Soluções para MS550, Métodos de Matemática Aplicada I, e F520, Métodos Matemáticos da Física I

Lista 4 - Funçã Gama

1. Mostre que $\Gamma(z) = \int_0^1 [\ln(1/t)]^{z-1}\, dt$.

   Solução: *Fazendo* $t = e^{-y}$ *temos*

$$\begin{aligned} \exp(y) &= 1/t \\ y &= \ln(1/t) \\ dy &= \frac{(-1/t^2)}{1/t} dt = \frac{-1}{t} dt. \end{aligned}$$

   Avaliando os extremos:

$$\begin{aligned} y \to 0, t \to 1, \\ y \to \infty, t \to 0. \end{aligned}$$

   Então

$$\begin{aligned} \Gamma(z) &= \int_1^0 t\, [\ln(1/t)]^{z-1} (-1/t)\, dt \\ &= \int_0^1 [\ln(1/t)]^{z-1}\, dt. \end{aligned}$$

2. Use a função gama para mostrar os seguintes resultados:

   (a) $\int_0^\infty \exp(-x^4)\, dx = \Gamma(5/5)$.

   Solução: *Fazendo* $x^4 = t$ *então* $x = t^{1/4}$. *E portanto* $dt = 4x^3 dx = 4t^{3/4} dx$ *implica que* $dx = (1/4)\, t^{-3/4} dt$. *Logo*

$$\begin{aligned} \int_0^\infty \exp(-x^4) dx &= \int_0^\infty \exp(-t) \frac{1}{4} t^{-3/4} dt \\ &= \frac{1}{4} \int_0^\infty \exp(-t) t^{-3/4} dt \\ &= \frac{1}{4} \Gamma(1/4). \end{aligned}$$

   Como $\Gamma(z+1) = z\Gamma(z)$, $(1/4)\,\Gamma(1/4) = \Gamma(5/4)$ *e portanto*

$$\int_0^\infty exp(-x^4) dx = \Gamma(5/4).$$

   (b) $\int_0^\infty x^{2s+1} \exp(-ax^2)\, dx = \Gamma(s+1_/\, (2a^{s+1})$.

   Solução: *Fazendo* $t = ax^2$, *temos* $dt = 2axdx$. *E avaliando os extremos:*

$$\begin{aligned} x \to 0, t \to 0, \\ x \to \infty, t \to \infty. \end{aligned}$$

Logo,

$$\begin{aligned}\int_0^\infty \left(x^2\right)^s \exp\left(-ax^2\right) x dx &= \int_0^\infty (t/a)^s \exp(-t) \frac{dt}{2a} \\ &= \frac{1}{2a^{s+1}} \int_0^\infty \exp(-t)\, t^s \\ &= \frac{1}{2a^{s+1}} \Gamma(s+1).\end{aligned}$$

(c) $\int_0^1 x^k \ln x dx = -1/(k+1)^2$.

Solução: *Fazendo* $x = \exp(-w)$, *temos* $w = \ln(1/x) = -\ln(x)$ *e consequentemente* $dw = \frac{-1}{x}dx = -\exp(w)\, dw$ *que implica em* $dx = -\exp(-w)\, dw$. *Avaliando os extremos,*

$$\begin{aligned} x \to 0, w \to \infty, \\ x \to 1, w \to 0. \end{aligned}$$

Logo,

$$\int_\infty^0 \exp(-wk)(-w)(-\exp(-w))\, dw = -\int_0^\infty \exp(-w(k+1))\, wdw.$$

Fazendo $w(k+1) = t$ *temos* $w = t/(k+1)$ *e consequentemente* $dt = (k+1)\, dw \to dw = dt/(k+1)$. *Avaliando os extremos*

$$\begin{aligned} w \to 0, t \to 0, \\ w \to \infty, t \to \infty. \end{aligned}$$

Logo,

$$\begin{aligned}-\int_0^\infty \exp(-t)(t/(k+1))(1/(k+1))\, dt &= (-1/(k+1)) \int_0^\infty \exp(-t)\, tdt \\ &= \frac{-1}{(k+1)^2}\Gamma(2) \\ &= \frac{-1}{(k+1)^2}\Gamma(1+1) \\ &= -\Gamma(1)/(k+1)^2 \\ &= -1/(k+1)^2.\end{aligned}$$

3. Mostre que $\lim_{x\to 0} \Gamma(ax)/\Gamma(x) = a^{-1}$.

Solução: *Usando a definição de Weierstrass, tem-se*

$$\frac{\Gamma(ax)}{\Gamma(x)} = \frac{x \exp(\gamma x) \prod_{n=1}^\infty (1 + x/n) \exp(-ax/n)}{ax \exp(\gamma ax) \prod_{n=1}^\infty (1 + ax/n) \exp(-ax/n)}$$

$$
\begin{aligned}
&= \frac{\exp(\gamma x)\prod_{n=1}^{\infty}(1+x/n)\exp(-x/n)}{(\exp(\gamma x))^{a}\,a\prod(1+ax/n)\exp(-ax/n}, \\
\lim_{x\to 0}\frac{\Gamma(ax)}{\Gamma(x)} &= \lim_{x\to 0}\left(\frac{\exp(\gamma x(1-a)}{a}\right)\lim_{x\to 0}\left(\frac{\prod_{n=1}^{\infty}(1+x/n)v\exp(-x/n)}{\prod_{n=1}^{\infty}(1+axqn)\exp(-x/n)}\right) \\
&= (1/a)\,.
\end{aligned}
$$

4. Mostre que $\mathrm{Res}_{z=-n}\Gamma(z) = (-1)^n/n!, \quad n = 0, 1, 2, \ldots$

   Solução: *Temos que*

$$
\begin{aligned}
\mathrm{Res}_{z=-n}\Gamma(z) &= \lim_{z\to -n}(z-(-n))\,\Gamma(z) \\
&= \lim_{z\to -n}(z+n)\lim_{k\to\infty}(k!k^z)/(z(z+1)\ldots) \\
&= \lim_{k\to\infty}\lim_{z\to -n}(z+n)k!k^z/(z(z+1)\ldots(z+n-1)(z+n)(z+n+1)\ldots) \\
&= \lim_{k\to\infty}k!/(k^n(-1)^n(n)(n-1)\ldots(1)(1)\ldots(k-n)) \\
&= \left((-1)^{/}n!\right)\lim_{k\to\infty}(k(k-1)\ldots(k-n)\ldots(1))/(k^n(k-n)\ldots(1)) \\
&= ((-1)^n/n!)\lim_{k\to\infty}(1-1/k)\ldots(1-(n-1)/k) \\
&= (-1)^n/n!
\end{aligned}
$$

5. Mostre que $|\,(ix)!|^2 = (\pi x)/(\sinh \pi x)$ onde $x! = \Gamma(x+1)$.

   Solução: *Como* $x! = \Gamma(x+1) = x\Gamma(x)$, *tem-se*

$$
\begin{aligned}
(ix)! &= \Gamma(ix+1) \\
&= ix\Gamma(ix), \\
|(ix)!|^2 &= (ix\Gamma(ix))(-ix\Gamma(-ix)) \\
&= ix\Gamma(ix)\Gamma(-ix+1) \\
&= (ix)(2\pi/(i(\exp(-i\pi ix)-\exp(i\pi ix))) && \textit{pela fórmula de reflexão} \\
&= (2\pi x)/(\exp(\pi x)-\exp(-\pi x)) \\
&= (2\pi x)/(2\sinh(\pi x))\,.
\end{aligned}
$$

   A fórmula de reflexão diz que

$$
\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) = 2\pi/(i(\exp(-i\pi z)-\exp(i\pi z)))\,.
$$

6. Mostre que $|x!| \geq |\,(x+iy)!|$, onde $x, y \in \mathbb{R}$, $x \neq -1, -2, \ldots$.

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Solução: *Temos que* $x! = \Gamma(x+1) = x\Gamma(x)$, *portanto* $|x!| = x\Gamma(x) = |\Gamma(x+1)|$. *E*

$$\begin{aligned}
(x+iy)! &= \Gamma(x+iy+1),\\
|(x+iy)!| &= |\Gamma(x+iy+1)|\\
&= \left|\int_0^\infty \exp(-t)t^{x+iy}dt\right|\\
&\leq \int_0^\infty |\exp(-t)t^{x+iy}|dy\\
&\leq \int_0^\infty |\exp(-t)t^x \exp(iy\ln t)|dt\\
&\leq \int_0^\infty |\exp(-t)t^x||\exp(iy\ln t)|dt\\
&\leq \int_0^\infty |\exp(-t)t^x dt & |\exp(iy\ln t)| = 1\\
&= x! \leq |x!|.
\end{aligned}$$

Como $|\int_0^\infty \exp(-t)t^x dt| = |\Gamma(x+1)| = |x!|$, *tem-se que* $|(x+iy)!| \leq |x!|$.

7. Seja $f(z) = (1+z)^\alpha$.

   (a) Mostre que

$$\left.\frac{d^n f}{dz^n}\right|_{z=0} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1}$$

   e use esse resultado para mostrar que $(1+z)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} z^n$, onde $\binom{\alpha}{n} = \alpha!/\left(n!\,(\alpha-n)!\right) = \Gamma(\alpha+1)/\left(n!\Gamma(\alpha-n+1)\right)$.

   Solução: *Temos que*

$$\begin{aligned}
\left[\frac{d^n f}{dz^n}\right]_{z=0} &= \left[\frac{d^n\left[(1+z)^\alpha\right]}{dz^n}\right]_{z=0}\\
&= \left[\alpha(\alpha-1)\dots(\alpha-n+1)(1+z)^{\alpha-n}\right]_{z=0}\\
&= \alpha(\alpha-1)\dots(\alpha-n+1) & z = 0.
\end{aligned}$$

   Como $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ *então* $\alpha = \Gamma(\alpha+1)/\Gamma(\alpha)$ *e portanto*

$$\begin{aligned}
\left[\frac{d^n f}{dz^n}\right]_{z=0} &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\Gamma(\alpha-1)}\dots\frac{\Gamma(\alpha-n-1}{\Gamma(\alpha-n)}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha-n+1}\\
&= \Gamma(\alpha+1)/\Gamma(\alpha-n+1).
\end{aligned}$$

   Expandindo $(1+z)^\alpha$ *em série de Taylor (em torno de* $z = 0$*):*

$$\begin{aligned}
(1+z)^\alpha &= \sum_{n=0}^\infty (1/n!)z^n\left[\frac{d^n(1+z)^\alpha}{dz^n}\right]_{z=0}\\
&= \sum_{n=0}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\frac{z^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} (\alpha/n) z^n,$$

onde $(\alpha/n) = \Gamma(\alpha+1)/\left(n!\Gamma(\alpha-n+1)\right.$.

(b) Generalize esse resultado para $a, b \in \mathbb{C}$ mostrando que $(a+b)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} a^n b^{\alpha-n}$.

Solução: *Para* $a, b \in \mathbb{C}$ *temos*

$$(a+b)^\alpha = \left[(1+a/b)b\right]^\alpha.$$

Fazendo $z = a/b$,

$$\begin{aligned}(a+b)^\alpha = (1+z)^\alpha b^\alpha \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n b^\alpha \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} (a^n/b^n) b^\alpha \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} a^n b^{\alpha-b}.\end{aligned}$$

(c) Mostre que se $\alpha = m \in \mathbb{Z}$ então essa série é truncada no termo $n = m$.

Solução: *Para* $\alpha = m \in \mathbb{Z}$, *no termo* $n = m$, *tem-se*

$$\binom{m}{m} b^m \alpha^{m-m} = b^m.$$

Analisando o termo seguinte da série, ou seja, $n = m+1$, *com* $\alpha = m$, *tem-se*

$$\begin{aligned}\binom{\alpha}{n} &= \binom{m}{m+1} \\ &= \Gamma(m+1)/\left((m+1)!\Gamma(m-(m+1)+1)\right. \\ &= \Gamma(m+1)/\Gamma(0).\end{aligned}$$

Como $\Gamma(0)$ *diverge, pois* 0 *é um dos pólos temos que a série é truncada em* $n = m$.

8. Mostre que $\int_{-1}^{n} (1+x)^a (1-x)^b \, dx = 2^{a+b+1} B(a+1, b+1)$, onde $B(a,b)$ é a função beta.

Solução: *Temos que a função beta corresponde a*

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

Fazendo $t = (1+x)/2$ *temos que* $x = 2t-1$ *e* $dt = dx/2$. *E os extremos ficam* $t = 0 \to x = -1$ *e* $= 1 \to x = 1$. *Substituindo na dfinição da função beta*

$$B(a,b) = \int_{-1}^{1} \left(\frac{1+x}{2}\right)^{a-1} \left(\frac{1-x}{2}\right)^{b-1} \frac{dx}{2}$$

$$
\begin{aligned}
&= \int_{-1}^{1} \frac{(1+x)^a}{2^2} \frac{(1-x)^b}{2^b} \frac{dx}{2} \\
&= \frac{1}{2^{a+b+1}} \int_{-1}^{1} (1+x)^a (1-x)^b dx.
\end{aligned}
$$

Portanto,

$$
\int_{-1}^{1} (1+x)^a (1-x)^b dx = 2^{a+b+1} B(a+1, b+1).
$$

9. Mostre que

   (a) $B(a,b) = B(a+1,b) + B(a,b+1)$.

   Solução: *Temos que*

$$
\begin{aligned}
B(a+1,b) + B(a,b+1) &= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} + \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)} \\
&= \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} + \frac{\Gamma(a)b\Gamma(b)}{(a+b)\Gamma(a+b)} \\
&= \frac{(a+b)\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
&= B(a,b).
\end{aligned}
$$

   (b) $B(a,b) = (b-1)\,/aB(a+1,b-1)$.

   Solução: *Temos que*

$$
\begin{aligned}
\frac{b-1}{a} B(a+1,b-1) &= \left(\frac{b-1}{a}\right) \frac{\Gamma(a+1)\Gamma(b-1)}{\Gamma(a+1+b-1)} \\
&= \left(\frac{b-1}{a}\right) \frac{\Gamma(a+1)\Gamma(b-1)}{\Gamma(a+b)} \\
&= \frac{(b-1)\Gamma(a+1)\Gamma(b-1)}{a\Gamma(a+b)} \\
&= \frac{(b-1)\Gamma(b-1)a\Gamma(a)}{a\Gamma(a+b)} \\
&= \frac{(b-1)\Gamma(b-1)\Gamma(a)}{\Gamma(a+b)} \\
&= \frac{\Gamma(b-1+1)\Gamma(a)}{\Gamma(a+b)} \\
&= \frac{\Gamma(b)\Gamma(a)}{\Gamma(a+b)} \\
&= B(a,b).
\end{aligned}
$$

10. Mostre que

$$\int_{-1}^{1} \left(1-x^2\right)^{1/2} x^{2n} dx = \begin{cases} \pi/2, & n=0; \\ \pi\left(2n-1\right)!!/\left(2n+2\right)!!, & n=1,2,\ldots \end{cases}$$

Solução: *Temos que*

$$\int_{-1}^{1} (1-x^2)^{1/2} x^{2n} dx = 2\int_{0}^{1} (1-x^2)^{1/2} x^2 dx.$$

Fazendo $x^2 = t$, $dx = dt/2x = dt/2t^{1/2}$. *Os extremos ficam com* $x = 0 \to t = 0$ *e* $x = 1 \to t = 1$. *Logo,*

$$\begin{aligned} 2\int_{0}^{1} (1-t)^{1/2} t^b t^{1/2}/2dt = \int_{0}^{1} (1-t)^{1/2} t^{n-1/2} dt \\ &= B(3/2, n+1/2) \\ &= B(n+1/2, 3/2). \end{aligned}$$

TODO

11. Mostre que $\int_t^z dx \, \left(\left(z-x\right)^{1-\alpha}\left(x-t\right)^{\alpha}\right) = \pi/\left(\sin \pi\alpha\right)$.

Solução:

12. Mostre que $d\left(z\right)_n/dz = \left(z\right)_n\left[\psi_0\left(z+n\right) - \psi_0\left(z\right)\right]$, onde $\left(z\right)_n = z\left(z+1\right)\cdots\left(z+n-1\right)$ é o símbolo de Pochammer e ψ_0 é a função digama.

Solução:

13. Mostre que $\ln\sin\pi z = \ln\pi z - \sum_{n=1}^{\infty}\zeta(2n)z^{2n}/n$, onde $\zeta(n)$ é a função zeta de Riemann.

Solução:

14. Mostre que $\psi_n(1) = \left(-1\right)^{n+1} n!\zeta(n+1)$, onde $\psi_n(z)$ é a função poligama (de ordem n).

Solução: *A função poligama correspondente a*

$$\psi_n(z+1) = (-1)^{n+1} n! \sum_{k=1}^{\infty} 1/(z+k)^{n+1}, n = 1, 2, \ldots$$

Para $z = 0$ *temos*

$$\psi_n(1) = (-1)^{n+1} n! \sum_{k=1}^{\infty} 1/k^{n+1}, n = 1, 2, \ldots$$

Como $\sum_{k=1}^{\infty} 1/k^{n+1} = \zeta(n+1)$ *temos* $\psi_n(1) = (-1)^{k+1} n!\zeta(n+1)$.

15. Mostre que $\psi_0(z) = -\gamma + \int_0^\infty dt/\left(e^t - 1\right) - \int_0^\infty \left(e^{(1-z)t}\right) dt/\left(e^t - 1\right)$.

Solução:

16. Mostre que $n!\zeta(n+1) = \int_0^\infty t^n dt/\left(e^t - 1\right)$.

Solução:

17. (E de 2011) Mostre que

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{1/4} dx = \frac{\pi}{\sqrt{2}}.$$

Solução: *Temos que*

$$\begin{aligned}
\int_{-1}^{1} (1+x)^{1/4}(1-x)^{-1/4} dx &= \int_0^1 2^{1/4} t^{1/4}(1-t)^{-1/4} 2dt \qquad\qquad t = (1+x)/2 \\
&= 2\int_0^1 t^{5/4-1}(1-t)^{3/4-1} dt \\
&= 2B(5/4, 3/4) \\
&= \frac{2\Gamma(5/4)\Gamma(3/4)}{\Gamma(2)} \\
&= \frac{2(1/4)\Gamma(1/4)\Gamma(1-1/4)}{1\Gamma(1)} \\
&= \frac{1}{2}\frac{\pi}{\sin(\pi/4)} \\
&= \pi/\sqrt{2}.
\end{aligned}$$