1. (2,5 pontos) Resolva por transformada de Laplace o seguinte PVI:

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi),$$

onde $y(0) = 1$ e $y'(0) = 0$. Calcule $y(\pi/2)$, $y(3\pi/2)$ e $y(3\pi)$.

Solução:

(0,3) $\mathcal{L}\{y'' + 4y\} = (s^2+4)Y(s) - s$,

(0,2) $\mathcal{L}\{\delta(t-\pi) - \delta(t-2\pi)\} = e^{-\pi s} - e^{-2\pi s}$,

$$(\mathbf{0,3})\ \ Y(s) = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4} + \frac{s}{s^2+4},$$

$$(\mathbf{0,3})\ \ \mathcal{L}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,$$

$$(\mathbf{0,3})\ \ \mathcal{L}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \frac{1}{2}u_\pi(t)\mathrm{sen}2(t-\pi) = \frac{1}{2}u_\pi(t)\mathrm{sen}2t,$$

$$(\mathbf{0,3})\ \ \mathcal{L}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\} = \frac{1}{2}u_{2\pi}(t)\mathrm{sen}2(t-2\pi) = \frac{1}{2}u_{2\pi}(t)\mathrm{sen}2t,$$

$$(\mathbf{0,2})\ \ y(t) = \cos 2t + \frac{1}{2}(u_\pi(t) - u_{2\pi}(t))\mathrm{sen}2t,$$

$$(\mathbf{0,2})\ \ y(\pi/2) = \cos 2(\pi/2) = \cos\pi = -1,$$

$$(\mathbf{0,2})\ \ y(3\pi/2) = -1,$$

$$(\mathbf{0,2})\ \ y(3\pi) = 1.$$

2. (1,5 pontos) Calcule a inversa da transformada de Laplace de $F(s)$:

$$F(s) = e^{-4s}\frac{2s-1}{s^2+4}.$$

Solução:

$$(\mathbf{0,5})\ \ \mathcal{L}^{-1}\left\{e^{-4s}\frac{2s-1}{s^2+4}\right\} = u_4(t)f(t-4),$$

$$(\mathbf{0,5})\ \ f(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4}\right\} = 2\cos 2t - \frac{1}{2}\mathrm{sen}2t,$$

$$(\mathbf{0,5})\ \ \mathcal{L}^{-1}\left\{e^{-4s}\frac{2s-1}{s^2+4}\right\} = u_4(t)\left(2\cos 2(t-4) - \frac{1}{2}\mathrm{sen}2(t-4)\right).$$

3. (3.0 pontos)
(a) Encontre a solução do sistema linear homogêneo de e.d.o.'s usando o método de autovalores e autovetores:

$$x^{'}(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x(t).$$

Solução:

$$(\mathbf{0,2})\ \ P(r) = \begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = r^2 - 1 = 0,\ \ r_1 = 1,\ \ r_2 = -1,$$

$$(\mathbf{0,3})\ \ r_1 = 1:\ \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(\mathbf{0,3})\ \ r_2 = -1:\ \ v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$(\mathbf{0,2})\ \ x_h(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

(b) Encontre a solução do sistema linear não-homogêneo (cujo sistema homogêneo associado está na parte (a)) utilizando o método de variação de parâmetros:

$$x^{'}(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x(t) + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}.$$

Solução:

$$(\mathbf{0,4})\ \ \Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix},\ \ \Psi^{-1}(t) \begin{pmatrix} 3e^{-t}/2 & -e^{-t}/2 \\ -e^t/2 & e^t/2 \end{pmatrix}$$

$$(\mathbf{0,4})\ \ u^{'}(t) = \Psi^{-1}(t) g(t) = \begin{pmatrix} 3e^{-t}/2 & -e^{-t}/2 \\ -e^t/2 & e^t/2 \end{pmatrix} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} 2 \\ -e^{2t} \end{pmatrix},$$

$$(\mathbf{0,4})\ \ u(t) = \begin{pmatrix} 2t \\ -e^{2t}/2 \end{pmatrix},$$

$$(\mathbf{0,4})\ \ x_p(t) = \Psi(t)u(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} 2t \\ -e^{2t}/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t - \frac{1}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t$$

$$(\mathbf{0,4})\ \ x(t) = x_h(t) + x_p(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t - \frac{1}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t$$

4. (3.0 pontos) Explique detalhadamente.

(a) (1.0) Estude a convergência da série:

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}.$$

Solução:

$$u_n = \frac{n!}{(2n!)}, \quad u_{n+1} = \frac{(n+1)!}{(2n+2)!},$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 \quad \Rightarrow conv.$$

(b) (1.0) Calcule a soma da série:

$$\sum_{n=0}^{\infty} \frac{3^{n-1} + 7(2^n)}{6^n}.$$

Solução:

$$\sum_{n=0}^{\infty} \frac{3^{n-1} + 7(2^n)}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3} \frac{3^n}{6^n} + 7\frac{2^n}{6^n} \right)$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + 7 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \frac{1}{1-\frac{1}{2}} + 7 \frac{1}{1-\frac{1}{3}} = \frac{67}{6}.$$

(c) (1.0) Usando o teste da integral determine se a série:

$$\sum_{n=2}^{\infty} \frac{1}{k(\ln k)^{2008}}.$$

converge ou diverge.

Solução:

$$\sum_{n=2}^{\infty} \frac{1}{k(\ln k)^{2008}}, \quad f(x) = \frac{1}{x(\ln x)^{2008}} > 0, \downarrow$$

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{dx}{x(\ln x)^{2008}} = \lim_{A\to\infty} \int_2^A \frac{dx}{x(\ln x)^{2008}}$$
$$= \lim_{A\to\infty} \int_{\ln 2}^A \frac{dz}{z^{2008}} = \lim_{A\to\infty} \frac{1}{-2008+1} z^{-2008+1}|_{\ln 2}^A$$
$$= \frac{1}{2007 \cdot (\ln 2)^{2007}} < \infty \Rightarrow conv.$$