Ambiguously-Typed Lambda Calculus

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Abstract

Here, we present the ambiguously-typed lambda calculus - a size-dependent type system measuring the *shape* of terms, based on their context, and an additional *substitution system*, facilitating the merge and sort of multiple terms' parameters.

[**?**]

1 Motivation

The Simply-Typed Lambda Calculus follows from the untyped lambda calculus in that there is structural assignment to parameters, and each "step" of arity is mechanically separated with $\rightarrow$. Values are given type labels, and arguments' types are checked one-for-one to the specification signature. Higher order function application, the true nature of lambda calculus, is retained through parameter specification (or type signature) nesting. The grammars are structured as follows:

Untyped Lambda Calculus	Simply-Typed Lambda Calculus
	$\tau ::= \tau \rightarrow \tau \mid T \quad \text{where} \quad T \in B$
$$\begin{aligned} e = \ & x \\ & \lambda x.e \\ & e \ \ e \end{aligned}$$	$$\begin{aligned} e = \ & x : \tau \\ & \lambda(x:\tau).e \\ & e \ \ e \\ & c \end{aligned}$$

c is a "term constant", such that c is an inhabitant of a type T included in our working set B.

The untyped lambda calculus gives us a foundation to base all others off of - it is the minimum embodyment of higher-order function application and abstraction. But, there is no beginning, and no end; it suffices only to provide action, and not results. This is what the simply-typed lambda calculus fills - it provides an encoding of the finite "end" of an expression in it's type, by utilizing $\rightarrow$ for each step.

The simply-typed lambda calculus makes a critical decision - it gives up infinite arity for the sake of traction and decidable termination. We present the ambiguously-typing scheme to give back our infinite arity, at the cost of detailed knowledge.

2 Overview

Our system encodes arity in the space of variables quantified over natural numbers, and constrained based on requirements induced by application and abstraction context. This is a size-dependent type system variant, similar to Cryptol. Indeed, our "size" of terms is ambiguous - it gives us no insight to how parameters are resolved. We additionally include a *parameter resulution system* - a method for unifying substitutions. We later shoe-horn a pseudo-monoid instance to our system, with the *union* of lambdas as our monoidal append.
Our type system also has decidable and total type inference; the size-dependent system initially assumes all terms to be polymorphic in arity, then, depending on how terms are used, minimum bounds are enforced in our sizes based on natural number literals.

2.1 Brief Example

$$x : \forall a \in \mathbb{N}. \Rightarrow \qquad a \tag{1}$$

$$f : \forall b \in \mathbb{N}. \Rightarrow \qquad b \tag{2}$$

$$f\, x : \forall a \in \mathbb{N}, b \in \mathbb{N}. \{a \geq 1\} \Rightarrow \qquad (a-1)+b \tag{3}$$

In our first examples 1 and 2, their sizes are purely polymorphic because there is no context telling us how the expression should behave. In 3, we can see some interesting ideas: because f was applied to x, we now have a constraint bound to it's type variable[1].

$$\Delta = \sum_{i=1}^{N} w_i (x_i - \bar{x})^2 \tag{4}$$

[1] A degenerate consequence of our structureless arity specification is that a type variable's reference to it's term must be syntactically in-order - $\forall\, a\, b$ over $x\, y$ will match x with a, and y with b.

Figure 1: Figure caption.

Figure 2: Figure caption

2.2 Subsection Heading

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3 Section Heading

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A Appendix Heading

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