( )F520 ( )MS550 - Exame Final - 11/07/2012

RA: ____________________ **Nome:** ____

(1) Seja a equação diferencial

$$x(1-x)y'' + (1-5x)y' - 4y = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que a equação indicial correspondente apresenta raízes reais iguais $r_1 = r_2 = 0$, e com isso obtemos que uma das soluções em forma de série dessa equação é

$$y_1(x) = \sum_{n=0}^{\infty} (n+1)^2 x^n.$$

Utilize o método de Frobenius para encontrar uma segunda solução $y_2(x)$ linearmente independente.

(2) Encontre os autovalores e autofunções do seguinte problema de Sturm-Liouville:

$$\begin{cases} xy'' + y' + \lambda xy = 0, \quad 0 < x < 1, \\ y(1) = 0, \quad \lim_{x \to 0^+} |y(x)| < \infty. \end{cases}$$

Escreva a relação de ortogonalidade satisfeita por essas autofunções.

Escolha e faça somente 3 das próximas 4 questões

(3) Use a função gama e/ou beta para mostrar que

$$\int_0^{\infty} \frac{\mathrm{d}t}{(1+t)\sqrt[6]{t}} = 2\pi.$$

(4) Sejam (r, θ, ϕ) as coordenadas esféricas, ou seja, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ e $z = r\cos\theta$. Mostre que

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}},$$

onde $P_n(\cdot)$ são os polinômios de Legendre e $n = 0, 1, 2, \ldots$

(5) Sejam $J_n(x)$ as funções de Bessel de primeira espécie e ordem n. Mostre que

$$\text{(i)} \quad J_n(u+v) = \sum_{m=-\infty}^{\infty} J_m(u) J_{n-m}(v),$$

$$\text{(ii)} \quad |J_0(x)| \leq 1, \qquad |J_n(x)| \leq 1/\sqrt{2} \quad (n = 1, 2, 3, \ldots).$$

(6) Seja ${}_2F_1(\alpha, \beta, \gamma; x)$ a função hipergeométrica. Mostre que

$${}_2F_1\left(\alpha, \frac{\alpha}{2}+1, \frac{\alpha}{2}; x\right) = (1+x)(1-x)^{-\alpha-1},$$

sendo $|x| < 1$.

ℹ Todas as questões tem o mesmo valor (2,0 pontos).

<u>FORMULÁRIO EVENTUALMENTE ÚTIL</u>

$$\nabla\cdot\mathbf{V}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3V_1)+\frac{\partial}{\partial q_2}(h_3h_1V_2)+\frac{\partial}{\partial q_3}(h_1h_2V_3)\right],\qquad \nabla\times\mathbf{V}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_{q_1} & h_2\mathbf{e}_{q_2} & h_3\mathbf{e}_{q_3}\\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3}\\ h_1V_1 & h_2V_2 & h_3V_3\end{vmatrix},$$

$$\nabla\cdot(f\mathbf{V})=\mathbf{V}\cdot\nabla f+f\nabla\cdot\mathbf{V},\qquad \nabla\times(f\mathbf{V})=f\nabla\times\mathbf{V}+\nabla f\times\mathbf{V},\qquad \nabla(fg)=f\nabla g+g\nabla f,$$

$$\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{e}_{q_1}+\frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{e}_{q_2}+\frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{e}_{q_3},\qquad h_r=1,\qquad h_\theta=r,\qquad h_\phi=r\sin\theta,\qquad \Gamma(1/2)=\sqrt{\pi},$$

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\,dt,\quad \Gamma(z+1)=z\Gamma(z),\quad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z),$$

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},\qquad B(z,w)=2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta,\qquad B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

$${}_2F_1(\alpha,\beta,\gamma;z)=\sum_{n=0}^\infty\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},\quad {}_2F_1(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

$$U(a,b;z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{-a+b-1}\,\mathrm{d}t,\quad \frac{\mathrm{d}^nU(a,b;z)}{\mathrm{d}z^n}=(-1)^n(a)_nU(a+n,b+n;z),$$

$$U(a,b;z)=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}\,{}_1F_1(a-b+1,2-b;z)+\frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a,b;z),\quad {}_1F_1(a,b;z)=\sum_{n=0}^\infty\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

$$J_\nu(x)=\sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},\quad J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x),\quad J_{\nu-1}(x)-J_{\nu+1}(x)=2J'_\nu(x),$$

$$\frac{d}{dx}(x^{-\nu}J_\nu(x))=-x^{-\nu}J_{\nu+1}(x),\quad \frac{d}{dx}(x^\nu J_\nu(x))=x^\nu J_{\nu-1}(x),\quad \mathrm{e}^{x(t-t^{-1})/2}=\sum_{k=-\infty}^{+\infty}t^kJ_k(x)$$

$$J_n(u+v)=\sum_{m=-\infty}^{+\infty}J_m(u)J_{n-m}(v)\quad J_n(x)=(-1)^nx^n\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^nJ_0(x)\quad J_0(x)=\frac{2}{\pi}\int_0^1\frac{\cos xt}{\sqrt{1-t^2}}\,dt$$

$$J_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta,\quad I_\nu(x)=i^{-\nu}J_\nu(ix)$$

$$P_n(x)={}_2F_1(-n,n+1,1;\tfrac{1-x}{2}),\quad P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}[(x^2-1)^n],\quad \frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^\infty P_n(x)t^n,$$

$$(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x),\quad (2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x),$$

$$P'_{n-1}(x)=-nP_n(x)+xP'_n(x),\quad (1-x^2)P'_n(x)=nP_{n-1}(x)-nxP_n(x),$$

$$(1-x^2)P'_n(x)=(n+1)xP_n(x)-(n+1)P_{n+1}(x),\quad P'_{n+1}(x)=(n+1)P_n(x)+xP'_n(x),$$

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\quad (1-x)^{-\alpha}=\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}x^n,\quad \int_{-1}^1P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{mn}.$$

(1) Veja correção do T2

(2) Veja correção do (1) da P2

(3) $y = \frac{1}{1+t}$ $\quad \therefore \; t=\infty \rightarrow y=0$

$\qquad\qquad\qquad\quad t=0 \rightarrow y=1$

$\Downarrow$

$y + yt = 1 \Rightarrow t = \frac{1-y}{y} \quad \therefore dt = -\frac{1}{y^2}dy$

$$I = \int_1^0 \left(-\frac{dy}{y^2}\right) y \frac{1}{\left(\frac{1-y}{y}\right)^{1/6}} = \int_0^1 dy\, y^{1-2+\frac{1}{6}} (1-y)^{-1/6}$$

$$= \int_0^1 dy\, y^{\frac{1}{6}-1} (1-y)^{\frac{5}{6}-1} = B\left(\frac{1}{6}, \frac{5}{6}\right) = \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma(1)}$$

$$= \Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right) = \frac{\pi}{\sin\frac{\pi}{6}} = 2\pi$$

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(4) $$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial(\cos\theta)}{\partial z}\frac{\partial}{\partial(\cos\theta)} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial(z/r)}{\partial z}\frac{\partial}{\partial(\cos\theta)}$$

mas: $$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \cos\theta$$

$$\frac{\partial(z/r)}{\partial z} = \frac{1}{r} - \frac{z}{r^2}\frac{\partial r}{\partial z} = \frac{1}{r} - \frac{1}{r}\cos^2\theta$$

$$\therefore \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} + \frac{(1-\cos^2\theta)}{r}\frac{\partial}{\partial(\cos\theta)}$$

$$\therefore \frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = \left[\cos\theta\frac{\partial}{\partial r} + \frac{(1-\cos^2\theta)}{r}\frac{\partial}{\partial(\cos\theta)}\right]\left(\frac{P_n(\cos\theta)}{r^{n+1}}\right) =$$

$$= \cos\theta\, P_n(\cos\theta)\frac{d}{dr}\left(\frac{1}{r^{n+1}}\right) + \frac{(1-\cos^2\theta)}{r^{n+2}}\frac{d\, P_n(\cos\theta)}{d(\cos\theta)}$$

$$= -\frac{(n+1)}{r^{n+2}}\cos\theta\, P_n(\cos\theta) + \frac{(1-\cos^2\theta)}{r^{n+2}}\, P_n'(\cos\theta) \qquad \circledast$$

mas do formulário, identificando $x = \cos\theta$, temos

$$(1-x^2)P_n'(x) - (n+1)x\,P_n(x) = -(n+1)\,P_{n+1}(x)$$

$$\therefore\ (1-\cos^2\theta)P_n'(\cos\theta) - (n+1)\cos\theta\, P_n(\cos\theta) = -(n+1)P_{n+1}(\cos\theta)$$

Em $\circledast$:

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = \frac{1}{r^{n+2}}\left(-(n+1)\,P_{n+1}(\cos\theta)\right) \qquad \checkmark$$

(5) Veja correção do T5

(6) $${}_2F_1\left(\alpha, \frac{\alpha}{2}+1, \frac{\alpha}{2}; x\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n \left(\frac{\alpha}{2}+1\right)_n}{\left(\frac{\alpha}{2}\right)_n}\frac{x^n}{n!} \qquad \circledast$$

mas: $$\left(\frac{\alpha}{2}+1\right)_n = \frac{\Gamma\left(\frac{\alpha}{2}+1+n\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} = \frac{\left(n+\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+n\right)}{\frac{\alpha}{2}\,\Gamma\left(\frac{\alpha}{2}\right)} = \left(\frac{2n}{\alpha}+1\right)\left(\frac{\alpha}{2}\right)_n$$

$$\therefore \circledast = \sum_{n=0}^{\infty}(\alpha)_n\left(1+\frac{2n}{\alpha}\right)\frac{x^n}{n!} = \underbrace{\sum_{n=0}^{\infty}(\alpha)_n\frac{x^n}{n!}}_{(1-x)^{-\alpha}} + \sum_{n=\cancel{0}\,1}^{\infty}(\alpha)_n\,\frac{2n}{\alpha}\,\frac{x^n}{n!}$$

$$= (1-x)^{-\alpha} + \sum_{n=0}^{\infty}(\alpha)_{n+1}\,\frac{2}{\alpha}\cdot\cancel{(n+1)}\,\frac{x^{n+1}}{\cancel{(n+1)!}_{\,n!}} = (1-x)^{-\alpha} + \frac{2}{\alpha}x\sum_{n=0}^{\infty}(\alpha)_{n+1}\frac{x^n}{n!} \qquad \circledast\circledast$$

mas: $(\alpha)_{n+1} = \alpha(\alpha+1)\dots(\alpha+n+1-1) = \alpha\cdot(\alpha+1)_n$

$$\circledast\circledast = (1-x)^{-\alpha} + 2x\sum_{n=0}^{\infty}(\alpha+1)_n\frac{x^n}{n!} = (1-x)^{-\alpha} + 2x(1-x)^{-\alpha-1} =$$

$$= (1-x)^{-\alpha}\left(1+\frac{2x}{1-x}\right) = (1-x)^{-\alpha}\left(\frac{1-x+2x}{1-x}\right) = (1-x)^{-\alpha-1}(1+x) \qquad \checkmark$$