

# Ambiguously-Typed Lambda Calculus

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## Abstract

Here, we present the ambiguously-typed lambda calculus - a size-dependent type system measuring the *shape* of terms, based on their context, and an additional *substitution system*, facilitating the merge and sort of multiple terms' parameters.

[?]

## 1 Motivation

The Simply-Typed Lambda Calculus follows from the untyped lambda calculus in that there is structural assignment to parameters, and each "step" of arity is mechanically separated with  $\rightarrow$ . Values are given type labels, and arguments' types are checked one-for-one to the specification signature. Higher order function application, the true nature of lambda calculus, is retained through parameter specification (or type signature) nesting. The grammars are structured as follows:

Untyped Lambda Calculus

$$\begin{aligned} e &= x \\ &\lambda x.e \\ &e\ e \end{aligned}$$

Simply-Typed Lambda Calculus

$$\tau ::= \tau \rightarrow \tau \mid T \quad \text{where } T \in B$$

$$\begin{aligned} e &= x : \tau \\ &\lambda(x : \tau).e \\ &e\ e \\ &c \end{aligned}$$

$c$  is a "term constant", such that  $c$  is an inhabitant of a type  $T$  included in our working set  $B$ .

The untyped lambda calculus gives us a foundation to base all others off of - it is the minimum embodiment of higher-order function application and abstraction. But, there is no beginning, and no end; it suffices only to provide action, and not results. This is what the simply-typed lambda calculus fills - it provides an encoding of the finite "end" of an expression in it's type, by utilizing  $\rightarrow$  for each step.

The simply-typed lambda calculus makes a critical decision - it gives up infinite arity for the sake of traction and decidable termination. We present the ambiguously-typing scheme to give back our infinite arity, at the cost of detailed knowledge.

## 2 Overview

Our system encodes arity in the space of variables quantified over natural numbers, and constrained based on requirements induced by application and abstraction context. This is a size-dependent type system variant, similar to Cryptol. Indeed, our "size" of terms is ambiguous - it gives us no insight to how parameters are resolved. We additionally include a *parameter resolution system* - a method for unifying substitutions. We later shoe-horn a pseudo-monoid instance to our system, with the *union* of lambdas as our monoidal append.

Our type system also has decidable and total type inference; the size-dependent system initially assumes all terms to be polymorphic in arity, then, depending on how terms are used, minimum bounds are enforced in our sizes based on natural number literals.

### 2.1 Brief Example

$$x : \forall a \in \mathbb{N}. \Rightarrow \quad \quad \quad a \quad (1)$$

$$f : \forall b \in \mathbb{N}. \Rightarrow \quad \quad \quad b \quad (2)$$

$$f x : \forall a \in \mathbb{N}, b \in \mathbb{N}. \{a \geq 1\} \Rightarrow \quad \quad \quad (a - 1) + b \quad (3)$$

$$(4)$$

In our first examples 1 and 2, their sizes are purely polymorphic because there is no context telling us how the expression should behave. In 3, we can see some interesting ideas: because  $f$  was applied to  $x$ , we now have a constraint bound to it's type variable<sup>1</sup>. Also, because  $x$  consumed one parameter in  $a$ , we must decrement it. Lastly, we take the left-over parameters in  $x$  and  $a - 1$  and combine them; in our (commutative) sized interpretation, this is simply addition<sup>2</sup>.

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<sup>1</sup>A degenerate consequence of our structureless arity specification is that a type variable's reference to it's term must be syntactically in-order -  $\forall a b$  over  $x y$  will match  $x$  with  $a$ , and  $y$  with  $b$ .

<sup>2</sup>This neglects the order that the parameters get combined intentionally.

Note that I didn't include the type of a lambda. Please be patient; we will find that a function's size depends on it's body.

## 2.2 Hand-Wave

Here is the grammar:

$$e = x \quad \text{term} \quad (5)$$

$$\lambda x. e \quad \text{abstraction} \quad (6)$$

$$[e \ e \quad \text{inner application} \quad (7)$$

$$e \ [e \quad \text{outer application} \quad (8)$$

$$e \ \diamond] \ e \quad \text{merge} \quad (9)$$

$$e \ [\diamond \ e \quad \text{contra - merge} \quad (10)$$

$$l \quad \text{literal} \quad (11)$$

$$\Delta = \sum_{i=1}^N w_i (x_i - \bar{x})^2 \quad (12)$$

## 2.3 Subsection Heading

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## 3 Section Heading

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Figure 1: Figure caption.

Figure 2: Figure caption

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## A Appendix Heading

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