( )F520 ( )MS550 - Primeira Prova - 07/05/2012

RA: ____________________ **Nome:** ____

(1) Sejam as coordenadas esféricas (r, θ, ϕ) dadas por

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

onde $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ e $0 \leq \phi < 2\pi$. Mostre que

$$\nabla \times (\cos\theta \nabla\phi) = \nabla\left(\frac{1}{r}\right).$$

(2) Use o método de Frobenius em torno do ponto $x_0 = 0$ para encontrar duas soluções linearmente independentes da equação diferencial

$$x(1-x)y'' + (1-5x)y' - 4y = 0.$$

(3) Considere a equação diferencial

$$z^n y'' + \alpha y' + \beta y = 0,$$

onde $z \in \bar{\mathbb{C}}$, α e β são constantes, e $n \in \mathbb{N}$. Determine os valores de n para os quais o ponto $z = \infty$ é um ponto singular regular dessa equação.

(4) Mostre que

$$\int_0^{\pi/2} \left(\frac{1}{\sin\theta} - 1\right)^{1/4} \frac{\cos\theta}{\sin^{1/2}\theta}\, d\theta = \frac{[\Gamma(1/4)]^2}{2\sqrt{\pi}}.$$

ℹ Valor das questões: (1) 2,0 (2) 4,0 (3) 2,0 (4) 2,0.

Formulário

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right], \qquad \nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_{q_1} & h_2\mathbf{e}_{q_2} & h_3\mathbf{e}_{q_3} \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix},$$

$$\nabla \cdot (f\mathbf{V}) = \mathbf{V} \cdot \nabla f + f\nabla \cdot \mathbf{V}, \qquad \nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + \nabla f \times \mathbf{V}, \qquad \nabla(fg) = f\nabla g + g\nabla f,$$

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{e}_{q_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{e}_{q_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{e}_{q_3}, \qquad h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta, \qquad \Gamma(1/2) = \sqrt{\pi},$$

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z),$$

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta\, d\theta, \qquad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt$$

(1) $\nabla \times (\cos\theta\, \nabla\phi) = \nabla(\cos\theta) \times \nabla\phi + \cos\theta\, \underbrace{\nabla \times (\nabla\phi)}_{=0}$

Mas: $\nabla(\cos\theta) = \frac{1}{h_r}\underbrace{\frac{\partial(\cos\theta)}{\partial r}}_{=0}\vec{e}_r + \frac{1}{h_\theta}\frac{\partial(\cos\theta)}{\partial\theta}\vec{e}_\theta + \frac{1}{h_\phi}\underbrace{\frac{\partial(\cos\theta)}{\partial\phi}}_{=0}\vec{e}_\phi$

$= \frac{1}{r}(-\sin\theta)\vec{e}_\theta$

$\nabla\phi = \frac{1}{h_r}\underbrace{\frac{\partial\phi}{\partial r}}_{=0}\vec{e}_r + \frac{1}{h_\theta}\underbrace{\frac{\partial\phi}{\partial\theta}}_{=0}\vec{e}_\theta + \frac{1}{h_\phi}\frac{\partial\phi}{\partial\phi}\vec{e}_\phi = \frac{1}{r\sin\theta}\vec{e}_\phi$

$\therefore \nabla \times (\cos\theta\, \nabla\phi) = -\frac{\sin\theta}{r}\vec{e}_\theta \times \frac{1}{r\sin\theta}\vec{e}_\phi = -\frac{1}{r^2}\vec{e}_r \quad (*)$ +1,0

Por outro lado:

$\nabla\left(\frac{1}{r}\right) = \frac{1}{h_r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\vec{e}_r + \frac{1}{h_\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)}_{=0}\vec{e}_\theta + \frac{1}{h_\phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)}_{=0}\vec{e}_\phi = -\frac{1}{r^2}\vec{e}_r \quad (**)$

$(*) = (**) \Rightarrow \nabla \times (\cos\theta\, \nabla\phi) = \nabla\left(\frac{1}{r}\right)$ +1,0

(2) $x(1-x)y'' + (1-5x)y' - 4y = 0 \;;\; y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$\therefore \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r}$

$+ \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} 5a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} 4a_n x^{n+r} = 0$

$\therefore a_0 r(r-1)x^{r-1} + a_0 r x^{r-1} + \sum_{n=0}^{\infty} a_n(n+r+1)(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_{n+1}(n+1+r)x^{n+r}$

$- \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} 5a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} 4a_n x^{n+r} = 0$

$\therefore \begin{cases} r^2 = 0 \Rightarrow r_1 = r_2 = 0 \\ a_{n+1}[(n+r+1)]^2 - a_n[\underbrace{\underbrace{(n+r)(n+r-1+5)+4}_{(n+r)^2+4(n+r)+4}}_{(n+r+2)^2}] = 0 \Rightarrow a_{n+1} = a_n\frac{(n+r+2)^2}{(n+r+1)^2} \end{cases}$ +1,0

$\boxed{r_1 = 0}$ $\quad a_{n+1} = a_n \frac{(n+2)^2}{(n+1)^2}$

$$\therefore a_1 = a_0 \cdot \frac{2^2}{1^2}$$

$$a_2 = a_1 \frac{3^2}{2^2} = a_0 \frac{3^2}{1^2}$$

$$a_3 = a_2 \frac{4^2}{3^2} = a_0 \frac{4^2}{1^2}$$

$$\vdots$$

$$a_n = a_0 (n+1)^2$$

$$\therefore \; y_1(x) = \sum_{n=0}^{\infty} (n+1)^2 x^n$$

+1,0

Segunda solução? $\Rightarrow$ vejá resolução do **T2**!

+2,0

(3) Vamos fazer a mudança de variável $z \longrightarrow t = \frac{1}{z}$ e ver o que acontece para $t = 0$.

$$\frac{d}{dz} = \frac{dt}{dz}\frac{d}{dt} = -t^2\frac{d}{dt} \quad ; \quad \frac{d^2}{dz^2} = -t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$$

$$\therefore ED \Rightarrow \frac{1}{t^n}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \alpha\left(-t^2\right)\frac{dy}{dt} + \beta y = 0$$

$$\therefore \frac{d^2y}{dt^2} + \underbrace{\left[\frac{2}{t} - \frac{\alpha}{t^{2-n}}\right]}_{P(t)}\frac{dy}{dt} + \underbrace{\frac{\beta}{t^{4-n}}}_{Q(t)} y = 0$$

+1,0

$t = 0$ é ponto singular regular se $P(t)$ tem no máximo um pólo de ordem 1 em $t = 0$ (A) e $Q(t)$ no máximo pólo de ordem 2 em $t = 0$. (B)

(A) $\Rightarrow \{ 2 - n \leq 1 \implies n \geq 1$

(B) $\Rightarrow \{ 4 - n \leq 2 \implies n \geq 2$

$\therefore$ para que essas condições sejam satisfeitas devemos ter $\boxed{n \geq 2}$

$\therefore n = 2, 3, 4, \ldots$

+1,0

(4) $$I = \int_0^{\pi/2} \left(\frac{1}{\sin\theta} - 1\right)^{1/4} \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$\sin\theta = t \quad \therefore \; dt = \cos\theta\, d\theta$

$$\therefore I = \int_0^1 \left(\frac{1}{t} - 1\right)^{1/4} \frac{dt}{t^{1/2}} = \int_0^1 t^{-3/4} (1-t)^{1/4} dt = \int_0^1 t^{\frac{1}{4}-1} (1-t)^{\frac{5}{4}-1} dt$$

$$= B\left(\frac{1}{4}, \frac{5}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{2\sqrt{\pi}}$$

+20

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