The Generalized Jug Problem

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Abstract

An example of a jug problem is, given a three-liter and a five-liter jug and plenty of water, can you get exactly four liters of water. Here we generalize the problem to a p-liter and q-liter jug and prove that if p and q are relatively prime, any integer amount of water between 1 and $p+q$ may be obtained. Furthermore, an algorithm for finding such amounts is given.

1 Introduction

The second problem of the IEEE Gamebook [1] is, "If you have a five-liter and a three-liter bottle and plenty of water, how can you get four liters of water in the five-liter bottle?" This problem hit popular culture fame in the movie *Die Hard 3*. A natural generalization of this problem is, "Given a p-liter and q-liter jug what amounts of water can be obtained?"

Throughout this paper, it will be assumed that $1 < p < q$. We prove that if p and q are relatively prime, then we can obtain all integer amounts of water between 1 and $p + q$. We begin by simplifying the problem, and then give an example to illustrate the algorithm that will be generalized in the proof. The proof follows and the paper ends with a conjecture about the case when p and q are not relatively prime.

2 Example

The problem may be simplified by noticing that if we can obtain liter amounts of water from one liter to $(p-1)$ liters of water, then we can obtain the other amounts up to $p+q$. Since we have only the two jugs to work with, the final amount of water must end up in one of the jugs or split between both of the jugs. For instance, $p + q - 1$ liters of water must be split between the two jugs. To find x liters of water where $p < x < q$, use the division algorithm to find r where $0 < r < p$ and $np + r = x$. Put r liters of water into the q-liter jug, which we assumed can be done, and add n fills of the p-liter jug. If $q < x < p + q$ then put $r = x - q < p$ liters of water into the p-liter jug and fill up the q-liter jug.

Recall that two numbers are relatively prime if they have no common divisors. We will use a five-liter and a twelve-liter jug to illustrate the algorithm. To obtain liter amounts of water between one and four we begin by putting ten liters of water into the twelve-liter jug. Fill the five-liter jug and top off the twelve-liter jug leaving three liters of water in the five-liter jug. We have now obtained three liters of water. Empty the twelve-liter jug and put into it the three liters of water currently in the five-liter jug. Now add five more liters to the twelve-liter jug leaving four liters of empty space. Fill the five-liter jug and top off the twelve-liter jug leaving one liter in the five-liter jug. If we continue on in this fashion, we claim that we will cycle through all the values between one and four.

3 The Proof

The following set of equations obtained by using the division algorithm, with $0 \leq r_i < p$, generalizes the above example:

$$\begin{aligned} n_1p + r_1 &= q \\ n_2p + r_2 &= q - (p - r_1) \\ n_3p + r_3 &= q - (p - r_2) \\ &\vdots \\ n_pp + r_p &= q - (p - r_{p-1}). \end{aligned} \tag{1}$$

Comparing this to the example, we see that r_1 represents the two liters of empty space in the twelve-liter jug after adding $n_1 = 2$ five-liter jugs of water. The $p - r_1$ represents what is left in the five-liter jug after topping off the twelve-liter jug. The r_2 is the four liters of empty space in the twelve-liter jug after it is filled with the $p - r_1 = 3$ liters of water, and then a full five ($n_2 = 1$). We continue in this fashion. What must be shown is that

$$\{p - r_1, p - r_2, \ldots, p - r_{p-1}\} = \{1, 2, \ldots, p - 1\}$$

or equivalently, since $0 \leq r_i < p$,

$$\{r_1, r_2, \ldots, r_{p-1}\} = \{1, 2, \ldots, p - 1\}.$$

Solving the equations in (1) for the r_i's yields

$$\begin{aligned} r_1 &= q - n_1p \\ r_2 &= q - (p - r_1) - n_2p = q + r_1 - (n_2 + 1)p \\ r_3 &= q - (p - r_2) - n_3p = q + r_2 - (n_3 + 1)p \\ &\vdots \\ r_p &= q - (p - r_{p-1}) - n_pp = q + r_{p-1} - (n_p + 1)p. \end{aligned}$$

If we "mod$p$" the equations become,

$$\begin{aligned} r_1 &\equiv q \bmod p \\ r_2 &\equiv q + r_1 \bmod p \\ r_3 &\equiv q + r_2 \bmod p \\ &\vdots \\ r_p &\equiv q + r_{p-1} \bmod p, \end{aligned}$$

and solving recursively we end up with,

$$\begin{aligned}
r_1 &\equiv q \bmod p \\
r_2 &\equiv 2r_1 \bmod p \\
r_3 &\equiv 3r_1 \bmod p \\
&\vdots \\
r_p &\equiv pr_1 \bmod p.
\end{aligned}$$

For now, let us suppose that r_1 and p are relatively prime. Then r_1 generates the group $\mathbb{Z}_p$ [2], so that

$$\{r_1, r_2, r_3, \ldots, r_{p-1}\} \equiv \{r_1, 2r_1, 3r_1, \ldots, (p-1)r_1\} \equiv \{1, 2, 3, \ldots, p-1\},$$

mod$p$. Further, because $r_i < p$ for all i,

$$\{r_1, r_2, r_3, \ldots, r_{p-1}\} = \{1, 2, 3, \ldots, p-1\},$$

which is what we needed to show.

Fortunately, r_1 and p are relatively prime. First, since p and q are relatively prime, the division algorithm gives $r_1 \neq 0$. If p and r_1 are relatively prime, we are done. If not, we can write $mr_1 = p$. Substituting this for p in $n_1 p + r_1 = q$ gives $(n_1 m + 1)r_1 = q$. In other words, r_1 divides p and q. This can only happen if $r_1 = 1$, in which case r_1 is still relatively prime to p.

If we look at the proof, we used the fact that p and q are relatively prime to get that r_1 generated all of $\mathbb{Z}_p$. Without this, r_1 will only generate a subgroup of $\mathbb{Z}_p$ and our algorithm fails. This leads us to conjecture that if p and q are not relatively prime then you can not obtain all liter amounts of water between 1 and $p+q$. In fact, you may only be able to obtain whatever you get from the subgroup generated by r_1, plus any multiples of p and q.

References

[1] Mack, D., *The Unofficial IEEE Brainbuster Gamebook*, The Institute of Electrical and Electronics Engineers, New York, 1992.

[2] Rotman, J., *An Introduction to the Theory of Groups* 3rd ed, Wm. C. Brown Publishers, Iowa, 1988.