Spectral Graph Approximation

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Abstract. TODO

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1 Introduction

With the rise of Big Data applications over the recent years, working with large graph structures also became more important. Algorithms like PageRank or spectral clustering are commonly used to analyze the web graph or social networks. In order to run such algorithms on large graphs however, optimizations are required.

For this purpose we will specifically look at graph coarsening. Coarsening reduces the size of a given graph while preserving its overall structure via some notion of graph similarity that will be defined later. Graph algorithms can then be run on the smaller coarsened graph. Afterwards the result for the coarsened graph can be iteratively refined to obtain an approximate result for the original graph. Figure 1 illustrates this approach.

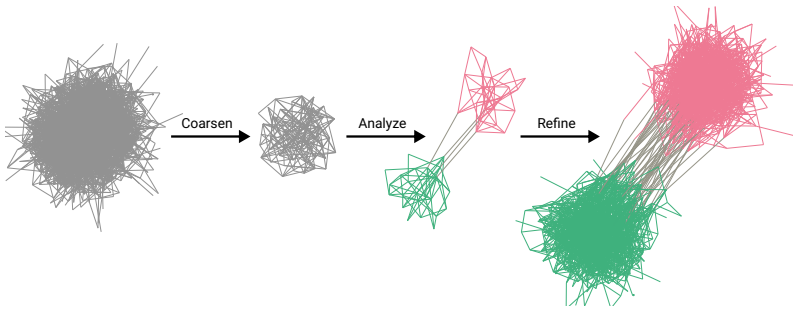

Fig. 1: Using graph coarsening to speed up graph algorithms, e. g. Min-Cut.

The goal of this paper is to show how graph coarsening works and how it affects the results of graph algorithms. This introduction consists of three sections, each of which aims to answer one main question:

1. *How can the structural properties of a graph be formally described?* To describe graph coarsening and its effects, the similarity between a graph G and its coarsened version G_c has to be quantified. We begin with an introduction to spectral

graph theory. It allows us to describe the structure of graphs and provides the notion of *spectral similarity*, a measure to compare the structure of graphs.

2. *How does graph coarsening work?* Using the notion of spectral similarity, we will look at a randomized coarsening algorithm and analyze its effects on the structure of the coarsened graph.
3. *How does coarsening affect the result of graph algorithms?* Finally we will put bounds on how much the described coarsening algorithm is expected to increase the error of spectral clustering.

2 Spectral Graph Theory

We start with an introduction to spectral graph theory, which will allow us to characterize and compare the structure of graphs. Graphs are most commonly described in their vertex base, i. e. the strength w_{ij} with which pairs (v_i, v_j) of vertices are connected.

$$G := (\mathcal{V}, \mathcal{E}, W) \text{ with } W \in \mathbb{R}^{N \times N}, N := |\mathcal{V}|$$

In this paper we will only consider undirected graphs with non-negative real weights; thus the adjacency matrix W is positive semi-definite.

The core idea of spectral graph theory essentially is to perform a change of basis and describe graphs in terms of their spectral base instead of their vertex base. To see what this means, we interpret the adjacency matrix W as a linear operator that operates on so called signals $x \in \mathbb{R}^N$. A signal x can be interpreted as a function $x : \mathcal{V} \to \mathbb{R}$ that assigns a signal strength to each vertex. By applying Wx the given signal strengths x_i are shifted according to the connection strenghts w_{ij} to neighboring vertices v_j. This interpretation of graphs is very similar to that of Markov chains where signals represent probability distributions.

2.1 Relating Graph Signals to Real Functions

Let us now compare discrete graph signals $x : \mathcal{V} \to \mathbb{R}$ to continuous real functions $f : \mathbb{R} \to \mathbb{R}$. Both only differ in their domain. The domain $\mathbb{R}$ of f has an inherent structure, the real number line, which provides a strict ordering of its elements and a notion of distance between them. The domain $\mathcal{V}$ of x however has no such inherent structure, i. e. $v_1 < v_2$ for two vertices v_1, v_2 does not have a clear meaning. The structure of $\mathcal{V}$ fully depends on the graph G that is acting on it. Intuitively graph signals can thus be understood as a discretized generalization of real functions, where the underlying structure of the input domain is not fixed but can be freely chosen. Figure 2 shows that all real functions f can be seen as signals x of the graph described by the real number line[1]. Both, real functions and graph signals, can be described as vectors in their time/vertex base:

$$f = \int_{\mathbb{R}} f(t) b_t dt \Rightarrow \langle f, b_t \rangle = f(t) \quad \text{and} \quad x = \sum_{i=1}^{N} x_i b_i \Rightarrow \langle x, b_i \rangle = x_i \tag{1}$$

[1] Technically this is not correct, since $\mathbb{R}$ is continuous whereas all vertex sets $\mathcal{V}$ have to be discrete. To build an intuition for graph signals, this detail can however be ignored.

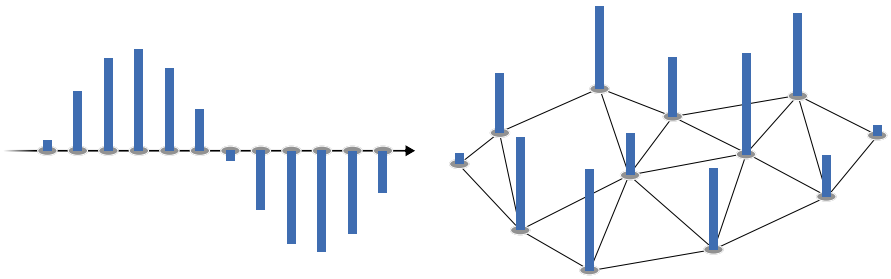

Fig. 2: Illustration of how the discretized real number line can be interpreted as an infinite linear graph, compared to some arbitrary finite non-linear graph. The blue bars show the signal strength at each vertex.

Here $\{b_i\}_{v_i \in \mathcal{V}}$ denotes the standard basis of the adjacency matrix W. Similarly $\{b_t\}_{t \in \mathbb{R}}$ denotes the infinite dimensional standard basis of the space of real functions, where $\langle \cdot, b_t \rangle := \delta_t(\cdot)$, with δ denoting the Dirac delta function.

2.2 Extending the Fourier Transform to Graphs

As mentioned at the beginning of this section, the core idea of spectral graph theory is to express graph signal vectors x in the spectral basis $\{u_i\}_{i=1}^{N}$ instead of the standard vertex basis $\{b_i\}_{v_i \in \mathcal{V}}$.

3 Graph Coarsening

TODO

4 Spectral Graph Similarity

TODO

5 Conclusion

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