment by a constant amount $\tau \geq 0$.

$$x(t,\tau) = x(t-\tau)$$

Our updated population model only has a discrete delay, and we will mostly focus on single discrete delay cases. Nonetheless, there are many other types of delay differential equations.

2.2.2 Time-Dependent Delays have the delay parameter $\tau(t)$ as a function of the current time.

2.2.3 State-Dependent Delays have the delay parameter $\tau(x(t))$ as a function of the current position.

2.2.4 Distributed Delay refers the historical value of x over a time interval rather than a single time instance. These delays take a weighted average of x over a historical continuous interval.

$$x(t,\tau) = \int_{t-\tau}^{t} x(s)k(t-s)ds = \int_{0}^{\tau} x(t-z)k(z)dz$$

The kernel $k(z)$ is a normalizing term with $\int_0^\tau k(s)ds = 1$
Such a model may prove more realistic. For instance a delayed population may be influenced by the past population over a segment of time rather than a single past instant. We can update our population model to:

$$\dot{N} = N'(t) = N(t)[r - \frac{r}{K}\int_{0}^{\infty} x(t-z)k(z)dz)]$$

2.2.5 Unbounded and Bounded Delays If we take τ to ∞ and if the kernel is not uniformly 0 on an unbounded set we have an **unbounded delay**. The case where we consider historical values of x over a bounded interval time, including the case of discrete delays are **bounded delays**.

2.2.6 Autonomous Delays Just like the case for ODEs, autonomous DDEs do not explicitly depend of the current time. Eg:

$$\dot{x} = x'(t) = f(x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_n))$$

Just like autonomous ODEs, autonomous DDEs experience **time invariance**. That is if $x(t)$ solves an autonomous DDE on interval $[a-\tau, b]$, then $x(t-c)$ is a solution on $[a+c-\tau, b+c]$.

2.2.7 Multiple Delays Seeing that the distributed delay case allows us to consider delays over an entire interval of time, we can also extend our discrete case to consider many different past instances.

$$\dot{x} = f(t, x(t), x(t, \tau)) = f(t, x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_k))$$

2.3 Initial History Problem

In ODE's, a initial value problem gives us a trajectory's value (position, rate of change, etc) at a single initial time instance, and we are able to find a solution for that trajectory on an interval over the inital time. With DDE's giving the initial value at a single time instance is not enough, since DDE's refer to previous values in time. As a result, we will be asked **initial history problems** for DDE's. We will be provided an **initial history function** $\phi(x)$ where we know $x(t) = \phi(t)$ on an initial time interval.
In our case of discrete delays, if τ^* is the largest discrete delay, and if t_0 is the initial time, then knowing the initial history on the interval $[t_0 - \tau^*, t_0]$ is sufficient for a solution beyond time t_0. More on this in section 3.1.

3 Theory

In this section, we will introduce the basic tools and theorems we will use to analyze DDE. This includes understanding the existence of solutions, and local linearization for stability analysis.

3.1 Solutions

We may first question whether a solution to a DDE with an initial history function would exist at all. In this section we will provide a basic solution existence theorem which will be sufficient for the equations we will analyze.

3.1.1 The Method of Steps
We will not prove the theorem in this write up, but to motivate the justification for this theorem, we will introduce a constructive technique to solving specific initial history function problems.

Suppose we had a simple Discrete Delay differential equation:

$$\dot{x} = f(t, x(t), x(t-\tau)), \quad \tau > 0$$

We also have an initial history function $\phi(t)$ defined on $[t_0 - \tau, t_0]$, where we know that on this interval, $x(t) = \phi(t)$.
We are wondering whether this set up will give rise to a unique solution. **The Method of Steps** helps motivate the intuition for why this is so:

1. First, If ϕ and f are continuous, we can properly define $x(t)$ on the interval $[t_0, t_0 + \tau]$ using values from the initial history function.

2. Iteratively, if $x(t)$ is defined on $[t_0 + n\tau, t_0 + (n + 1)\tau], n \in \mathbb{N}$ we can define $x(t)$ on $[t_0 + (n + 1)\tau, t_0 + (n + 2)\tau]$ using our history of values.

This process may be terminated early at some $T > t_0$ if our solution "blows up" to either ∞. In this case, we would have a uniquely defined solution for our DDE with initial history over $[t_0 - \tau, T)$. Otherwise, we can define our solution $x(t)$ on an arbitrarily long interval starting from $t_0 - \tau$.
This method, however has a few limitations which restrict its application.

1. The Method of steps applies only to DDE's of discrete delays.

2. We require that ϕ and f be continuous. (Furthermore, we also need f_x to be continuous, but this can be improved to a Lipschitz condition)

3. The Method of Steps does not give us a method to extend our solution backwards in time. In general, DDEs do not allow us to go backwards in time as each time value refers to values even further in the past rather than the future.

Nonetheless, the method of steps does help justify an existence theorem on DDEs that will be enough for the scope of this project.

3.1.2 Existence Theorem

Consider a discrete DDE and initial history function ϕ in the form

$$\dot{x} = f(t, x(t, \tau)) \quad \tau \geq 0, \quad x \in \mathbb{R}^n$$

and $x(t, \tau)$ is a discrete delay history segment. Our initial history function ensures that:

$$x(t) = \phi(t), \forall t \in [t_0 - \tau, t_0], \quad t_0 \in \mathbb{R}$$

For the cases of multiple delays and delay parameters $\tau = [\tau_k]_{k=1}^{K}$, our initial history function will have to be defined on $[t_0 - \tau^*, t_0]$ where τ^* is the largest delay.

Existence Theorem: If f is continuous and satisfies the global Lipschitz condition (Lip), if $M > 0$, and if ϕ is continuous with $||\phi|| \leq M$, then there exists $T > 0$ which depends only on M such that a unique solution of the DDE initial history problem exists on the interval $[t_0 - \tau, t_0 + T]$

Lipschitz Condition (Lip): $\forall a, b \in \mathbb{R}^n$ and $\forall M > 0$, $\exists L > 0$ such that $\forall t \in [a, b]$ and ϕ, ψ with $||\phi||, ||\psi|| < M$ (ϕ, ψ are each arbitrary choices for the $x(t, \tau)$ parameter).

$$||f(t, \phi) - f(t, \psi)|| \leq L||\phi - \psi||$$

A more generalized existence theorem can extend these results to DDEs of distributed delays, state dependent delays, etc. However, the method of steps is not used in the proof due to the

limited nature of an initial history segment as a fixed interval. Furthermore, the generalized existence theorem does not allow backwards extension of the solution.
Indeed, the inability for solutions of most DDE's to extend backwards in time is a key difference of DDEs from ODEs where the Picard-Lindelöf existence theorem gives a solution interval around the initial condition.

3.2 Linearization

We we focus on a local linearization theory for the autonomous single discrete delay case, although these methods can be extended to other types of delays. Consider the DDE:

$$\dot{x}(t) = f(x(t), x(t-\tau)), \quad \tau \geq 0$$

Suppose x^* was fixed point. Let $x(t) = x^* + \epsilon(t)$ where $\epsilon > 0$ is a small deviation from the fixed point. We can expand and use Taylor's theorem to estimate:

$$\begin{aligned}
\dot{x}(t) = (x^* + \epsilon(t))' = \dot{\epsilon}(t) &= f(x(t), x(t-\tau)) \\
&= f(x^* + \epsilon(t), x^* + \epsilon(t-\tau)) \\
&\approx f(x^*(t), x^*(t-\tau)) + (x^* + \epsilon(t) - x^*)f_{x(t)} + (x^* + \epsilon(t-\tau) - x^*)f_{x(t-\tau)} \\
&= 0 + \epsilon(t)A + \epsilon(t-\tau)B \\
\dot{\epsilon}(t) &= A\epsilon(t) + B\epsilon(t-\tau)
\end{aligned}$$

Thus, we can assume local solution of form $\epsilon(t) = e^{\lambda t} \implies \dot{\epsilon}(t) = \lambda e^{\lambda t}$. This gives us a **characteristic equation**: $\lambda e^{\lambda t} = A\epsilon(t) + B\epsilon(t-\tau)$.
Looking at specific solutions, we can see $\Re(\lambda) < 0$ implies exponential decay. If all solutions have $\Re(\lambda) < 0$, we have asymptotic stability. Meanwhile, $\Re(\lambda) > 0$ implies an unstable fixed point.

4 Example

We will start with the simplest example of a DDE.

Recall the first dynamical system introduced in Math 376:

$$\dot{x} = x'(t) = -\mu x(t), \quad \mu \in \mathbb{R}$$

We have a single steady state at $x = 0$. Furthermore, if $\mu > 0$ the origin is stable because dynamics are positive when $x < 0$, and change in x is negative when $x > 0$. Conversely, if $\mu < 0$ the origin is unstable.

We now introduce our first DDE by introducing a **discrete delay parameter** τ:

$$\dot{x} = x'(t) = -\mu x(t-\tau), \quad \mu \in \mathbb{R} \quad \tau \geq 0$$

The steady states occurs when $\dot{x} = 0 \implies x = 0$. We will now analyze stability at the origin.

4.1 Stability Analysis

As with ODEs, lets assume that we have a solution of the form $x(t) = e^{\lambda t}$
So $x'(t) = \lambda e^{\lambda t} = \dot{x} = -\mu e^{\lambda(t-\tau)}$

$$\implies \lambda e^{\lambda t} = -\mu e^{\lambda(t-\tau)} \implies \lambda = -\mu e^{-\lambda\tau}$$

This defines a **characteristic equation** for our DDE. Roots of the characteristic equation $\lambda = \alpha + i\beta$ will determine the stability and oscillations. We now describe different cases.

4.1.1 Case of $\tau = 0$ is the ODE
First, in the case of no delay where $\tau = 0$, we get $\lambda = -\mu$. The real part of the solution is negative when $\mu > 0$ and the steady state is stable. Conversely, if $\mu < 0$ the steady state is unstable. This matches our initial analysis of the ODE prior to adding the delay.

4.1.2 Case of $\mu < 0$ is not stable
To handle the case where $\mu < 0$, consider the characteristic equation rewritten as a function on λ, $f(\lambda) = \lambda + \mu e^{-\lambda\tau}$. We seek λ such that $f(\lambda) = 0$. We will use intermediate value theorem to show that $\lambda > 0$ when $\mu < 0$. $f(0) = \mu < 0$, and $\lim_{\lambda\to\infty} f(\lambda) = \lim_{\lambda\to\infty}(\lambda + \mu e^{-\lambda\tau}) = \infty - 0 = \infty$. By intermediate value theorem, $f(\lambda)$ has a real root in $\lambda \in (0, \infty)$. Therefore, if $\mu < 0$, the origin admits a trajectory that is repelled from it $\implies$ the origin is not stable.

4.1.3 Case of $0 < \mu\tau < \frac{\pi}{2}$ is asymptotically stable
More generally, (for $\tau > 0$ and $\mu > 0$), $\lambda = \alpha + i\beta$. We have:

$$\begin{aligned}
\lambda &= -\mu e^{-\lambda\tau} \\
\alpha + i\beta &= -\mu e^{-(\alpha+i\beta)\tau} \\
\alpha + i\beta &= -\mu e^{-\alpha\tau}(\cos(\beta\tau) - i\sin(\beta\tau)) \\
&\implies \Re(\lambda) = \alpha = -\mu e^{-\alpha\tau}\cos(\beta\tau), \quad \Im(\lambda) = \beta = \mu e^{-\alpha\tau}\sin(\beta\tau)
\end{aligned}$$

Now we can show that if $0 < \mu\tau < \frac{\pi}{2}$ we have $\alpha < 0$ by proving the contrapositive:
Let $\tau > 0$ and $\mu > 0$ as we have handled the cases of $\tau = 0$, and $\mu \leq 0$.
Suppose there exists a root to the characteristic equation with $\alpha \geq 0$. We can also look at $\beta \geq 0$ because α is an even function for values of β, so any root with $\beta < 0$ gives a corresponding root with $\beta > 0$. $\alpha \geq 0$ and $\beta\tau \geq 0$ mean that:
$\cos(\beta\tau) \leq 0 \implies \beta\tau \in \bigcup_{k=0}^{\infty}\{[\frac{\pi}{2}, \frac{3\pi}{2}) + 2\pi k\} \implies \beta\tau \geq \frac{\pi}{2} \iff \beta \geq \frac{\pi}{2\tau}$
Lets rewrite the imaginary part of the characteristic equation:

$$\beta = \mu e^{-\alpha\tau} sin(\beta\tau) \iff \frac{e^{\alpha\tau}}{\mu} = \frac{\sin(\beta\tau)}{\beta}$$

Given that $\beta \geq \frac{\pi}{2\tau}$ and $\alpha \geq 0$ we can provide a bound on this:

$$\frac{1}{\mu} \leq \frac{e^{\alpha\tau}}{\mu} = \frac{\sin(\beta\tau)}{\beta} \leq \frac{1}{\frac{\pi}{2\tau}} = \frac{2\tau}{\pi} \implies \mu \geq \frac{\pi}{2\tau} \iff \mu\tau \geq \frac{\pi}{2}$$

We conclude that if $0 < \mu\tau < \frac{\pi}{2}$, then all roots of the characteristic equation have $\alpha \leq 0$ implying asymptotic stability in this region.

4.1.4 Case of $\mu\tau > \frac{\pi}{2}$
We can also show that if $\mu\tau > \frac{\pi}{2}$, then there exists a solution with $\alpha > 0$, and $\beta\tau \in (\frac{\pi}{2}, \pi)$. Lets preform a polar substitution: $\lambda = \frac{r}{\tau}e^{i\theta}$. We can rewrite the characteristic equation:

$$\begin{aligned}
\lambda &= -\mu e^{-\lambda\tau} \\
\frac{r}{\tau}e^{i\theta} &= -\mu e^{-(\alpha+i\beta)\tau} \\
\frac{r}{\tau}[\cos(\theta) + i\sin(\theta)] &= -\mu e^{-\alpha\tau}(\cos(\beta\tau) - i\sin(\beta\tau)) \\
\frac{r}{\tau}[\cos(\theta) + i\sin(\theta)] &= -\mu e^{-\alpha\tau}(\cos(-\beta\tau) + i\sin(-\beta\tau)) \\
\frac{r}{\tau}[\cos(\theta - \pi) + i\sin(\theta - \pi)] &= \mu e^{-\alpha\tau}(\cos(-\beta\tau) + i\sin(-\beta\tau))
\end{aligned}$$

From this we can see that $\frac{r}{\tau} = \mu e^{-\alpha\tau}$, and $\beta\tau + 2\pi k = \pi - \theta$. Since we are looking for solutions on $\beta\tau \in (\frac{\pi}{2}, \pi)$ we will take $k = 0$ and $\theta \in (0, \frac{\pi}{2})$.
Given our substitution and the fact that we are looking for solutions on $\alpha > 0 \implies \alpha\tau > 0$ and $\beta\tau > 0$, we can apply trigonometry on the complex plane to find that $\tan(\theta) = \frac{\beta}{\alpha}$. This gives $\alpha = \beta\cot(\theta) = (\pi - \theta)\cot(\theta)$ which is strictly decreasing on $\theta \in (0, \frac{\pi}{2})$.
$\frac{r}{\tau} = \mu e^{-\alpha\tau}$ and $\frac{r}{\tau}\sin(\theta) = \beta\tau$ gives us $\mu\tau = \frac{\beta\tau e^{\alpha\tau}}{\sin(\theta)}$ which is strictly decreasing on $\theta \in (0, \frac{\pi}{2})$.
Now we have $\mu\tau = \frac{\beta\tau e^{\alpha\tau}}{\sin(\theta)}$, $\alpha\tau = (\pi - \theta)\cot(\theta)$, and $\beta\tau = \pi - \theta$ on $\theta \in (0, \frac{\pi}{2})$.
As $\theta \to 0$, $\alpha\tau \to \infty$, $\beta\tau \to \pi$, $\mu\tau \to \infty$, and as $\theta \to \frac{\pi}{2}$, $\alpha\tau \to 0$, $\beta\tau \to \frac{\pi}{2}$, $\mu\tau \to \frac{\pi}{2}$.
So $\alpha > 0$ as $\mu\tau \in (\frac{\pi}{2}, \infty)$. So the origin is not stable on this region of $\mu\tau$.

4.1.5 Cases of $\mu = 0$ and $\mu\tau = \frac{\pi}{2}$ can be seen to boundary cases given the continuous nature of the equations and the previous results.

4.1.6 Stability Summary: From these results we have a general idea of the stability of the origin over different values μ and τ. The origin is asymptotically stable when $\mu > 0$ and $\mu\tau < \frac{\pi}{2}$. The graph shown in Figure 1 represents the stability analysis of the simplest delay differential equation (DDE) at the origin.

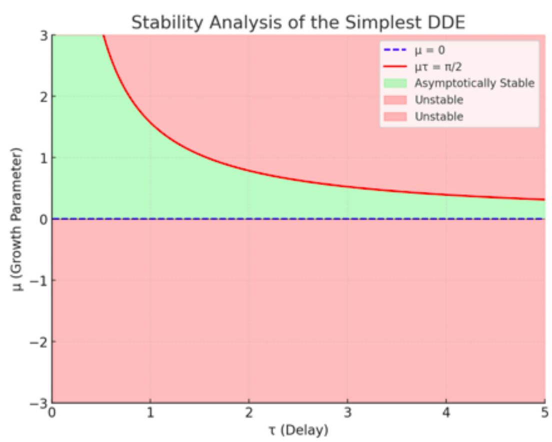

Figure 1: Stability of the Simplest DDE.

4.2 Observations

4.2.1 Oscillations and the effect of delays
After a cumbersome analysis of the transcendental characteristic equation, our stability analysis gives us insight into the effect the delay parameter.

To start, $\mu > 0$ now has an unstable region for large values of τ.
In addition, the roots of our characteristic equation have imaginary parts, raising oscillatory behavior in our system. This is particularly striking because our simple DDE is an autonomous differential equation on one variable x which would have no oscillations were it an ODE.

4.2.2 Infinite dimensional problem

The introduction of the delay τ effectively makes our equation an infinite dimensional problem. This becomes clear when considering the initial history problem. Unlike 1-D ODEs, which require one initial value, our DDE requires an initial history segment, an interval of values, making our problem infinite dimensional. This extra flexibility gives our system room to exhibit oscillations and even chaos in the one variable case.

4.2.3 Numerical solutions

To get a clearer picture of the dynamics of the system, we will employ numerical solutions. The following simulations use MATLAB's built-in DDEs solver, dde23.

Figure 2 shows the solution of our simple DDE over various choices of μ and τ. In each case, the initial history segment is a constant stretch of one. The time span simulated is $t \in [0, 50]$.
We experimented with the following cases:

1. $(\mu = -1, \tau = 0.125)$ is unstable.
2&3. $\mu = 1$ and $\tau \in \{0.125, 1\}$ are in the stable region, but we can observe the larger delay parameters introducing oscillations, and we see that increasing the delay increases the oscillations too.
3. $(\mu = 1, \tau = \pi/2)$ The delay is perfectly large enough such that the oscillations are a closed orbit around the origin.
5. $(\mu = 1, \tau = 1)$ is unstable and we see the instability as a result of the very large oscillations from the large delay.

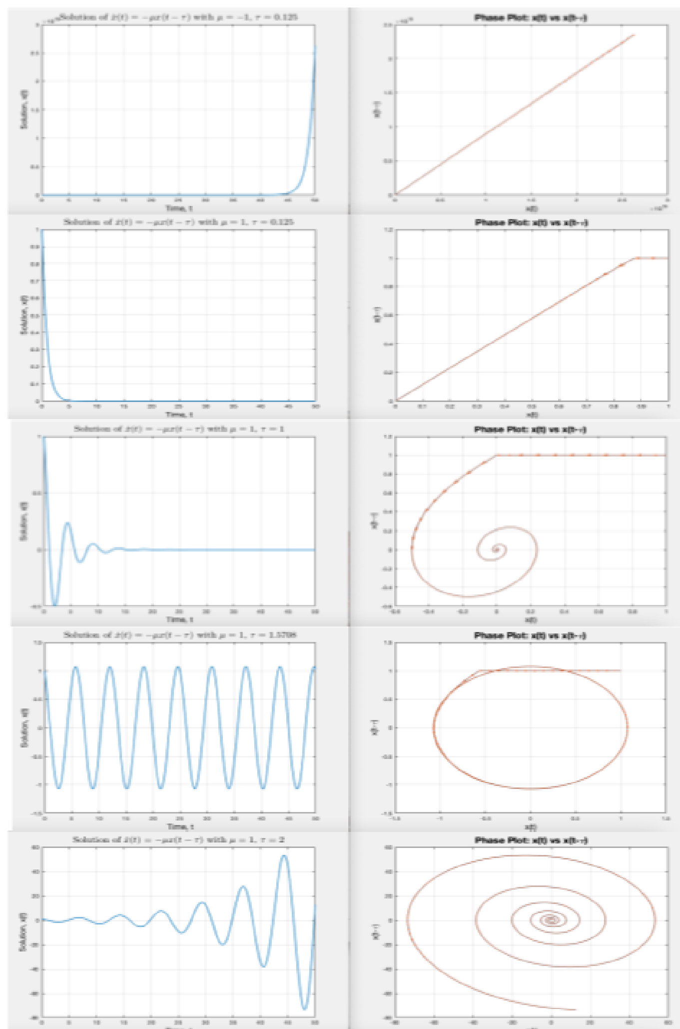

Figure 2: Dynamics of the Simplest DDE. The left shows position over time. The right is a phase portrait projection on $x(t)$ vs $x(t - \tau)$.

The straight line at the start of each phase portrait the inital evaluation with our constant initial history segment.

5 Applications

In this section we will explore the effect of delay in other models using numerical simulations rather than going through the complex stability analysis.

5.1 The Logistic Model

We will begin by returning to our delayed logistic model:

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right)$$

with $r, K > 0$ and $\tau \geq 0$. Figure 3 shows our simulation of $r = 0.5$, $K = 10000$, and $\tau \in \{0.5, 2.1\}$. We can see that very small delays do not change our model from the standard logistic curve very much. However, a larger delay introduces oscillations Both solutions nonetheless convincingly converge to K within $t \in [0, 50]$.

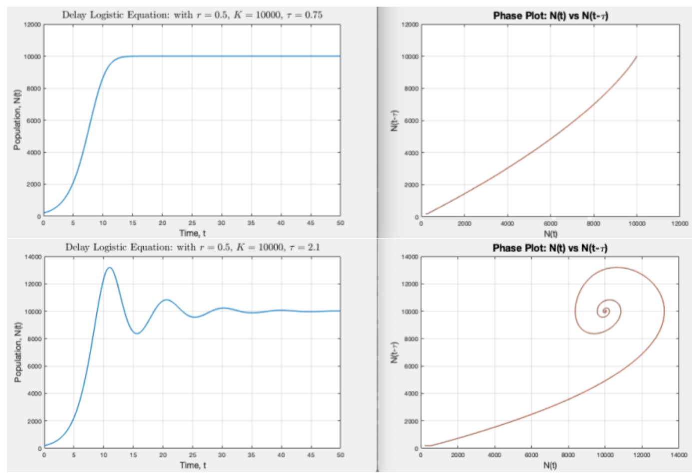

Figure 3: Dynamics of the Logistic DDE with small delays.

As a population model, we can see that past baby booms result of future baby booms delayed by τ as the population settles to carrying capacity.

Figure 3 shows our simulation larger delays. Keeping r and K the same, we have $\tau \in \{4, 6\}$. Unlike the simple DDE where solutions diverge from the steady state, out solutions settle into a limit cycle around K. This suggests that a Hopf bifurcation occurs for some $2.1 < \tau < 4$.

In each above simulation, we used an constant of 200 as the initial history function.

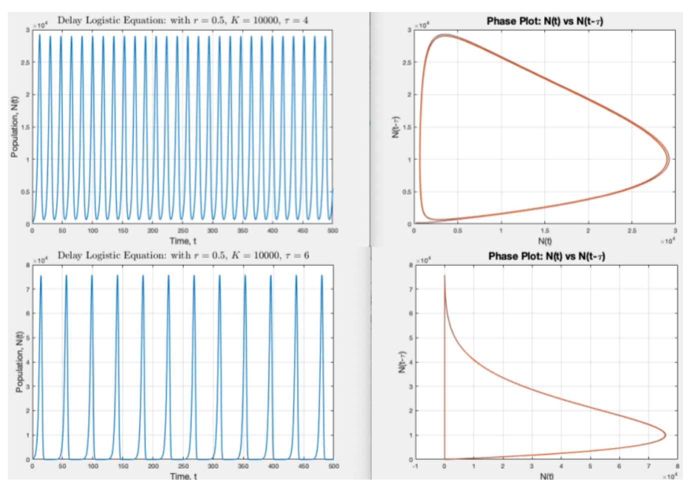

Figure 4: Dynamics of the Logistic DDE with larger delays.

5.2 The Mackey Glass Model

The Mackey Glass Equations were developed to model changes in complex physiological systems. Mackey and Glass suspected that many physiological diseases correspond to bifurcations in cell dynamics. Applications include hematology, psychiatry, neurology among other fields. These equations remain an open topic of research[2].

The follwoing is our example of a Mackey-Glass Equation:

$$\dot{P}(t) = \beta \frac{P(t-\tau)}{1 + P^n(t-\tau)} - \gamma P(t) \quad \gamma, n, \beta > 0 \quad \tau \geq 0$$

Keeping $\beta = 0.25$, $\gamma = 0.1$, and $n = 10$, we now investigate the effect of increasing $\tau \geq 0$. Additionally we will simulate on $t \in [0, 1000]$ and our initial history segment will be the constant 0.5.

Our first simulation takes $\tau \approx 0$. Our MATLAB function requires positive parameters. The results are shown in Figure 5. We can see a straight convergence towards a fixed point just above 1.

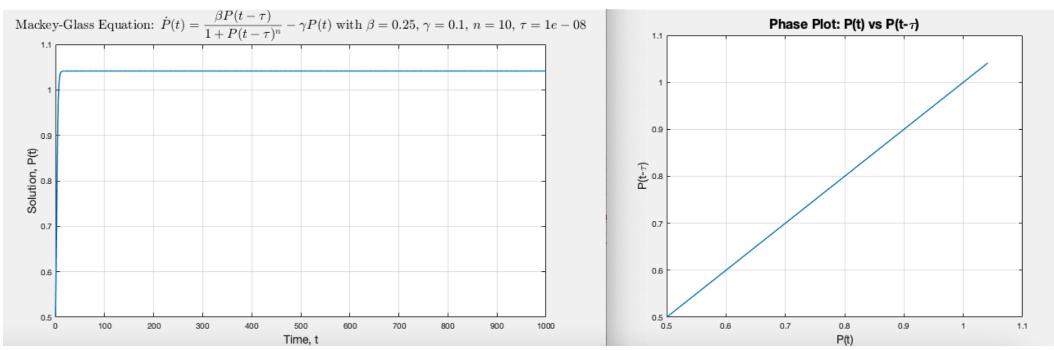

Figure 5: Mackey-Glass with $\tau \approx 0$

Our second simulation takes $\tau = 3$ and is shown in Figure 6. Here we observe that the dynamics of our system converges in a spiral towards the stable fixed point.

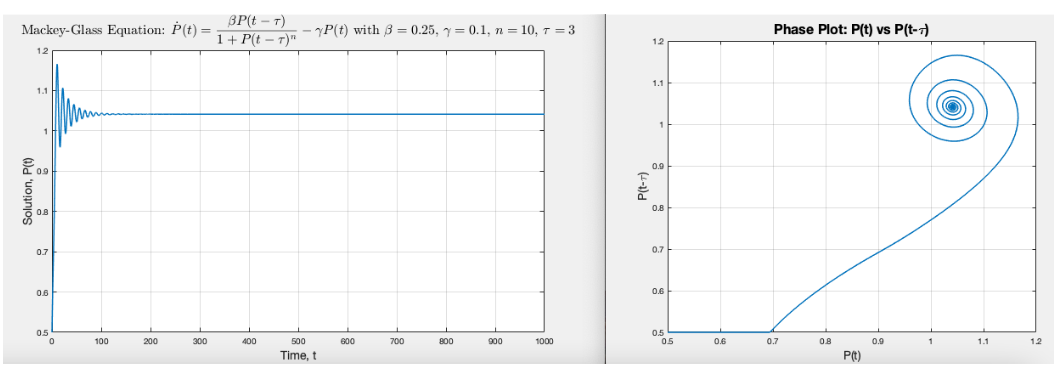

Figure 6: Mackey-Glass with $\tau = 3$

Figure 7 shows a simulation with $\tau = 6$. Here we observe another change in the dynamics of our system. Akin to the logistic system, our solution settled into a limit cycle around the fixed point, raising a Hopf bifurcation for some critical delay value between 3 and 6.

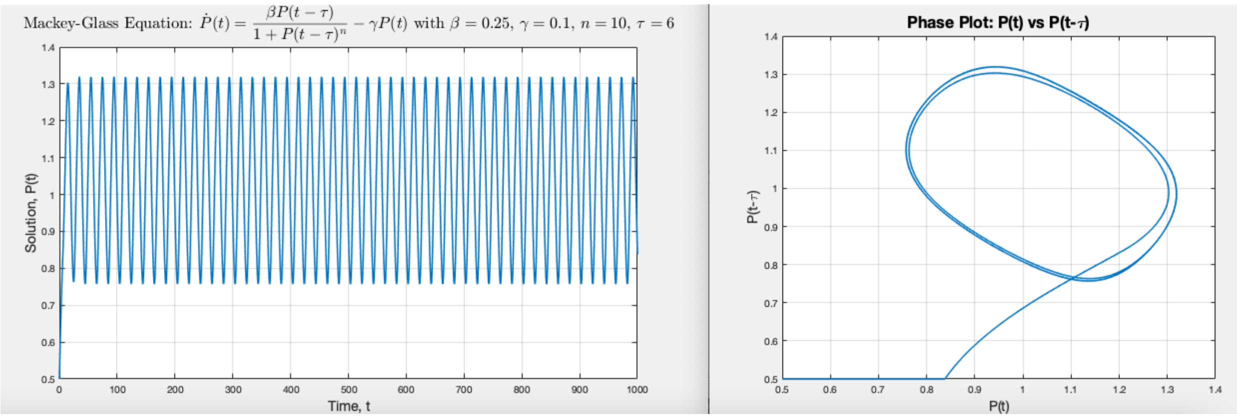

Figure 7: Mackey-Glass with $\tau = 6$

Our last simulation is for $\tau = 18$. Figure 8 shows chaotic dynamics. As stated previously, the infinite dimensional nature of the initial history problem allows the flexibility for the one variable dynamical system to exhibit chaos. The phase portrait projection gives a sketch of a strange attractor in this system.

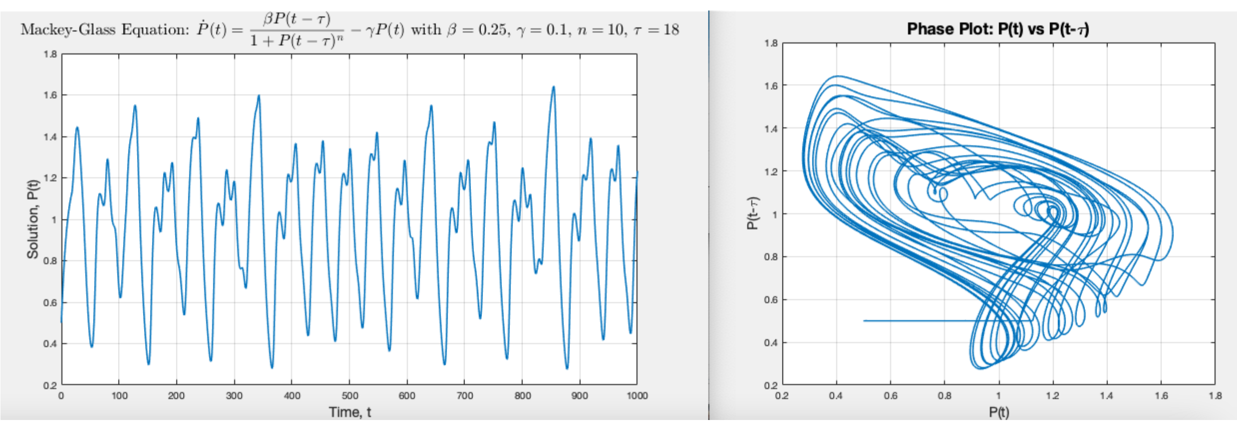

Figure 8: Mackey-Glass with $\tau = 18$

This chaotic behavior makes the Mackey-Glass model useful in Time Series Analysis model evaluation. More on this can be found in [3].

6 Conclusion

In this project, we examined Delay Differential Equations (DDEs), an extension of Ordinary Differential Equations which incorporate a trajectory's history in its current dynamics. This necessitates an initial history segment for any solution to be found. The introduction of a delay allows our dynamical system model to be more expressive, but we may have more difficulty analyzing our model's dynamics. Among all the different ways to express delay, we focused on DDEs of single discrete delay, and we justified an existence theorem for this case with the method of steps. To demonstrate our local linearization theory, we preformed stability analysis on a simple DDE. After a cumbersome examination of a transcendental characteristic equation, we were able to anticipate the behavior of our system. Further numerical simulation verified that the introduction of delay can result in oscillations which, when large enough, could make an initially stable fixed point unstable. The appearance of oscillations in a one variable autonomous equation is explained by the infinite dimensional nature of the initial history problem. Furthermore, the infinite dimensions allow trajectories to exhibit limit cycles, and chaos. We examined these behaviors in subsequent simulations of the logistic DDE and the Mackey-Glass Equation.

A Appendix: References

[1] Hal Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, 2011.

[2] M.C. Mackey and L. Glass, *The Mackey-Glass Equation: History and Applications*, McGill University, 2009. Available at: *https://www.mcgill.ca/mathematical-physiology-lab/files/mathematical-physiology-lab/2009dec_mackey_glass_equation.pdf*.

[3] Markus Thill, *MGAB: Mackey-Glass Artificial Benchmark*. GitHub repository, 2024. Available at: *https://github.com/MarkusThill/MGAB?tab=readme-ov-file*.

B Appendix: Code Listings

Simple DDE simulation:

```
function plotSimpleDelay(mu, tau, tspan)
    % Function to simulate and plot the solution of a DDE
    % Parameters:
    %   mu - parameter in the DDE
    %   tau - time delay
    %   tspan - simulation time span, eg [0, 20]

    % History function (x(t) for t <= 0)
    history = @(t) 1; % Constant initial value, can be modified

    % DDE definition
    dde = @(t, x, x_delayed) -mu * x_delayed;

    % Solver options for more points near the start
    options = ddeset('RelTol', 1e-6, 'AbsTol', 1e-8, 'InitialStep', 1e-4, 'MaxStep', 0.1);

    % Solve DDE
    sol = dde23(dde, tau, history, tspan, options);
```

```

    % Plot x(t) vs t
    figure;
    plot(sol.x, sol.y, 'LineWidth', 1.5);
    xlabel('Time, t', 'FontSize', 14);
    ylabel('Solution, x(t)', 'FontSize', 14);
    title(['Solution of $$\dot{x}(t) = -\mu x(t-\tau)$$ with $$\mu = ', num2str(mu), '$$, $$\tau = ', num2str(tau), '$$'], 'Interpreter', 'latex', 'FontSize', 16);
    grid on;

    % Phase plot: x(t) vs x(t-tau)
    figure;
    x_delayed = zeros(size(sol.x));
    for i = 1:length(sol.x)
        if sol.x(i) <= tau
            x_delayed(i) = history(sol.x(i) - tau);  % Use history function
        else
            x_delayed(i) = interp1(sol.x, sol.y, sol.x(i) - tau, 'spline', 'extrap'); % Interpolate delayed values
        end
    end
    plot(sol.y, x_delayed, 'LineWidth', 1.5);
    hold on;
    dx = diff(sol.y); % Change in x(t)
    dy = diff(x_delayed); % Change in x(t-tau)
    quiver(sol.y(1:end-1), x_delayed(1:end-1), dx, dy, 'AutoScale', 'off', 'MaxHeadSize', 0.2);
    xlabel('x(t)', 'FontSize', 14);
    ylabel('x(t-\tau)', 'FontSize', 14);
    title('Phase Plot: x(t) vs x(t-\tau)', 'FontSize', 16);
    grid on;
end

%plotSimpleDelay(-1, 0.125, [0, 50])
%plotSimpleDelay(1, 0.125, [0, 50])
%plotSimpleDelay(1, 1, [0, 50])
%plotSimpleDelay(1, pi/2, [0, 50])
%plotSimpleDelay(1, 2, [0, 50])

```

Logistic DDE simulation:

```
function plotLogisticDelay(r, K, tau, tspan)
    % Function to solve and plot the delay logistic differential equation
    % Parameters:
    %   r   - Growth rate
    %   K   - Carrying capacity
    %   tau - Delay
    %   tspan - time span

    % History function (N(t) for t <= 0)
    history = @(t) 200; % Initial population (can be adjusted)

    % DDE definition
    dde = @(t, N, N_delayed) r * N * (1 - N_delayed / K);

    % Solver options
    options = ddeset('RelTol', 1e-6, 'AbsTol', 1e-8, 'InitialStep', 1e-4, 'MaxStep', 0.1);

    % Solve DDE
    sol = dde23(dde, tau, history, tspan, options);

    % Plot N(t) vs t
    figure;
    plot(sol.x, sol.y, 'LineWidth', 1.5);
    xlabel('Time, t', 'FontSize', 14);
    ylabel('Population, N(t)', 'FontSize', 14);
    title(['Delay Logistic Equation: ' ...
        ' with $$r = ', num2str(r), '$$, $$K = ', num2str(K), '$$, $$\tau = ', num2str(tau), '$$'], ...
        'Interpreter', 'latex', 'FontSize', 16);
    grid on;

    % Phase plot: N(t) vs N(t-tau)
    figure;
    N_delayed = zeros(size(sol.x));
    for i = 1:length(sol.x)
        if sol.x(i) <= tau
            N_delayed(i) = history(sol.x(i) - tau);  % Use history function
        else
            N_delayed(i) = interp1(sol.x, sol.y, sol.x(i) - tau, 'spline', 'extrap'); % Interpolate delayed values
        end
    end
    plot(sol.y, N_delayed, 'LineWidth', 1.5);
    hold on;
    dN = diff(sol.y); % Change in N(t)
    dN_delayed = diff(N_delayed); % Change in N(t-tau)
    quiver(sol.y(1:end-1), N_delayed(1:end-1), dN, dN_delayed, 'AutoScale', 'off', 'MaxHeadSize', 0.2);
    xlabel('N(t)', 'FontSize', 14);
    ylabel(['N(t-\tau)'], 'FontSize', 14);
    title('Phase Plot: N(t) vs N(t-\tau)', 'FontSize', 16);
    grid on;
end
```

```matlab

%plotLogisticDelay(0.5,10000,0.75,[0,50])
%plotLogisticDelay(0.5,10000,2.1,[0,50])
%plotLogisticDelay(0.5,10000, 4,[0,500])
%plotLogisticDelay(0.5,10000,6,[0,500])
```

Mackey-Glass DDE simulation:

```matlab
function plotMackeyGlass(beta, gamma, n, tau, tspan)
    % Function to solve and visualize the Mackey-Glass equation
    % Parameters:
    %   beta  - Growth rate parameter
    %   gamma - Decay rate parameter
    %   n     - Nonlinearity parameter
    %   tau   - Time delay
    %   tspan - Time span for simulation eg. [0,200]

    % History function (P(t) for t <= 0)
    history = @(t) 0.5;  % Initial condition for P(t) (can be adjusted)

    % Define the Mackey-Glass DDE
    dde = @(t, P, P_delayed) beta * P_delayed / (1 + P_delayed^n) - gamma * P;

    % Solver options
    options = ddeset('RelTol', 1e-6, 'AbsTol', 1e-8, 'InitialStep', 1e-4, 'MaxStep', 0.1);

    % Solve the DDE
    sol = dde23(dde, tau, history, tspan, options);

    % Plot P(t) vs t
    figure;
    plot(sol.x, sol.y, 'LineWidth', 1.5);
    xlabel('Time, t', 'FontSize', 14);
    ylabel('Solution, P(t)', 'FontSize', 14);
    title(['Mackey-Glass Equation: $$\dot{P}(t) = \frac{\beta P(t-\tau)}{1 + P(t-\tau)^n} - \gamma P(t)$$' ...
           ' with $$\beta = ', num2str(beta), '$$, $$\gamma = ', num2str(gamma), '$$, $$n = ', num2str(n), '$$, $$\tau = ', num2str(tau), '$$'], ...
           'Interpreter', 'latex', 'FontSize', 16);
    grid on;

    % Phase plot: P(t) vs P(t-tau)
    figure;
    P_delayed = zeros(size(sol.x));
    for i = 1:length(sol.x)
        if sol.x(i) <= tau
            P_delayed(i) = history(sol.x(i) - tau);  % Use history function
        else
            P_delayed(i) = interp1(sol.x, sol.y, sol.x(i) - tau, 'spline', 'extrap'); % Interpolate delayed values
        end
    end
    plot(sol.y, P_delayed, 'LineWidth', 1.5);
    xlabel('P(t)', 'FontSize', 14);
    ylabel('P(t-\tau)', 'FontSize', 14);
    title('Phase Plot: P(t) vs P(t-\tau)', 'FontSize', 16);
    grid on;
end
%plotMackeyGlass(0.25, 0.1, 10, 0.00000001, [0,1000])
%plotMackeyGlass(0.25, 0.1, 10, 3, [0,1000])
%plotMackeyGlass(0.25, 0.1, 10, 6, [0,1000])
%plotMackeyGlass(0.25, 0.1, 10, 18, [0,1000])

```