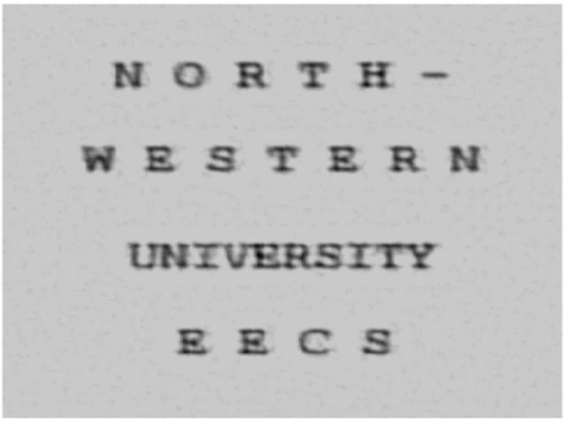
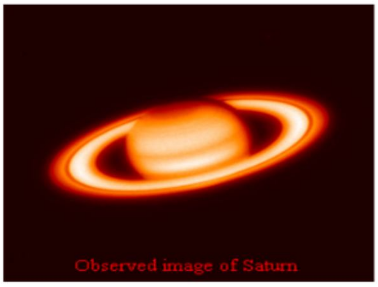
Introduction

- Recovery Examples
- Restoration Solution Approaches
  - Deterministic
  - Stochastic
- Other Recovery Problems

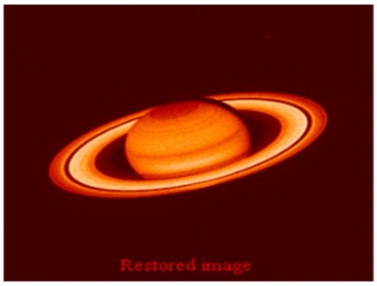
Defocussing



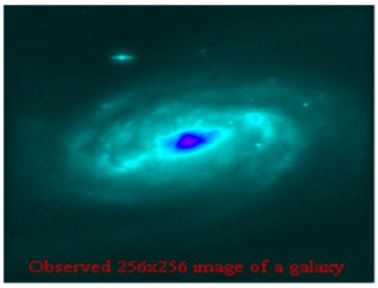
Hubble Space Telescope Image Restoration



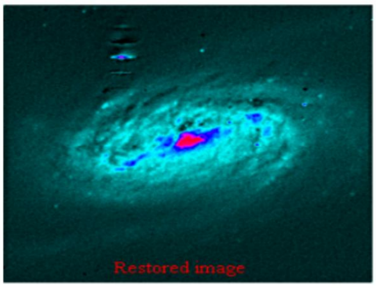
Observed image of Saturn



Restored image



Observed 256x256 image of a galaxy



Restored image

Images courtesy of NASA

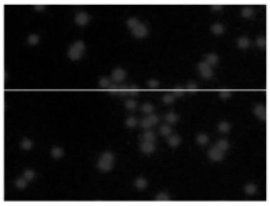
Cross-Channel Degradation

original

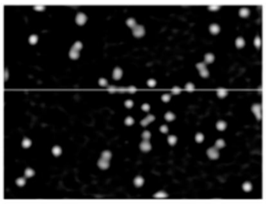
degraded

restored

Degraded Images

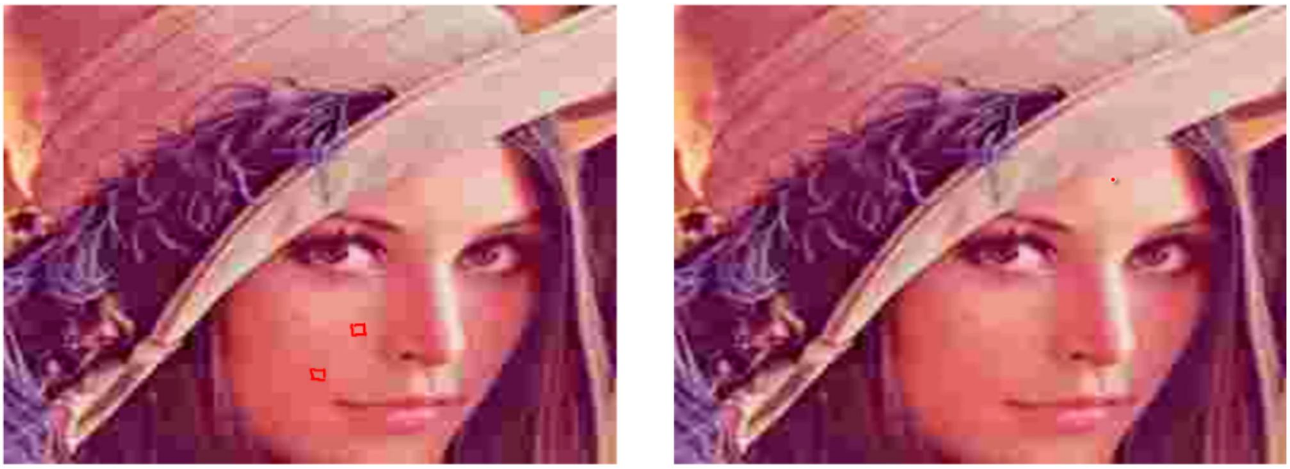


Restored Images



Blind Spatially Varying Restoration

Blocking Artifact Removal

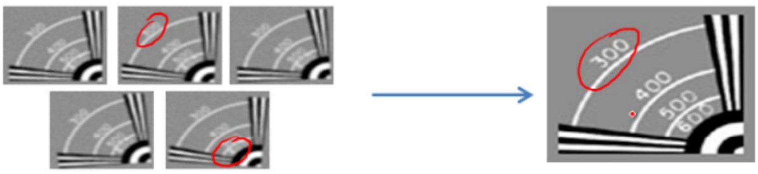


Video Blocking Artifact Removal

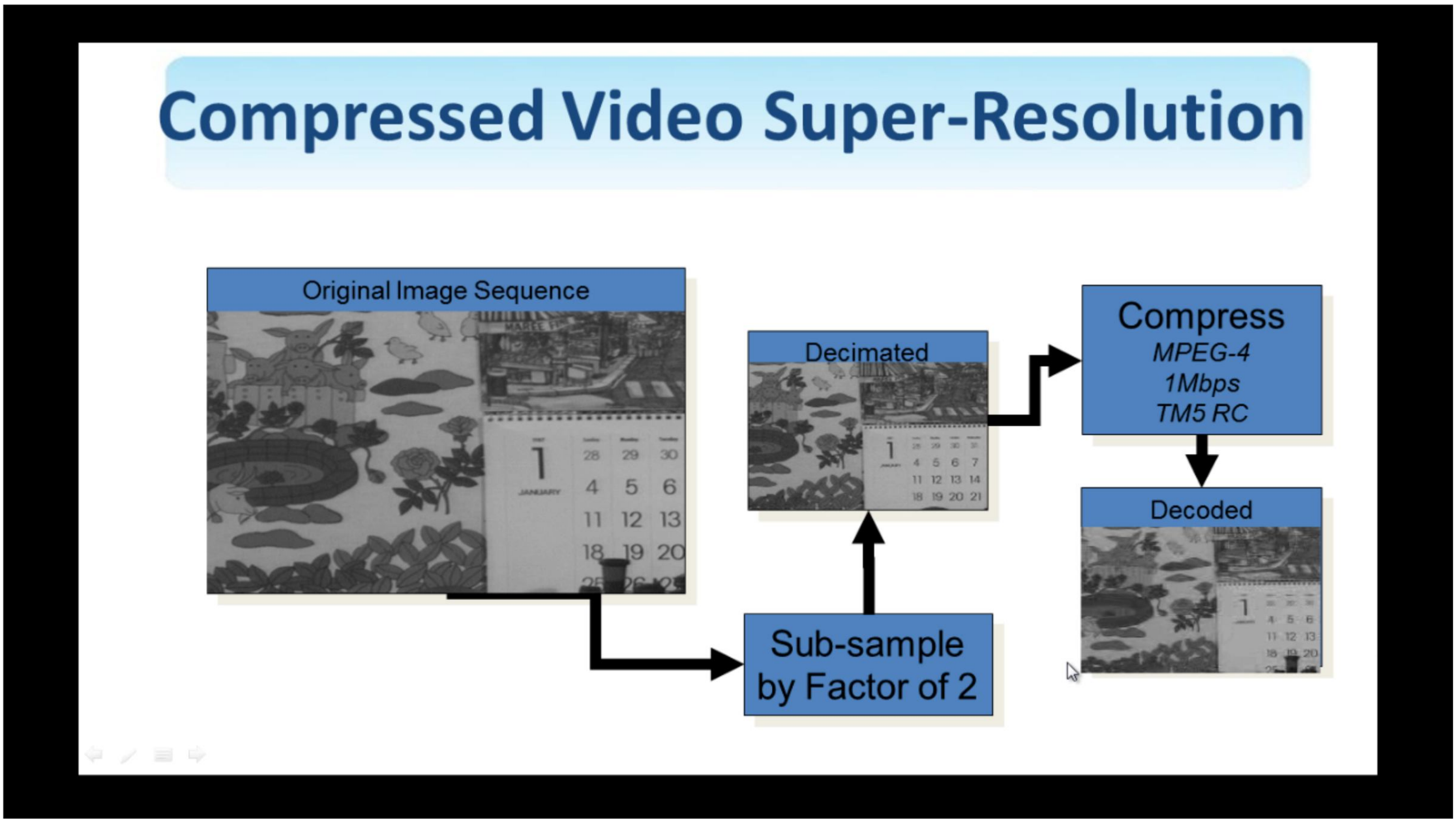
Error Concealment

Inpainting

Image Super-Resolution



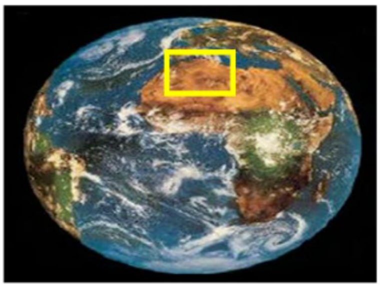
4 images


Compressed Video Super-Resolution
Original Image Sequence
Decimated
Compress
MPEG-4
1Mbps
TM5 RC
Decoded
Sub-sample
by Factor of 2

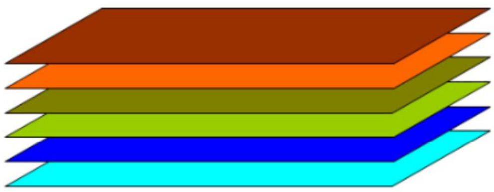

Compressed Video Super-Resolution
Bi-Linear Interpolation
Proposed Algorithm

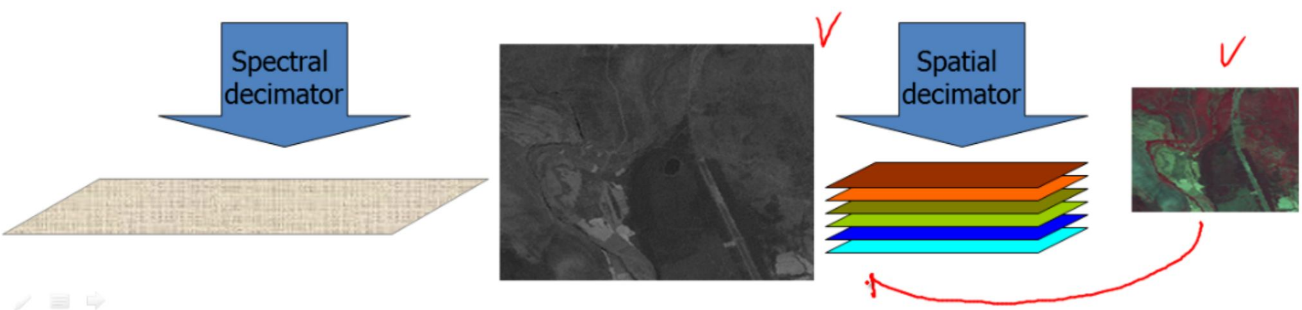
Dual Exposure Restoration

Pansharpening Problem

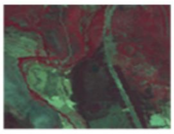


With an ideal sensor we would have high resolution multispectral images

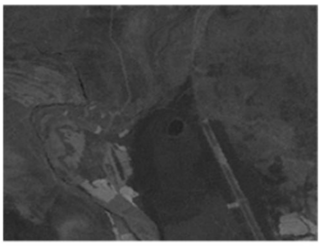




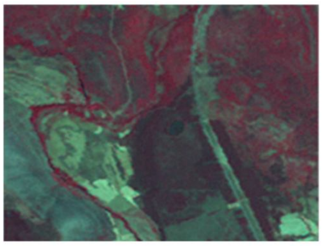
Pansharpening Example



LR Spectral

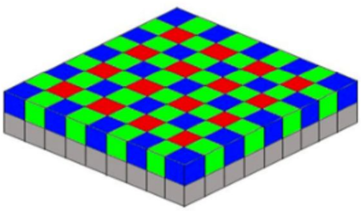


Panchromatic

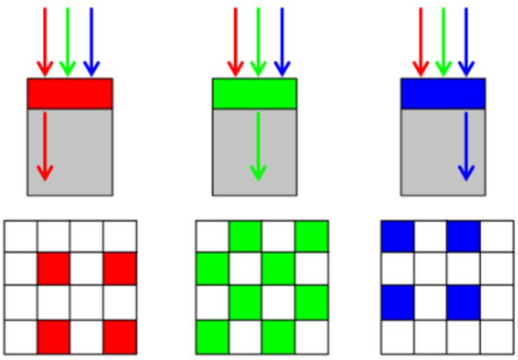


Pansharpened

Demosaicking



CFA: Bayer



R 75%

G 50%

B 75%

Tracking Blurred Objects

Mean Shift

Our algorithm

Tracking Blurred Objects

Mean Shift

Our algorithm

Sources of Degradation

- motion
- atmospheric turbulence
- out-of-focus lens
- limitations of acquisition systems (optics, physics, cost, etc)
- finite resolution of sensors
- quantization errors
- transmission errors
- noise

Forms of the Recovery Problem

- Noise Smoothing
- Restoration/Deconvolution (1D, 2D, 3D)
  - Multi-spectral, multi-channel
- Removal of Compression Artifacts
- Super-Resolution
  - Pansharpening
  - Demosaicking
- Inpainting, Concealment
- Dual Exposure Imaging
- Reconstruction from Projections
- Compressive Sensing
- Light-field Reconstruction

Single vs. multiple observations

Applications

- space exploration, HST (*atmospheric turbulence*)
- remote sensing (*pansharpening*)
- surveillance (*super-resolution*)
- medicine (**diagnostic x-rays, sinograms**)
- neuroimaging (**source localization**)
- nondestructive testing
- commercial, digital photography (*out-of-focus*)
- (video) printing
- microscopy (*diffraction limit*)
- molecular and cellular imaging
- multimedia communications (***quantization, transmission errors***)

Inverse Problems

$$f \longrightarrow \boxed{H} \longrightarrow g$$

Known	Problem Type
H, g	recovery -- an inverse problem
g	blind recovery
g, H partially	semi-blind recovery
f, g	system identification
f, H	system implementation

Parameter Estimation

Motion estimation
Disparity estimation
Boundary detection through differentiation

Inverse problems

Historical Notes

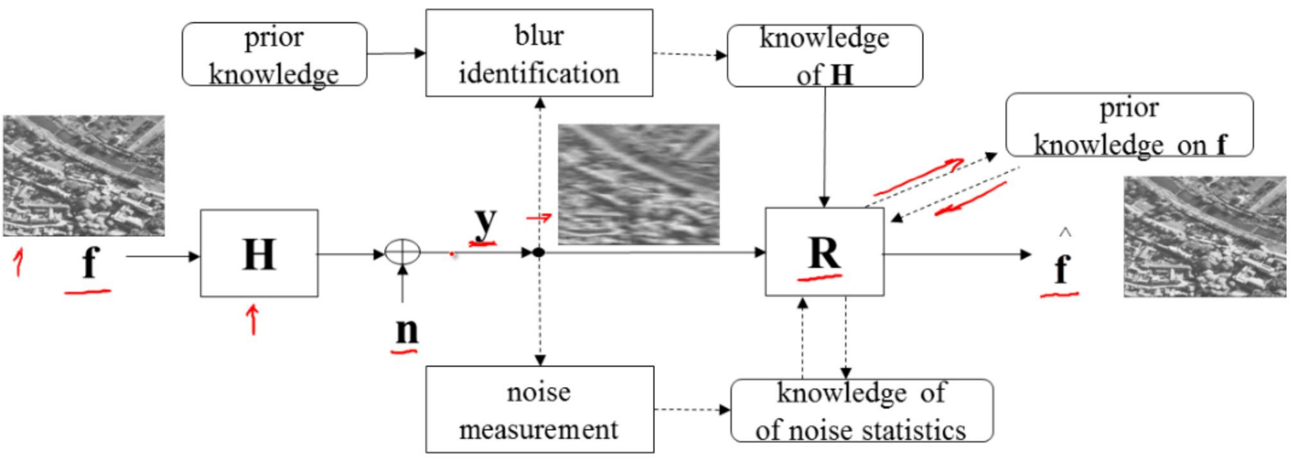
US and Former Soviet Union space programs in 1950's and 1960's

- The 22 images produced during the Mariner IV flight to Mars in 1964 cost $10M

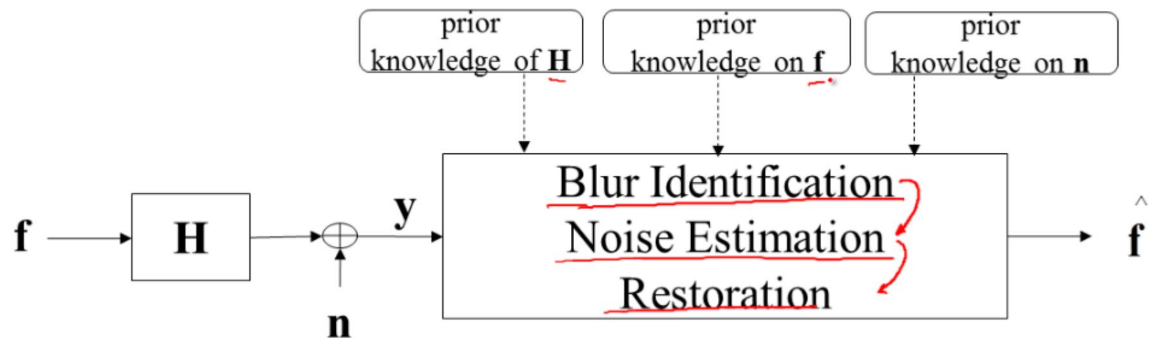
Active area of R&D; activity in waves, driven by mishaps (i.e., HST), new applications (i.e., SR of images and videos for UHD displays), and new mathematical developments (i.e., sparsity)

Notoriety in the media ("JFK"- Zapruder 8mm film-, "no way out", "rising sun")

Degradation / Restoration System



Degradation/ Restoration System



Degradation Model

$$y(i,j) = H[f(i,j)] + n(i,j)$$

restoration

noisy-blurred observed image

degradation operator

source or original image

noise component

In many applications H[ ] can be well approximated by an LSI system and the noise by an additive and signal independent process

deconvolution

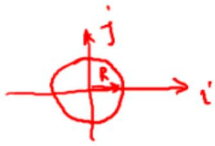
$$y(i,j) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(i-m, j-n) + n(i,j) = f(i,j) ** h(i,j) + n(i,j)$$

Representative Degradations

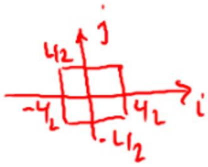
- 1-D Motion: $$h(i) = \begin{cases} \dfrac{1}{L+1} & \text{for } -\dfrac{L}{2} \le i \le \dfrac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Atmospheric turbulence: $$h(i,j) = K \exp\left(-\frac{i^2+j^2}{2\sigma_R^2}\right)$$

- Out of focus: $$h(i,j) = \begin{cases} \dfrac{1}{\pi R} & \text{for } \sqrt{i^2+j^2} \le R \\ 0 & \text{otherwise} \end{cases}$$



- Pill-box:



$$h(i,j) = \begin{cases} \dfrac{1}{(L+1)^2} & \text{for } -\dfrac{L}{2} \le i, j \le \dfrac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

Objective degradation/restoration metrics

- Blurred Signal-to-Noise Ratio (BSNR)

$$\text{BSNR} = 10\log_{10}\left\{\frac{\frac{1}{MN}\sum_i\sum_j\left[g(i,j)-\bar{g}(i,j)\right]^2}{\sigma_n^2}\right\}$$

$$y(i,j) = \underbrace{f(i,j) ** h(i,j)}_{g(i,j)} + n(i,j)$$

$g(i,j) = y(i,j) - n(i,j)$ $\quad$ σ_n^2 : noise variance

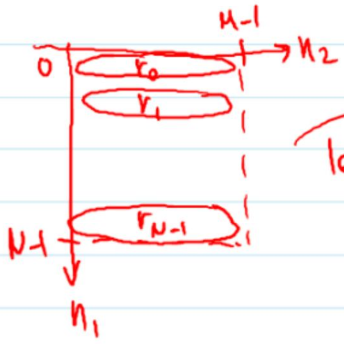
$\bar{g}(i,j) = E\{g(i,j)\}$

- Improvement in Signal-to-Noise Ratio (ISNR)

$$\text{ISNR} = 10\log_{10}\left\{\frac{\sum_i\sum_j\left[f(i,j)-y(i,j)\right]^2}{\sum_i\sum_j\left[f(i,j)-\hat{f}(i,j)\right]^2}\right\}$$

original

Vector Notation for Images



stacking

lexicogra·

$$\mathbf{x} = \begin{bmatrix} r_0^T \\ r_1^T \\ \vdots \\ r_{N-1}^T \end{bmatrix} \quad NM \times 1$$

$\underline{x}$

$\bar{x}$

$x(n_1, n_2)$ $\quad 0 \le n_1 \le N-1$

$0 \le n_2 \le M-1$

Matrix-Vector Notation

$$y(n) = x(n) * h(n) = \sum_k x(k) \underline{h(n-k)}$$

$x(n): [0, \dots, N-1]$

$h(n): [0, \dots, L-1]$

$y(n): [0, \dots, N+L-2] \leftarrow$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(N+L-2) \end{bmatrix} = \begin{bmatrix} h(0) & & & \emptyset \\ h(1) & h(0) & & \\ h(2) & h(1) & h(0) & \\ \vdots & & \ddots & h(0) \\ h(L-1) & & & \\ & \emptyset & & h(L-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$(N+L-1) \times 1 \qquad (N+L-1) \times N \qquad N \times 1$

$$y = H \; x$$

$$y = H x$$

Toeplitz

Matrix-Vector Notation

$$y(n) = x(n) \; (N+L-1) \; h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \\ y(N+L-2) \end{bmatrix} = \begin{bmatrix} h(0) & & & h(L-1) \cdots h(2) & h(1) \\ h(1) & \ddots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset & h(L-1) \\ h(L-1) & & \ddots & \ddots & \\ & \ddots & & \ddots & \\ \emptyset & & h(L-1) \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} (L-1) \text{ zeros} \end{matrix}$$

$$y = H x$$

Eigen-values/vectors of Circulant Matrices

$$H = \begin{bmatrix} h(0) & \cdots & h(M-1) \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} | & | & & | \\ w_0 & w_2 & \cdots & w_{M-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{M-1} \end{bmatrix} \begin{bmatrix} | & | & & | \\ w_0 & w_1 & \cdots & w_{M-1} \\ | & | & & | \end{bmatrix}^{-1}$$

e-vector, e-values

$$H w_n = \lambda_n w_n \, , \quad n = 1, .., M$$

$$\Rightarrow H = W D W^{-1}$$

$$w_n = \left[1 \quad e^{j\frac{2\pi}{M}n} \quad e^{j\frac{2\pi}{M}2n} \cdots e^{j\frac{2\pi}{M}(M-1)n} \right]^T$$

$$\{\lambda_0, .., \lambda_{M-1}\} = M \cdot \text{DFT}\{h(0) \ldots h(M-1)\}$$

Eigen-values/vectors of Circulant Matrices

$$y(n) = x(n) \, Ⓝ \, h(n) \quad \longleftrightarrow \quad y = Hx$$

DFT

$$Y(k) = X(k) \cdot H(k) \qquad = WDW^{-1}x$$

$k = 0, \dots, N-1$

$$\Rightarrow \underbrace{W^{-1}y}_{\text{DFT of } y} = D \underbrace{W^{-1}x}_{\text{DFT of } x}$$

$$\begin{bmatrix} Y(0) \\ \vdots \\ Y(N-1) \end{bmatrix} = \begin{bmatrix} H(0) & & \\ & \ddots & \\ & & H(N-1) \end{bmatrix} \begin{bmatrix} X(0) \\ \vdots \\ X(N-1) \end{bmatrix}$$

Matrix-vector notation

- By stacking (lexicographically) the observed image into a vector

$$\underline{\mathbf{y} = \mathbf{Hf} + \mathbf{n}} \qquad f \rightarrow \boxed{H} \rightarrow \oplus \rightarrow y, \quad n \uparrow$$

- LSI degradation model

$$y(i,j) = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} f(m,n) h(i-m, j-n) + n(i,j) = f(i,j) ** h(i,j) + n(i,j)$$

  - $\mathbf{H}$ is a ***block circulant*** matrix

j: h_{M-1} --- $h(i,j) \rightarrow H_{M-1}$

$h_1 \rightarrow H_1$

$h_0 \rightarrow H_0$

i

circulant

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_{M-1} & \cdots & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \cdots & \mathbf{H}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_0 \end{bmatrix}$$

Spectral Properties of Block Circulant Matrices

$$\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1} \Rightarrow \mathbf{D} = \mathbf{W}^{-1}\mathbf{H}\mathbf{W}$$

diagonal with elements the stacked values of the 2D DFT of $h(i, j)$

stacked 2D DFT of the image $f(i, j)$

$$\mathbf{y} = \mathbf{H}\mathbf{f} + \mathbf{n} \Rightarrow \mathbf{y} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1}\mathbf{f} + \mathbf{n} \Rightarrow \mathbf{W}^{-1}\mathbf{y} = \mathbf{D}\mathbf{W}^{-1}\mathbf{f} + \mathbf{W}^{-1}\mathbf{n} \Rightarrow Y = DF + N$$

Discrete Frequency Domain Representation

$$Y(u,v) = H(u,v)F(u,v) + N(u,v), \qquad u, v = 0, 1, \ldots, M-1$$

Elements of Linear Algebra

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad \|x\|_2^2 = x^T x = \sum_{i=1}^{N} x_i^2$$

$$\|y - Hx\|_2^2 = (y-Hx)^T(y-Hx) = y^T y - y^T H x - x^T H^T y + x^T H^T H x$$

$$= y^T y - 2x^T H^T y + x^T H^T H x$$

$$\nabla_x \|x\|_2^2 = \begin{bmatrix} \frac{\partial}{\partial x_1}\|x\|_2^2 \\ \vdots \\ \frac{\partial}{\partial x_N}\|x\|_2^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}\sum_i x_i^2 \\ \vdots \\ \frac{\partial}{\partial x_N}\sum_i x_i^2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ \vdots \\ 2x_N \end{bmatrix} = 2x$$

$$\nabla_x (x^T y) = y \quad , \quad \nabla_x (x^T H^T H x) = 2H^T H x$$

Inverse Filter

Degradation equation: $\mathbf{y} = \mathbf{H}\mathbf{f} + \mathbf{n}$

$$\underset{\mathbf{f}}{\text{Minimize}} \quad J(\mathbf{f}) = \|\mathbf{y} - \mathbf{H}\mathbf{f}\|_2^2$$

$$\nabla_f J(f) = \frac{\partial J(\mathbf{f})}{\partial \mathbf{f}} = 0 \Rightarrow \frac{\partial}{\partial f}\left(\cancel{y^Ty} - 2f^TH^Ty + f^TH^THf\right) = -2H^Ty + 2H^THf = 0$$

generalized inverse

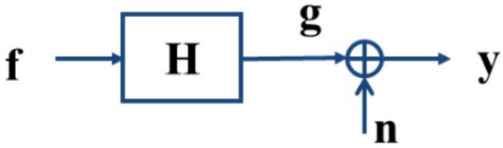
$$\mathbf{H}^T\mathbf{H}\mathbf{f} = \mathbf{H}^T\mathbf{y} \Rightarrow \mathbf{f} = (\mathbf{H}^T\mathbf{H})^{+}\mathbf{H}\mathbf{y}$$

H Block circulant:

$$\frac{H^*(u,v)Y(u,v)}{H^*(u,v)H(u,v)} = \frac{Y(u,v)}{H(u,v)} = \frac{H(u,v)F(u,v)}{H(u,v)} + \frac{N(u,v)}{H(u,v)}$$

$$F(u,v) = \begin{cases} \dfrac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} & |H(u,v)| \neq 0 \ (\geq T) \\ 0 & |H(u,v)| = 0 \ (< T) \end{cases}$$

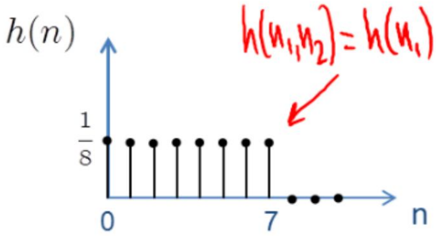
1D Motion Blur



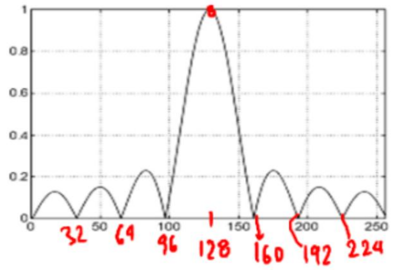
Motion Blur over 8 pixels
BSNR=20 dB

1D Motion Blur

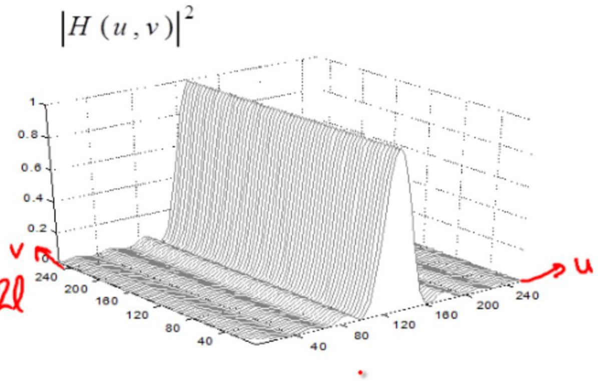
$h(n_1,n_2) = h(n_1)\cdot\delta(n_2)$



$|H(u,v)|$



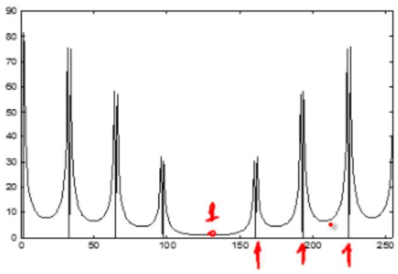
$\frac{256}{8} l = 32l$

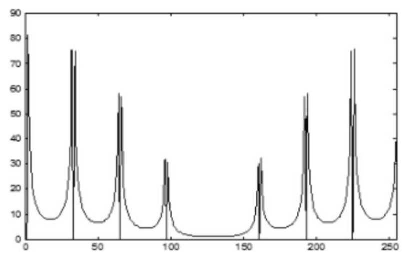


Thresholded Inverse Filter

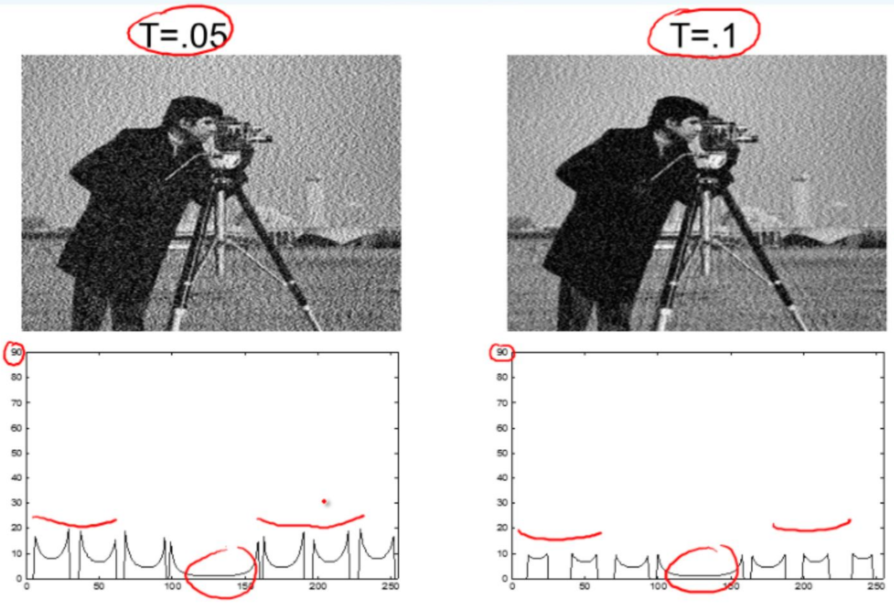
$T=10^{-16}$

$T=.01$

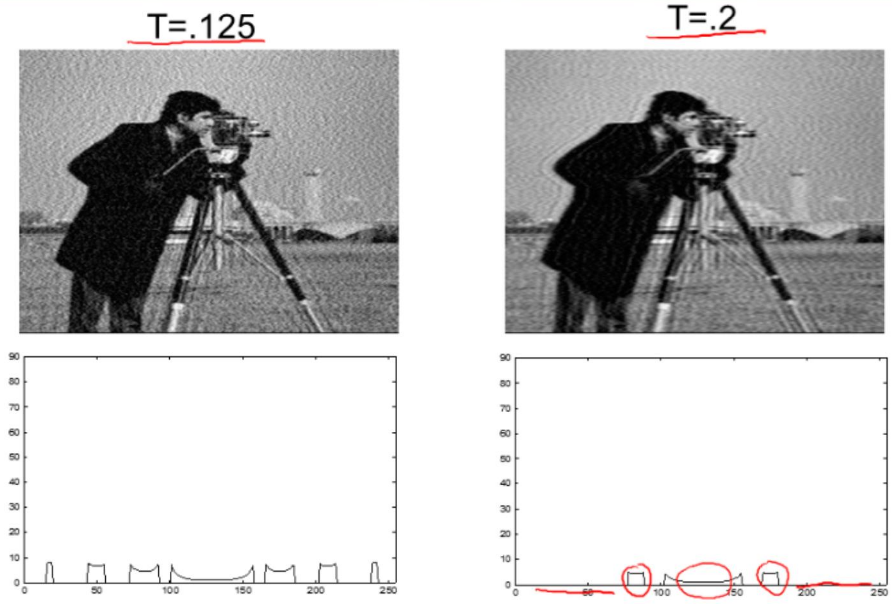




Thresholded Inverse Filter



Thresholded Inverse Filter



Constrained Least-Squares Filter

$$\underset{\mathbf{f}}{\text{Minimize}} \quad J(\mathbf{f}) = \|\mathbf{y} - \mathbf{H}\mathbf{f}\|_2^2 \quad \leftarrow \text{fidelity to the data}$$

$$\text{subject to} \quad \|\mathbf{C}\mathbf{f}\|_2^2 < \varepsilon \quad \leftarrow \text{prior knowledge}$$

generalized inverse

$$\min_{\mathbf{f}} \left(\|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2 + \alpha \|\mathbf{C}\mathbf{f}\|^2 \right) \Rightarrow \mathbf{f} = \left(\mathbf{H}^T\mathbf{H} + \alpha \mathbf{C}^T\mathbf{C}\right)^+ \mathbf{H}^T\mathbf{y}$$

regularization parameter

Lagrange multiplier

- $\mathbf{C}$ is a high-pass filter, such as the 2D Laplacian
- for $\mathbf{H}$ and $\mathbf{C}$ block circulant:

2D Laplacian

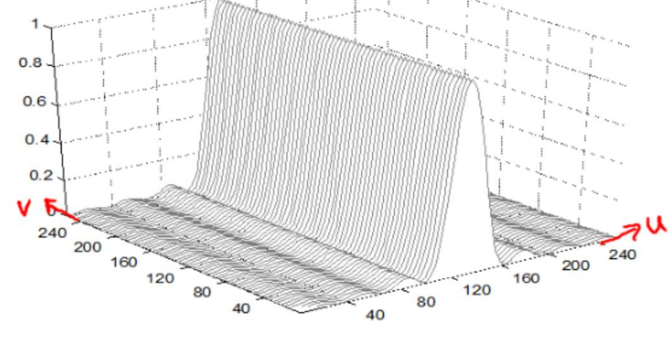
$$\mathbf{C} = \begin{bmatrix} 0.00 & 0.25 & 0.00 \\ 0.25 & -1.00 & 0.25 \\ 0.00 & 0.25 & 0.00 \end{bmatrix}$$

$$\boxed{F(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \alpha |C(u,v)|^2} Y(u,v)} \leftarrow \|Cf(\alpha)\|^2 < \epsilon$$

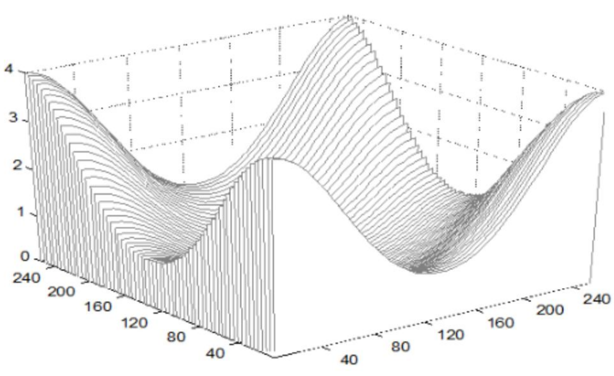
$\alpha = 0 \quad \text{CLS} \rightarrow \text{LS}$

Frequency Responses

$|H(u,v)|^2$



$|C(u,v)|^2$

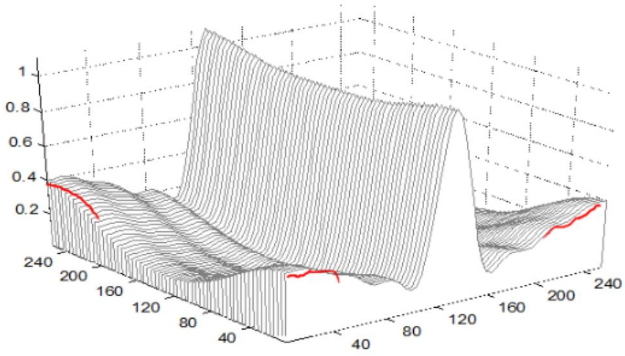


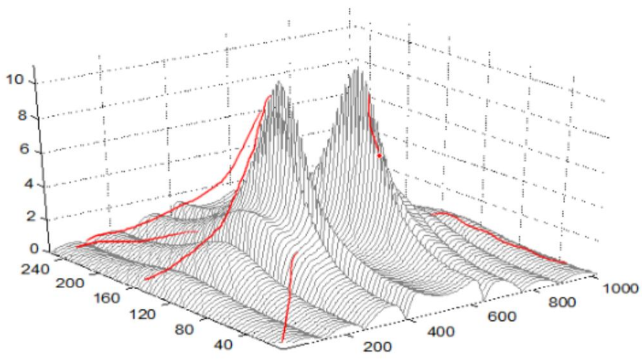
Constrained Least Squares Filter

$\alpha = 0.1$

$$\underline{\left|H(u,v)\right|^2} + .1 \cdot \underline{\left|C(u,v)\right|^2}$$

$$\left|\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + .1 \cdot \left|C(u,v)\right|^2}\right|$$



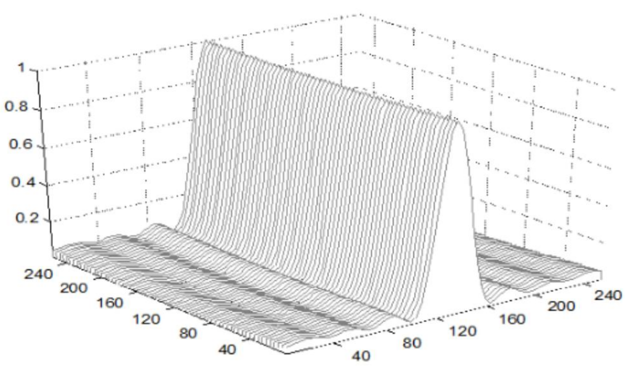


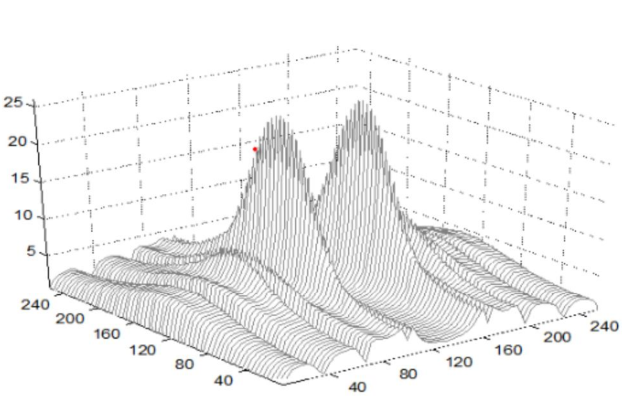
Constrained Least Squares Filter

$\alpha = 0.01$

$$\underline{\left|H(u,v)\right|^2} + .01 \cdot \left|C(u,v)\right|^2$$

$$\left|\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + .01 \cdot \left|C(u,v)\right|^2}\right|$$





Constrained Least Squares Filter

$\alpha = 1$

$\alpha = 0.1$

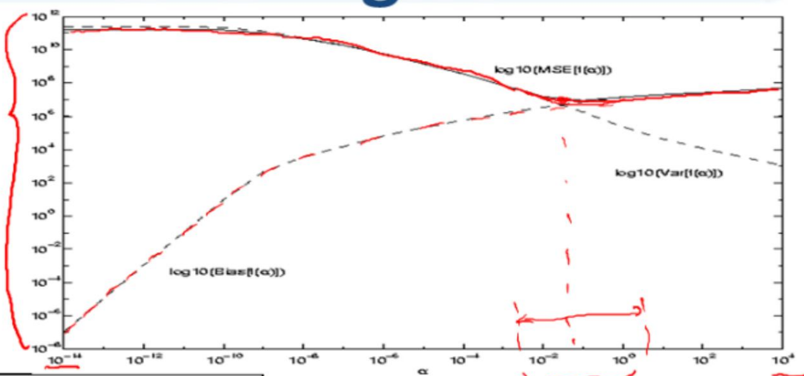
$\alpha = 0.01$

$\|Cf\|_2^2 < \varepsilon$

Effect of Regularization Parameter in CLS Algorithm

original image

$$E \,\| \mathbf{f}(\alpha) - \mathbf{f} \|^2 = Bias\ (\mathbf{f}(\alpha)) + Var\ (\mathbf{f}(\alpha))$$



$$Var\ (\alpha) = \sigma_n^2 \sum_{u=0}^{M} \sum_{v=0}^{N} \frac{|H(u,v)|^2}{\left(|H(u,v)|^2 + \alpha |C(u,v)|^2\right)^2}$$

$$Bias\ (\alpha) = \sigma_n^2 \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \frac{|F(u,v)|^2 \alpha^2 |C(u,v)|^4}{\left(|H(u,v)|^2 + \alpha |C(u,v)|^2\right)^2}$$

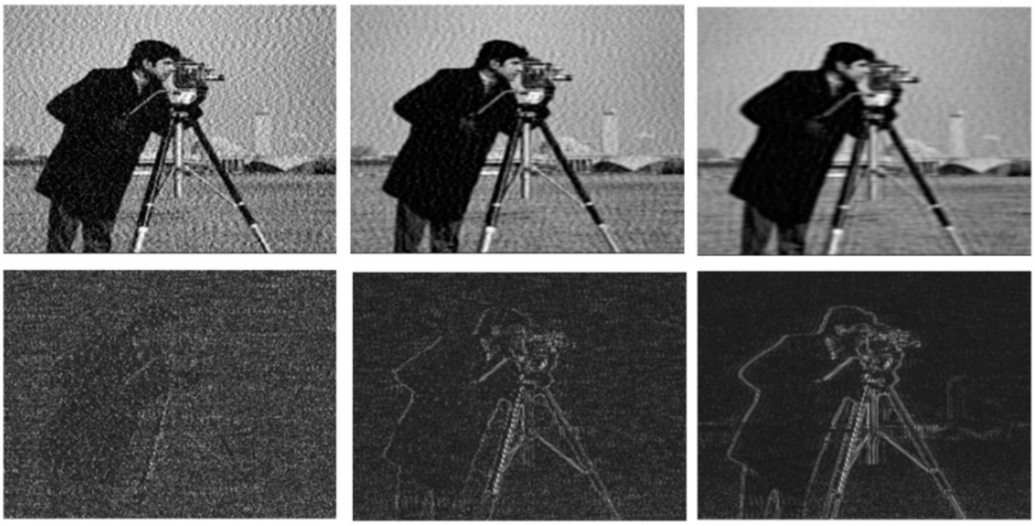
- N. P. Galatsanos, and A. K. Katsaggelos, "Methods for Choosing the Regularization Parameter and Estimating the Noise Variance in Image Restoration and their Relation", *IEEE Trans. Image Processing*, vol. 1, pp. 322-336, July 1992.

Effect of Regularization Parameter in CLS Algorithm

$\alpha = 0.01$ $\alpha = 0.1$ $\alpha = 1$

restored

|original-restored|
mapped in [32-255]



Regularization Parameter Choice

- Knowledge of the noise variance

$\min \|Cx\|^2$
s.t. $\|y - Hf\| = \epsilon^2$

$\|y - Hf(\alpha)\|^2 \approx \epsilon^2$

noise variance unknown

- Visual Inspection
- The L-curve
- Generalized cross-validation

N. P. Galatsanos, and A. K. Katsaggelos, "Methods for Choosing the Regularization Parameter and Estimating the Noise Variance in Image Restoration and their Relation", *IEEE Trans. Image Processing*, vol. 1, pp. 322-336, July 1992.

Non-Quadratic Regularization

CLS: $\hat{\mathbf{f}}(\alpha)_{CLS} = \arg\min_{\mathbf{f}} \|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2 + \alpha\|\mathbf{C}\mathbf{f}\|^2$ ℓ_2-Norms

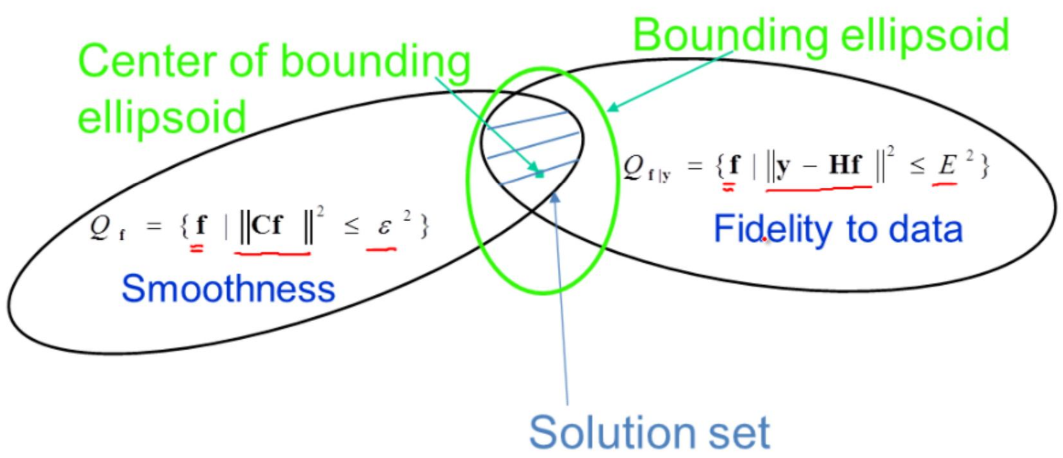
Non-Quadratic Regularization: $\hat{\mathbf{f}}(\alpha) = \arg\min_{\mathbf{f}} J_1(\mathbf{f}, \mathbf{y}) + \alpha J_2(\mathbf{f})$

Maximum-Entropy Regularization: $\hat{\mathbf{f}}(\alpha)_{ME} = \arg\min_{\mathbf{f}} \|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2 + \alpha \sum_{i=1}^{N} f_i \log(f_i)$

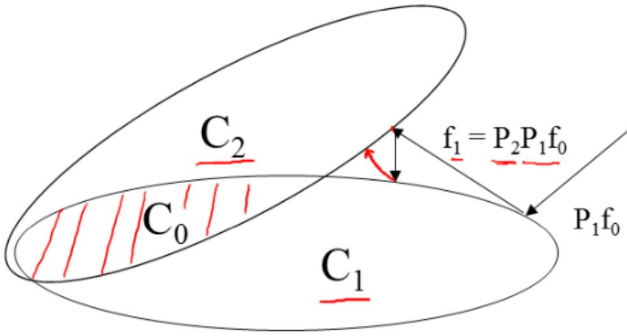
Total Variation Regularization: $\hat{\mathbf{f}}(\alpha)_{TV} = \arg\min_{\mathbf{f}} \|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2 + \alpha \sum_{i=1}^{N} |[\mathbf{\Delta f}]_i|$

l_p Norms: $J(\mathbf{z}) = \|\mathbf{z}\|_p^p = \sum_{i=1}^{N} |z_i|^p, \quad 1 \le p \le 2$

Set Theoretic Approach Principle



POCS Algorithm



For C_i, $i = 1, 2, \dots, m$, convex and closed

$$\mathbf{f}_k = P_m \dots P_2 P_1 \mathbf{f}_{k-1}, \quad \mathbf{f}_0 \text{ arbitrary,}$$

then

$$\mathbf{f}_k \rightarrow \mathbf{f}^* \in C_0 \equiv \bigcap_{i=1}^{m} C_i .$$

Set Theoretic Restoration Approach

- Find a solution belonging to both sets

C1 $$Q_{\mathbf{f}|\mathbf{y}} = \{ \mathbf{f} \mid \| \mathbf{y} - \mathbf{Hf} \|^2 \le E^2 \}$$

C2 $$Q_{\mathbf{f}} = \{ \mathbf{f} \mid \| \mathbf{Cf} \|^2 \le \varepsilon^2 \}$$

- Solution Approach 1: Center of ellipsoid bounding the intersection

CLS $$\mathbf{f} = \left(\mathbf{H}^T \mathbf{H} + \alpha \mathbf{C}^T \mathbf{C} \right)^{+} \mathbf{H}^T \mathbf{y} \qquad \alpha = (\varepsilon / E)^2$$

- Solution Approach 2: Alternate projections onto convex sets

$$\mathbf{f}_{k+1} = P_1 P_2 \mathbf{f}_k$$

$$P_1 \mathbf{f} = \mathbf{f} + \lambda_1 \left(\mathbf{I} + \lambda_1 \mathbf{H}^T \mathbf{H} \right)^{-1} \mathbf{H}^T (\mathbf{y} - \mathbf{Hf})$$

$$P_2 \mathbf{f} = [\mathbf{I} - \lambda_2 \left(\mathbf{I} + \lambda_2 \mathbf{C}^T \mathbf{C} \right)^{-1} \mathbf{C}^T \mathbf{C}] \mathbf{f}$$

Iterative Restoration Algorithms

- There is no need to explicitly implement the inverse of an operator.
- The restoration process is monitored as it progresses.
- The number of iterations can be used as a means of regularization.
- The effects of noise can be controlled in each iteration.
- They can be applied to cases of spatially varying or nonlinear degradations or when the type of degradation is unknown (blind restoration).

Basic Approach

Find root(s) of $\Phi(\mathbf{f})$

Successive Approximations Iteration

$$\begin{aligned} \mathbf{f}_0 &= \mathbf{0} \\ \mathbf{f}_{k+1} &= \mathbf{f}_k + \beta\,\Phi(\mathbf{f}_k) \\ &= \Psi(\mathbf{f}_k) \end{aligned}$$

$f_k : \Phi(f_k) = 0$

$f_{k+1} = f_k + \beta\Phi(f_k)^{0} = f_k$

Successive Approximations Iteration with Constraints

$$\begin{aligned} \mathbf{f}_0 &= \mathbf{0} \\ \tilde{\mathbf{f}}_k &= C\,\mathbf{f}_k \\ \mathbf{f}_{k+1} &= \tilde{\mathbf{f}}_k + \beta\,\Phi(\tilde{\mathbf{f}}_k) = \Psi(C\,\mathbf{f}_k) \end{aligned}$$

Constraint or projection operator

Basic Iteration

$$\mathbf{y} = \mathbf{Hf} + \mathbf{n}$$

$$\Phi(\mathbf{f}) = \mathbf{y} - \mathbf{Hf}$$

$$\mathbf{f}_{k+1} = \mathbf{f}_k + \beta(\mathbf{y} - \mathbf{Hf}_k) = \beta\,\mathbf{y} + (\mathbf{I} - \beta\,\mathbf{H})\mathbf{f}_k$$

Frequency Domain Iteration (H block circulant)

$$F_{k+1}(u,v) = \beta\,Y(u,v) + (1 - \beta\,H(u,v))F_k(u,v)$$

Convergence

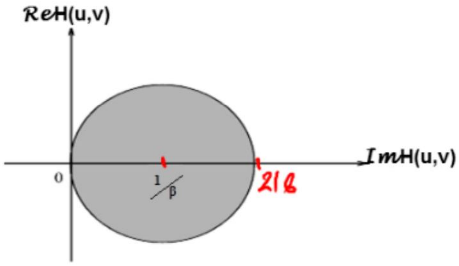
$$F_k(u,v) = R_k(u,v)Y(u,v) \qquad R_k(u,v) = \beta \sum_{l=0}^{k-1} (1 - \beta H(u,v))^l$$

$F_0 = 0$; $F_1 = \beta Y$; $F_2 = \beta Y + (1-\beta H)F_1 = \beta Y + (1-\beta H)\beta Y$

$F_3 = \beta Y + (1-\beta H)\beta Y + (1-\beta H)^2 \beta Y$

if $\quad |1 - \beta H(u,v)| < 1$

Special case $\quad 0 < \beta < \dfrac{2}{H_{max}(u,v)}$



$$\lim_{k\to\infty} R_k(u,v) = \lim_{k\to\infty} \beta\,\frac{1 - (1 - \beta\,H(u,v))^k}{1 - (1 - \beta\,H(u,v))} = \begin{cases} \dfrac{1}{H(u,v)} & H(u,v) \neq 0 \\ k\beta & H(u,v) = 0 \end{cases}$$

Least Squares (LS) Iteration

$$\Phi(\mathbf{f}) = \frac{1}{2}\nabla_{\mathbf{f}} \|\mathbf{y} - \mathbf{Hf}\|^2$$
$$\mathbf{f}_{k+1} = \mathbf{f}_k + \beta \underbrace{\mathbf{H}^T(\mathbf{y} - \mathbf{Hf}_k)}$$
$$= \beta \mathbf{H}^T\mathbf{y} + (\mathbf{I} - \beta \mathbf{H}^T\mathbf{H})\mathbf{f}_k$$

Frequency Domain Iteration (H block circulant)

$$F_{k+1}(u,v) = \beta H^*(u,v)Y(u,v) + \left(1 - \beta|H(u,v)|^2\right)F_k(u,v)$$

Convergence

$$R_k(u,v) = \beta\sum_{l=0}^{k-1}\left(1 - \beta|H(u,v)|^2\right)^l H^*(u,v) = \beta\frac{1-(1-\beta|H(u,v)|^2)^k}{1-(1-\beta|H(u,v)|^2)}H^*(u,v)$$

sufficient condition for convergence

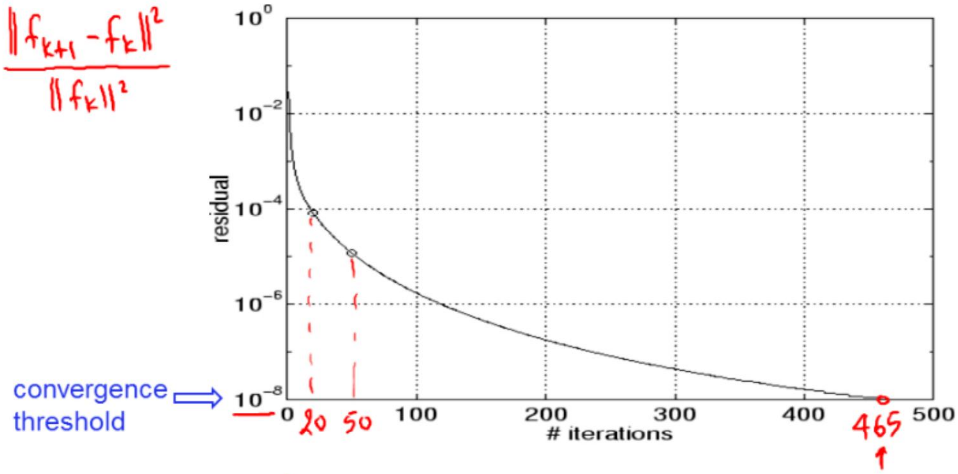
$$\left|1 - \beta|H(u,v)|^2\right| < 1, \quad \text{or} \quad 0 < \beta < \frac{2}{\max_{u,v}|H(u,v)|^2}$$

0<β<2

$$\lim_{k\to\infty} R_k(u,v) = \lim_{k\to\infty}\beta\frac{1-(1-\beta|H(u,v)|^2)^k}{1-(1-\beta|H(u,v)|^2)}H^*(u,v) = \begin{cases}\dfrac{1}{H(u,v)} & H(u,v) \neq 0\\ 0 & H(u,v) = 0\end{cases}$$

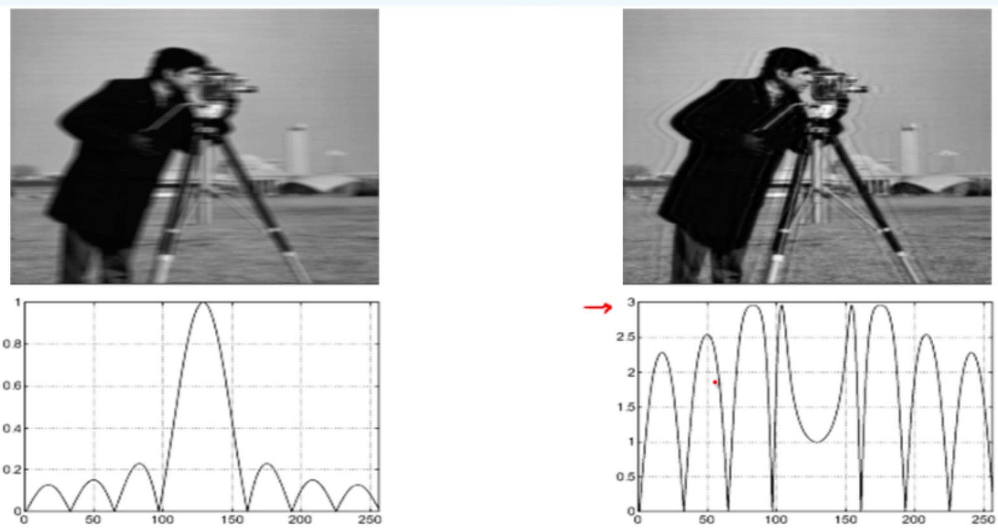
Residual Error

$$\frac{\|f_{k+1}-f_k\|^2}{\|f_k\|^2}$$



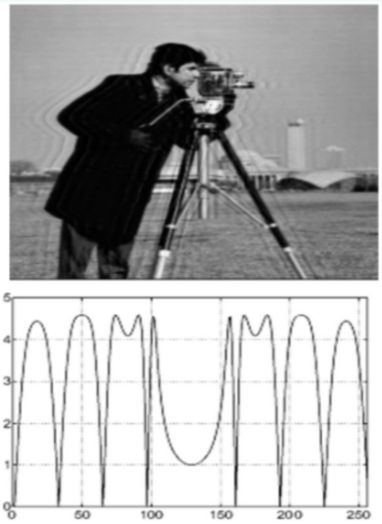
Residual error versus number of iterations for the iterative LS algorithm; 1D motion blur over 8 pixels, no noise.

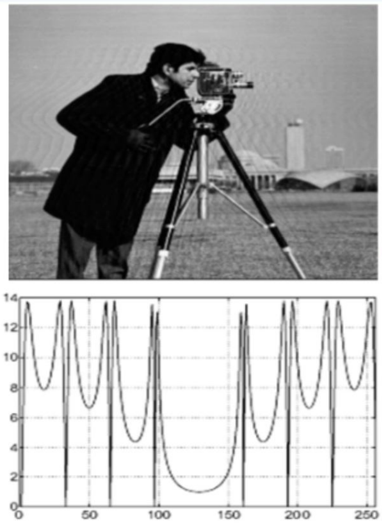
Experimental Results



(l) 1D motion blur over 8 pixels; (r) iterative LS restoration, k=20, ISNR=4.03 dB.

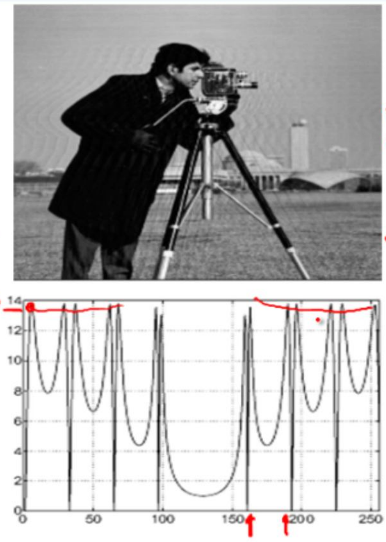
Experimental Results

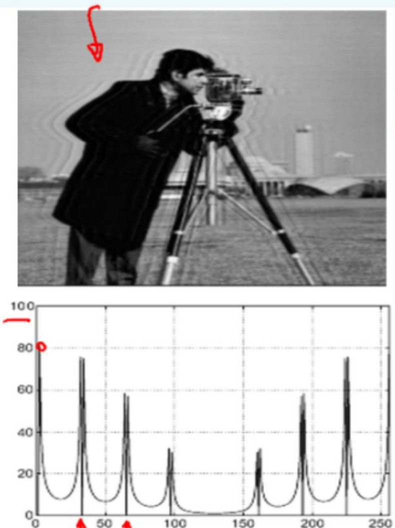




Iterative LS restorations: (l) k=50, ISNR=6.22 dB; (r) k=465, ISNR=11.58 dB.

Experimental Results

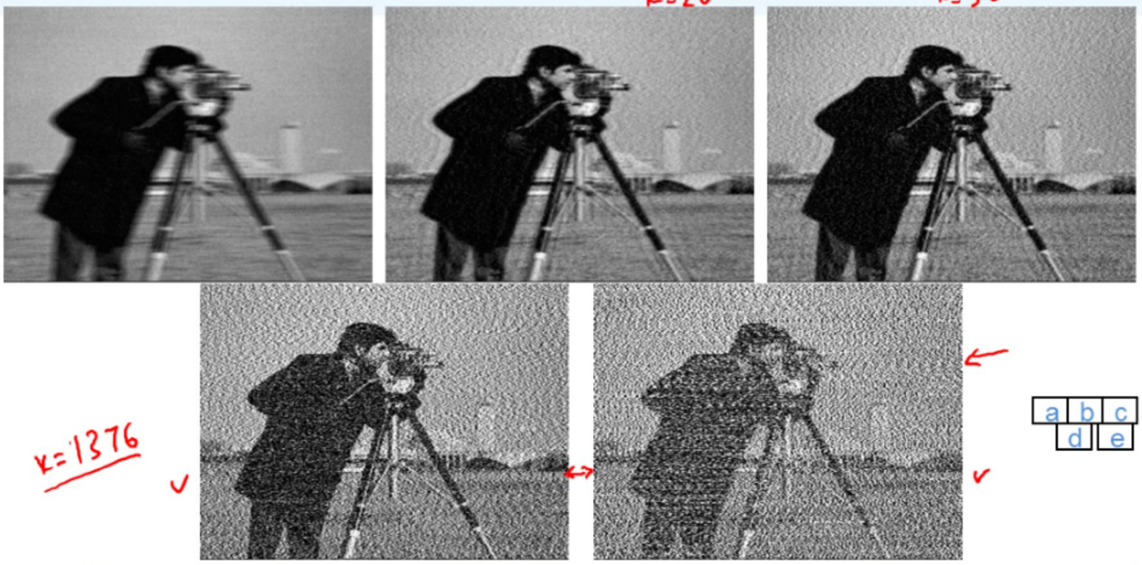




(l) Iterative LS restoration, k=465, ISNR=11.58 dB; (r) direct inverse, ISNR=15.50 dB.

Experimental Results

k=20 k=50



k=1376

(a) 1D motion blur over 8 pixels, BSNR=20dB; (b)-(d) iterative LS restorations: (b) k=20, ISNR=1.83 dB; (c) k=50, ISNR=-0.30 dB; (d) k=1376, ISNR=-9.06 dB; (e) direct inverse, ISNR=-12.09 dB.

CLS Iteration

$$\Phi(\mathbf{f}) = \frac{1}{2}\nabla_{\mathbf{f}}\left(\|\mathbf{y}-\mathbf{Hf}\|^2 + \alpha\|\mathbf{Cf}\|^2\right)$$

$$\mathbf{f}_{k+1} = \beta\mathbf{H}^T\mathbf{y} + (\mathbf{I} - \beta(\mathbf{H}^T\mathbf{H} + \alpha\mathbf{C}^T\mathbf{C}))\mathbf{f}_k$$

Frequency Domain Iteration (H, C block circulant)

$$F_{k+1}(u,v) = \beta H^*(u,v)Y(u,v) + \left(1 - \beta\left(|H(u,v)|^2 + \alpha|C(u,v)|^2\right)\right)F_k(u,v)$$

Convergence

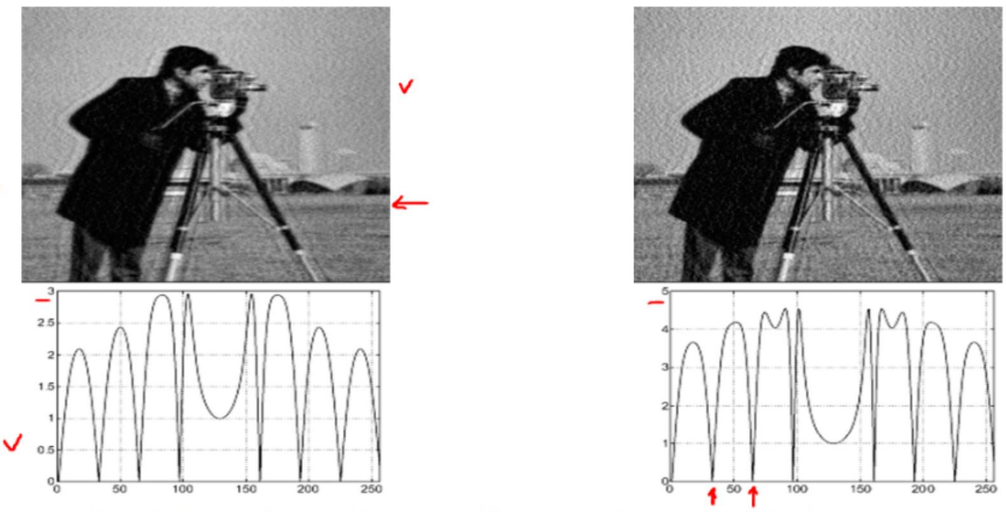
$$R_k(u,v) = \beta \sum_{l=0}^{k-1} \left(1 - \beta\left(|H(u,v)|^2 + \alpha |C(u,v)|^2\right)\right)^l H^*(u,v)$$

sufficient condition for convergence

$$\boxed{\left|1 - \beta\left(|H(u,v)|^2 + \alpha |C(u,v)|^2\right)\right| < 1}$$

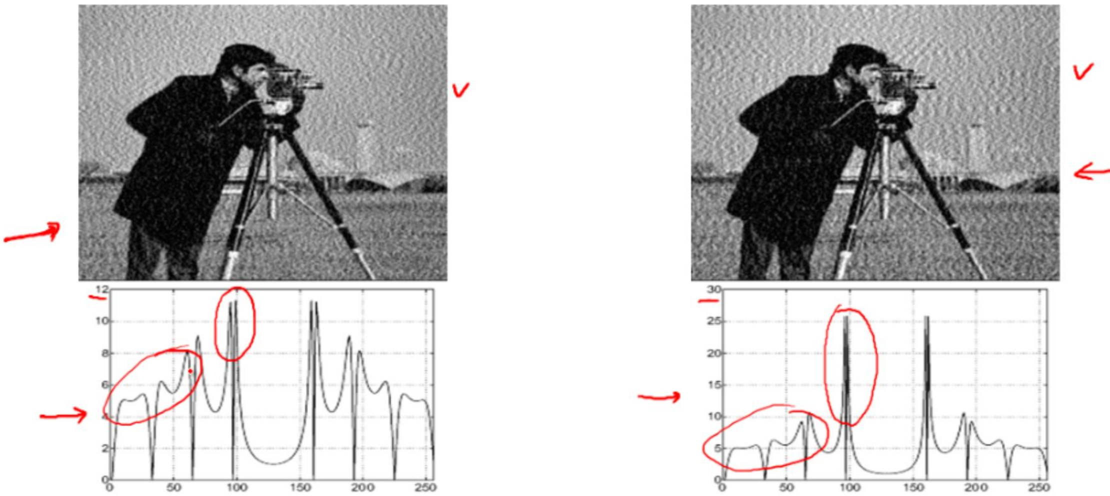
$$\lim_{k \to \infty} R_k(u,v) = \begin{cases} \dfrac{H^*(u,v)}{|H(u,v)|^2 + \alpha |C(u,v)|^2} & |H(u,v)|^2 + \alpha |C(u,v)|^2 \neq 0 \\ 0 & |H(u,v)|^2 + \alpha |C(u,v)|^2 = 0 \end{cases}$$

Experimental Results



Restorations of a noisy-blurred image (1D motion blur over 8 pixels, BSNR=20dB) iterative CLS restorations, with C a 2D Laplacian and α=0.01 and corresponding $|R_k(u,0)|$; (l) k=20, ISNR=2.12 dB; (r) k=50, ISNR=0.98 dB.

Experimental Results



Restorations of a noisy-blurred image (1D motion blur over 8 pixels, BSNR=20dB) and corresponding $|R_k(u,0)|$ by a CLS filter with C a 2D Laplacian and α =0.01; (l) iterative k=330, ISNR=-1.01 dB; (r) direct ISNR=-1.64 dB.

Spatially Adaptive CLS Iteration

$$\Phi(\mathbf{f}) = \frac{1}{2}\nabla_{\mathbf{f}}\left(\|\mathbf{y}-\mathbf{Hf}\|^2_{\mathbf{W}_1} + \alpha\,\|\mathbf{Cf}\|^2_{\mathbf{W}_2}\right)$$

$$\mathbf{f}_{k+1} = \beta\,\mathbf{H}^T\mathbf{W}_1^T\mathbf{W}_1\mathbf{y} + (\mathbf{I} - \beta(\mathbf{H}^T\mathbf{W}_1^T\mathbf{W}_1\mathbf{H} + \alpha\,\mathbf{C}^T\mathbf{W}_2^T\mathbf{W}_2\mathbf{C}))\,\mathbf{f}_k$$

Choice of weights

$\mathbf{W}_2 = \mathbf{V}$ (the visibility matrix)

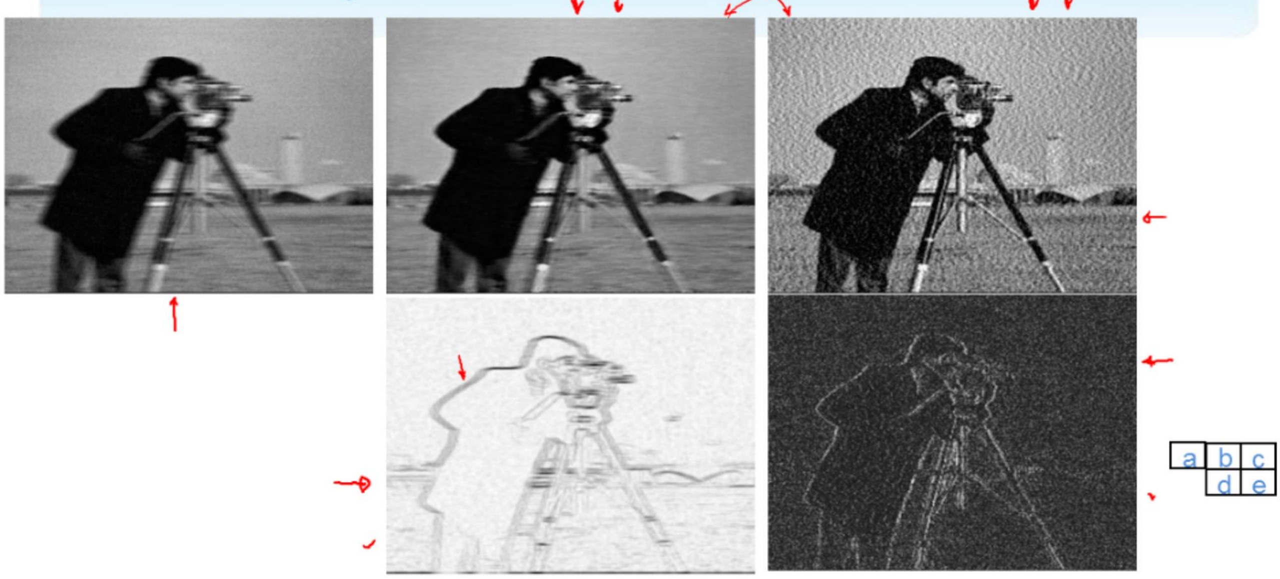
$\mathbf{V} \approx 1/(\sigma_f^2)$ (measure of the local activity)

$\mathbf{W}_1 = \mathbf{1} - \mathbf{W}_2$

$$\|x\|^2_W = x^T W^T W x = \sum_{i=1}^{N} w_i^2 x_i^2$$

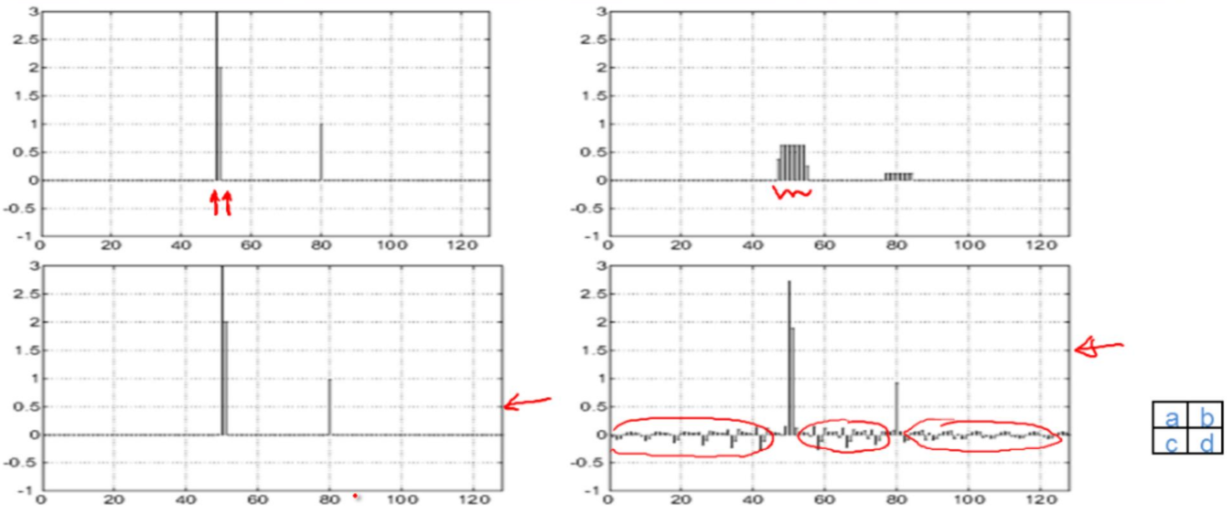
$$W = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_N \end{bmatrix},\quad w_i > 0$$

Experimental Results



(a) noisy-blurred image; 1D motion blur over 8 pixels, BSNR=20dB; Restorations (b) iterative adaptive CLS; (c) iterative CLS; (d) entries of visibility matrix and (e) |fig. b - fig. c| both linearly mapped in [32,255] range.

Positivity Constraint



(a) original signal; (b) blurred signal by 1D motion blur over 8 samples; (c) iterative LS with positivity constraint; (d) iterative LS without positivity constraint.

Ringing Artifacts

impulse response of overall system

impulse response of restoration filter

$$s_{all}(i,j) = h(i,j) * r(i,j)$$

restored image

original image

$$\hat{f}(i,j) = s_{all}(i,j) * f(i,j)$$

Ideally

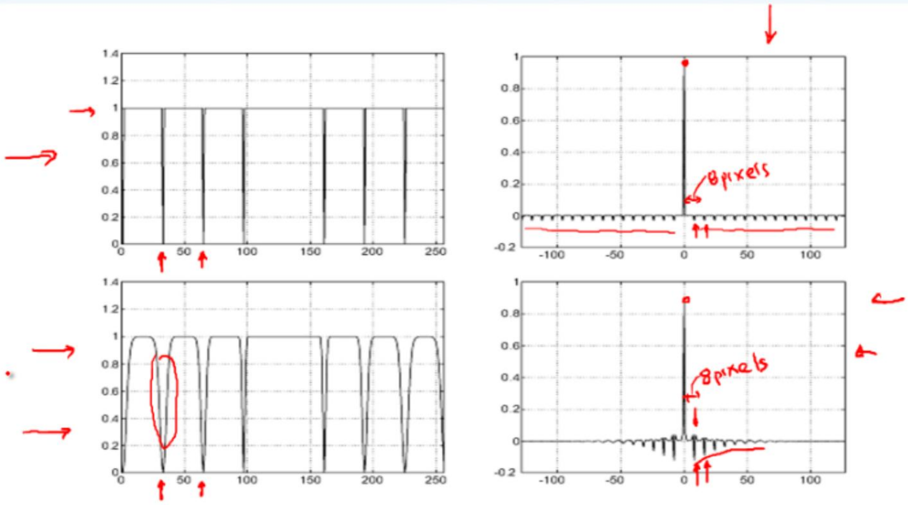
discrete impulse

$$s_{all}(i,j) = \delta(i,j)$$

Or

$$S_{all}(u,v) = 1, \quad \forall\ (u,v)$$

Experimental Results



a	b
c	d

1D motion blur over 8 pixels; (a), (b): $S_{all}(u,0)$ and $s_{all}(i,0)$ for the direct inverse filter; (c) and (d): $S_{all}(u,0)$ and $s_{all}(i,0)$ for the iterative LS restoration algorithm.