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# Modeling stock prices by multifractional Brownian motion: an improved estimation of the pointwise regularity

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This paper deals with the problem of estimating the pointwise regularity of multifractional Brownian motion, assumed as a model of stock price dynamics. We (a) correct the shifting bias affecting a class of absolute moment-based estimators and (b) build a data-driven algorithm in order to dynamically check the local Gaussianity of the process. The estimation is therefore performed for three stock indices: the Dow Jones Industrial Average, the FTSE 100 and the Nikkei 225. Our findings show that, after the correction, the pointwise regularity fluctuates around 1/2 (the sole value consistent with the absence of arbitrage), but significant deviations are also observed.

*Keywords*: Multifractional Brownian motion; Pointwise Hölder exponent; Asset price dynamics; time varying parameter; Applied finance

*JEL Classification*: C1, C2, C5, C13, C22, C51, G1, G14

## 1. Introduction and motivation

In recent years the modeling of stock returns has become one of the central issues of finance. The growing awareness of the limits of standard financial theory originates from both the huge number of empirical contributions produced in the last quarter century and the recent financial crises. The former provided significant evidence that seriously weakens the Brownian motion paradigm and its variants. The latter generates shocks whose magnitude is several times greater than those foreseen by the dominant paradigm. They both call for a rethink of the very bases of the theory, which is increasingly unable to account for the actual complexity of financial markets. New insights are likely to come from framing the notion of efficiency in a dynamic perspective, which means including market inefficiencies in the models rather than considering them as pathological outliers to be hidden under the carpet of an equilibrium model.

Therefore, the challenge becomes to preserve the elegant and theoretically grounded no-arbitrage theory of the Brownian world and while encompassing the turbulence displayed by the empirical dynamics. Many models have been proposed with this aim, relaxing the hypothesis of independence or the assumption of an identical distribution of the log price variations. Although necessary to infer the overall probabilistic properties of a model, the assumption of stationarity is nevertheless one of the most hard to digest sources of unrealism; as recent history teaches, if artlessly used by practitioners, it can lead to dramatic mistakes in terms of both risk assessment and the consequent decision making. In the standard Gaussian framework, stationarity mostly means assuming

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that the process pointwise regularity (quantified by the Hölder exponent†) is time-invariant; at the same time, independence means assuming that the Hölder parameter is almost everywhere equal to 1/2. This is claimed as the sole value consistent with the arbitrage principle, the cornerstone of all modern financial theory (Rogers 1997).

The model we look at in this paper relaxes both these assumptions: the multifractional Brownian motion (mBm) is a non-stationary-dependent Gaussian process that generalizes the well-known fractional Brownian motion by allowing the process pointwise regularity to change over time, even abruptly. It is worth emphasizing that, in spite of the assonance, this process should not be confused with the more popular multifractal models studied, for example, by Calvet *et al.* (1997), Arneodo *et al.* (1998) and Riedi (2002). In fact, unlike the mBm, multifractal models are mostly based on measures deforming calendar (or physical) time. Therefore, by definition, they belong to the family of time-changed processes. Generally, the mBm is not a multifractal process. Ayache (2000) provide technical conditions under which generalized multifractional Brownian motion (which extends the mBm) can be multifractal. An analysis of the use and limits in finance of multifractality detection techniques is proposed by Bianchi and Pianese (2007).

Although the difficulty of inferring global probabilistic properties discourages the adoption of mBm in financial modeling, it is a very intriguing and powerful stochastic process for several reasons.

- Choosing an appropriate functional parameter $H_t$ makes the mBm able to reproduce the stylized facts that characterize financial time series: the absence of autocorrelation of returns and significant autocorrelation in the absolute/square returns, unconditional and conditional heavy tails, gain/loss asymmetry, aggregational Gaussianity, intermittency, volatility clustering, and volume/volatility correlation (see Cont 2001 for a discussion of the stylized empirical facts emerging in various types of financial markets).
- The time-varying Hölder exponent of the mBm can harmonize the many inconsistent estimates that the literature provides of the long-range dependence parameter (see Baillie 1996 and Henry and Zaffaroni 2003 for a survey). In fact, as noted by Bianchi and Pianese (2008), it can easily be shown that detecting dependence or not using asymptotic estimators strongly relies on the segment of data one looks at, even for the same time series, which basically indicates non-stationarity.
- It is not only about the capability of the mBm to mimic financial dynamics: most importantly, this process is flexible enough to give a rationale to the market mechanism. To see this, one should consider that it is obtained by replacing the constant parameter $H$ of the fractional Brownian motion by the function $H_t$, ranging in (0, 1]. It should be emphasized that the central value 1/2 recovers, as a particular case, the Brownian motion, whereas values larger (less) than 1/2 indicate persistence (antipersistence); the more intense, the more $H_t$ deviates from 1/2. Therefore, the path designed by $H_t$ can describe the state of the market. It can capture both the trends characterizing 'bull' and 'bear' periods and the mean-reversion, symptomatic of the frenetic buy-and-sell activity affecting the markets during financial crises. In other words, the model can potentially gather the multifaceted investors' trading behavior, in particular their reactions to financial crashes, and seize the asymmetric impact that the information flow has on prices: the confidence that investors nourished in the past is a slowly increasing process, whereas destroying expectations is generally the instantaneous consequence of shocks that bring investors to suddenly reconsider the weight they ascribe to the past. If so, we should expect what indeed some recent contributions observe: the long-range dependence parameter fluctuates around 1/2 (Costa and Vasconcelos 2003, Jiang *et al.* 2007, Alvarez-Ramirez *et al.* 2008) with nosedives systematically followed by gradual upward movements that tend to restore the previous regularity level (Bianchi 2005). In short, the mBm—probabilistically well-founded and conceptually very parsimonious—could be able to harmonize the different approaches to the study of financial dynamics: the efficient market theories (in that $\mathbb{E}(H_t) = 1/2$), the agent-based models and behavioral finance. The ambition of this synthesis is balanced by the complexity of the mBm, the understanding of which is in its early stages.

Due to the reasons mentioned above, the starting point of an mBm-based modelization relies on obtaining good estimates of its functional parameter $H_t$, a task that in the last 15 years has attracted many contributions in fields other than finance. The problem is sketched by Péltier and Lévy Véhel (1994, 1995), who define an estimator based on the average variation of the sampled process. Using the method defined by Benassi *et al.* (1998b) for filtered white noise, Istas and Lang (1997) and

†Remember that the Hölder exponent measures the degree of irregularity of the graph of a function. Given the function $f(x)$, if there exists a constant $C$ and a polynomial $P_n$ of degree $n < h$ such that $|f(x) - P_n(x - x_0)| \le C|x - x_0|^h$, the Hölder exponent $H(x_0)$ is defined as the supremum of all $h$s such that the above relation holds. The polynomial $P_n$ is often associated with the Taylor expansion of $f$ around $x_0$, but the relation is valid even if such an expansion does not exist.

Benassi *et al.* (1998a) introduce generalized quadratic variations,† an estimator of a continuously differentiable function, whose advantage is to allow a Gaussian limiting distribution with a $\sqrt{N}$ rate of convergence, $N$ being the sampling points along the process path. The result was extended by Coeurjolly (2005), who considers an estimator based on a local estimation of the second-order moment of a unique discretized filtered path. This allows the consideration of Hölderian functions (of arbitrary positive order) and provides limit theorems for the functional estimators. A semi-parametric estimator for a piece-wise constant $H_t$ was proposed by Benassi *et al.* (1999, 2000) with the aim of detecting abrupt changes of the Hölder exponent for Gaussian processes with almost surely continuous paths. To identify the functional parameter of an even more general mBm (the generalized mBm), Ayache and Lévy Véhel (2004) and Ayache *et al.* (2005, 2007) use the generalized quadratic variation and derive a central limit theorem for their estimator. With a specific look at financial applications, Bianchi (2005) reported an extension to the mBm, a class of estimators introduced for the fBm by Péltier and Lévy Véhel (1994), and studied its Gaussian limiting distribution with a $(\sqrt{\delta}\log N)$ rate of convergence, with $\delta$ and $N$, respectively, being the length of the estimation window and the number of sampling points. More recently, Loutridis (2007) proposed an algorithm based on the scaled window variance method for estimating both global and local scaling exponents and reported its simplicity and computational efficiency with respect to other techniques. Finally, a class of consistent estimators based on convex combinations of sample quantiles of discrete variations was proposed by Coeurjolly (2008), who also derives the almost certain convergence and the asymptotic normality of the estimators.

In this work we improve the estimation technique of $H_t$ proposed by Bianchi (2005). Specifically, we correct the shifting bias that affects the estimator and introduce a local normality testing procedure. Section 2 recalls the main properties of the mBm and of the estimator. Section 3 deals with the improvements mentioned above. An application to actual financial data is presented in section 4. Finally, section 5 concludes.

## 2. Multifractional Brownian motion as a model of financial dynamics

The model we discuss is an extension of the well-known fractional Brownian motion (fBm) introduced by Mandelbrot and Van Ness (1968). In turn, this generalizes the ordinary Brownian motion to model long-range dependence. In order to appreciate the capability of multifractional Brownian motion in financial modeling, it is useful to recall briefly the main features of the fBm. This is the only Gaussian zero mean $H$-sssi‡ stochastic process $B_H(t)$ with covariance function given by

$$\mathbb{E}(B_H(s)B_H(t)) = \frac{K^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \tag{2.1}$$

where $K^2 = \mathrm{var}(B_H(1))$ and $t, s \in \mathbb{R}^+$. The fBm admits the following moving average representation:

$$B_H(t) = KV_H \int_{\mathbb{R}} \Big[ |t-s|^{H-(1/2)} 1_{]-\infty,t]}(s) - |s|^{H-(1/2)} 1_{]-\infty,0]}(s) \Big] \, \mathrm{d}B(s), \tag{2.2}$$

where $V_H = \sqrt{\Gamma(2H+1)\sin(\pi H)}/\Gamma(H+(1/2))$ is a normalizing factor, 1 denotes the indicator function and $B(\cdot)$ is ordinary Brownian motion. Its increment process, called fractional Gaussian noise (fGn), is such that $\mathbb{E}(\{B_H(t) - B_H(s)\}^2) = K^2|t-s|^{2H}$.

For $H \in (1/2, 1]$, the motion displays more and more persistent paths as $H$ increases. It reduces to standard Brownian motion when $H = 1/2$ and, finally, if $H \in (0, 1/2)$ the sequences display mean-reverting behavior, which becomes more evident as $H$ decreases.

Since fBm is characterized by the value of the parameter $H$, the most immediate generalization can be obtained by replacing $H$ by a proper function of time $H_t$. This extension, referred to as *multifractional Brownian motion* (mBm) (Péltier and Lévy Véhel 1995, Benassi *et al.* 1997, Ayache and Lévy Véhel 2000), is able to describe the dynamics of signals whose regularity changes over time. The cost of the increased flexibility of the model resides in the fact that the increments of mBm are generally no longer stationary nor self-similar.§ Having accounted for the function $H_t$, the stochastic integral can be written as

$$X_{H_t}(t) = KV_{H_t} \int_{\mathbb{R}} \Big[ |t-s|^{H_t-(1/2)} 1_{]-\infty,t]}(s) - |s|^{H_t-(1/2)} 1_{]-\infty,0]}(s) \Big] \, \mathrm{d}B(s), \tag{2.3}$$

where, as above, $V_{H_t} = \sqrt{\Gamma(2H_t+1)\sin(\pi H_t)}/\Gamma(H_t + (1/2))$ is a normalizing factor.

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†Basically, with regard to the process $X$ sampled $N$ times, the generalized quadratic variation is defined as

$$V_N = \sum_{p=0}^{N-2} \left( X\left(\frac{p+2}{N}\right) - 2X\left(\frac{p+1}{N}\right) + X\left(\frac{p}{N}\right) \right)^2.$$

The variation serves to define the estimator $\hat{h}_N = (1/2)[1 - (\ln V_N / \ln N)]$, which, under certain assumptions on the function $H_t$, satisfies $\lim_{N\to+\infty} \hat{h}_N = \inf_{t\in(0,1)} H_t$ almost surely. The result stated for the infimum provides a way to estimate $H$ itself at any point $t \in (0, 1)$. In fact. choosing an appropriate $(\epsilon, N)$-neighborhood of $t$, one can calculate the generalized quadratic variation $V_{\epsilon,N}$ and hence the estimator $\hat{h}_{\epsilon,N}$ that satisfies $\lim_{N\to+\infty} \hat{h}_{\epsilon,N} = \inf_{|s-t|<\epsilon} H_s$ (a.s.). Letting $\epsilon$ tend to zero, one obtains an estimate of $H_t$.

‡Here, $H$-sssi is the self-similar of parameter $H$ with stationary increments (see Samorodnitsky and Taqqu 1994 for a detailed discussion).

§Indeed, Ayache and Taqqu (2005) provide sufficient conditions for a multifractional process with a random functional parameter to be self-similar (in the sense of its marginal distributions) or to have stationary increments.

**Remark 1:** The moving average representation (2.3) of the mBm is due to Péltier and Lévy Véhel (1995). Benassi *et al*. (1997) provides a different formulation, also convenient for simulation purposes, based on a spectral approach. The two representations are equivalent up to a multiplicative deterministic function of time (Cohen 1999).

**Remark 2:** Unlike the generalization considered by Coeurjolly (2000) that assumes $K$ to depend on time, here we maintain as a constant the scale parameter and explain the whole process variability in terms of its exponent $H_t$.

The covariance of the process is given by (Ayache *et al.* 2000)

$$\mathbb{E}(X_{H_t}(t)X_{H_s}(s)) = K^2 D(H_t, H_s)(t^{H_t+H_s} + s^{H_t+H_s} - |t-s|^{H_t+H_s}), \tag{2.4}$$

where

$$D(H_t, H_s) = \frac{\sqrt{\Gamma(2H_t+1)\Gamma(2H_s+1)\sin(\pi H_t)\sin(\pi H_s)}}{2\Gamma(H_t+H_s+1)\sin\{\pi[(H_t+H_s)/2]\}}.$$

Figure 1 shows the effect of the function $H_t$ on a path of the process, simulated using the improved Chan and Wood (1998) algorithm. Note the increasing jaggedness for decreasing values of $H_t$ (or, symmetrically, the increasing smoothness for increasing values).

A sufficient condition preserving the continuity of motion requires $H : [0, \infty) \to (0, 1]$ to be a Hölderian function. This restriction is relaxed by Ayache and Lévy Véhel (2000), who define a Gaussian process, Generalized Multifractional Brownian Motion, that extends the mBm and allows the functional parameter $H_t$ to belong to a set of functions larger than the space of the Hölder functions. The result is a continuous process with the Hölder regularity given by even very irregular functions.

**Remark 3:** Note that $H_t$ is the pointwise Hölder exponent of mBm at point $t$. This means that, in the neighborhood of $t$, the process is asymptotically self-similar with parameter $H_t$, in the sense stated by Benassi *et al*. (1997). Denote by $Y(t, au) = X_{H_{t+au}}(t+au) - X_{H_t}(t)$ the increment process, it holds that

$$\lim_{a\to 0^+} a^{-H_t} Y(t, au) \stackrel{d}{=} B_{H_t}(u), \quad u \in \mathbb{R}. \tag{2.5}$$

Equality (2.5) states that, at any point $t$, there exists an fBm with parameter $H_t$ 'tangent' to the mBm. From (2.1) it follows that $\mathrm{var}(B_{H_t}(u)) = K^2 u^{2H_t}$. Therefore, recalling that fBm is a Gaussian process, the infinitesimal increment of mBm at time $t$, normalized by $a^{H_t}$, is normally distributed with mean 0 and variance $K^2 u^{2H_t}$ ($u \in \mathbb{R}$, $a \to 0^+$). The estimator of $H_t$ discussed in the following section is based on this result.

The above remark is the motivation that justifies the use of mBm as a model of financial dynamics. Its financial meaning will be discussed in section 4.

### 2.1. Estimating the pointwise regularity

Exploiting remark **3**, Bianchi (2005) proposes a class of moving-window estimators of the functional parameter $H_t$. When the process (2.3) is sampled at discrete times $t_i = i/(n-1)$ $(i = 0, \ldots, n-1)$, setting $X_{i,n} = X_{H_{t_i}}(t_i)$ for brevity, one has

$$X_{j+q,n} - X_{j,n} \stackrel{d}{\simeq} \mathcal{N}\left(0, K^2\left(\frac{q}{n-1}\right)^{2H_{t_i}}\right), \tag{2.6}$$

for $i = \delta+1, \ldots, n-q$, $j = (n-1)t_i - \delta, \ldots, (n-1)t_i - q$ and $q = 1, \ldots, \delta$.

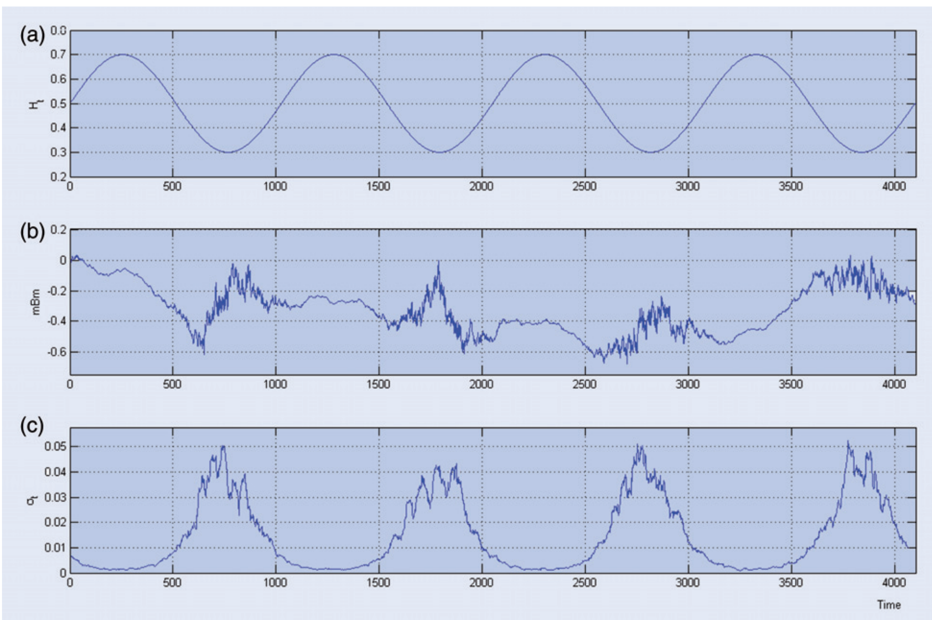


Figure 1. Surrogate mBm. (a) Sinusoidal $H_t$. (b) Surrogate process. (c) Volatility process.

Note that the variance in (2.6) is a consequence of assuming a smooth $H_t$. In fact, from (2.4) it is easy to calculate the variance of the increments of mBm:

$$\begin{aligned}\operatorname{var}(X_t - X_s) &= \mathbb{E}((X_t - X_s)^2) - (\mathbb{E}(X_t - X_s))^2 \\ &= \mathbb{E}(X_t^2 + X_s^2 - 2X_tX_s) \\ &= K^2 t^{2H_t} + K^2 s^{2H_s} - 2\mathbb{E}(X_tX_s) \\ &= K^2[t^{2H_t} + s^{2H_s} - 2D(H_t, H_s) \\ &\quad \times (t^{H_t+H_s} + s^{H_t+H_s} - |t-s|^{H_t+H_s})].\end{aligned}$$

Since $\lim_{|H_t - H_s| \to 0} D(H_t, H_s) = 1/2$, whenever $H_t \approx H_s$,

$$\operatorname{var}(X_t - X_s) \approx K^2 |t-s|^{2H_t}.$$

Recalling that mBm is sampled in discrete time over $n$ points, one has

$$\begin{aligned}\operatorname{var}(X_{j+q,n} - X_{j,n}) &\cong K^2 \left|\frac{j+q}{n-1} - \frac{j}{n-1}\right|^{2H_{t/(n-1)}} \\ &= K^2 \left(\frac{q}{n-1}\right)^{2H_{t/(n-1)}},\end{aligned}$$

which is assumption (2.6) for the variance.

The construction of the estimator† starts from the formula providing the $k$th absolute moment of $Y \sim \mathcal{N}(0, \sigma^2)$,

$$\mathbb{E}(|Y|^k) = \frac{2^{k/2}\Gamma((k+1)/2)}{\Gamma(1/2)} \sigma^k. \tag{2.7}$$

Exploiting (2.6), one can define the quantity

$$\mathbb{S}^k_{\delta,q,n,K}(t) = \frac{1}{\delta - q + 1} \sum_{j=t-\delta}^{t-q} |X_{j+q,n} - X_{j,n}|^k, \quad t = \delta+1, \ldots, n,$$

which, by (2.7), yields

$$\begin{aligned}\mathbb{E}(\mathbb{S}^k_{\delta,q,n,K}(t)) &= \mathbb{E}\left(\frac{1}{\delta - q + 1} \sum_{j=t-\delta}^{t-q} |X_{j+q,n} - X_{j,n}|^k\right) \\ &= \frac{2^{k/2}\Gamma((k+1)/2)}{\Gamma(1/2)} K^k \left(\frac{q}{n-1}\right)^{kH_t}.\end{aligned} \tag{2.8}$$

Since the ratio

$$\frac{\mathbb{S}^k_{\delta,q,n,K}(t)}{\mathbb{E}(\mathbb{S}^k_{\delta,q,n,K}(t))} = \frac{\sqrt{\pi}\mathbb{S}^k_{\delta,q,n,K}(t)}{2^{k/2}\Gamma((k+1)/2)K^k(q/(n-1))^{kH_t}} \tag{2.9}$$

tends to 1 in probability as $\delta$ tends to infinity, we can write

$$\frac{\sqrt{\pi}\mathbb{S}^k_{\delta,q,n,K}(t)}{2^{k/2}\Gamma((k+1)/2)K^k} \xrightarrow{P} \left(\frac{q}{n-1}\right)^{kH_t}, \tag{2.10}$$

and hence

$$\frac{\log(\sqrt{\pi}\mathbb{S}^k_{\delta,q,n,K}(t)/(2^{k/2}\Gamma((k+1)/2)K^k))}{k\log(q/(n-1))} \xrightarrow{P} H_t. \tag{2.11}$$

By (2.11) the class of estimators directly follows:

$$H^k_{\delta,q,n,K}(t) = \frac{\left\{\begin{array}{l}\log([\sqrt{\pi}/(\delta-q+1)]\sum_{j=t-\delta}^{t-1}|X_{j+q,n} - X_{j,n}|^k/ \\ \quad (2^{k/2}\Gamma((k+1)/2)K^k))\end{array}\right\}}{k\log(q/(n-1))}. \tag{2.12}$$

With regard to the estimator's distribution and provided that $H_t < 3/4$, it can be proved that

$$\begin{aligned}&k\log\left(\frac{n-1}{q}\right)\sqrt{\delta - q + 1}(H_t - H^k_{\delta,q,n,K}(t)) \\ &\overset{d}{\simeq} \mathcal{N}\left(0, \frac{\pi}{2^k\Gamma^2((k+1)/2)}\sigma^2\right),\end{aligned} \tag{2.13}$$

where $\sigma^2$ is the limit variance of a series of normalized nonlinear functions of a stationary Gaussian sequence with slowly decaying autocorrelation function.

**Remark 4 :** Relation (2.13) provides conditions useful for setting two of the four parameters of the estimator, $q$ and $k$. Concerning the former, as the estimator's variance increases with $q$, a natural choice consists of setting it equal to 1, the minimal admissible value.

The optimal choice of $k$ is a bit more complicated to set and involves writing the variance in (2.13) for $H^k_{\delta,1,n,1}(t) = H = 1/2$; toilsome computations show that, in this case, one obtains

$$\begin{aligned}\operatorname{var}(H^k_{\delta,1,n,1}(t)) &= \frac{\sqrt{\pi}}{\delta k^2 \log^2(n-1)\Gamma^2((k+1)/2)} \\ &\quad \cdot \left[\Gamma\left(\frac{2k+1}{2}\right) - \frac{1}{\sqrt{\pi}}\Gamma^2\left(\frac{k+1}{2}\right)\right],\end{aligned}$$

which, minimized, leads to the optimal $k (k = 2)$. The parameter $\delta$ and the unknown value $K$ are the remaining quantities that have to be determined in order to obtain a reliable estimation of $H_t$ and this is the main purpose of this paper.

In the general case, relationship (2.13) is hard to calculate numerically because of the term $\sigma^2$. For our purposes, we will be content with estimating its values by simulation. Figure 2 displays the estimator's standard deviation for different values of $H$ and $\delta$, obtained by Monte Carlo simulation, for 1000 samples of length $N = 4096$ (for different sample sizes, only the surface height changes, therefore their representation is omitted). The surface is drawn by interpolating the data using a locally weighted smoothing quadratic regression, whose $r$ square does not differ significantly from one, irrespective of the sample. From (2.13), it is possible to calculate the confidence intervals, displayed for some values in table 1, for each estimated $H_t$.

An idea of how the estimator (2.12) works is provided in figure 3, which displays the functional parameter $H_t$ estimated by $H^2_{30,1,4096,1}(t)$ for the surrogate time series of figure 1. Note that the goodness of fit (i.e. the almost perfect shadowing) is a consequence of both the very

†The detailed proofs recalled in this section can be found in Bianchi (2005).

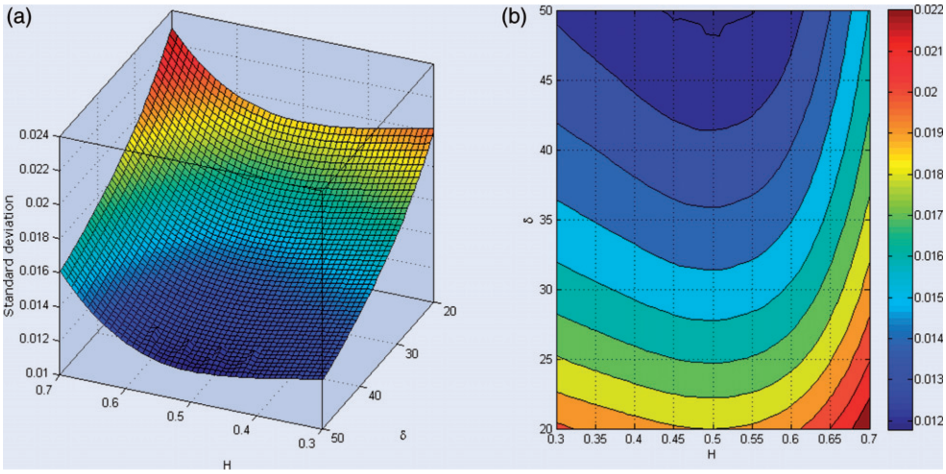


Figure 2. Standard deviation of the estimator. (a) Surface fit by locally weighted smoothing quadratic regression. (b) Contour plot ($N = 4096$, $K = 1$, $k = 2$, $q = 1$).

Table 1. Confidence intervals (at a significance level of $\alpha = 0.05$) for different values of $H_t$ and lags $\delta$.

| | | $N$ | | | |
|---|---|---|---|---|---|
| $H$ | $\delta$ | 1024 | 2048 | 4096 | 8192 |
| 0.3 | 20 | [0.253,0.347] | [0.256,0.344] | [0.260,0.340] | [0.261,0.339] |
| | 30 | [0.262,0.338] | [0.265,0.335] | [0.267,0.333] | [0.268,0.332] |
| | 40 | [0.267,0.333] | [0.270,0.330] | [0.272,0.328] | [0.272,0.328] |
| | 50 | [0.271,0.329] | [0.273,0.327] | [0.275,0.325] | [0.274,0.326] |
| 0.4 | 20 | [0.354,0.446] | [0.357,0.443] | [0.361,0.439] | [0.363,0.437] |
| | 30 | [0.363,0.437] | [0.366,0.434] | [0.369,0.431] | [0.370,0.430] |
| | 40 | [0.368,0.432] | [0.370,0.430] | [0.373,0.427] | [0.374,0.426] |
| | 50 | [0.372,0.428] | [0.374,0.426] | [0.376,0.424] | [0.376,0.424] |
| 0.5 | 20 | [0.455,0.545] | [0.458,0.542] | [0.462,0.538] | [0.465,0.535] |
| | 30 | [0.464,0.536] | [0.466,0.534] | [0.469,0.531] | [0.471,0.529] |
| | 40 | [0.469,0.531] | [0.471,0.529] | [0.474,0.526] | [0.475,0.525] |
| | 50 | [0.472,0.528] | [0.474,0.526] | [0.476,0.524] | [0.478,0.522] |
| 0.6 | 20 | [0.554,0.646] | [0.557,0.643] | [0.561,0.639] | [0.563,0.637] |
| | 30 | [0.563,0.637] | [0.565,0.635] | [0.568,0.632] | [0.569,0.631] |
| | 40 | [0.568,0.632] | [0.570,0.630] | [0.572,0.628] | [0.573,0.627] |
| | 50 | [0.572,0.628] | [0.573,0.627] | [0.575,0.625] | [0.576,0.624] |
| 0.7 | 20 | [0.649,0.751] | [0.652,0.748] | [0.655,0.745] | [0.655,0.745] |
| | 30 | [0.658,0.742] | [0.660,0.740] | [0.662,0.738] | [0.661,0.739] |
| | 40 | [0.664,0.736] | [0.665,0.735] | [0.666,0.734] | [0.665,0.735] |
| | 50 | [0.667,0.733] | [0.668,0.732] | [0.669,0.731] | [0.667,0.733] |

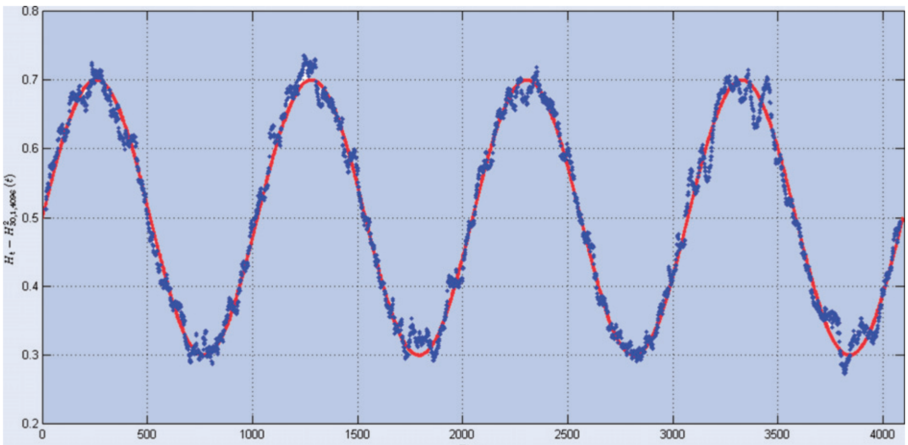


Figure 3. Estimation of the surrogate time series of figure 1. (——) $H_t$. (●) $H^2_{30,1,4096,1}(t)$.

regular function $H_t$ and the knowledge of $K$, which in our simulations is set equal to 1.

## 3. Improving the estimator

We concentrate on two refinements of the estimator recalled in the previous section.

As noted, reliable estimates are obtained if one knows the actual $K$. In fact, the trick used to obtain the almost perfect matching between $H_t$ and its estimation in figure 3 relies on performing (2.12) with the same $K$ used to generate the simulated series ($K=1$). Obviously, in actual time series, the value of $K$ is unknown. As we will see below, this is reflected in the shift of $H^k_{\delta,q,n,K}(t)$ with respect to $H_t$.

The second issue concerns the window $\delta$. The hypothesis is that the observations are normally distributed with mean zero and variance depending on the parameter $H_t$. In the multifractional framework, normality of the data results from the constancy of $H_t$ within the window $\delta$ itself. When dealing with actual time series, this assumption cannot be made since its effectiveness probably depends on the size of $\delta$, which should be allowed to vary over time. This is particularly true when financial data are considered. In fact, since $H_t$ is assumed to change as a consequence of new information, maintaining a constant size of the estimation window would mean assuming, unrealistically, that the arrival of information, no matter how destabilizing, always forces $H_t$ to change at the same rate. Therefore, in order to improve the estimator's capability of shadowing $H_t$, it seems reasonable to change the size of the window according to the local marginal normality of the price variations. This problem will be faced in section 3.2.

Before applying the estimator to financial time series, we will test the improvements on surrogate data as a benchmark. Since our goal is to provide a working tool, we will consider very jagged $H_t$'s and not a smooth function as the sine used for the previous example. This accomplishes a twofold task: (a) we will put the estimator under stress to evaluate its effectiveness in shadowing non-smooth trajectories, and (b) the simulated series will be designed to comply with the actual financial time series as much as possible.

Although the simulation of an mBm is quite a complicated issue that would require a separate discussion, we summarize below the main steps, omitting the technicalities involved at each stage.

As recalled in section 2, a sufficient condition that entails the mBm to be continuous is requiring $H_t$ to be a Hölderian function. Although this assumption can be relaxed to include more general cases (Ayache and Lévy Véhel 2000, Benassi *et al.* 2000), it is weak enough to generate sufficiently jagged $H_t$'s. Exploiting the fact that the fBm of parameter $H$ is almost surely $H$-Hölderian (see e.g. Benassi *et al.* 2004), the basic idea consists of assuming $H_t$ to be the appropriately rescaled sample path of an fBm of given parameter $H$. The jaggedness of the functional parameter $H_t$ can therefore be controlled simply by choosing a sufficiently small $H$ for the simulated fBm. This idea follows directly from the definition of a *Multifractional Process with Random Exponent* (MPRE) (Ayache and Taqqu 2005).

Let $H_m$ and $H_M$ denote, respectively, the minimum and the maximum $H_t$ we want to surrogate. The simulation procedure works as follows.

- Generate a $B_H(t)$ with $H<1/2$. Choosing $H$ (significantly) less than 1/2 ensures that the surrogated $H_t$ will display sufficiently non-smooth behavior.†
- Rescale the simulated path by setting

$$H_t := B^*_H(t) = \frac{B_H(t) - \min_t B_H(t)}{\max_t B_H(t) - \min_t B_H(t)}(H_M - H_m) + H_m.$$

  In this way, the simulated $H_t$ is forced to belong to the interval $[H_m, H_M]$.
- Simulate a multifractional Brownian motion with functional parameter $H_t$. We will use the modified Chan and Wood algorithm to surrogate the series.

Figure 4 shows the steps of the procedure described above.

### 3.1. Unknown K

As recalled above, when actual data are taken into consideration, the parameter $K$ to be valued in (2.12) is generally unknown and a misleading $K$ causes a shift of the estimated sequence. This can easily be seen from relation (2.12), written as follows:

$$H^k_{\delta,q,n,K}(t) = \frac{\left\{\begin{array}{l}\log([\sqrt{\pi}\sum_{j=t-\delta}^{t-1}|X_{j+q,n} - X_{j,n}|^k]/ \\ [(\delta-q+1)2^{k/2}\Gamma((k+1)/2)])\end{array}\right\}}{k\log(q/(n-1))} - \frac{\log K}{\log(q/(n-1))}. \quad (3.1)$$

The problem is relevant because the shift can be significant even when $n$ is large, with the logarithm slowly varying at infinity.‡ In order to estimate the right $K$, note that, once an estimation has been obtained using an arbitrary $K^*$, relation (2.6) implies the random variables collected in the set

$$V_q = \{X_{j+q,n} - X_{j,n} : H^k_{\delta,q,n,K}(t) \in (H^* - \epsilon, H^* + \epsilon)\} \quad (3.2)$$

to be normally distributed with mean zero and variance equal to $K^2(q/(n-1))^{2H}$ as $\epsilon \to 0$. Note that $H = H^* + \{(\log K^*/K)/[\log q/(n-1)]\}$, where the error term

†In order to simulate fBm we used the wavelet-based algorithm introduced by Sellan (1995), implemented by Abry and Sellan (1996) and revised by Bardet *et al.* (2003), in order to remove the many high-frequency components.

‡Remember that the function $L$ is slowly varying at infinity if $\lim_{t\to\infty}(L(\alpha t)/L(t)) = 1$ for some $\alpha \in \mathbb{R}^+$.

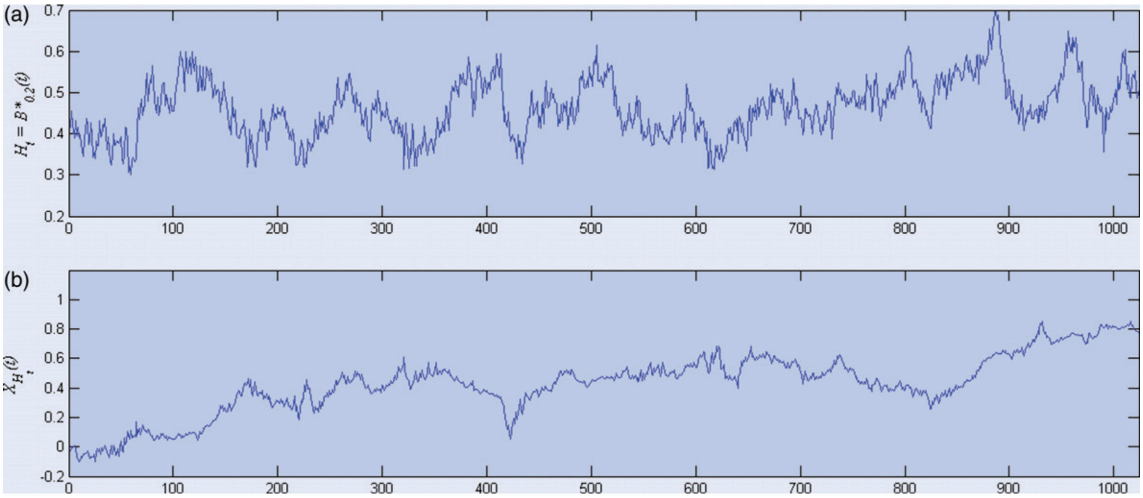


Figure 4. Example of a surrogate mBm. (a) $H_t$ is set equal to the rescaled surrogate $B^*_{0.2}(t)$. (b) The mBm is simulated with functional parameter $H_t$.

$(\log K^*/K)/[\log q/(n-1)]$ accounts for the shifting bias, which is trivially 0 if and only if $K^* = K$.

Therefore, applying (2.7), for each fixed $H^*$ and for $\epsilon \to 0$, it is straightforward to obtain

$$\log \mathbb{E}(|X_{j+q,n} - X_{j,n}|^s) = \log \frac{2^{s/2}\Gamma((s+1)/2)}{\Gamma(1/2)} \hat{K}^s + s\hat{H} \log\left(\frac{q}{n-1}\right). \tag{3.3}$$

**Remark 5:** As the bias caused by a wrong $K$ in (2.12) turns into a shift of the estimated sequence $H_t$, the difference $H^* - \hat{H}$ is expected to be equal to $(\log K^*/K)/[\log q/(n-1)]$ for any $H^*$. Therefore, relation (3.3) provides testable moment conditions to estimate $K$, which can be evaluated through the intercept of a simple linear fit in the plane $(\log [q/(n-1)], \log \mathbb{E}(|X_{j+q,n} - X_{j,n}|^s))$ for increasing $q$.

More precisely, the correction can be made using the following five steps.

(1) The estimation provided (2.12) is run with an arbitrary $K^*$.
(2) The empirical density function $f_{H^k_{\delta qnK^*}(t)}(x)$ of the estimates is calculated.†
(3) The value $H^* = \arg\max f_{H^k_{\delta qnK^*}(t)}(x))$ is taken as the center of the interval (whose inf and sup are the extremes of the bin that $H^k_{\delta,q,n,K^*}$ belongs to) that serves to define the set in (3.2).
(4) Once $V_q$ is settled, the log linear fit (3.3) is performed. The parameter $K$ is therefore estimated by deducing $\hat{K}$ from the first term on the right-hand side of (3.3).
(5) Finally, the estimation is rerun with $\hat{K}$.

The procedure itemized above (1–5) is particularly effective when $f_{H^k_{\delta qnK^*}(t)}(x)$ is suitably large for some $x$, so as to guarantee the set $V_q$ having a sufficient number of elements to perform the log-linear regression in a reliable way. In fact, as we will see in the application, the larger the number of data points in the set $V_q$, the more accurate the estimate of $K$ will be.

Figure 5 provides an idea of the results generated by the correcting procedure: figure 5(a) displays the signal $H_t$ used to generate a sample of multifractional Brownian motion with $K = 1/2$ (blue line) and the values $H^2_{25,1,4096,2}(t)$ estimated by (2.12) (red line). Note that setting $K^* = 2$ causes the estimated sequence to lay upward with respect to the actual data (a downward bias would occur for $K^* < K$). Figure 5(b) represents the log-linear fit for the second absolute moment. As expected from (3.3), the coefficient of determination is not significantly far from 1. Figure 5(c) shows the sequence re-estimated by $H^2_{25,1,4096,\hat{K}}(t)$. The bias affecting the former estimation has almost disappeared, as one can see from figure 5(d), which shows the absolute error with respect to the actual data set.

The results of a more thorough analysis are summarized in table 2, which refers to 1000 sampled mBms with $K = 1/2$. As the linear fit (3.3) holds for any order $s$, it was run for the first five moments and the variance. Furthermore, the estimation was performed by considering the (average) $\hat{K}$ obtained by (A) the maximal frequency bin, (B) the three maximal frequencies bins, and (C) all the bins of the empirical frequency histogram. The table reproduces the average $\hat{K}$ (over 1000 samples and, when necessary, over the bins), the standard deviation of the estimates and the consequent bias. From table 2, none of the three choices performs univocally better than the other two. If choice (C)—which is likely to

†In the application, we use Scott's rule to determine the number of bins. Scott (1979) provides a formula for the optimal histogram bin width that asymptotically minimizes the integrated mean squared error: $h_n = 3.49 \cdot \sigma \cdot n^{-1/3}$, where $\sigma$ is an estimate of the data standard deviation.

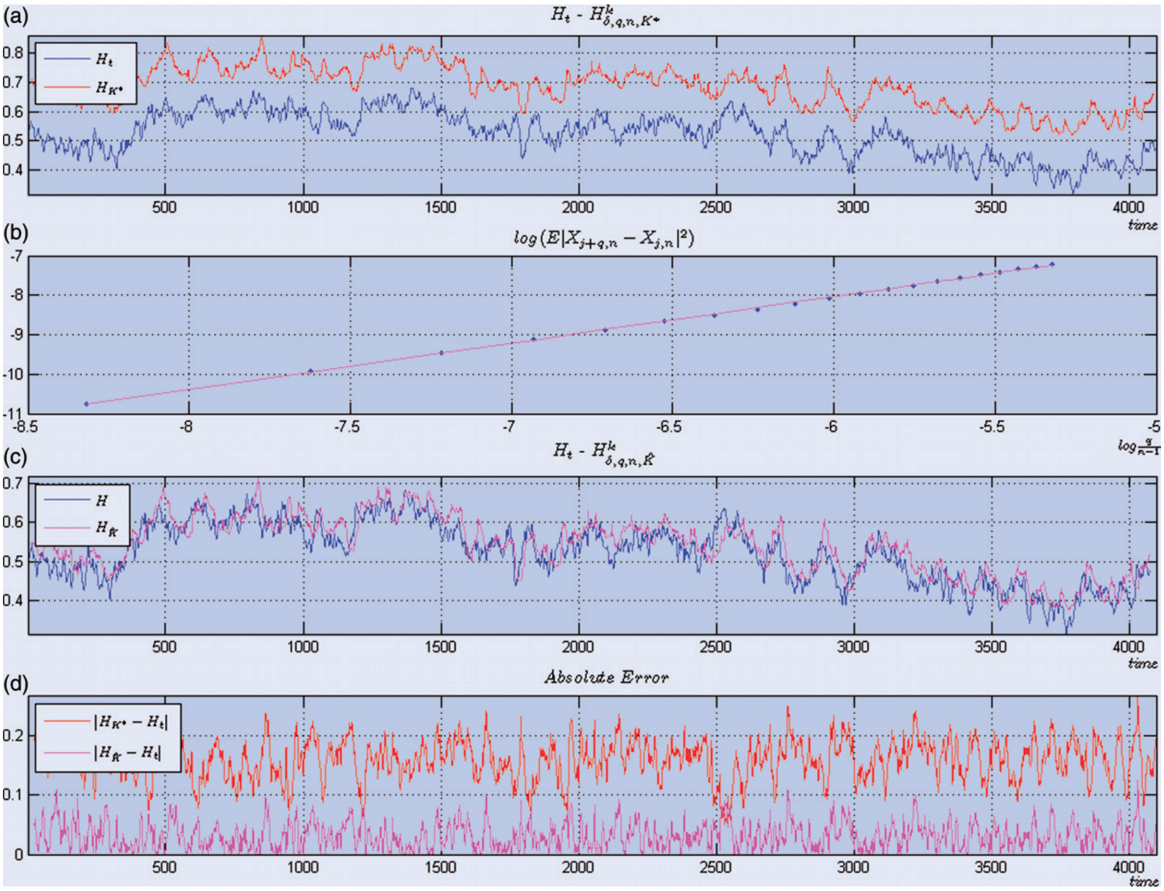

Figure 5. Example of correction for $K$. (a) $H_t$ of an mBm simulated with $K = 0.5$ (blue line) and $H^k_{\delta,q,n,K^*}$ with $K^* = 2$ (red line). (b) Log-linear fit of the second absolute moment $\hat{K} = \exp(\beta/2)$, where $\beta$ is the regression intercept. (c) $H_t$ (blue line) and $H^k_{\delta,q,n,\hat{K}}$ with $\hat{K} \approx 0.56$ (magenta line). (d) Absolute error of $H^k_{\delta,q,n,2}$ and of $H^k_{\delta,q,n,0.56}$.

be the set that better reflects all real data†—definitely has the advantage of presenting the lower variance in the distribution of the estimated $K$, it (a) provides estimates higher than those obtained with choice (A) for small samples, and (b) it strongly depends on the moment, with respect to which it displays larger ranges. A close inspection of the data suggests choosing the single bin (A) and the second-order moment, corrected for the mean (namely, the variance). In fact, (B) almost always underperforms in terms of the estimated $K$ and choices (A) and (C) even out for $N = 2048$ and $N = 8192$, but whereas (A) is much more accurate for $N = 1024$, (C) is barely more accurate for $N = 4096$.

### 3.2. Floating window

Estimator (2.12) requires the log-variations of the process to be normally distributed with mean zero within a window of length $\delta$. With regard to financial time series, this assumption is likely to deteriorate the more quickly the more $H_t$ departs from its central value of 1/2. In fact, consistent with the financial interpretation discussed in section 1, the more $H_t$ departs from its normal level the more quickly the market mechanism will tend to restore the equilibrium, that is to bring back $H_t$ to 1/2. This means that local normality is expected to last for significant spans of time when $H_t$ is close to 1/2 and is unlikely to overstay as the difference $|H_t - (1/2)|$ increases. In previous work (Bianchi 2005, Eom *et al.* 2008), the length of the window was fixed once for all (i.e. for the whole time series) in a naive way, considering 'reasonable' thresholds of about one trading month (25–30 data points). Here, we propose a time-varying $\delta$, depending on the local Gaussianity of the increment sequence $X_{j+q,n} - X_{j,n}$, based on the algorithm described in figure 6.

Let $\delta_{\max}$ be the maximal window length (in the case of financial time series, this can be linked, for example, to the time horizon the trader looks at in taking his decisions) and $\delta_{\min}$ the minimal window length (tied, for example, to the power of the statistics used to test for normality). The algorithm is initialized with $\delta_{\min}$ because it is relevant to obtain timely information about the dynamics of $H_t$. If the normality test is passed, the window is increased by one unit and the test looped until it fails or $\delta$ reaches $\delta_{\max}$. In the former case, if $\delta > \delta_{\min}$, we

†Owing to the closeness of the values, taking one or three bins does not suffice to exclude, in principle, that the variability of the estimates could be ascribed solely to the estimator variance.

Table 2. Estimation of $K$ via relation (3.3) for 1000 sampled mBms. Fit over (A) the maximal frequency bin, (B) the three maximal frequencies bins, and (C) all the bins of the empirical frequency histogram.

| | | $s=1$ | $s=2$ | $s=3$ | $s=4$ | $s=5$ | Variance |
|---|---|---|---|---|---|---|---|
| $N=1024$ | | | | | | | |
| (A) | $\hat{K}$ | 0.5768 | 0.5695 | 0.5614 | 0.5498 | 0.5353 | 0.5255 |
| | $\sigma_{\hat{K}}$ | 0.2734 | 0.2518 | 0.2384 | 0.2299 | 0.2242 | 0.2476 |
| | Bias | −0.0206 | −0.0188 | −0.0167 | −0.0137 | −0.0099 | −0.0072 |
| (B) | $\hat{K}$ | 0.6709 | 0.6425 | 0.6171 | 0.5932 | 0.5703 | 0.5847 |
| | $\sigma_{\hat{K}}$ | 0.1374 | 0.1213 | 0.1148 | 0.1125 | 0.1114 | 0.1174 |
| | Bias | −0.0424 | −0.0362 | −0.0304 | −0.0247 | −0.0190 | −0.0226 |
| (C) | $\hat{K}$ | 0.6775 | 0.6397 | 0.6087 | 0.5809 | 0.5555 | 0.5764 |
| | $\sigma_{\hat{K}}$ | 0.1311 | 0.1116 | 0.1035 | 0.1003 | 0.0986 | 0.1146 |
| | Bias | −0.0438 | −0.0356 | −0.0284 | −0.0216 | −0.0152 | −0.0205 |
| $N=2048$ | | | | | | | |
| (A) | $\hat{K}$ | 0.5341 | 0.5209 | 0.5118 | 0.5023 | 0.4916 | 0.4929 |
| | $\sigma_{\hat{K}}$ | 0.2058 | 0.1874 | 0.1811 | 0.1805 | 0.1818 | 0.1762 |
| | Bias | −0.0087 | −0.0054 | −0.0031 | −0.0006 | 0.0022 | 0.0019 |
| (B) | $\hat{K}$ | 0.5807 | 0.5518 | 0.5305 | 0.5124 | 0.4959 | 0.5148 |
| | $\sigma_{\hat{K}}$ | 0.1235 | 0.1127 | 0.1116 | 0.1137 | 0.1159 | 0.1026 |
| | Bias | −0.0196 | −0.0129 | −0.0078 | −0.0032 | 0.0011 | −0.0038 |
| (C) | $\hat{K}$ | 0.5806 | 0.5509 | 0.5277 | 0.5074 | 0.4888 | 0.4997 |
| | $\sigma_{\hat{K}}$ | 0.1003 | 0.0890 | 0.0872 | 0.0869 | 0.0871 | 0.0795 |
| | Bias | −0.0196 | −0.0127 | −0.0071 | −0.0019 | 0.0030 | 0.0000 |
| $N=4096$ | | | | | | | |
| (A) | $\hat{K}$ | 0.5818 | 0.5893 | 0.5982 | 0.6036 | 0.6042 | 0.5657 |
| | $\sigma_{\hat{K}}$ | 0.1809 | 0.1771 | 0.1890 | 0.2050 | 0.2192 | 0.1718 |
| | Bias | −0.0182 | −0.0197 | −0.0216 | −0.0226 | −0.0228 | −0.0148 |
| (B) | $\hat{K}$ | 0.6224 | 0.6239 | 0.6247 | 0.6221 | 0.6162 | 0.5922 |
| | $\sigma_{\hat{K}}$ | 0.1310 | 0.1413 | 0.1551 | 0.1664 | 0.1743 | 0.1353 |
| | Bias | −0.0263 | −0.0266 | −0.0268 | −0.02627 | −0.0251 | −0.0204 |
| (C) | $\hat{K}$ | 0.6133 | 0.5977 | 0.5859 | 0.5741 | 0.5615 | 0.5543 |
| | $\sigma_{\hat{K}}$ | 0.0804 | 0.0773 | 0.0796 | 0.0827 | 0.0851 | 0.07378 |
| | Bias | −0.0246 | −0.0215 | −0.0191 | −0.0166 | −0.0140 | −0.0124 |
| $N=8192$ | | | | | | | |
| (A) | $\hat{K}$ | 0.6749 | 0.6685 | 0.6654 | 0.6628 | 0.6595 | 0.6409 |
| | $\sigma_{\hat{K}}$ | 0.2566 | 0.2470 | 0.2503 | 0.2593 | 0.2702 | 0.2384 |
| | Bias | −0.0333 | −0.0322 | −0.0317 | −0.0313 | −0.0307 | −0.0275 |
| (B) | $\hat{K}$ | 0.6656 | 0.6665 | 0.6694 | 0.6702 | 0.6677 | 0.6322 |
| | $\sigma_{\hat{K}}$ | 0.1674 | 0.1671 | 0.1782 | 0.1920 | 0.2042 | 0.1605 |
| | Bias | −0.0317 | −0.0319 | −0.0324 | −0.0325 | −0.0321 | −0.0260 |
| (C) | $\hat{K}$ | 0.7253 | 0.7021 | 0.6839 | 0.6675 | 0.6513 | 0.6488 |
| | $\sigma_{\hat{K}}$ | 0.1206 | 0.1169 | 0.1151 | 0.1141 | 0.1134 | 0.1124 |
| | Bias | −0.0413 | −0.0377 | −0.0348 | −0.0321 | −0.0293 | −0.0289 |

calculate $H^k_{\delta-1,q,n,K}(t)$, otherwise the estimate does not take place and $t$ is incremented by one unit. In the latter case, we simply calculate $H^k_{\delta-1,q,n,K}(t)$. Clearly, (a) the estimator's variance will change along the path with the size of $\delta$ according to (2.13), and (b) the sequence $H^k_{\delta,q,n,K}(t)$ will contain blanks for the time subsets in which normality is rejected.

## 4. Application to financial data

Using the procedure described in the previous section, the pointwise Hölder exponent was estimated for three main indices representative of the US (Dow Jones Industrial Average), European (FTSE 100) and Japanese (Nikkei 225) stock markets. Table 3 summarizes the data set, which approximately covers the last 25 years.

Table 4 shows the estimated parameters for the three indices. The estimation starts by assigning to $K$ the initial (arbitrary) value 1, corresponding to which (lines 'a') the mean values 0.5403 (DJIA), 0.5398 (FTSE100) and 0.5062 (N225) are obtained for $H_t$. The table also shows the standard deviation, the skewness and the kurtosis of the functional parameter distributions. Note that, once the estimation is corrected for $K$ (0.6987 for the DJIA, 0.6946 for the FTSE100 and 0.7982 for the N225), the means translate downwards (lines 'b'), whereas the other moments remain unchanged, consistent with the shifting nature of the bias. Furthermore, it is worth emphasizing that, after the adjustment for the local normality (lines 'c'):

- the standard deviation decreases. In fact, while local normality generally holds corresponding to small local variations of $H_t$, it is unlikely to

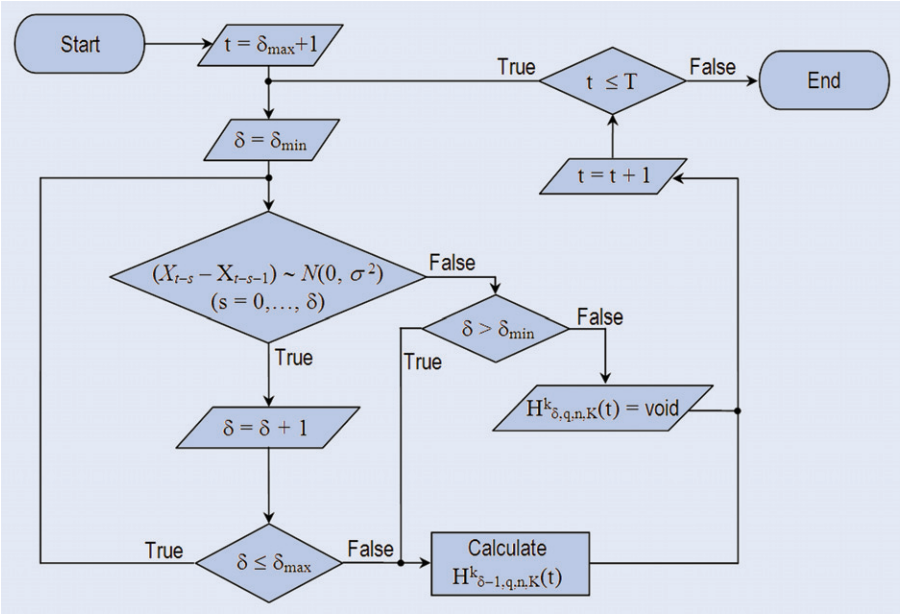


Figure 6. Algorithm for adaptive $\delta$.

Table 3. The data set.

| | DJIA | FTSE100 | N225 |
|---|---|---|---|
| Starting date | 01/02/1986 | 01/02/1986 | 01/06/1986 |
| Ending date | 06/01/2010 | 06/01/2010 | 06/01/2010 |
| No. obs. | 6157 | 6168 | 6003 |
| Log variations | | | |
| Mean | 3.04e−4 | 2.09e−4 | −4.93e−5 |
| Std | 0.01182 | 0.01143 | 0.01513 |
| Skewness | −1.80512 | −0.39680 | −0.20663 |
| Kurtosis | 46.16581 | 11.94622 | 10.88002 |
| Max | 0.10508 | 0.09384 | 0.13234 |
| Min | −0.25631 | −0.13029 | −0.16137 |

hold when large and sudden variations are observed. Since these generally involve extreme values of the Hölder exponent, correcting for local normality removes the extremes of the $H_t$ distribution, causing a decrease of the standard deviation;

- the negative skewness increases. In fact, large local variations tend to occur when the Hölder parameter is larger than 1/2 (which is roughly the mean value) and, as noted above, they are removed because of their inconsistency with local normality. As a consequence, the right-hand side of the distribution is lightened with respect to the left-hand side; and
- the kurtosis increases. In fact, from the distributions the extreme variations are removed, resulting in the weight of the central values increasing.

Figures 7–9 display the dynamics of the estimated $H_t$ during the whole time period. Some remarks should be made.

- In all cases, the estimated pointwise Hölder exponent seems to proceed erratically around 1/2, with fluctuations that are generally bounded between 0.4 and 0.6. The mean stationarity of $H_t$ is in principle far from obvious and, if widespread in the financial context, it can lead to a relevant simplification of the model, whose non-stationarity depends on the non-stationarity of $H_t$ itself. This facet seems to be particularly relevant for its potential to harmonize the no-arbitrage principle of standard financial theory with the turbulence that indeed affects real financial markets. In fact, assuming the constancy of the Hölder exponent over time (which it is not!), one could estimate one value for the whole time series and conclude that it roughly equals 1/2, the sole value consistent with the absence of arbitrage. From a more thorough examination, things look a little bit more complicated and perhaps they should be framed in a dynamic perspective, which is what we suggest doing here.
- Periods can be observed in which the functional parameter stays significantly above 1/2, but generally the greater the distance the faster the return to the central value, as an elastic force law. Again, this behavior looks self-consistent in that it describes the tendency of markets to move towards equilibrium.
- The behavior of $H_t$ reflects the asymmetric role played by new information. Very large and sudden downward variations tend to occur when the Hölder exponent is larger than 1/2; the reverse (large upward variations) occurs only shortly after the fall of the exponent. Also, this behavior looks strongly self-consistent if one recalls the meaning of $H_t$: the pointwise regularity is a proxy for the confidence traders nourished in the past. In fact, the larger $H_t$ the

Table 4. Parameter estimation.

| | Mean | Std | Skewness | Kurtosis | $K$ |
|---|---|---|---|---|---|
| DJIA | $0.5403^{a}$ | $0.0510^{a,b}$ | $-0.8667^{a,b}$ | $4.6973^{a,b}$ | $1.0000^{a}$ |
| | $0.4992^{b}$ | | | | $0.6987^{b,c}$ |
| | $0.4972^{c}$ | $0.0496^{c}$ | $-0.9847^{c}$ | $5.2218^{c}$ | |
| FTSE100 | $0.5398^{a}$ | $0.0490^{a,b}$ | $-0.8788^{a,b}$ | $4.0145^{a,b}$ | $1.0000^{a}$ |
| | $0.4980^{b}$ | | | | $0.6946^{b,c}$ |
| | $0.4963^{c}$ | $0.0478^{c}$ | $-0.9620^{c}$ | $4.2128^{c}$ | |
| N225 | $0.5062^{a}$ | $0.0513^{a,b}$ | $-0.1170^{a,b}$ | $3.4265^{a,b}$ | $1.0000^{a}$ |
| | $0.4803^{b}$ | | | | $0.7982^{b,c}$ |
| | $0.4777^{c}$ | $0.0500^{c}$ | $-0.1299^{c}$ | $3.6080^{c}$ | |

[a]1986–1993. [b]1994–2001. [c]2002–2010.

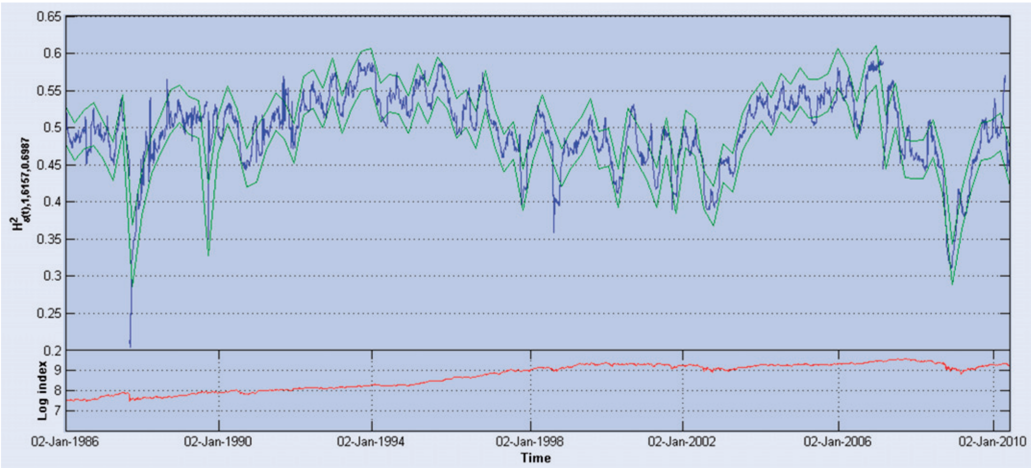


Figure 7. DJIA. Estimated $H_t$ (blue); 95% confidence intervals (green).

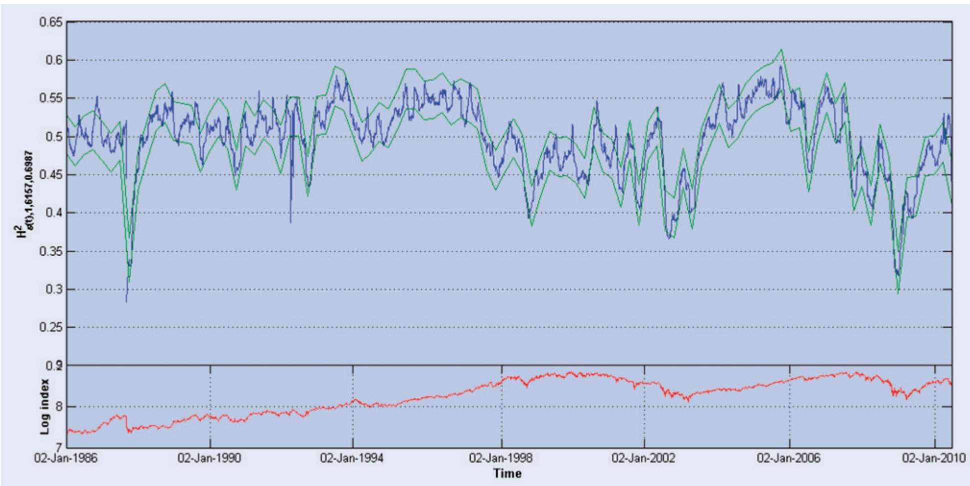


Figure 8. FTSE100. Estimated $H_t$ (blue); 95% onfidence intervals (green).

smoother the price process and the smoother the price process the more evident the trend is. This ultimately means that the larger $H_t$ the more past information weighs in the traders' assessment of future prices (irrespective of the direction of the trend). Therefore, it makes sense to observe large and sudden downward movements—that is 'memory' destruction—only when $H_t$ is larger than 1/2. The reason why financial crises do not limit themselves to resetting the 'memory' ($H_t = 1/2$) but cause mean-reversion—that is values of $H_t$ decidedly lower than 1/2—resides in the frenetic buy-and-sell activity that typically follows crashes as a consequence of the unpredictability of short-term trends. Also note that nosedives are generally followed by more gradual upward movements that tend to restore the pre-existing regularity level. If confirmed by more extensive analyses, this mechanism could be useful in

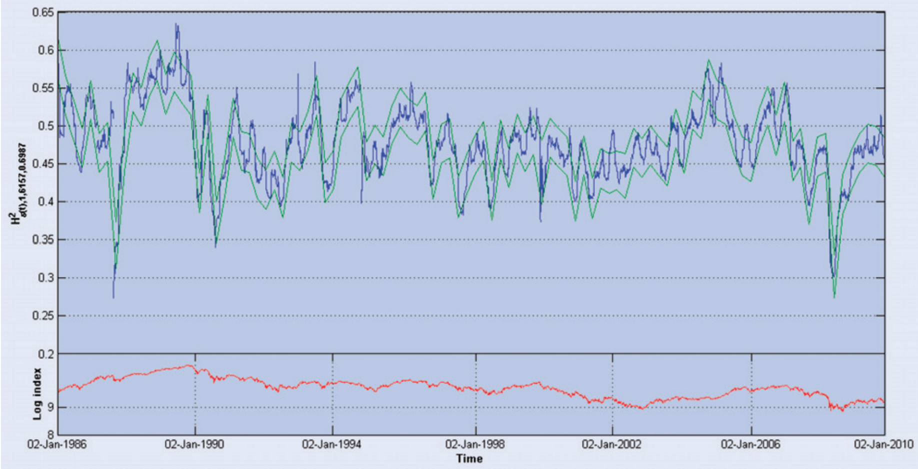


Figure 9. N225. Estimated $H_t$ (blue); 95% confidence intervals (green).

characterizing the degree of efficiency of a market.

- The analysis confirms the recent findings of Bianchi and Pantanella (2010): the pointwise Hölder exponents of the DJIA and FTSE100 evolve very similarly, whereas they both differ significantly with respect to the N225. Equipped with the financial interpretation of $H_t$ discussed above, the pairwise correlations of the regularity exponents are informative concerning how similarly markets process and discount new information (in fact, the correlation we are referring to concerns $H_t$ and not prices or returns or even volatility). Therefore, it is no surprise that Western markets show very highly correlated Hölder exponents and that lower values characterize the Japanese market. But there is more: the decomposition of the whole 25-year period into subperiods of about eight years each is of particular interest because one realizes that the level of correlation has increased over time, passing from 0.74 to 0.92 (DJIA–FTSE), from 0.20 to 0.76 (DJIA–N225) and from 0.46 to 0.78 (FTSE–N225) (see table 5). We ascribe this significant increment to the globalization process, which induces homogenization of the impact of new information. This is also reflected in the convergence of the pairwise correlations: in the first subperiod the correlation between the FTSE100 and the N225 (0.46) was more than twice that between the DJIA and the N225 (0.20), but in the following periods this difference became practically negligible (0.51 versus 0.53 in the second period and 0.76 versus 0.78 in the third period).

Table 5. Pairwise $H_t$ correlations.

| | DJIA | FTSE100 | N225 |
|---|---|---|---|
| DJIA | 1.0000 | 0.7418[a]<br>0.8121[b]<br>0.9173[c] | 0.2013[a]<br>0.5071[b]<br>0.7647[c] |
| FTSE100 | | 1.0000 | 0.4633[a]<br>0.5275[b]<br>0.7826[c] |
| N225 | | | 1.0000 |

[a]1986–1993. [b]1994–2001. [c]2002–2010.

## 5. Conclusion

In this paper we have focused on a new model of stock prices, the multifractional Brownian motion (mBm). We demonstrate its robustness both in mimicking the price dynamics and in providing, in a parsimonious way, a rationale for the market mechanism. Since the model is fully characterized by its pointwise Hölder exponent—a measure of the instantaneous regularity of the process paths—we accomplish its estimation, improving significantly on the estimator proposed by Bianchi (2005). The refinements pertain to (a) the elimination of the shifting bias that affected the previous estimator, and (b) the introduction of a local normality test procedure which removes the unreliable estimates. By simulation, we also derive the confidence intervals of the estimator when $H_t \neq 1/2$ and demonstrate the goodness of the estimates even when the local Hölder exponent changes very erratically over time. This allows us to analyse three main stock indices: the Dow Jones Industrial Average, the FTSE 100 and the Nikkei 225. For all three series we find evidence that the pointwise regularity fluctuates around a unique value consistent with the arbitrage principle, but large deviations are evident corresponding to financial crises and turbulent periods. We leave to future studies the detailed analysis of this aspect. Here we are content in stressing that this complex and consistent behavior seems to reconcile standard financial theory and the turbulence of empirical markets.

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## References


Abry, P. and Sellan, F., The wavelet-based synthesis for the fractional Brownian motion proposed by F. Sellan and Y. Meyer: remarks and fast implementation. *Appl. Comp. Harmon. Anal.*, 1996, **3**(4), 377–383.

Alvarez-Ramirez, J., Alvarez, J., Rodriguez, E. and Fernandez-Anaya, G., Time-varying Hurst exponent for US stock markets. *Physica A*, 2008, **387**(24), 6159–6169.

Arneodo, A., Bacry, E. and Muzy, J.F., Random cascades on wavelet dyadic trees. *J. Math. Phys.*, 1998, **39**, 4142–4164.

Ayache, A., *The Generalized Multifractional Brownian Motion can be Multifractal*, Vol. 22, 2000 (Publications du Laboratoire de statistique et probabilités).

Ayache, A., Benassi, S.S.C. and Lévy Véhel, J., Regularity and identification of generalized multifractional Gaussian processes. *Séminaire de Probabilités XXXVIII, Lecture Notes in Mathematics*, 2005, **1857**, 290–312.

Ayache, A., Bertrand, P. and Lévy Véhel, J., A central limit theorem for the quadratic variations of the step fractional Brownian motion. *Statist. Infer. Stochast. Process.*, 2007, **10**(1), 1–27.

Ayache, A., Cohen, S. and Léevy Véehel, J., The covariance structure of multifractional Brownian motion, with application to long range dependence. In *ICASSP '00: Proceedings of Acoustics, Speech, and Signal Processing, 2000. An IEEE International Conference*, pp. 3810–3813, 2000 (IEEE Computer Society: Washington, DC).

Ayache, A. and Lévy Véhel, J., The generalized multifractional Brownian motion. *Statist. Infer. Stochast. Process.*, 2000, **3**(1), 7–18.

Ayache, A. and Lévy Véhel, J., On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion. *Stochast. Process. Applic.*, 2004, **111**(1), 119–156.

Ayache, A. and Taqqu, M., Multifractional processes with random exponent. *Publicacions Matemàtiques*, 2005, **49**, 459–486.

Baillie, R., Long memory processes and fractional integration in econometrics. *J. Econometr.*, 1996, **73**, 5–59.

Bardet, J.-M., Lang, G., Oppenheim, G., Philippe, A. and Stoev, S., *Generators of Long-range Dependence Processes: A Survey*, 2003 (Birkhäuser).

Benassi, A., Bertrand, P., Cohen, S. and Istas, J., Identification d'un processus Gaussien multifractionnaire avec des ruptures sur la fonction d'échelle. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 1999, **329**(5), 435–440.

Benassi, A., Bertrand, P., Cohen, S. and Istas, J., Identification of the Hurst index of a step fractional Brownian motion. *Statist. Infer. Stochast. Process.*, 2000, **3**(1/2), 101–111.

Benassi, A., Cohen, S. and Istas, J., Identifying the multifractional function of a Gaussian process. *Statist. Probab. Lett.*, 1998a, **39**, 337–345.

Benassi, A., Cohen, S. and Istas, J., On roughness indices for fractional fields. *Bernoulli*, 2004, **10**(2), 357–373.

Benassi, A., Cohen, S., Istas, J. and Jaffard, S., Identification of filtered white noises. *Stochast. Process. Applic.*, 1998b, **75**, 31–49.

Benassi, A., Jaffard, S. and Roux, D., Elliptic Gaussian random processes. *Rev. Mat. Iberoam.*, 1997, **13**, 19–90.

Bianchi, S., Pathwise identification of the memory function of multifractional Brownian motion with application to finance. *Int. J. Theor. Appl. Financ.*, 2005, **8**(2), 255–281.

Bianchi, S. and Pantanella, A., Pointwise regularity exponents and market cross-correlations. *Int. Rev. Bus. Res. Pap.*, 2010, **6**(2), 39–51.

Bianchi, S. and Pianese, A., Modelling stock price movements: Multifractality or multifractionality? *Quant. Financ.*, 2007, **7**, 301–319.

Bianchi, S. and Pianese, A., Multifractional properties of stock indices decomposed by filtering their pointwise Hölder regularity. *Int. J. Theor. Appl. Financ.*, 2008, **11**(06), 567–595.

Calvet, L., Fischer, A. and Mandelbrot, B., A multifractal model of asset returns. Cowles Foundation Discussion Paper 1164, Yale University, 1997.

Chan, G. and Wood, A., Simulation of multifractional Brownian motion. *Proceedings in Computational Statistics*, 1998, 233–238.

Coeurjolly, J., Inférence statistique pour les mouvements Browniens fractionnaires et multifractionnaires. PhD Thesis, 2000.

Coeurjolly, J., Identification of the multifractional Brownian motion. *Bernoulli*, 2005, **11**(6), 987–1008.

Coeurjolly, J., Hurst exponent estimation of locally self-similar Gaussian processes using sample quantiles. *Ann. Statist.*, 2008, **36**(3), 1404–1434.

Cohen, S., From self-similarity to local self-similarity: the estimation problem. In *Fractals: Theory and Applications in Engineering*, edited by M. Dekking, J. Lévy-Véhel, E. Lutton, and C. Tricot, pp. 3–16, 1999 (Springer: Berlin).

Cont, R., Empirical properties of asset returns: stylized facts and statistical issues. *Quant. Financ.*, 2001, **1**(2), 223–236.

Costa, R.L. and Vasconcelos, G.L., Long-range correlations and nonstationarity in the Brazilian stock market. *Physica A*, 2003, **329**(1/2), 231–248.

Eom, C., Choi, S., Oh, G. and Jung, W.-S., Hurst exponent and prediction based on weak-form efficient market hypothesis of stock markets. *Physica A*, 2008, **387**(18), 4630–4636.

Henry, M. and Zaffaroni, P., The long range dependence paradigm for macroeconomics and finance. In *Theory and Applications of Long-range Dependence*, edited by P. Doukhan, G. Oppenheim, and M.S. Taqqu, pp. 417–438, 2003 (Birkhäuser).

Istas, J. and Lang, G., Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. Henri Poincaré*, 1997, **33**(4), 407–436.

Jiang, J., Ma, K. and Cai, X., Non-linear characteristics and long-range correlations in Asian stock markets. *Physica A*, 2007, **378**(2), 399–407.

Loutridis, S.J., An algorithm for the characterization of time-series based on local regularity. *Physica A*, 2007, **381**, 383–398.

Mandelbrot, B. and Van Ness, J., Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 1968, **10**(4), 422–437.

Péltier, R. and Lévy Véhel, J., A new method for estimating the parameter of fractional Brownian motion. *Rapport de recherche INRIA n2396*, 1994 (Robotique, image et vision – Action Fractales). Programme 4.

Péltier, R. and Lévy Véhel, J., Multifractional Brownian motion: definition and preliminary results. Technical Report RR-2645, CiteSeerX – Scientific Literature Digital Library and Search Engine, 1995. Available online at: http://citeseerx.ist.psu.edu/oai2

Riedi, R.H., Multifractal Processes. In *Long Range Dependence: Theory and Applications*, edited by P. Doukhan, G. Oppenheim, and M. Taqqu, pp. 625–715, 2002 (Birkhäuser).

Rogers, L., Arbitrage with fractional Brownian motion. *Math. Finance*, 1997, **7**, 95–105.

Samorodnitsky, G. and Taqqu, M., *Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance*, 1994 (Chapman and Hall: London).

Scott, D., On optimal and data-based histograms. *Biometrika*, 1979, **66**(3), 605–610.

Sellan, F., Synthèse de mouvements Browniens fractionnaires à l'aide de la transformation par ondelettes. *Comptes rendus de l'Académie des sciences, Série 1, Mathématique*, 1995, **321**, 351–358.