Minimisation of (history-)deterministic generalised (co-)Büchi automata

joint work with Antonio Casares, Denis Kuperberg, Olivier Idir and Aditya Prakash

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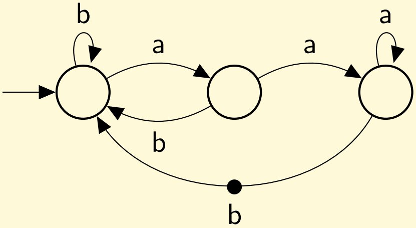
Büchi automaton



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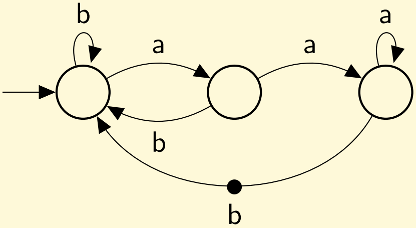

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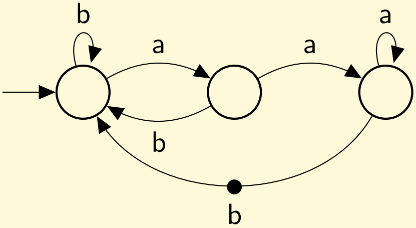

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Deterministic Büchi automata recognise *Büchi languages*.

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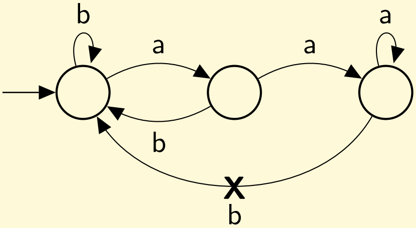

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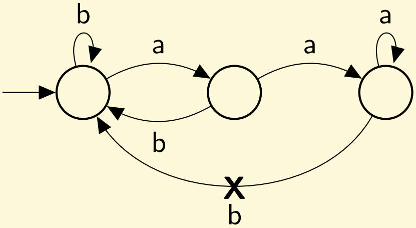

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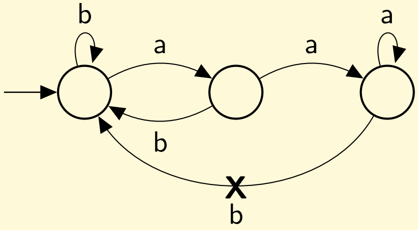

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Deterministic co-Büchi automata recognise *co-Büchi languages*.
= complements of Büchi languages.

History-determinism [Henzinger, Piterman 2006]

$\mathcal{A}$ is history-deterministic if there is a *resolver* $\sigma : \Delta^* \times \Sigma \to \Delta$ such that for all $w \in L(\mathcal{A})$, applying σ while reading w yields an accepting run.

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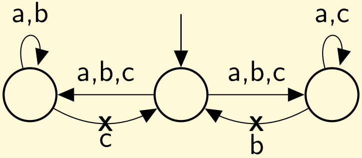

History-deterministic co-Büchi automaton.

Finitely many b or finitely many c.

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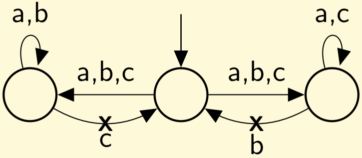

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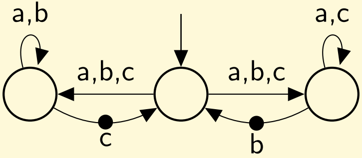

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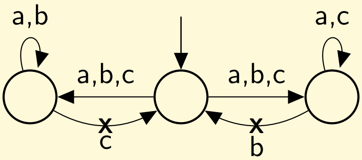

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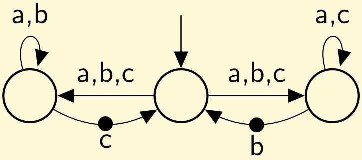

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[Kuperberg, Skrzypczak 2015]

History-deterministism can be tested in PTIME for Büchi and co-Büchi automata.

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Minimise $\mathcal{A}$ = find $\mathcal{B}$ of the same type, the same language and with as few states as possible.

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But history-deterministic Büchi and co-Büchi are very different!

[Kuperberg, Skrzypczak 2015]

- History-deterministic co-Büchi automata can be exponentially smaller than deterministic ones,
- Every history-deterministic Büchi automata has an equivalent deterministic one of size $O(n^2)$.

[Schewe 2010]

It is NP-complete to minimise deterministic (co-)Büchi automata.

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[Abu Radi, Kupferman 2019]

We can minimise history-deterministic co-Büchi automata in polynomial time when the acceptance condition is on the transitions.

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We can minimise history-deterministic co-Büchi automata in polynomial time when the acceptance condition is on the transitions.

Can we minimise (history-)deterministic (co-)Büchi automata in polynomial time?

In the transition-based world

	Büchi	Co-Büchi
Deterministic	???	???
History-deterministic	???	PTIME

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	Generalised Büchi	Generalised co-Büchi
Deterministic	NP-complete	NP-complete
History-deterministic	NP-complete	PTIME

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Set of colours $C = \{\color{blue}\bullet, \color{green}\bullet, \color{red}\bullet, \ldots\}$, colouring function $col : \Delta \to 2^C$

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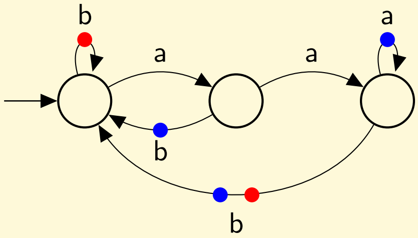

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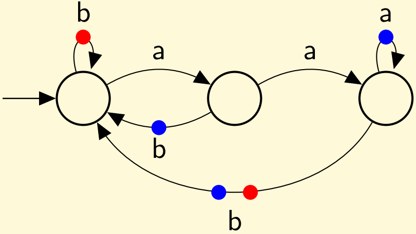
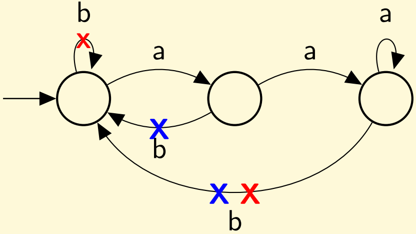


Infinitely many *aab* or infinitely many *ab* and b^2

Generalised co-Büchi: avoid some colour indefinitely after sone point.



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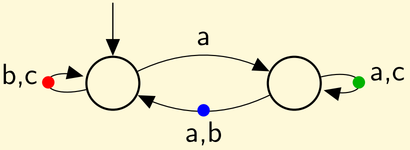
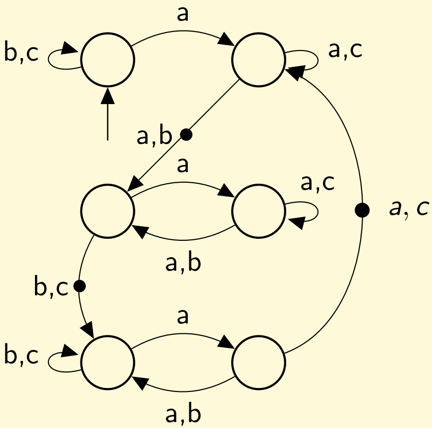
From GBA to BA



Generalised Büchi with n states and k colours.

$\rightsquigarrow$



Büchi with nk states.

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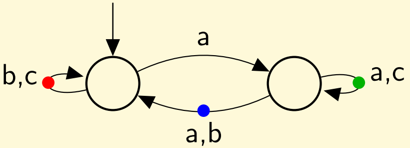

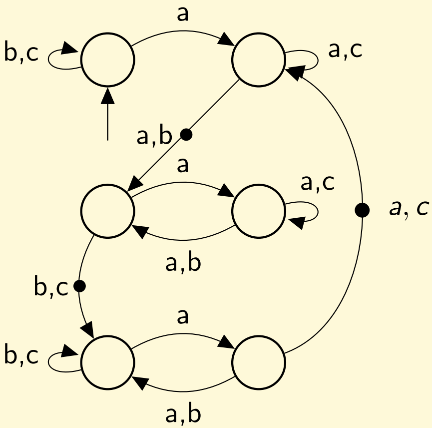

Generalised Büchi with n states and k colours.

Büchi with nk states.

Preserves history-determinism!

Minimisation of gen. HD co-Büchi in polynomial time

Step 1: Apply Abu Radi-Kupferman

Given a generalised history-deterministic co-Büchi recognising L,

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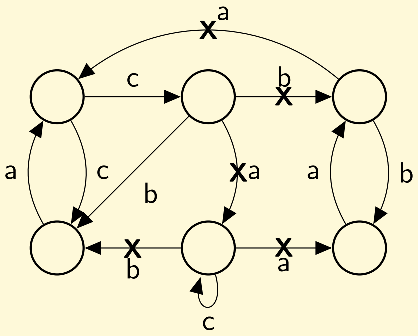
Given a generalised history-deterministic co-Büchi recognising L,

- Compute an equivalent history-deterministic co-Büchi
- We can minimise it $\to \mathcal{A}^L_{min}$ using Abu Radi and Kupferman's algorithm

Step 2: Merge safe components

Suppose the language is *prefix-independent*, i.e., all states have the same residual.
(= the language is stable under prefix modification)



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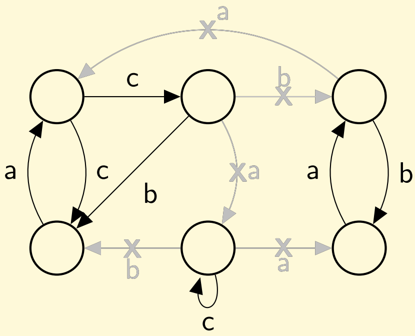

Accepted = stay in a safe component eventually

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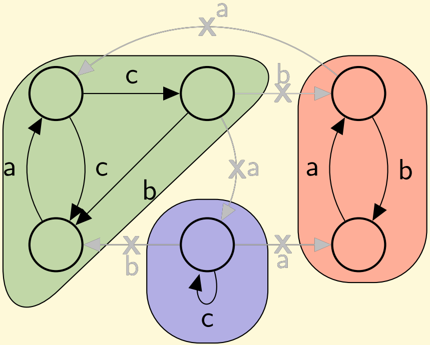

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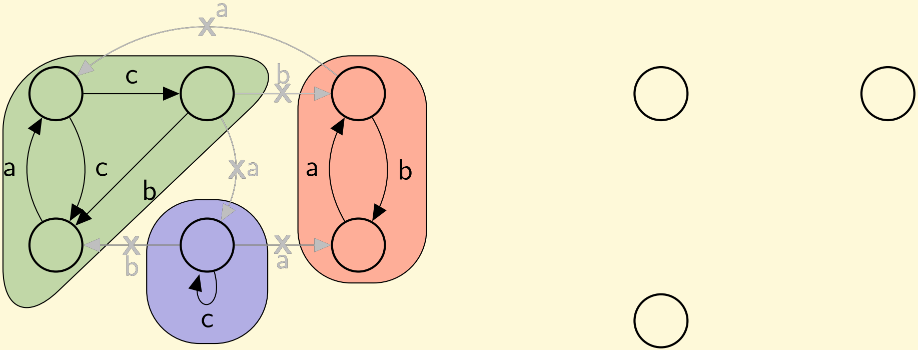

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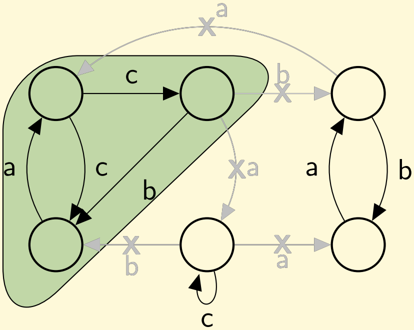

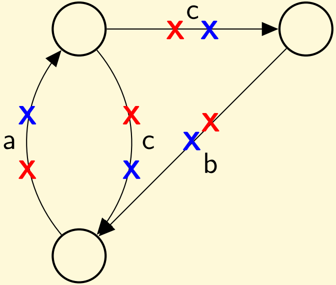

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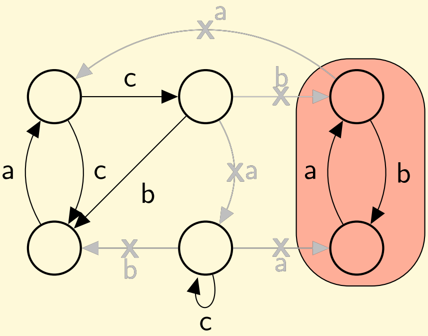

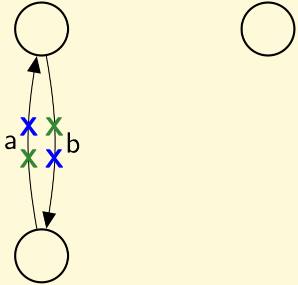

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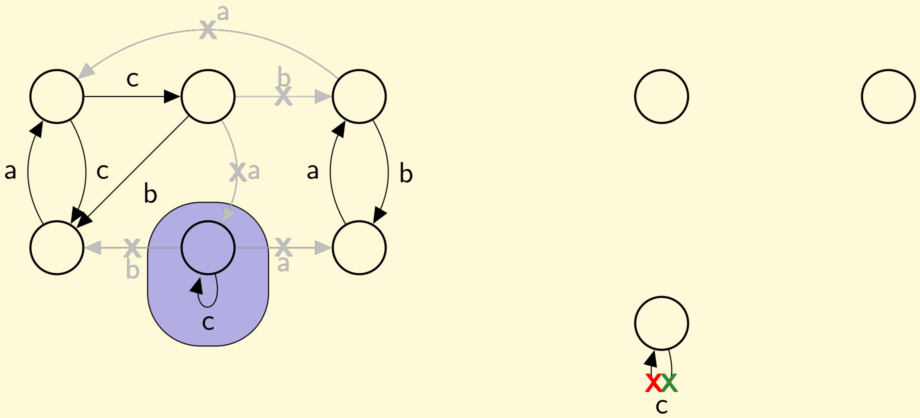

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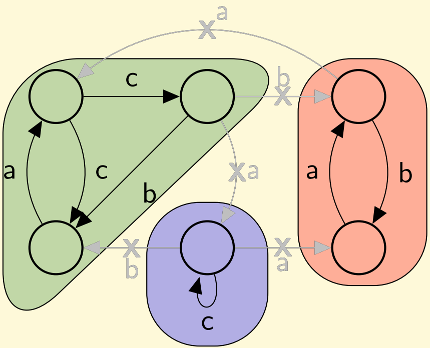

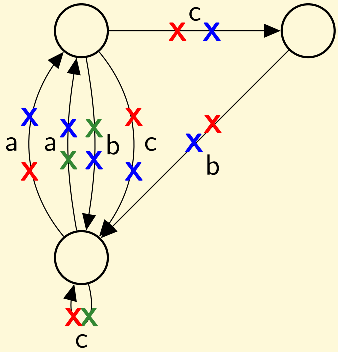

+ all other transitions, with X X X

Why it is minimal

[Abu Radi, Kupferman 2019]

For all equivalent HD co-Büchi automaton $\mathcal{C}$ there is an injection $\eta : \mathit{SafeComp}(\mathcal{A}^L_{min}) \to \mathit{SafeComp}(\mathcal{C})$ such that $|\eta(S)| \geq |S|$ for all $S \in \mathit{SafeComp}(\mathcal{C})$.

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If we had a smaller HD gen. co-Büchi we could unfold it to get an HD co-Büchi where all components have size $< m$.

This work

We can minimize generalised HD co-Büchi automata in polynomial time.

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If not prefix-independent $\to \sim$ apply the procedure for each residual.

A sketch of NP-completeness

NP upper bound

$\mathcal{A}$ a (history-)deterministic gen. (co-)Büchi automaton with n states and k colours.
Is there an automaton of the same type with $\leq m$ states equivalent to $\mathcal{A}$?

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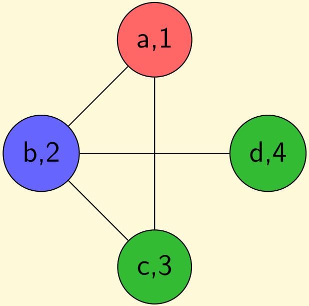
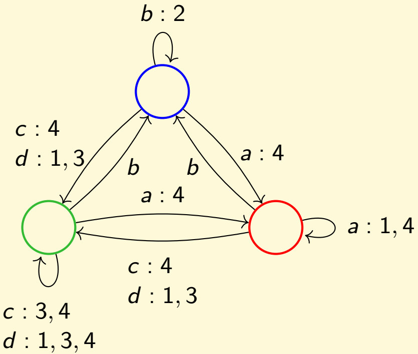
This work

If $\mathcal{B}$ exists then it can be recoloured to use $\leq \mathcal{O}(|\mathcal{A}|km)$ colours.

NP lower bound

From graph 3-colouring.
Suitable language: $L_G = \bigcap_{v \in V} (V^* vv)^\omega \cup V^*(V \setminus N(v))^\omega$.





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This problem is NP-complete:
Given a (history-)deterministic Büchi automaton $\mathcal{A}$ and $k \in \mathbb{N}$,
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This problem is NP-complete:
Given a (history-)deterministic Büchi automaton $\mathcal{A}$ with $\boxed{4}$ states,
is there an equivalent one with $\leq \boxed{3}$ states?

Minimising colours

This work

It is NP-complete to minimise both the number of states and colours for (history-)deterministic gen. (co-)Büchi automata.

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Proof idea:

Minimising colours

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[Casares, M. 2024] $\rightarrow$ study of the complexity of simplifying conditions on ω-automata.

What is left to do

- Minimisation of Büchi, parity automata
- Are HD gen. Büchi automata more succinct than deterministic ones?
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Thanks!