( )F520 ( )MS550 - Segunda Prova - 24/06/2013

RA: ____________________ **Nome:** ____

(1) Mostre que

$$\int_0^{\pi/2} J_1(x\cos\theta)\,\mathrm{d}\theta = \frac{1-\cos x}{x},$$

onde $J_1(\cdot)$ denota a função de Bessel de primeira espécie e ordem um.

(2) Calcule, para $n \geq 1$, a integral

$$\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x)\,\mathrm{d}x,$$

onde $P_n(\cdot)$ denota o polinômio de Legendre de ordem n.

(3) (i) Encontre os autovalores e autofunções do problema

$$\begin{cases} x^2y'' + 3xy' = -\lambda y, & 1 < x < \mathrm{e}, \\ y(1) = 0, \quad y(\mathrm{e}) = 0. \end{cases}$$

(ii) Escreva a relação de ortogonalidade envolvendo estas autofunções.

(4) (i) Encontre os autovalores e autofunções do problema

$$\begin{cases} (1-x^2)y'' - 2xy' = -\lambda y, & 0 < x < 1, \\ y(0) = 0, \quad \lim\limits_{x\to 1} |y(x)| < \infty. \end{cases}$$

(ii) Sejam $y_n(x)$ $(n = 0, 1, 2, \ldots)$ estas autofunções. Calcule $y_n'(0)$.

ℹ Valor das questões: (1) 2,5 (2) 2,5 (3) 2,5 (4) 2,5.

Formulário (eventualmente útil)

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\,dt,\quad \Gamma(z+1)=z\Gamma(z),\quad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z),$$

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},\qquad B(z,w)=2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta,\qquad B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

$${}_2F_1(\alpha,\beta,\gamma;z)=\sum_{n=0}^\infty\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},\quad {}_2F_1(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

$$U(a,b;z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{-a+b-1}\,\mathrm{d}t,\quad \frac{\mathrm{d}^nU(a,b;z)}{\mathrm{d}z^n}=(-1)^n(a)_nU(a+n,b+n;z),$$

$$U(a,b;z)=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}\,{}_1F_1(a-b+1,2-b;z)+\frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a,b;z),\quad {}_1F_1(a,b;z)=\sum_{n=0}^\infty\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

$$J_\nu(x)=\sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},\quad J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x),\quad J_{\nu-1}(x)-J_{\nu+1}(x)=2J_\nu'(x),$$

$$\frac{d}{dx}(x^{-\nu}J_\nu(x))=-x^{-\nu}J_{\nu+1}(x),\quad \frac{d}{dx}(x^\nu J_\nu(x))=x^\nu J_{\nu-1}(x),\quad \mathrm{e}^{x(t-t^{-1})/2}=\sum_{k=-\infty}^{+\infty}t^kJ_k(x)$$

$$J_n(u+v)=\sum_{m=-\infty}^{+\infty}J_m(u)J_{n-m}(v)\quad J_n(x)=(-1)^nx^n\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^nJ_0(x)\quad J_0(x)=\frac{2}{\pi}\int_0^1\frac{\cos xt}{\sqrt{1-t^2}}\,dt$$

$$J_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta,\quad \cos x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n}}{(2n)!},\quad \sin x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

$$P_\nu(x)={}_2F_1(-\nu,\nu+1,1;\tfrac{1-x}{2}),\quad Q_\nu(x)=\frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)(2x)^{\nu+1}}\,{}_2F_1\left(\frac{\nu+2}{2},\frac{\nu+1}{2},\nu+\frac{3}{2};\frac{1}{x^2}\right),$$

$$P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}[(x^2-1)^n],\quad \frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^\infty P_n(x)t^n,$$

$$(2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x),\quad (2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x),$$

$$P_{n-1}'(x)=-nP_n(x)+xP_n'(x),\quad (1-x^2)P_n'(x)=nP_{n-1}(x)-nxP_n(x),$$

$$P_{n+1}'(x)=(n+1)P_n(x)+xP_n'(x),\quad (1-x)^{-\alpha}=\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}x^n,\quad \int_{-1}^1P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{mn}.$$

① Veja correção do $\boxed{T5}$. Pontos: (+1,0); (+1,0); (+0,5).

② Como o produto de duas funções pares ou duas funções ímpares é uma função par, temos:

$$I = \int_0^1 x^2 P_{n+1}(x)\, P_{n-1}(x)\, dx = \frac{1}{2}\int_{-1}^{+1} x^2 P_{n+1}(x)\, P_{n-1}(x)\, dx$$

Usando: $(2n+1)\,x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$, temos:

$$I = \frac{1}{2}\int_{-1}^{+1} \frac{\left[((n+1)+1) P_{n+2}(x) + (n+1) P_n(x)\right]}{(2(n+1)+1)} \cdot \frac{\left[((n-1)+1) P_n(x) + (n-1) P_{n-2}(x)\right]}{(2(n-1)+1)}\, dx =$$

(+1,0)

$$= \frac{1}{2(2n+3)(2n-1)} \left[(n+2)n \underbrace{\int_{-1}^{+1} P_{n+2}(x) P_n(x)\, dx}_{=0} + (n+2)(n-1) \underbrace{\int_{-1}^{+1} P_{n+2}(x) P_{n-2}(x)\, dx}_{=0} \right.$$

$$\left. + (n+1)n \underbrace{\int_{-1}^{+1} P_n(x)\, P_n(x)\, dx}_{\frac{2}{2n+1}} + (n+1)(n-1) \underbrace{\int_{-1}^{+1} P_n(x)\, P_{n-2}(x)\, dx}_{=0} \right] =$$

$$= \frac{2n(n+1)}{2(2n+3)(2n-1)(2n+1)} = \frac{n(n+1)}{(2n+3)(4n^2-1)}$$

(+1,5)

③ $x^2 y'' + 3xy' + \lambda y = 0 \Rightarrow$ eq. de Euler

$y = x^r \Rightarrow r(r-1) + 3r + \lambda = r^2 + 2r + \lambda = 0$

$\therefore r = -1 \pm \sqrt{1-\lambda}$

(A) $\boxed{1-\lambda > 0}$: $1-\lambda = k^2 \quad (k>0)$

$\therefore r = -1 \pm k$

$\therefore y = A x^{-1+k} + B x^{-1-k}$

$y(1) = 0 = A + B = 0 \Rightarrow A = -B \Rightarrow B = 0$

$y(e) = A\underbrace{(e^{-1+k} - e^{-1-k})}_{\neq 0} = 0 \Rightarrow A = 0$

$\therefore$ apenas solução trivial (+0.5)

(b) $\boxed{1-\lambda=0}$

$$\therefore y = A'x + B'x\ln x$$
$$y(1) = A' = 0$$
$$y(e) = B'e\ln e = 0 \Rightarrow B'=0$$

$\therefore$ apenas solução trivial (+0,5)

(c) $\boxed{1-\lambda<0}$ $\quad 1-\lambda = -k^2 \quad (k>0)$

$$\therefore r = -1 \pm \sqrt{-k^2} = -1 \pm ik$$

$$\therefore y = A''x^{-1+ik} + B''x^{-1-ik} = x^{-1}(A''x^{ik} + B''x^{-ik})$$
$$= x^{-1}(A''e^{ik\ln x} + B''e^{-ik\ln x})$$
$$= x^{-1}(A'''\cos(k\ln x) + B'''\sin(k\ln x))$$

$$\therefore y(1) = 0 = 1^{-1}(A'''\underbrace{\cos(k\underbrace{\ln 1}_{=0})}_{1} + B'''\underbrace{\sin(k\underbrace{\ln 1}_{=0})}_{=0}) \Rightarrow A'''=0$$

$$y(e) = 0 = e^{-1}B'''\sin(k\underbrace{\ln e}_{1}) \Rightarrow \sin k = 0$$
$$\therefore k = n\pi, \quad n = 1,2,3,\ldots$$

$$\boxed{\begin{aligned} \lambda_n &= 1 + n^2\pi^2, \quad n=1,2,3\ldots \\ y_n(x) &= x^{-1}\sin(n\pi\ln x) \end{aligned}}$$

(+1,0)

(ii) ortogonalidade?

forma auto-adjunta: $\mu x^2y'' + \mu 3xy' + \lambda\mu y = 0$

tal que $3\mu x = (\mu x^2)' = \mu' x^2 + \mu 2x$

$$\therefore \mu x = \mu' x^2 \Rightarrow \frac{\mu'}{\mu} = \frac{1}{x} \therefore \boxed{\mu = x}$$

$$\therefore \quad x^3y'' + 3x^2y' + \lambda xy = 0 \Rightarrow \underline{\rho(x) = x}$$

$$\therefore \int_1^e y_n(x)\,y_m(x)\,.\,x\,dx = 0 \quad , m\neq n$$

(+0,5)

4 Escrevendo λ na forma $\lambda = \nu(\nu+1)$, vemos que a equação é a eq. de Legendre:

$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$

cuja solução geral é:

$$y(x) = A\,P_\nu(x) + B\,Q_\nu(x)$$ +0,5

A condição $\lim\limits_{x\to 1} |y(x)| < \infty$ implica

$$B = 0 \quad \text{e} \quad \nu = n$$ +0,5

pois $P_n(x)$ é um polinômio.

A condição $P_n(0) = 0$ implica que n é ímpar, pois $P_{2k}(0) \neq 0$, $P_{2k+1}(0) = 0$. Escrevendo $n = 2k+1$ $(k = 0, 1, \ldots)$ temos:

$$\boxed{\begin{aligned} \lambda_k &= (2k+1)(2k+2), \\ y_k(x) &= P_{2k+1}(x), \quad k = 0, 1, 2, \ldots \end{aligned}}$$ +0,5

(ii) $P'_{2k+1}(0) = \,?$

função geratriz: $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

derivando c/ relação a x: $$\frac{(-\frac{1}{2})(-2t)}{(1-2xt+t^2)^{3/2}} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\,t^n$$

$x=0$ $$\frac{t}{(1+t^2)^{3/2}} = t(1+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(0)\,t^n$$

$$\therefore t\sum_{k=0}^{\infty} \frac{(3/2)_k}{k!}(-t^2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k (3/2)_k}{k!}\,t^{2k+1} = \sum_{n=0}^{\infty} P_n'(0)\,t^n$$

$\therefore$ $P_{2k}'(0) = 0$ e: $$P_{2k+1}'(0) = \frac{(-1)^k (3/2)_k}{k!}$$ +1,0