T_6 ( ) F520 ( ) MS550 · Nome: ______________________ RA: _________

Sejam $P_n(x)$ $(n = 0, 1, 2, \ldots)$ os polinômios de Legendre. Mostre que

$$\text{(a)}\quad P_n'(1) = \frac{n(n+1)}{2},$$

$$\text{(b)}\quad P_n'(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \cdots + \Delta_n,$$

onde $\Delta_n = 3P_1(x)$ se n for par e $\Delta_n = P_0(x)$ se n for ímpar.

(a) derivando (★) $\Rightarrow \dfrac{t}{[1-2xt+t^2]^{3/2}} = \sum_{n=0}^{\infty}{}' P_n'(x)\, t^n$

Logo, para $x=1$: $\dfrac{t}{[1-2t+t^2]^{3/2}} = \dfrac{t}{(1-t)^3} = \sum_{n=0}^{\infty}{}' P_n'(1)\, t^n$

+2,0

e usando (★★):

$$\sum_{n=0}^{\infty}{}' P_n'(1)\, t^n = t\sum_{n=0}^{\infty}{}' \frac{(3)_n}{n!} t^n = \sum_{n=0}^{\infty}{}' \frac{(3)_n\, t^{n+1}}{n!} = \sum_{n=0}^{\infty}{}' \frac{(n+2)!}{2 \cdot n!} t^{n+1}$$

$$\therefore P_0'(1) + \sum_{n=0}^{\infty}{}' P_{n+1}'(1)\, t^{n+1} = \sum_{n=0}^{\infty}{}' \frac{(n+2)(n+1)}{2} t^{n+1} \Rightarrow \begin{cases} P_0'(1) = 0 \\ P_{n+1}'(1) = \dfrac{(n+1)(n+2)}{2} \end{cases}$$

$$\therefore P_n'(1) = \frac{n(n+1)}{2}, \quad n = 0, 1, 2, \ldots$$

+3,0

(b) De (★★★) p/ $n \to n-1 \Rightarrow P_n'(x) = (2n-1)P_{n-1}(x) + P_{n-2}'(x)$ (i)

De (i) p/ $n \to n-2 \Rightarrow P_{n-2}'(x) = (2n-5)P_{n-3}(x) + P_{n-4}'(x)$ (ii)

De (ii) p/ $n \to n-2 \to P_{n-4}'(x) = (2n-9)P_{n-5}(x) + P_{n-6}'(x) \ldots$ (iii)

$\cdots\cdots$

Somando: $P_n'(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \cdots$

+3,0

FORMULÁRIO EVENTUALMENTE ÚTIL

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad (n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x), \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \text{ (★)}$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x), \text{ (★★★)} \quad (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0,$$

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x), \quad P_n(-x) = (-1)^n P_n(x), \quad (1-z)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n, \text{ (★★)} \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}\,dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z).$$

Se n é par, a última relação é:

$$P_2(x) = 3P_1(x) + \underbrace{P_0'(x)}_{=0} = 3P_1(x)$$

Se n é ímpar, a última relação é:

$$P_1(x) = P_0(x) + 0 = P_0(x)$$

$$\therefore \Delta_n = \begin{cases} 3P_1(x), & n \text{ par} \\ P_0(x), & n \text{ ímpar} \end{cases}$$

+2,0