( )F520 ( )MS550 - Exame - 10/07/2013

RA: ____________________ **Nome:** ____

(1) (i) Encontre os autovalores e autofunções do problema

$$\begin{cases} xy'' + y' - \dfrac{1}{x}y = -\lambda xy, \quad 0 < x < 1, \\ \lim\limits_{x\to 0}|y(x)| < \infty, \quad y(1) = 0. \end{cases}$$

(ii) Escreva a relação de ortogonalidade envolvendo estas autofunções.

Escolha e faça somente 3 das próximas 5 questões

(2) Mostre que

$$J_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^{\pi/2} \cos(x\sin\theta)\cos^{2\nu}\theta\, d\theta,$$

onde $J_\nu(x)$ denota a função de Bessel de primeira espécie e ordem ν.

(3) Sejam $P_n(x)$ $(n = 0, 1, 2, \ldots)$ os polinômios de Legendre e $Q_n(x)$ $(n = 0, 1, 2, \ldots)$ as funções de Legendre de segunda espécie. Mostre que eles satisfazem:

$$\text{(i)} \qquad P_n'(1) = \frac{n(n+1)}{2},$$
$$\text{(ii)} \qquad xQ_n'(x) - Q_{n-1}'(x) = nQ_n(x).$$

(4) (i) Encontre os autovalores e autofunções do problema

$$\begin{cases} (xy')' = -\lambda x^{-1}y, \quad 1 < x < \mathrm{e}^{2\pi}, \\ y'(1) = 0, \quad y'(\mathrm{e}^{2\pi}) = 0. \end{cases}$$

(ii) Escreva a relação de ortogonalidade envolvendo estas autofunções.

(5) Seja ${}_2F_1(\alpha, \beta, \gamma; z)$ a função hipergeométrica. Mostre que

$$\text{(i)} \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\phi}\, \mathrm{d}\phi = \frac{\pi}{2}\, {}_2F_1(-1/2, 1/2, 1; k^2),$$
$$\text{(ii)} \quad {}_2F_1(\alpha, \beta, \beta - \alpha + 1; -1) = \frac{\Gamma(1-\alpha+\beta)\Gamma(1+\beta/2)}{\Gamma(1+\beta)\Gamma(1-\alpha+\beta/2)}.$$

(6) Seja a equação diferencial

$$x^2y'' + (x^2 - x)y' + y = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que a equação indicial correspondente apresenta raízes iguais $r_1 = r_2 = 1$, e com isso obtemos que uma das soluções em forma de série dessa equação diferencial é

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1}.$$

Use o método de Frobenius para encontrar uma segunda solução $y_2(x)$ linearmente independente.

ℹ Todas as questões tem o mesmo valor (2,5 pontos).

Formulário (eventualmente útil)

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\,dt,\quad \Gamma(z+1)=z\Gamma(z),\quad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z),$$

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},\qquad B(z,w)=2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta,\qquad B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

$${}_2F_1(\alpha,\beta,\gamma;z)=\sum_{n=0}^\infty\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},\quad {}_2F_1(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

$$U(a,b;z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{-a+b-1}\,\mathrm{d}t,\quad \frac{\mathrm{d}^nU(a,b;z)}{\mathrm{d}z^n}=(-1)^n(a)_nU(a+n,b+n;z),$$

$$U(a,b;z)=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}\,{}_1F_1(a-b+1,2-b;z)+\frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a,b;z),\quad {}_1F_1(a,b;z)=\sum_{n=0}^\infty\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

$$J_\nu(x)=\sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},\quad J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x),\quad J_{\nu-1}(x)-J_{\nu+1}(x)=2J'_\nu(x),$$

$$\frac{d}{dx}(x^{-\nu}J_\nu(x))=-x^{-\nu}J_{\nu+1}(x),\quad \frac{d}{dx}(x^\nu J_\nu(x))=x^\nu J_{\nu-1}(x),\quad \mathrm{e}^{x(t-t^{-1})/2}=\sum_{k=-\infty}^{+\infty}t^kJ_k(x)$$

$$J_n(u+v)=\sum_{m=-\infty}^{+\infty}J_m(u)J_{n-m}(v)\quad J_n(x)=(-1)^nx^n\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^nJ_0(x)\quad J_0(x)=\frac{2}{\pi}\int_0^1\frac{\cos xt}{\sqrt{1-t^2}}\,dt$$

$$J_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta,\quad \cos x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n}}{(2n)!},\quad \sin x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

$$P_\nu(x)={}_2F_1(-\nu,\nu+1,1;\tfrac{1-x}{2}),\quad Q_\nu(x)=\frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)(2x)^{\nu+1}}\,{}_2F_1\left(\frac{\nu+2}{2},\frac{\nu+1}{2},\nu+\frac{3}{2};\frac{1}{x^2}\right),$$

$$P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}[(x^2-1)^n],\quad \frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^\infty P_n(x)t^n,$$

$$(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x),\quad (2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x),$$

$$P'_{n-1}(x)=-nP_n(x)+xP'_n(x),\quad (1-x^2)P'_n(x)=nP_{n-1}(x)-nxP_n(x),$$

$$P'_{n+1}(x)=(n+1)P_n(x)+xP'_n(x),\quad (1-x)^{-\alpha}=\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}x^n,\quad \int_{-1}^1P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{mn},$$

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\quad Q_n(x)=\frac{1}{2}\int_{-1}^1\frac{P_n(y)}{x-y}\,\mathrm{d}y,$$

(1) Veja NOTAS DE AULA, CAP 5, pg 207, EXEMPLO 5.12
com $\underline{\underline{a=1}}$ e $\underline{\underline{\nu=1}}$.

(2) $$I=\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta=\sum_{n=0}^{\infty}(-1)^n\int_0^{\pi/2}\frac{x^{2n}\sin^{2n}\theta}{(2n)!}\cos^{2\nu}\theta\,d\theta=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\frac{1}{2}B(n+\tfrac{1}{2},\nu+\tfrac{1}{2})=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\frac{\Gamma(n+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(n+\nu+1)}$$

$$=\frac{\Gamma(\nu+\frac{1}{2})}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(n+\nu+1)}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(2n+1)}$$

mas da fórmula de duplicação com $z=n+\frac{1}{2}\Rightarrow\frac{\Gamma(n+\frac{1}{2})}{\Gamma(2n+1)}=\frac{\sqrt{\pi}}{\Gamma(n+1)2^{2n}}$

$$\therefore I=\frac{\Gamma(\nu+\frac{1}{2})}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(n+\nu+1)}\frac{\sqrt{\pi}}{n!\,2^{2n}}=\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\,\Gamma(n+\nu+1)}=$$

$$=\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{2(x/2)^{\nu}}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}=\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{2(x/2)^{\nu}}J_\nu(x)$$ ✓

(3) (i) Da equação de Legendre para $y=P_n(x)$,

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$$

tomando $x=1$, temos

$$\underbrace{(1-1)}_{=0}P_n''(1)-2\cdot 1\,P_n'(1)+n(n+1)\underbrace{P_n(1)}_{=1}=0$$

$$\therefore P_n'(1)=\frac{n(n+1)}{2}$$ ✓

Outro método ⇒ derivando função geratriz em x e tomando $x=1$

(ii) $n\,Q_n(x) = \frac{n}{2}\int_{-1}^{+1} \frac{P_n(y)}{x-y}\,dy = \frac{1}{2}\int_{-1}^{+1} \frac{[y P_n'(y) - P_{n-1}'(y)]}{x-y}\,dy =$

$$= \frac{1}{2}\int_{-1}^{+1} \frac{[(x-(x-y))P_n'(y) - P_{n-1}'(y)]\,dy}{x-y} =$$

$$= \frac{1}{2}\left[-\int_{-1}^{+1} P_n'(y)\,dy + x\int_{-1}^{+1} \frac{P_n'(y)}{x-y}\,dy - \int_{-1}^{+1} \frac{P_{n-1}'(y)}{x-y}\,dy\right]$$

$$= \frac{1}{2}\left[-P_n(y)\Big|_{-1}^{+1} + x\left[\frac{P_n(y)}{x-y}\Big|_{-1}^{+1} - \int_{-1}^{+1} \frac{P_n(y)}{(x-y)^2}\,dy\right] - \left[\frac{P_{n-1}(y)}{x-y}\Big|_{-1}^{+1} - \int_{-1}^{+1} \frac{P_{n-1}(y)}{(x-y)^2}\,dy\right]\right] =$$

$$= \frac{1}{2}\Big[-1 + (-1)^n + x\,\frac{1}{x-1} - \frac{x(-1)^n}{x+1} + x\,\frac{d}{dx}\int_{-1}^{+1} \frac{P_n(y)}{x-y}\,dy$$

$$- \frac{1}{x-1} + \frac{(-1)^{n-1}}{x+1} - \frac{d}{dx}\int_{-1}^{+1} \frac{P_{n-1}(y)}{x-y}\,dy\Big]$$

$$= \frac{1}{2}\left[\cancel{-1} + \frac{\cancel{(x-1)}}{\cancel{x-1}} + (-1)^n - \frac{\cancel{(x+1)}(-1)^n}{\cancel{x+1}}\right] + x\,Q_n(x) - Q_{n-1}'(x)$$

④ $(xy')' = xy'' + y' = -\lambda x^{-1} y$

$$x^2 y'' + xy' + \lambda y = 0$$

$y = x^r$: $r(r-1) + r + \lambda = 0 \Rightarrow \boxed{r^2 = -\lambda}$

(i) $\boxed{\lambda < 0}$ $\lambda = -k^2$: $r^2 = k^2$

$$y = Ax^k + Bx^{-k} \Rightarrow y' = Akx^{k-1} - Bkx^{-k-1}$$

$y'(1) = 0 = Ak - Bk \Rightarrow A = B$

$y'(e^{2\pi}) = 0 = Ak\big(\underbrace{e^{2\pi(k-1)} - e^{2\pi(-k-1)}}_{\neq 0}\big) \Rightarrow A = 0 : B = 0$

apenas solução trivial

(ii) $\lambda = 0$ $\therefore y = A' + B' \ln x$ $\therefore y' = B' \frac{1}{x}$

$y'(1) = B' = 0$

$y'(e^{\alpha\pi}) = 0$ ✓ OK!

$\therefore y = A' = cte$ ' é solução!

$\lambda = 0$ //

(iii) $\boxed{\lambda > 0}$ $\lambda = k^2$ $\therefore r = \pm ik$

$\therefore$

$$y = A'' x^{ik} + B'' x^{-ik} = A'' e^{ik\ln x} + B'' e^{-ik\ln x}$$

$$= C\cos(k\ln x) + D\sin(k\ln x)$$

$$y' = -C\sin(k\ln x)\frac{k}{x} + D\cos(k\ln x)\frac{k}{x}$$

$y'(1) = D.k = 0 \Rightarrow D = 0$

$$y'(e^{\alpha\pi}) = -C\sin(k\ln e^{2\pi})\frac{k}{e^{\alpha\pi}} = -\frac{Ck}{e^{\alpha\pi}}\sin(2\pi k) = 0$$

$\therefore 2\pi k = n\pi$, $n = 1, 2, \ldots$

$\therefore k = n/2$

$\therefore \lambda_n = (n/2)^2$

$y_n(x) = \cos\left(\frac{n\ln x}{2}\right)$, $n = 1, 2, 3 \ldots$ //

$\therefore$ juntando os dois casos $\Rightarrow$ $\begin{cases} y_n(x) = \cos\left(\frac{n\ln x}{2}\right) \\ \lambda_n = \frac{n^2}{4} \end{cases}$, $n = 0, 1, 2, 3 \ldots$ ✓

5 (i) $E(K) = \int_0^{\pi/2} \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n}{n!} (K^2 \sin^2\phi)^n \, d\phi =$

$$= \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n}{n!} K^{2n} \int_0^{\pi/2} \sin^{2n}\phi \, d\phi = \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n}{n!} \frac{K^{2n}}{2} B(n+1/2, 1/2) =$$

$$= \frac{1}{2} \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n K^{2n}}{n!} \frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)} = \frac{1}{2} \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n K^{2n}}{n!\,(1)_n} \frac{\Gamma(n+1/2)}{\Gamma(1/2)} [\Gamma(1/2)]^2 =$$

$$= \frac{\pi}{2} \sum_{n=0}^{\infty}{}' \frac{(-1/2)_n K^{2n}}{n!\,(1)_n} (1/2)_n = \frac{\pi}{2} \, {}_2F_1(-1/2, 1/2, 1; K^2) \quad ✓$$

(ii) $${}_2F_1(\alpha, \beta, \beta-\alpha+1; -1) = \frac{1}{B(\beta, \beta-\alpha+1-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\beta-\alpha+1-\beta-1} (1+t)^{-\alpha} \, dt =$$

$$= \frac{1}{B(\beta, 1-\alpha)} \int_0^1 t^{\beta-1} (1-t^2)^{-\alpha} \, dt = \frac{1}{2B(\beta, 1-\alpha)} \int_0^1 (y^{1/2})^{\beta-1} (1-y)^{-\alpha} \frac{dy}{2y^{1/2}} =$$

$$= \frac{1}{2B(\beta, 1-\alpha)} B\left(\frac{\beta}{2}, 1-\alpha\right) = \frac{1}{2} \frac{\Gamma(\beta/2)\,\cancel{\Gamma(1-\alpha)}}{\Gamma(1-\alpha+\beta/2)} \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\cancel{\Gamma(1-\alpha)}}$$

$$= \frac{\beta}{2} \frac{\Gamma(\beta/2)\,\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha+\beta/2)\,\beta\,\Gamma(\beta)} = \frac{\Gamma(1+\beta/2)\,\Gamma(1-\alpha+\beta)}{\Gamma(1+\beta)\,\Gamma(1-\alpha+\beta/2)} \quad ✓$$

6 Veja a SEGUNDA SOLUÇÃO do Teste T2