( )F520 ( )MS550 - Exame Final - 11/07/2011

RA: ___________________ **Nome:** ____

(1) Seja a equação diferencial

$$x^2y'' + x(x-1)y' + y = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que a equação indicial correspondente apresenta raízes iguais $r_1 = r_2 = 1$. Podemos também notar que uma solução dessa equação diferencial é

$$y_1(x) = x\mathrm{e}^{-x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n!}.$$

Utilize o método de Frobenius para encontrar uma segunda solução $y_2(x)$ linearmente independente.

(2) Encontre os autovalores e autofunções do seguinte problema de Sturm-Liouville:

$$x(xy')' + \lambda y = 0, \qquad 1 < x < \mathrm{e}^{2\pi},$$
$$y'(1) = 0, \quad y'(\mathrm{e}^{2\pi}) = 0.$$

Escreva a relação de ortogonalidade satisfeita por essas autofunções.

Escolha e faça somente 3 das próximas 4 questões

(3) Mostre que

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/4}\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

(4) Sejam $P_n(x)$ os polinômios de Legendre ($n = 0, 1, 2, \ldots$). Mostre que

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

(5) Sejam $J_n(x)$ as funções de Bessel de primeira espécie e ordem n ($n = 0, 1, 2, \ldots$). Mostre que

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}J_n(x) = J_0\left(\sqrt{x^2-2xt}\right).$$

(6) Seja ${}_2F_1(\alpha,\beta,\gamma;x)$ a função hipergeométrica. Mostre que

$${}_2F_1\left(\alpha, \frac{\alpha}{2}+1, \frac{\alpha}{2}; x\right) = (1+x)(1-x)^{-\alpha-1}.$$

ℹ Valor das questões: (1) 2,5 (2) 2,5 (3) 2,0 (4) 2,0 (5) 2,0 (6) 2,0.

Formulário eventualmente útil

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n+\nu}}{\Gamma(n+\nu+1)n!}, \quad \mathrm{e}^{z(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty}J_n(z)t^n, \quad J_m(z) = \frac{1}{2\pi i}\oint_c \frac{\mathrm{e}^{z(t-t^{-1})/2}}{t^{m+1}}\mathrm{d}t,$$

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \quad (n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x), \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n,$$

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad (1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0,$$

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x), \quad P_n(-x) = (-1)^nP_n(x), \quad (1-z)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

$$\Gamma(z) = \int_0^{\infty}\mathrm{e}^{-t}t^{z-1}\,dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z),$$

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad B(z,w) = 2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta, \qquad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

$${}_2F_1(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \quad {}_2F_1(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

(1) $x^2y'' + (x^2-x)y' + y = 0 \quad (*)$

$r_1 = r_2 = 1 \quad , \quad y_1(x) = xe^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!}$

Frobenius $\Rightarrow y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n x^n$

$\therefore y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} a_n x^{n+1}$

$\therefore y_2'(x) = y_1'\ln x + y_1\frac{1}{x} + \sum_{n=1}^{\infty}(n+1)a_n x^n$

$y_2''(x) = y_1''\ln x + 2y_1'\frac{1}{x} - y_1\frac{1}{x^2}\sum_{n=1}^{\infty} n(n+1)a_n x^{n-1}$

$(*) \Rightarrow 2xy_1' - y_1 + \sum_{n=1}^{\infty} n(n+1)a_n x^{n+1} + xy_1 + \sum_{n=1}^{\infty}(n+1)a_n x^{n+2}$

$- y_1 - \sum_{n=1}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=1}^{\infty} a_n x^{n+1} = 0$

e usando a expressão para $y_1(x)$:

$2\sum_{n=1}^{\infty}(-1)^n \frac{n x^{n+1}}{n!(n-1)!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!} + \sum_{n=1}^{\infty} a_n(n^2-1+1)x^{n+1} + \sum_{n=1}^{\infty}(n+1)a_n x^{n+2} = 0$

$-\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+2}}{n!} + a_1x^2 + \sum_{n=1}^{\infty}\left[a_{n+1}(n+1)^2 + a_n(n+1)\right]x^{n+2} = 0$

$$\therefore \begin{cases} a_1 = 1 \\ a_{n+1}(n+1)^2 + a_n(n+1) - \frac{(-1)^n}{n!} = 0 \Rightarrow a_{n+1} = \frac{-a_n}{n+1} + \frac{(-1)^n}{(n+1)(n+1)!} \end{cases}, n=1,2,\ldots$$

+1,0

$n=1 \Rightarrow a_2 = -\frac{a_1}{2} - \frac{1}{2.2!} = -\frac{1}{2}\left(1+\frac{1}{2}\right)$

$n=2 \Rightarrow a_3 = -\frac{a_2}{3} + \frac{(-1)^2}{3.3!} = \frac{1}{3!}\left(1+\frac{1}{2}\right) + \frac{1}{3!}\frac{1}{3} = \frac{1}{3!}\left(1+\frac{1}{2}+\frac{1}{3}\right)$

$n=3 \Rightarrow a_4 = -\frac{a_3}{4} - \frac{1}{4.4!} = -\frac{1}{4!}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{4!4} = \frac{-1}{4!}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$

de onde pode-se concluir que

$$a_n = \frac{(-1)^{n+1}}{n!}\Big(\underbrace{\sum_{k=1}^{n}\frac{1}{k}}_{H_n}\Big)$$

+1,0

$$\therefore \boxed{y_2(x) = y_1(x)\ln x + x\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n!}x^n}$$

+0,5

onde $y_1(x) = xe^{-x}$, $H_n = \sum_{k=1}^{n}\frac{1}{k}$.

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(2) $\begin{cases} x(xy')' + \lambda y = x^2y'' + xy' + \lambda y = 0 \ , \ 1<x<e^{2\pi} \\ y'(1) = y'(e^{2\pi}) = 0 \end{cases}$

$y = x^r \Rightarrow r(r-1)+r+\lambda = 0 \quad \therefore r = \pm\sqrt{-\lambda}$

(i) $\underline{\lambda<0}$: $\lambda = -k^2 \ (k>0) \quad \therefore r = \pm k$

$y = Ax^k + Bx^{-k}$, $y' = Akx^{k-1} - Bkx^{-k-1}$

$y'(1)=0 \Rightarrow A+B=0$ $\quad \therefore$ apenas solução trivial +0,5

$y'(e^{2\pi})=0 \Rightarrow A=0$

(ii) $\underline{\lambda=0}$: $y = A_1 + B_1\ln x$, $y' = \frac{B_1}{x}$

$y'(1)=0 \Rightarrow B_1=0$ $\quad \therefore y = A_1 = \text{cte.}$ +0,5

$y'(e^{2\pi}) = 0$ ✓ $\quad \lambda = 0$

(iii) $\underline{\lambda>0}$, $\lambda = k^2 \ (k>0)$ $\quad r = \pm ik$

$y = A_2x^{ik} + B_2x^{-ik} = A_2e^{ik\ln x} + B_2e^{-ik\ln x} = C_1\cos(k\ln x) + C_2\sin(k\ln x)$

$$y' = -\frac{k}{x} c_1 \sin(k \ln x) + \frac{k}{x} c_2 \cos(k \ln x)$$

$$y'(1) = 0 \Rightarrow c_2 = 0$$

$$y'(e^{2\pi}) = 0 \Rightarrow \sin(k \ln e^{2\pi}) = \sin(k 2\pi) = 0$$

$$\therefore \quad 2\pi k = n\pi \quad , n = 1, 2, \dots$$

$$\therefore \quad y_n = \cos\left(\frac{n \ln x}{2}\right) \qquad n = 1, 2, \dots$$

$$\lambda_n = n^2/4$$

+1,0

$$\therefore \boxed{(ii) + (iii) \Rightarrow \begin{cases} y_n = \cos\left(\frac{n \ln x}{2}\right) \\ \lambda_n = n^2/4 \end{cases} \quad , n = 0, 1, 2, 3, \dots}$$

$$ED \Rightarrow (x y')' + \frac{\lambda}{x} y = 0 \Rightarrow p(x) = \frac{1}{x}$$

$$\therefore \text{ortogonalidade} \Rightarrow \int_1^{e^{2\pi}} \cos\left(\frac{n \ln x}{2}\right) \cos\left(\frac{m \ln x}{2}\right) \frac{1}{x} dx = N_n \delta_{m,n}$$

onde $m, n = 0, 1, 2, \dots$

+0,5

③ $$I = \int_{-1}^{+1} (1+x)^{1/4} (1-x)^{-1/4} dx = \int_0^1 2^{1/4} t^{1/4} 2^{-1/4} (1-t)^{-1/4} \cdot 2\, dt$$

mudança de variável: $\boxed{\frac{1+x}{2} = t}$

$$= 2 \int_0^1 t^{\frac{5}{4}-1} (1-t)^{\frac{3}{4}-1} dt = 2 B\left(\frac{5}{4}, \frac{3}{4}\right)$$

+1,0

$$= \frac{2 \Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma(2)} = \frac{2 \cdot \frac{1}{4} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right)}{1 \cdot \Gamma(1)} =$$

$$= \frac{1}{2} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2 \frac{\sqrt{2}}{2}} = \frac{\pi}{\sqrt{2}}$$

+1,0

(4) $g(x,t) = \dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Derivando com relação a t:

$$\frac{\partial g(x,t)}{\partial t} = \left(-\frac{1}{2}\right)\frac{[-2x+2t]}{[1-2xt+t^2]^{3/2}} = \frac{(x-t)}{(1-2xt+t^2)}\, g(x,t)$$

+0,5

$$\therefore (x-t)\, g(x,t) = (1-2xt+t^2)\frac{\partial g(x,t)}{\partial t}$$

$$\therefore (x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x)\, t^{n-1}$$

$$\sum_{n=0}^{\infty} x P_n(x)\, t^n - \sum_{n=0}^{\infty} P_n(x)\, t^{n+1} = \sum_{n=0}^{\infty} n P_n(x)\, t^{n-1} - \sum_{n=0}^{\infty} 2nx P_n(x)\, t^n + \sum_{n=0}^{\infty} n P_n(x)\, t^{n+1}$$

$$\sum_{n=0}^{\infty} (2n+1)x P_n(x)\, t^n - \sum_{n=1}^{\infty} P_{n-1}(x)\, t^n = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x)\, t^n + \sum_{n=1}^{\infty} (n-1) P_{n-1}(x)\, t^n$$

$$x P_0(x) + \sum_{n=1}^{\infty} (2n+1)x P_n(x)\, t^n = P_1(x) + \sum_{n=1}^{\infty} (n+1) P_{n+1}(x)\, t^n + \sum_{n=1}^{\infty} n P_{n-1}(x)\, t^n$$

$$\therefore \begin{cases} x P_0(x) = P_1(x) \\ (2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \end{cases}, \quad n = 1, 2, \ldots$$

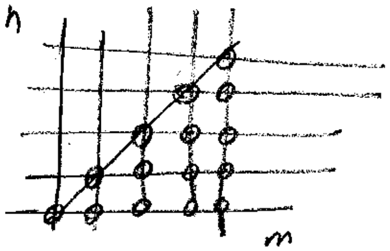
+1,5

(5) $J_0\left(\sqrt{x^2-2xt}\right) = \sum_{m=0}^{\infty} \dfrac{(-1)^m \left(\frac{1}{2}\sqrt{x^2-2xt}\right)^{2m}}{(m!)^2} =$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}(x^2-2xt)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}\sum_{n=0}^{\infty} \binom{m}{n}(x^2)^{m-n}(-2xt)^n$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}\, x^{2m-n}\, t^n}{2^{2m-n}\, m!\, n!\, (m-n)!} \quad (*)$$

+0,5



$$\sum_{m=0}^{\infty}\sum_{n=0}^{m} = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}$$

$$(*) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{m+n} x^{2m-n} t^n}{2^{2m-n} m!\, n!\, (m-n)!} \overset{\boxed{m-n=k}}{=} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+2n} x^{2(n+k)-n} t^n}{2^{2(n+k)-n} (n+k)!\, n!\, k!} =$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} k!\, (k+n)!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} J_n(x)$$ +1,5

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6 $${}_2F_1(\alpha, \alpha/2+1, \alpha/2; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\frac{\alpha}{2}+1)_n}{(\alpha/2)_n} \frac{x^n}{n!} = (*)$$

mas: $$\left(\frac{\alpha}{2}+1\right)_n = \frac{\Gamma(\frac{\alpha}{2}+1+n)}{\Gamma(\frac{\alpha}{2}+1)} = \frac{(\frac{\alpha}{2}+n)\Gamma(\frac{\alpha}{2}+n)}{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})} = \left(1+\frac{2n}{\alpha}\right)(\alpha/2)_n$$ +0,5

$$(*) = \sum_{n=0}^{\infty} (\alpha)_n \frac{(1+2n/\alpha)\cancel{(\alpha/2)_n}}{\cancel{(\alpha/2)_n}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + \frac{2}{\alpha} \sum_{n=1}^{\infty} n (\alpha)_n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + \frac{2}{\alpha} \sum_{n=0}^{\infty} (\alpha)_{n+1} \frac{x^{n+1}}{n!} = (**)$$ +0,5

mas: $(\alpha)_{n+1} = \alpha(\alpha+1)\cdots(\alpha+n) = \alpha(\alpha+1)_n$

$$(**) = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + 2x \sum_{n=0}^{\infty} (\alpha+1)_n \frac{x^n}{n!}$$

$$= (1-x)^{-\alpha} + 2x(1-x)^{-\alpha-1}$$

$$= (1-x)^{-\alpha-1}(1-x+2x) = (1+x)(1-x)^{-\alpha-1}$$ +1,0