MA 327 A/B PROVA 3 29/11/11

RA: ϕ NOME: Gabarito

1) Considere $V = M_{2\times 2}(\mathbb{R})$ munido do produto interno $\langle A, B\rangle = tr(B'A)$.

Seja $W = [A_1, A_2, A_3]$ onde $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ e $A_3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

a) (0,5) Qual é a dimensão de W ?

b) (2,0) Encontre uma base para $W^{\perp}$

a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\dim W = 2$ pois $\{A_1, A_2, A_3\}$ é LD mas $\{A_1, A_2\}$ é LI.
$(A_3 = 2A_1 - A_2)$

b) $\dim W^{\perp} = 2$ Seja $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $A \in W^{\perp} \iff$

$\langle A, A_1\rangle = \langle A, A_2\rangle = 0$

$\langle A, A_1\rangle = tr(A_1^t A) = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = tr\left(\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}\right) =$
$= a+b+c+d$

$\langle A, A_2\rangle = tr(A_2^t A) = tr\left(\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = tr\left(\begin{bmatrix} a-c & b-d \\ 2a+c & 2b+d \end{bmatrix}\right) =$
$= a+2b-c+d$

$\begin{cases} a+b+c+d = 0 & (1) \\ a+2b-c+d = 0 & (2) \end{cases} \therefore \begin{cases} 2a+3b+2d = 0 & (1)+(2) \\ b-2c = 0 & (-1)+(2) \end{cases} \Longrightarrow$

$$\begin{cases} 2a+6c+2d=0 \\ b=2c \end{cases} \Rightarrow \begin{cases} d=-a-3c \\ b=2c \end{cases}$$

$$W^{\perp}=\left\{ \begin{bmatrix} a & 2c \\ c & -a-3c \end{bmatrix} : a,c \in \mathbb{R} \right\}$$

$$= \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \right]$$

base para $W^{\perp}$: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \right\}$

2) Considere $P_2(\mathbb{R})$ munido do produto interno $\langle p,q\rangle = \int_0^1 p(x)\, q(x)\, dx$

a) (1,5) Usando o processo de Gram-Schmidt encontre uma base ortonormal para $W = [1, -x]$

b) (1,0) Se $q(x) = 1 - 2x + x^2$ determine a decomposição

$q(x) = p(x) + r(x)$ com $p(x) \in W$ e $r(x) \in W^{\perp}$

a) $u_1 = 1$

$$u_2 = -x - \frac{\langle -x, 1\rangle}{\|1\|^2} \cdot 1$$

$$\langle -x, 1\rangle = \int_0^1 -x\, dx = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$$

$$\|1\|^2 = \langle 1, 1\rangle = \int_0^1 1\, dx = x\Big|_0^1 = 1 \Rightarrow \|1\| = 1$$

$$\therefore u_2 = -x - \frac{-\frac{1}{2}}{1} \cdot 1 = \boxed{-x + \frac{1}{2}}$$

$$\|u_2\|^2 = \langle u_2, u_2\rangle = \int_0^1 \left(\frac{1}{2} - x\right)^2 dx = \int_0^1 \left(\frac{1}{4} - x + x^2\right) dx =$$

$$= \left(\frac{1}{4}x - \frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{2} + \frac{1}{3} = \frac{3 - 6 + 4}{12} = \frac{1}{12}$$

$$\therefore \|u_2\| = \frac{1}{\sqrt{12}}$$

base ortonormal p/ $W = \left\{ \overset{u_1'}{\overset{\shortparallel}{1}}, \overset{u_2'}{\overset{\shortparallel}{\sqrt{12}\left(-x + \frac{1}{2}\right)}} \right\}$

b) $p(x) = \mathrm{proj}_W\, q =$

$= \langle q, u_1' \rangle u_1' + \langle q, u_2' \rangle u_2'$

$= \langle 1-2x+x^2, 1 \rangle . 1 + \langle 1-2x+x^2, \sqrt{12}\left(\frac{1}{2}-x\right) \rangle \sqrt{12}\left(\frac{1}{2}-x\right)$

$\int_0^1 (1-2x+x^2)\,dx = \left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1 = \cancel{1} - \cancel{1} + \frac{1}{3} = \frac{1}{3}$

$\int_0^1 (1-2x+x^2)\left(\frac{1}{2}-x\right)dx = \sqrt{12}\int_0^1 \left(\frac{1}{2} - x - x + 2x^2 + \frac{x^2}{2} - x^3\right)dx =$

$= \sqrt{12}\left(\frac{1}{2}x - x^2 + \frac{5}{2}\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \sqrt{12}\left(\frac{1}{2} - 1 + \frac{5}{6} - \frac{1}{4}\right)$

$= \sqrt{12}\left(\frac{6-12+10-3}{12}\right) = \frac{\sqrt{12}}{12}$

$\therefore\ p(x) = \frac{1}{3} + \frac{\sqrt{12}}{12}\cdot\sqrt{12}\left(\frac{1}{2}-x\right) = \frac{1}{3} + \frac{1}{2} - x = \boxed{\frac{5}{6} - x}$

Daí: $r(x) = q(x) - p(x) = 1 - 2x + x^2 - \frac{5}{6} + x =$

$= \boxed{\frac{1}{6} - x + x^2}$

3) (1,5) Seja V um espaço vetorial com base $\alpha = \{v_1, v_2, v_3\}$. Seja

$\beta = \{\varphi_1, \varphi_2, \varphi_3\}$ a base dual da base α .

Se $[v]_\alpha = \begin{bmatrix} 3a \\ a \\ -2a \end{bmatrix}$ e $[\varphi]_\beta = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ encontre o valor de a para que

$\varphi(v) = -12$.

$$v = 3a v_1 + a v_2 - 2a v_3$$

$$\varphi = 2\varphi_1 - \varphi_2 + 3\varphi_3$$

$$\varphi(v) = (2\varphi_1 - \varphi_2 + 3\varphi_3)(3a v_1 + a v_2 - 2a v_3) =$$

$$= 6a\,\varphi_1(v_1) - a\,\varphi_2(v_2) - 6a\,\varphi_3(v_3) = -a$$

$$\varphi(v) = -12 \Rightarrow -a = -12 \Rightarrow a = 12$$

4) (2,0) Considere $\mathbb{R}^2$ com o produto interno usual. Encontre a expressão

de $T^*: \mathbb{R}^2 \to \mathbb{R}^2$ se $T(1,1) = (1,2)$ e $T(-1,1) = (3,0)$

$$(x,y) = a(1,1) + b(-1,1)$$

$$\begin{cases} x = a - b \\ y = a + b \end{cases} \Rightarrow \boxed{\frac{x+y}{2} = a} \qquad b = y - \frac{x+y}{2} = \boxed{\frac{y-x}{2}}$$

$$T(x,y) = \left(\frac{x+y}{2}\right)(1,2) + \left(\frac{y-x}{2}\right)(3,0) = \left(\frac{x+y}{2} + \frac{3y-3x}{2}, x+y\right)$$

$$= (-x+2y, x+y)$$

$$T(1,0) = (-1,1)$$

$$T(0,1) = (2,1)$$

$\{(1,0),(0,1)\}$ é base ortonormal em relação ao produto usual

$$[T]_{can}^{can} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$[T^*]_{can}^{can} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \boxed{T^*(x,y) = (-x+y, 2x+y)}$$

5) Sejam $T, R, S : V \to V$ lineares tais que T é unitária, R é auto-adjunta e S é anti-adjunta, isto é, $S^* = -S$.

a) (0,7) Se $R(v_0) = u_0$ e $S(u_0) = w_0$ calcular $(R \circ S)^*(v_0)$

b) (0,8) Se $S(u) = v$ e $T^{-1}(u) = w$ calcular $(4iS + T)^*(u)$

c) (1,0) Se λ é um autovalor de S mostre que $\lambda = 0$ ou λ é imaginário puro (isto é: $\lambda = bi$, $b \in \mathbb{R}$, $b \neq 0$)

a) $(R \circ S)^*(v_0) = (S^* \circ R^*)(v_0) = (-S)(R(v_0)) = (-S)(u_0) = -S(u_0) =$

$= \boxed{-w_0}$

b) $(4iS + T)^*(u) = ((-4i)(S^*) + T^*)(u) =$

$= 4i\,S(u) + T^{-1}(u) = \boxed{4iv + w}$

c) $S(v) = \lambda v \quad , v \neq 0$

$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle S(v), v \rangle = \langle v, S^*(v) \rangle = \langle v, -S(v) \rangle =$

$= -\langle v, \lambda v \rangle = -\bar{\lambda} \langle v, v \rangle$

$\therefore \lambda = -\bar{\lambda} \quad \xrightarrow{\lambda = a+bi} \quad a + bi = -(a - bi)$

$\cancel{a} + bi = -\cancel{a} + bi$

$\boxed{a = 0} \qquad b \in \mathbb{R}$ qualquer

Se $b = 0$ $\boxed{\lambda = 0}$

Se $b \neq 0$ $\boxed{\lambda = bi}$