( )F520 ( )MS550 - Segunda Prova - 27/06/2011

RA: ________________ **Nome:** ____________________________________

(1) Sejam $P_n(x)$ os polinômios de Legendre ($n = 0, 1, 2, \ldots$). Mostre que

$$\text{(i) } P_n(1) = 1, \qquad \text{(ii) } \int_0^1 P_{2n}(x)\mathrm{d}x = 0 \quad (n \neq 0).$$

(2) Sejam $J_n(x)$ as funções de Bessel de primeira espécie e ordem n ($n = 0, 1, 2, \ldots$). Mostre que

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} J_n(x) = J_0\left(\sqrt{x^2 - 2xt}\right).$$

(3) Seja ${}_2F_1(\alpha, \beta, \gamma; x)$ a função hipergeométrica. Mostre que

$${}_2F_1(\alpha, \beta, \beta - \alpha + 1; -1) = \frac{\Gamma(1 + \beta - \alpha)\Gamma(1 + \beta/2)}{\Gamma(1 + \beta)\Gamma(1 + \beta/2 - \alpha)}.$$

(4) Encontre os autovalores e autofunções do seguinte problema de Sturm-Liouville:

$$\begin{aligned} &x(xy')' + \lambda y = 0, \qquad 1 < x < \mathrm{e}^{2\pi}, \\ &y'(1) = 0, \quad y'(\mathrm{e}^{2\pi}) = 0. \end{aligned}$$

Escreva a relação de ortogonalidade satisfeita por essas autofunções.

ℹ Valor das questões: (1) i - 1,0; ii - 1,5 (2) 2,5 (3) 2,5 (4) 3,5.

Formulário eventualmente útil

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{\Gamma(n + \nu + 1)n!}, \quad \mathrm{e}^{z(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(z)t^n, \quad J_m(z) = \frac{1}{2\pi i}\oint_c \frac{\mathrm{e}^{z(t-t^{-1})/2}}{t^{m+1}}\mathrm{d}t,$$

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n, \quad (n + 1)P_n(x) = P'_{n+1}(x) - xP'_n(x), \quad \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

$$(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad (1 - x^2)P''_n(x) - 2xP'_n(x) + n(n + 1)P_n(x) = 0,$$

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x), \quad P_n(-x) = (-1)^n P_n(x), \quad (1 - z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n, \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)},$$

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\,dt, \quad \Gamma(z + 1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z + 1/2) = \sqrt{\pi}\Gamma(2z),$$

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}, \qquad B(z, w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta\,d\theta, \qquad B(z, w) = \int_0^1 t^{z-1}(1 - t)^{w-1}\,dt$$

$${}_2F_1(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \quad {}_2F_1(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)}\int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1}(1 - tz)^{-\alpha}\,dt$$

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i) $g(x,t) = \dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

$\underline{x=1} \Rightarrow \sum_{n=0}^{\infty} P_n(1) t^n = \dfrac{1}{\sqrt{1-2t+t^2}} = \dfrac{1}{1-t} = \sum_{n=0}^{\infty} t^n$

$\therefore \boxed{P_n(1) = 1}$ +1,0

ii) $\int_0^1 P_{2n}(x)\,dx = \int_0^1 \left[\dfrac{P'_{2n+1}(x) - P'_{2n-1}(x)}{2(2n)+1} \right] dx$

$= \dfrac{1}{(4n+1)} \left[P_{2n+1}(x)\Big|_0^1 - P_{2n-1}(x)\Big|_0^1 \right] = \dfrac{1}{(4n+1)} \left[\cancel{P_{2n+1}(1)} - P_{2n+1}(0) - \cancel{P_{2n-1}(1)} + P_{2n-1}(0) \right]$ (*)

$\underline{x=0} \Rightarrow \sum_{n=0}^{\infty} P_n(0) t^n = \dfrac{1}{\sqrt{1+t^2}} = \sum_{m=0}^{\infty} \dfrac{(1/2)_m (-1)^m (t^2)^m}{m!} \Rightarrow \underline{P_{2n+1}(0) = 0}$

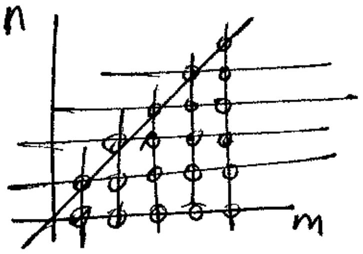
$(*) = \dfrac{1}{(4n+1)} \left[P_{2n-1}(0) - P_{2n+1}(0) \right] = \dfrac{1}{(4n+1)} (0-0) = 0$ //

+1,5

② $$J_0\left(\sqrt{x^2-2xt}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}\sqrt{x^2-2xt}\right)^{2m}}{(m!)^2} =$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}\left(x^2-2xt\right)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} \sum_{n=0}^{m} \binom{m}{n} (x^2)^{m-n} (-2xt)^n$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{m+n}\, x^{2m-n}\, t^n}{2^{2m-n}\, m!\, n!\, (m-n)!} \qquad (*)$$



$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty}$$

+1,0

$$(*) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{m+n}\, x^{2m-n}\, t^n}{2^{2m-n}\, m!\, n!\, (m-n)!} \;=\; \boxed{m-n=k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+2n}\, x^{2(n+k)-n}\, t^n}{2^{2(n+k)-n}\, (n+k)!\, n!\, k!} =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+n}}{2^{2k+n}\, k!\, (k+n)!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} J_n(x)$$

+1,5

(3)

$$ {}_2F_1(\alpha, \beta, \beta-\alpha+1; -1) = \frac{1}{B(\beta, (\beta-\alpha+1)-\beta)} \int_0^1 t^{\beta-1} (1-t)^{(\beta-\alpha+1)-\beta-1} (1-t(-1))^{-\alpha} dt $$

$$ = \frac{1}{B(\beta, 1-\alpha)} \int_0^1 t^{\beta-1} (1-t)^{-\alpha} (1+t)^{-\alpha} dt $$

$$ = \frac{1}{B(\beta, 1-\alpha)} \int_0^1 t^{\beta-1} (1-t^2)^{-\alpha} dt = $$

$$ = \frac{1}{B(\beta, 1-\alpha)} \int_0^1 (y^{1/2})^{\beta-1} (1-y)^{-\alpha} \frac{1}{2} y^{-1/2} dy = $$

$$ = \frac{1}{2B(\beta, 1-\alpha)} \int_0^1 y^{\frac{\beta}{2}-1} (1-y)^{-\alpha} dy = \frac{B(\beta/2, -\alpha+1)}{2B(\beta, 1-\alpha)} $$

+1,5

$$ = \frac{1}{2} \frac{\Gamma(\beta/2)\, \cancel{\Gamma(1-\alpha)}\, \Gamma(1-\alpha+\beta)}{\Gamma(1-\alpha+\beta/2)\, \Gamma(\beta)\, \cancel{\Gamma(1-\alpha)}} = $$

$$ = \frac{\beta}{2} \frac{\Gamma(\beta/2)\, \Gamma(1+\beta-\alpha)}{\beta\, \Gamma(\beta)\, \Gamma(1+\beta/2-\alpha)} = \frac{\Gamma(1+\beta/2)\, \Gamma(1+\beta-\alpha)}{\Gamma(1+\beta)\, \Gamma(1+\beta/2-\alpha)} $$

+1,0

④ $\begin{cases} x(xy')' + \lambda y = x^2y'' + xy' + \lambda y = 0, & 1<x<e^{2\pi} \\ y'(1) = y'(e^{2\pi}) = 0 \end{cases}$

$y = x^r \Rightarrow r(r-1)+r+\lambda = 0 = r^2+\lambda \quad \therefore \quad r = \pm\sqrt{-\lambda}$

(i) $\underline{\underline{\lambda<0}}$; $\lambda = -k^2 \ (k>0) \quad \therefore \ r = \pm k$

$y = Ax^k + Bx^{-k}$, $y' = Akx^{k-1} - Bkx^{-k-1}$

$y'(1) = 0 \Rightarrow A+B=0$ $\quad \therefore$ apenas solução trivial $\quad$ +0,5

$y'(e^{2\pi}) = 0 \Rightarrow A = 0$

(ii) $\underline{\underline{\lambda=0}}$

$y = A_1 + B_1 \ln x$; $y' = \frac{B_1}{x}$

$y'(1)=0 \Rightarrow B_1 = 0$ $\qquad \therefore \ y = A_1 = $ etc.

$\lambda = 0$ $\quad //$ $\quad$ +1,0

$y'(e^{2\pi}) = 0$ ✓

(iii) $\underline{\underline{\lambda>0}}$; $\lambda = k^2 \ (k>0)$, $r = \pm ik$

$y = A_2x^{ik} + B_2x^{-ik} = A_2e^{ik\ln x} + B_2e^{-ik\ln x} = C_1\cos(k\ln x) + C_2\sin(k\ln x)$

$y' = -\frac{k}{x}C_1\sin(k\ln x) + \frac{k}{x}C_2\cos(k\ln x)$

$y'(1)=0 \Rightarrow C_2 = 0$

$y'(e^{2\pi}) = 0 \Rightarrow \sin(k\ln e^{2\pi}) = \sin(k.2\pi) = 0 \Rightarrow 2\pi k = n\pi$, $n = 1,2,\ldots$

$\therefore \ y_n = \cos\left(\frac{n\ln x}{2}\right) \quad n = 1,2,\ldots$

$\lambda_n = n^2/4$ $\quad //$ $\quad$ +1,0

De (ii) + (iii) $\Rightarrow \begin{cases} y_n = \cos\left(\frac{n\ln x}{2}\right) \\ \lambda_n = \frac{n^2}{4} \quad , \ n = 0,1,2,\ldots \end{cases}$ $\quad$ +0,5

ED: $(xy')' + \frac{\lambda}{x}y = 0 \Rightarrow \rho(x) = \frac{1}{x}$

$\therefore$ relacão de ortogonalidade $\Rightarrow \displaystyle\int_1^{e^{2\pi}} \cos\left(\frac{n\ln x}{2}\right)\cos\left(\frac{m\ln x}{2}\right)\cdot\frac{1}{x}\,dx = N_n\,\delta_{mn}$ $\quad$ +0,5

$m,n = 0,1,2,\ldots$