T_2 ( ) F520 ( ) MS550 · Nome: ______________________ RA: _________

Encontre a solução geral, na forma de série, da equação diferencial

$$x(1-x)y'' - 3xy' - y = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad ED \Rightarrow xy'' - x^2y'' - 3xy' - y = 0$$

$$\therefore \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r}$$

$$- \sum_{n=0}^{\infty} 3a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$a_0 r(r-1)x^{r-1} + \sum_{n=0}^{\infty} x^{n+r}\Big[a_{n+1}(n+r+1)(n+r) - a_n \underbrace{[(n+r)(n+r-1)+3(n+r)+1]}_{\substack{(n+r)^2+2(n+r)+1 \\ (n+r+1)^2}}\Big] = 0$$

$$\therefore r(r-1) = 0 \Rightarrow \begin{cases} r_1 = 1 \\ r_2 = 0 \end{cases}$$

$$a_{n+1}(n+r)(n+r+1) = a_n(n+r+1)^2 \underset{n+r+1\neq 0}{\Longrightarrow} (n+r)a_{n+1} = (n+r+1)a_n$$

$$(n = 0, 1, 2, \dots)$$

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$\boxed{r = r_1 = 1}$

$$a_{n+1} = \frac{n+2}{n+1}a_n \quad : \quad a_1 = 2a_0 \ ; \ a_2 = \frac{3}{2}a_1 = 3a_0 \ ; \ a_3 = \frac{4}{3}a_2 = 4a_0$$

$$\therefore a_n = (n+1)a_0 \qquad , n = 0, 1, 2, \dots$$

$$\therefore \boxed{y_1 = \sum_{n=0}^{\infty}(n+1)x^{n+1}}$$

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$\boxed{r = r_2 = 0}$

$$y_2 = K y_1 \ln x + \underbrace{x^{r_2}}_{1}\Big[1 + \sum_{n=1}^{\infty} c_n x^n\Big]$$

$$y_2' = K y_1' \ln x + K y_1 \frac{1}{x} + \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y_2'' = K y_1'' \ln x + 2K y_1' \frac{1}{x} - K y_1 \frac{1}{x^2} + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}.$$

Usando essas expressões na ED mais a expressão para $y(x)$:

$$2K\sum_{n=0}^{\infty}(n+1)^2x^n - K\sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=1}^{\infty}c_{n+1}(n+1)n\,x^n - 2K\sum_{n=0}^{\infty}(n+1)^2x^{n+1}$$
$$- 2K\sum_{n=0}^{\infty}(n+1)x^{n+1} - \sum_{n=2}^{\infty}c_n n(n-1)x^n - 3\sum_{n=1}^{\infty}c_n n x^n - 1 - \sum_{n=1}^{\infty}c_n x^n = 0$$

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$\therefore$

$$2K - K + 2K\,4x - K\,2x + 2K\sum_{n=2}^{\infty}(n+1)^2x^n - K\sum_{n=2}^{\infty}(n+1)x^n + c_2\,2x$$
$$+\sum_{n=2}^{\infty}c_{n+1}(n+1)n\,x^n - 2Kx - 2K\sum_{n=2}^{\infty}n^2x^n - 2Kx - 2K\sum_{n=2}^{\infty}n\,x^n$$
$$-\sum_{n=2}^{\infty}c_n n(n-1)x^n - 3c_1x - 3\sum_{n=2}^{\infty}c_n n x^n - 1 - c_1x - \sum_{n=2}^{\infty}c_n x^n = 0$$

$$\therefore \begin{cases} K-1=0 \Rightarrow K=1 \\ -4c_1+2c_2+2K=0 \Rightarrow c_2 = 2c_1-1 \\ 2K(n+1)^2 - K(n+1) + c_{n+1}(n+1)n - 2Kn^2 - 2Kn - c_n n(n-1) - 3c_n n - c_n = 0 \\ \quad (n=2,3,\ldots) \end{cases}$$

Usando $K=1 \Rightarrow c_{n+1}\,n(n+1) = c_n(n+1)^2 - (n+1)$, $n=2,3,\ldots$

$$\therefore\quad c_{n+1} = \frac{(n+1)}{n}c_n - \frac{1}{n}$$

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$$c_3 = \frac{3}{2}c_2 - \frac{1}{2} = \frac{3}{2}(2c_1-1) - \frac{1}{2} = 3c_1 - \frac{3}{2} - \frac{1}{2} = 3c_1 - 2$$
$$c_4 = \frac{4}{3}c_3 - \frac{1}{3} = \frac{4}{3}(3c_1-2) - \frac{1}{3} = 4c_1 - \frac{8}{3} - \frac{1}{3} = 4c_1 - 3$$
$$c_5 = \frac{5}{4}c_4 - \frac{1}{4} = \frac{5}{4}(4c_1-3) - \frac{1}{4} = 5c_1 - \frac{15}{4} - \frac{1}{4} = 5c_1 - 4$$

$\therefore\ c_{n+1} = (n+1)c_1 - n$, $n = 0, 1, 2, \ldots$ (vale tb. $n=0$!)

$$\therefore y_2 = 1y_1\ln x + 1 + \sum_{n=1}^{\infty}c_nx^n = y_1\ln x + 1 + \sum_{n=0}^{\infty}c_{n+1}x^{n+1}$$
$$= y_1\ln x + 1 + \sum_{n=0}^{\infty}[(n+1)c_1 - n]x^{n+1}$$
$$= y_1\ln x + 1 - \sum_{n=0}^{\infty}n\,x^{n+1} + \overset{\text{cte}}{c_1}\underbrace{\sum_{n=0}^{\infty}(n+1)x^{n+1}}_{y_1(x)}$$

$$\therefore \boxed{y_2(x) = y_1\ln x + 1 - x\sum_{n=1}^{\infty}n\,x^n}$$

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