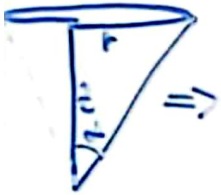
Calcule la trayectoria que da la distancia mas corta entre dos puntos sobre la superficie de un cono invertido, con ángulo de vertice α. Use coordenadas cilíndricas.

En coordenadas cilindricas, el elemento de linea es $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$



$\Rightarrow \tan\alpha = \frac{r}{z}$

$\hookrightarrow r = z\tan\alpha$

$dr = \tan\alpha\, dz$

$$ds^2 = \tan^2\alpha\, dz^2 + z^2\tan^2\alpha\, d\theta^2 + dz^2$$

$$ds^2 = z^2\tan^2\alpha\, d\theta + (\tan^2\alpha + 1)\, dz^2$$

$$ds^2 = z^2\tan^2\alpha\, d\theta^2 + \sec^2\alpha\, dz^2$$

$$ds = \sqrt{z^2\tan^2\alpha\, d\theta^2 + \sec^2\alpha\, dz^2}$$

$$S = \int ds = \int_{\theta_1}^{\theta_2} \sqrt{z^2\tan^2\alpha + \sec^2\alpha\, \dot{z}^2}\, d\theta \qquad \mathcal{L} = \sqrt{z^2\tan^2\alpha + \sec^2\alpha\, \dot{z}^2}$$

$$\mathcal{L}^2 = z^2\tan^2\alpha + \sec^2\alpha\, \dot{z}^2$$

Usando las ecuaciones de euler-lagrange y hallando el cuadrado de la distancia

$$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{z}} = 2\sec^2\alpha\, \dot{z} \Rightarrow \frac{d}{d\theta}(2\sec^2\alpha\, \dot{z}) = 2\sec^2\alpha\, \ddot{z}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2z\tan^2\alpha$$

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$$\frac{2\sec^2\alpha\,\ddot{z}}{2\sec^2\alpha} - \frac{2z\tan^2\alpha}{2\sec^2\alpha} = 0$$

$$\ddot{z} - \operatorname{sen}^2\alpha\, z = 0$$

Ec. auxiliar: $r^2 - \operatorname{sen}^2\alpha = 0 \Rightarrow r = \pm \operatorname{sen}\alpha$

$$z(\theta) = C_1 e^{\theta \operatorname{sen}\alpha} + C_2 e^{-\theta \operatorname{sen}\alpha}$$

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Calcule el valor mínimo de la integral

$$I = \int_0^1 \left[(y')^2 + 12xy \right] dx$$

donde la función $y(x)$ satisface $y(0) = 0$ y $y(1) = 1$

Usando la ecuación de euler

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \text{, donde } f(y(x), y'(x), x) = (y')^2 + 12xy$$

$$\hookrightarrow \frac{\partial f}{\partial y'} = 2y' \wedge \frac{\partial f}{\partial y} = 12x \text{, sustituyendo:}$$

$$\frac{d}{dx}(2y') - 12x = 0 \Rightarrow 2y'' - 12x = 0 \Rightarrow y'' = 6x$$

$$\int y''(x)\,dx = \int 6x\,dx \Rightarrow y'(x) = 3x^2 + C_1 \Rightarrow \int y'(x)\,dx = \int \left[3x^2 + C_1\right] dx$$

$$y(x) = x^3 + C_1 x + C_2 \begin{cases} \to y(0) = 0 \Rightarrow 0 = (0)^3 + C_1(0) + C_2 \Rightarrow C_2 = 0 \\ \to y(1) = 1 \Rightarrow 1 = (1)^3 + C_1(1) \Rightarrow C_1 = 0 \end{cases} \quad \therefore y(x) = x^3$$

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Sustituyendo $y(x) = x^3$ en la integral:

$$I = \int_0^1 \left[(3x^2)^2 + 12x(x^3) \right] dx = \int_0^1 \left[9x^4 + 12x^4 \right] dx = \int_0^1 21x^4 \, dx = \frac{21x^5}{5} \Big|_0^1$$

$$I = \frac{21}{5}$$

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3) $P_1 = (a, 0, 0)$ $\qquad$ $P_2 = (-a, 0, \pi)$

$x^2 + y^2 - a^2 = 0 \rightarrow$ Radio a constante.

$$S = \int_{P_1}^{P_2} dr \Rightarrow \quad dr^2 = dx^2 + dy^2 + dz^2$$

$$\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \\ z &= z \end{aligned} \rightarrow \begin{aligned} dx &= -a\sin\theta\, d\theta \\ dy &= a\cos\theta\, d\theta \\ dz &= dz \end{aligned}$$

$$dr^2 = (-a\sin\theta d\theta)^2 + (a\cos\theta d\theta)^2 + dz^2$$

$$dr^2 = a^2 d\theta^2 (\sin^2\theta + \cos^2\theta) + dz^2$$

$$dr = \sqrt{a^2 d\theta^2 + dz^2}$$

$$S = \int_{P_1}^{P_2} \sqrt{(a^2 d\theta^2 + dz^2)} = \int_{P_1}^{P_2} \sqrt{dz^2\left[\left(a^2 \frac{d\theta^2}{dz^2}\right) + 1\right]}$$

$$S = \int_{P_1}^{P_2} \sqrt{a^2\left(\frac{d\theta}{dz}\right)^2 + 1}\, dz \rightarrow \quad \mathcal{L} = \sqrt{a^2\dot{\theta}^2 + 1}$$

→ Se resuelve:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \cancel{\frac{\partial \mathcal{L}}{\partial \theta}}^{=0} = 0$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\sqrt{a^2\dot{\theta}^2 + 1}\right) = \frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2 + 1}}$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\underbrace{\left(\frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2 + 1}}\right)}_{= cte} = 0$$

$$\Rightarrow \frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2 + 1}} = C \rightarrow \text{Se despeja } \dot{\theta}$$

$$\rightarrow \left(\frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2 + 1}}\right)^2 = C^2 \Rightarrow \quad a^4\dot{\theta}^2 = C^2(a^2\dot{\theta}^2 + 1)$$

$$a^4\dot{\theta}^2 - C^2 a^2\dot{\theta}^2 = C^2$$

$$\dot{\theta}^2 = \frac{C^2}{a^4 - a^2C^2}$$

$$\Rightarrow \frac{d\theta}{dz} = \frac{C}{\sqrt{a^4 - a^2C^2}} \Rightarrow \int d\theta = \int \frac{C\,dz}{\sqrt{a^4 - a^2C^2}}$$

$$\theta(z) = \frac{C}{a\sqrt{a^2 - C^2}} z + D$$

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Problemas de frontera

$\theta(0) = 0$ $\qquad\qquad$ $\theta(\pi) = \pi$

Coordenadas P_1 y P_2 $\qquad\qquad$ Se dice que vale π porque el radio varía de a a $-a$, es decir, π.

i) $\theta(0) = \dfrac{c}{a\sqrt{a^2 - c^2}}(0) + D = 0 \Rightarrow D = 0$

ii) $\theta(\pi) = \dfrac{c}{a\sqrt{a^2 - c^2}}\pi + 0 = \pi \Rightarrow \dfrac{c}{a\sqrt{a^2 - c^2}} = 1$

Se despeja c:

$$c = a\sqrt{a^2 - c^2}$$

$$c^2 = a^2(a^2 - c^2)$$

$$c^2 + a^2c^2 = a^4$$

$$c^2 = \frac{a^4}{a^2+1} \Rightarrow c = \frac{a^2}{\sqrt{a^2+1}}$$

$$\Rightarrow \theta(z) = \frac{\dfrac{a^2}{\sqrt{a^2+1}}}{a\sqrt{a^2 - \left(\dfrac{a^2}{\sqrt{a^2+1}}\right)^2}}$$

$$\rightsquigarrow a^2 - \left(\frac{a^4}{a^2+1}\right) = \frac{a^4 + a^2 - a^4}{a^2+1} = \frac{a^2}{a^2+1}$$

$$\Rightarrow \theta(z) = \frac{\dfrac{a^2}{\sqrt{a^2+1}}}{a\sqrt{\dfrac{a^2}{a^2+1}}} z = \frac{\dfrac{\cancel{a^2}}{\cancel{\sqrt{a^2+1}}}}{\dfrac{\cancel{a^2}}{\cancel{\sqrt{a^2+1}}}} z = z$$

$$\boxed{\theta(z) = z}$$

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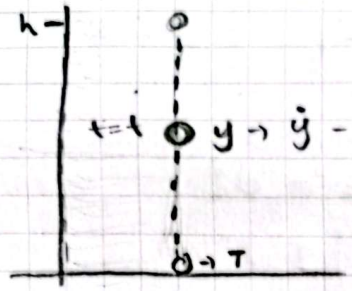
4)



$$y = h - g_1 t$$
$$y = h - \frac{1}{2} g_2 t^2$$
$$y = h - \frac{1}{4} g_3 t^3$$

Energías

$$\mathcal{L} = T - V \quad \rightarrow \quad T = \frac{1}{2} m V^2 = \frac{1}{2} m \dot{y}^2$$
$$V = mgh = mgy$$

$$\mathcal{L} = \frac{1}{2} m \dot{y}^2 - mgy$$

$$S = \int_0^T \mathcal{L}\, dt = \int_0^T \left(\frac{1}{2} m \dot{y}^2 - mgy \right) dt$$

Para obtener la ecuación de movimiento se resuelve

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left(\frac{1}{2} m \dot{y}^2 - mgy \right) = m\dot{y}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) = \frac{d}{dt}\left(m\dot{y} \right) = m\ddot{y}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y} \left(\frac{1}{2} m \dot{y}^2 - mgy \right) = -mg$$

→ Se reemplaza:

$$m\ddot{y} - (-mg) = 0 \quad \Rightarrow \quad m(\ddot{y} + g) = 0$$

→ Se integra:

i) $\int d\dot{y} = \int -g\, dt \quad \Rightarrow \quad \dot{y}(t) = -gt + \dot{y}_0$

ii) $\int dy = \int (-gt + \dot{y}_0)\, dt \quad \Rightarrow \quad y(t) = -\frac{1}{2} g t^2 + \dot{y}_0 t + y_0$

→ Como se deja caer desde $y=0$, entonces:

$$\dot{y}(0) = 0 \quad \wedge \quad y(0) = h$$

$$\Rightarrow \quad y(t) = h - \frac{1}{2} g t^2$$

→ Es el movimiento que hace la partícula y que corresponde a de mínima acción.

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El Lagrangiano de una partícula con masa m es:

$$\mathcal{L} = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)$$

Donde $f(x)$ es una función diferenciable de x. Encuentre la ec. de movimiento.

Usando la ecuación de euler-lagrange:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}f(x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = m\dot{x}^2 f'(x) - 2f(x)f'(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}^2 f'(x)$$

Sustituyendo:

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}^2 f'(x) - m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$$

$$\Rightarrow m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$$

$$\Rightarrow m\ddot{x}(m\dot{x}^2 + 2f(x)) + f'(x)(m\dot{x}^2 + 2f(x)) = 0$$

$$\Rightarrow (m\dot{x}^2 + 2f(x))(m\ddot{x} + f'(x)) = 0$$

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6. Comprobar que las ecuaciones de Lagrange de

$$\mathcal{L} = \frac{1}{2} g_{ab}(q_c)\, \dot{q}^a \dot{q}^b$$

son

$$\ddot{q}^d + \Gamma^d_{bc} \dot{q}^b \dot{q}^c = 0$$

donde

$$\Gamma^d_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right)$$

→ Para obtener las ecuaciones se resuelve:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

⇒ Primera Parte:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial}{\partial \dot{q}^i}\left(\frac{1}{2} g_{ab}(q_c)\, \dot{q}^a \dot{q}^b\right) \rightarrow \text{Regla de la cadena}$$

$$= \frac{1}{2} g_{ab}(q_c) \left[\frac{\partial}{\partial \dot{q}^i}\left(\dot{q}^a\right) \dot{q}^b + \dot{q}^a \frac{\partial}{\partial \dot{q}^i}\left(\dot{q}^b\right) \right]$$

Se cambian los índices, según la suma de Einstein.

$$= \frac{1}{2} g_{ab} \left(\delta^a{}_i\, \dot{q}^b + \delta^b{}_i\, \dot{q}^a \right) = \frac{1}{2}\left(g_{ib}\, \dot{q}^b + g_{ai}\, \dot{q}^a \right)$$

$$= \frac{1}{2}\left(g_{ab}\, \dot{q}^b + g_{ab}\, \dot{q}^b \right) = g_{ab}(q_c)\, \dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = \frac{d}{dt}\left(g_{ab}(q_c)\, \dot{q}^b \right)$$

$$= \frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b + g_{ab} \ddot{q}^b$$

⇒ Segunda Parte:

$$\frac{\partial \mathcal{L}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(\frac{1}{2} g_{ab}(q_c)\, \dot{q}^a \dot{q}^b\right)$$

$$= \frac{1}{2} \frac{\partial g_{ab}}{\partial q^c} \dot{q}^a \dot{q}^b \underset{\text{suma de Einstein}}{=} \frac{1}{2} \frac{\partial g_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c$$

→ Se reemplazan en la ecuación:

$$\frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b + g_{ab} \ddot{q}^b - \frac{1}{2} \frac{\partial g_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c = 0$$

$$\cancel{g^{da} g_{ab}}^{\,\delta^d_b}\, \ddot{q}^b + g^{da} \frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b - \frac{1}{2} \frac{\partial g_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c g^{da} = 0$$

$$\ddot{q}^b + g^{ad}\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2} \frac{\partial g_{bc}}{\partial q^a} \right) \dot{q}^b \dot{q}^c = 0$$

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$$\ddot{q}^b + g^{ad}\left(\frac{1}{2}\frac{\partial g_{ab}}{\partial q^c} + \frac{1}{2}\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^b\dot{q}^c = 0$$

→ Para que quede Γ^a_{bc} igual, entonces se usa la suma de Einstein para cambiar subíndices, respetando que cuando se repiten son sumas:

$$\ddot{q}^a + \frac{1}{2}g^{ad}\left(\frac{\partial g_{db}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^b\dot{q}^c = 0$$

reemplazando Γ^a_{bc} por el dado en el planteamiento del problema.

$$\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0$$

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