T_5 ( ) F520 ( ) MS550 · Nome: ____________ RA: ________

Mostre que

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} e^{izt}\, dt.$$

$$I = \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}} e^{izt} = \int_{-1}^{+1} \underbrace{(1-t^2)^{\nu-\frac{1}{2}} \cos zt}_{\text{função par}}\, dt + \underbrace{i\int_{-1}^{+1} \underbrace{(1-t^2)^{\nu-\frac{1}{2}} \sin zt}_{\text{função ímpar}}\, dt}_{=0}$$

$$= 2\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos zt\, dt = 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} t^{2n}\, dt \stackrel{\boxed{t^2=y}}{=}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \int_0^1 (1-y)^{\nu-\frac{1}{2}} y^{n-\frac{1}{2}}\, dy \stackrel{(*)}{=} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} B\left(\nu+\tfrac{1}{2}, n+\tfrac{1}{2}\right) =$$

+4,0

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \frac{\Gamma(\nu+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(\nu+n+1)} \stackrel{(**)}{=} \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}\, \Gamma(2n)}{(2n)!\,\Gamma(n)\, 2^{2n-1}\Gamma(\nu+n+1)} =$$

$$= \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}\, \cancel{\Gamma(2n)}}{(2n)\cancel{\Gamma(2n)}\,\Gamma(n)\, 2^{2n-1}\Gamma(\nu+n+1)} = \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}$$

$$= \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) (z/2)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\,\Gamma(\nu+n+1)} = \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) (z/2)^{-\nu} J_\nu(z)$$

+6,0

Formulário eventualmente útil

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{\Gamma(n+\nu+1)n!}, \quad e^{z(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(z) t^n, \quad J_m(z) = \frac{1}{2\pi i}\oint_c \frac{e^{z(t-t^{-1})/2}}{t^{m+1}} dt,$$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z), \; (**)$$

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta\, d\theta, \qquad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt. \; (*)$$