Soluções para MS550, Métodos de Matemática Aplicada I, e F520, Métodos Matemáticos da Física I

Lista 1 - Sistemas de coordenadas

Algumas expressões úteis na resolução das questões:

$$\frac{\partial \vec{r}}{\partial q_i} = \frac{\partial x}{\partial q_i}\vec{e}_x + \frac{\partial y}{\partial q_i}\vec{e}_y + \frac{\partial z}{\partial q_i}\vec{e}_z, \tag{1}$$

$$h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}, \tag{2}$$

$$\vec{e}_{q_i} = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i} \tag{3}$$

$$\vec{e}_{q_i} = \sum_j \left(\vec{e}_{q_i}\cdot\vec{e}_{p_j}\right)\vec{e}_{p_j} \tag{4}$$

$$\text{grad } f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\vec{e}_{q_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\vec{e}_{q_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\vec{e}_{q_3}, \tag{5}$$

$$\text{div } \vec{V} = \nabla\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial h_2h_3V_1}{\partial q_1} + \frac{\partial h_3h_1V_2}{\partial q_2} + \frac{\partial h_1h_2V_3}{\partial q_3}\right], \tag{6}$$

$$\text{rot } \vec{V} = \nabla\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_{q_1} & h_2\vec{e}_{q_2} & h_3\vec{e}_{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}, \tag{7}$$

$$\begin{aligned}\nabla^2 f = \frac{1}{h_1h_2h_3}&\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\right)\right. \\ &\left.+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right] f,\end{aligned} \tag{8}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}, \qquad \text{coordenadas cilíndricas} \tag{9}$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}. \qquad \text{coordenadas esféricas} \tag{10}$$

1. Seja $\vec{V} = z\vec{i} - 2x\vec{j} + y\vec{k}$. Mostre que as componentes de $\vec{V}$ em coordenadas cilíndricas circulares são dadas por

$$V_\rho = z\cos\theta - 2\rho\cos\theta\sin\theta,$$
$$V_\theta = -z\sin\theta - 2\rho\cos^2\theta,$$
$$V_z = \rho\sin\theta$$

Solução: *Usando* (9) *escrevemos* $\vec{V}$ *como*

$$\vec{V} = z\vec{i} - 2\rho\cos\theta\vec{j} + \rho\sin\theta\vec{k}.$$

Calculamos os fatores de escala utilizando (2)*:*

$$h_p = \sqrt{\cos^2\theta + \sin^2\theta} = 1,$$

$$h_\theta = \sqrt{\rho^2 \sin^2\theta + \rho^2\cos^2\theta} = \rho,$$
$$h_z = \sqrt{1} = 1$$

e com eles os vetores os vetores tangentes unitários dados por (3)*:*

$$\begin{aligned}\vec{e}_\rho &= \cos\theta\vec{j} + \sin\theta\vec{k},\\ \vec{e}_\theta &= -\sin\theta\vec{j} + \cos\theta\vec{k},\\ \vec{e}_z &= \vec{i}.\end{aligned}$$

Utilizando (4) *escrevemos* $\vec{i}$, $\vec{j}$ e $\vec{k}$ *em função de* $\vec{e}_\rho$, $\vec{e}_\theta$ e $\vec{e}_z$*:*

$$\begin{aligned}\vec{i} &= \vec{e}_z,\\ \vec{j} &= \cos\theta\vec{e}_\rho - \sin\theta\vec{e}_\theta,\\ \vec{k} &= \sin\theta\vec{e}_\rho + \cos\theta\vec{e}_\theta.\end{aligned}$$

Reescrevendo $\vec{V}$ *em função dos vetores* $\vec{e}_\rho$, $\vec{e}_\theta$ e $\vec{e}_z$ *obtemos*

$$\begin{aligned}\vec{V} &= z\vec{e}_z - 2\rho\cos\theta\left(-2\cos\theta\vec{e}_\rho + 2\sin\theta\vec{e}_\theta\right) + \rho\sin\theta\left(\sin\theta\vec{e}_\rho + \cos\theta\vec{e}_\theta\right)\\ &= \left(4\rho\cos^2\theta + \rho\sin^2\theta\right)\vec{e}_\rho + \left(-2\rho\sin\theta\cos\theta + \rho\sin\theta\cos\theta\right)\vec{e}_\theta + z\vec{e}_z\\ &= \left(3\rho\cos^2\theta + \rho\right)\vec{e}_\rho - \rho\sin\theta\cos\theta\vec{e}_\theta + z\vec{e}_z.\end{aligned}$$

2. Seja o campo vetorial
$$\vec{V} = V_\rho(\rho,\theta)\vec{e}_\rho + V_\theta(\rho,\theta)\vec{e}_\theta.$$
Mostre que $\nabla\times\vec{V} = \text{rot }\vec{V}$ possui componente apenas na direção z.

Solução: *Utilizando* (7) *temos*

$$\begin{aligned}\nabla\times\vec{V} &= \begin{vmatrix}\vec{e}_\rho & \vec{e}_\theta & \vec{e}_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ V_\rho & V_\theta & 0\end{vmatrix}\\ &= -\frac{\partial V_\theta}{\partial z}\vec{e}_\rho + \frac{\partial V_\rho}{\partial z}\vec{e}_\theta + \left(\frac{\partial V_\theta}{\partial p} - \frac{\partial V_\rho}{\partial\theta}\right)\vec{e}_z\\ &= \left(\frac{\partial V_\theta}{\partial p} - \frac{\partial V_\rho}{\partial\theta}\right)\vec{e}_z.\end{aligned}$$

3. Seja o campo vetorial $\vec{V} = \rho\vec{e}_z$. Mostre que
$$\nabla\times\vec{V} = -\vec{e}_\theta, \qquad \nabla\times(\vec{V}\times(\nabla\times\vec{V})) = 0.$$

Solução: *Para mostrar que* $\nabla \times \vec{V} = -\vec{e}_\theta$ *utilizamos* (7)*:*

$$\begin{aligned}\nabla \times \vec{V} &= \begin{vmatrix} \vec{e}_\rho & \vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \rho \end{vmatrix} \\ &= \frac{\partial\rho}{\partial\theta}\vec{e}_\rho - \frac{\partial\rho}{\partial\rho}\vec{e}_\theta \\ &= -\vec{e}_\theta.\end{aligned}$$

E para mostrar $\nabla \times \left(\vec{V} \times \left(\nabla \times \vec{V}\right)\right) = 0$ *começamos utilizando o resultado anterior, i.e.,* $\nabla \times \vec{V} = -\vec{e}_\rho$*:*

$$\begin{aligned}\vec{V} \times \left(\nabla \times \vec{V}\right) &= \vec{V} \times (-\vec{e}_\theta) \\ &= \rho\vec{e}_z \times (-\vec{e}_\theta) \\ &= 0 \\ \nabla \times \left(\vec{V} \times \left(\nabla \times \vec{V}\right)\right) &= \nabla \times 0 \\ &= 0.\end{aligned}$$

4. Calcule $\frac{\partial \vec{e}_{q_i}}{\partial q_j}$ com $i, j = 1, 2, 3$ quando q_i são as coordenadas cilíndricas e mostre que as únicas derivadas não nulas são

$$\begin{aligned}\frac{\partial \vec{e}_\rho}{\partial\theta} &= \vec{e}_\theta, \\ \frac{\partial \vec{e}_\theta}{\partial\theta} &= -\vec{e}\rho.\end{aligned}$$

Solução: *Utilizando* (9) *e* (2) *temos*

$$\begin{aligned}h_\rho &= \sqrt{\cos^2\theta + \sin^2\theta} = 1, \\ h_\theta &= \sqrt{\rho^2\sin^2\theta + \rho\cos^2\theta} = \rho, \\ h_z &= \sqrt{1} = 1.\end{aligned}$$

Calculando os vetores tangentes unitários temos

$$\begin{aligned}\vec{e}_\rho &= \cos\theta\vec{i} + \sin\theta\vec{j}, \\ \vec{e}_\theta &= \frac{1}{\rho}\left(-\rho\sin\theta\vec{i} + \rho\cos\theta\vec{j}\right) = -\sin\theta\vec{i} + \cos\theta\vec{j}, \\ \vec{e}_z &= \vec{k}.\end{aligned}$$

Então

$$\frac{\partial \vec{e}_\rho}{\partial \rho} = 0, \qquad \frac{\partial \vec{e}_\rho}{\partial \theta} = -\sin\theta\vec{i} + \cos\theta\vec{j} = \vec{e}_\theta, \qquad \frac{\partial \vec{e}_\rho}{\partial z} = 0,$$
$$\frac{\partial \vec{e}_\theta}{\partial \rho} = 0, \qquad \frac{\partial \vec{e}_\theta}{\partial \theta} = -\cos\theta\vec{i} - \sin\theta\vec{j} = -\vec{e}_\rho, \qquad \frac{\partial \vec{e}_\theta}{\partial z} = 0,$$
$$\frac{\partial \vec{e}_z}{\partial \rho} = 0, \qquad \frac{\partial \vec{e}_z}{\partial \theta} = 0, \qquad \frac{\partial \vec{e}_z}{\partial z} = 0.$$

5. Sejam as coordenadas esferoidais achatadas (ξ, η, ϕ) dadas por

$$\begin{aligned} x &= a\cosh\xi\cos\eta\cos\phi, \\ y &= a\cosh\xi\cos\eta\sin\phi, \\ z &= a\sinh\eta\sin\eta, \end{aligned}$$

onde $\xi \geq 0$, $-\pi/2 \leq \eta \leq \pi/2$, $0 \leq \phi \leq 2\pi$.
Mostre que os fatores de escala são dados por

$$\begin{aligned} h_\xi = h\eta &= a\sqrt{\sinh^2\xi + \sin^2\eta}, \\ h_\phi &= a\cosh\xi\cos\eta. \end{aligned}$$

Solução: *Utilizando* (2) *vamos calcular os fatores de escala:*

$$\begin{aligned} h_\xi &= \sqrt{(a\sinh\xi\cos\eta\cos\phi)^2 + (a\sinh\xi\cos\eta\sin\phi)^2 + (a\cosh\xi\sin\eta)^2} \\ &= a\sqrt{\sinh^2\xi\cos^2\eta\left(\cos^2\phi + \sin^2\phi\right) + \cosh^2\xi\sin^2\eta} \\ &= a\sqrt{\sinh^2\xi\left(\cos^2\eta + \sin^2\eta\right) + \sin^2\eta} \\ &= a\sqrt{\sinh^2\xi + \sin^2\eta}, \\ h_\eta &= \sqrt{(a\cosh\xi\sin\eta\cos\phi)^2 + (a\cosh\xi\sin\eta\sin\phi)^2 + (a\sinh\xi\cos\eta)^2} \\ &= a\sqrt{\cosh^2\xi\sin^2\eta\left(\cos^2\phi + \sin^2\phi\right) + \sinh^2\xi\cos^2\eta} \\ &= a\sqrt{\left(1 + \sinh^2\xi\right)\sin^2\eta + \sinh^2\cos^2\eta} \\ &= a\sqrt{\sin^2\eta + \sinh^2\xi}, \\ h_\phi &= \sqrt{(a\cosh\xi\cos\eta\sin\phi)^2 + (a\cosh\xi\cos\eta\cos\phi)^2 + 0} \\ &= a\sqrt{\cosh^2\xi\cos^2\eta\left(\sin^2\phi + \cos^2\phi\right)} \\ &= a\cosh\xi\cos\eta. \end{aligned}$$

6. Sejam (u, v, z) as coordenadas cilíndricas parabólicas, definidas como

$$x = \frac{1}{2}(u^2 - v^2),\ y = uv,\ z = z,$$

com $-\infty < u < +\infty$, $v \geq 0$, $-\infty < z < +\infty$, e $\vec{r}$ o vetor posição, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Mostre que em coordenadas cilíndricas parabólicas temos

$$\vec{r} = \frac{1}{2}\sqrt{u^2 + v^2}(u\vec{e}_u + v\vec{e}_v) + z\vec{e}_z.$$

Usando essas coordenadas, mostre que div $\vec{r} = 3$.

Solução: *Sendo*

$$\vec{r} = \frac{1}{2}\left(u^2 - v^2\right)\vec{i} + uv\vec{j} + z\vec{k}$$

começamos utilizando (2) *para calcular os fatores de escala:*

$$\begin{aligned} h_u &= \sqrt{u^2 + v^2}, \\ h_v &= \sqrt{u^2 + v^2}, \\ h_z &= 1. \end{aligned}$$

Em seguida utilizamos (1) *para calcular os vetores tangentes:*

$$\begin{aligned} \vec{e}_u &= \frac{1}{\sqrt{u^2 + v^2}}\left(u\vec{i} + v\vec{j}\right), \\ \vec{e}_v &= \frac{1}{\sqrt{u^2 + v^2}}\left(-v\vec{i} + u\vec{j}\right), \\ \vec{e}_z &= \vec{k}. \end{aligned}$$

Utilizando (4), *escrevemos $\vec{i}$, $\vec{j}$, $\vec{k}$ em função de $\vec{u}$, $\vec{v}$* e *$\vec{z}$:*

$$\begin{aligned} \vec{i} &= \frac{1}{\sqrt{u^2 + v^2}}\left(u\vec{e}_u - v\vec{e}_v\right), \\ \vec{j} &= \frac{1}{\sqrt{u^2 + v^2}}\left(v\vec{e}_u + u\vec{e}_v\right), \\ \vec{k} &= \vec{e}_z. \end{aligned}$$

Substituindo $\vec{i}$, $\vec{j}$ e *$\vec{k}$ em $\vec{r}$ temos*

$$\begin{aligned} \vec{r} &= \frac{u^2 - v^2}{2\sqrt{u^2 + v^2}}\left(u\vec{e}_u - v\vec{e}_v\right) + \frac{uv}{\sqrt{u^2 + v^2}}\left(v\vec{e}_u + u\vec{e}_v\right) + z\vec{e}_z \\ &= \frac{1}{2\sqrt{u^2 + v^2}}\left(u^3\vec{e}_u - u^2v\vec{e}_v - uv^2\vec{e}_u + v^3\vec{e}_v + 2uv^2\vec{e}_u + 2u^2v\vec{e}_v\right) + z\vec{e}_z \\ &= \frac{1}{2\sqrt{u^2 + v^2}}\left(u^3\vec{e}_u + u^2v\vec{e}_v + uv^2\vec{e}_u + v^3\vec{e}_v\right) + z\vec{e}_z \\ &= \frac{1}{2\sqrt{u^2 + v^2}}\left(u^2\left(u\vec{e}_u + v\vec{e}_v\right) + v^2\left(u\vec{e}_u + v\vec{e}_v\right)\right) + z\vec{e}_z \end{aligned}$$

$$
\begin{aligned}
&= \frac{u^2+v^2}{2\sqrt{u^2+v^2}}\left(u\vec{e}_u + v\vec{e}_v\right) + z\vec{e}_z \\
&= \frac{\sqrt{u^2+v^2}}{2}\left(u\vec{e}_u + v\vec{e}_v\right) + z\vec{e}_z
\end{aligned}
$$

Por último, utilizando (6) temos

$$
\begin{aligned}
\text{div } \vec{r} &= \frac{1}{h_u h_v h_z}\left[\frac{\partial h_v h_z V_u}{\partial u} + \frac{\partial h_z h_u V_v}{\partial v} + \frac{\partial h_u h_v V_z}{\partial q_z}\right] \\
&= \frac{1}{u^2+v^2}\left(\frac{\partial\left(u^2+v^2\right) u\ 2}{\partial u} + \frac{\partial\left(u^2+v^2\right) v\ 2}{\partial v} + \frac{\partial\left(u^2+v^2\right) z}{\partial z}\right) \\
&= \frac{1}{u^2+v^2}\left(\frac{2u^2+u^2+v^2}{2} + \frac{2v^2+u^2+v^2}{2} + \left(u^2+v^2\right)\right) \\
&= \frac{1}{u^2+v^2}\left(\frac{6\left(u^2+v^2\right)}{2}\right) \\
&= 3
\end{aligned}
$$

7. Sejam (u, v, z) as coordenadas cilíndricas parabólicas e o campo vetorial

$$
\vec{V} = \frac{1}{4}\sqrt{u^2+v^2}(-v\vec{e}_u + u\vec{e}_v).
$$

Mostre que rot $\vec{V} = \vec{e}_z$.

Solução: *Para utilizar (7) precisamos do fator de escala e das componentes dos vetores tangentes. Primeira vamos calcular os fatores de escala utilizando (2):*

$$
\begin{aligned}
h_u &= \sqrt{u^2+v^2}, \\
h_v &= \sqrt{v^2+u^2}, \\
h_z &= 1.
\end{aligned}
$$

Para calcular as componentes dos vetores tangentes note que

$$
V_i = \vec{V}\vec{e}_i.
$$

Logo,

$$
\begin{aligned}
V_u &= -\frac{v}{4}\sqrt{u^2+v^2}, \\
V_v &= \frac{u}{4}\sqrt{u^2+v^2}, \\
V_z &= 0.
\end{aligned}
$$

Por fim, calculando rot $\vec{V}$ temos

$$
\begin{aligned}
rot\ \vec{V} &= \frac{1}{u^2+v^2}\begin{vmatrix} \sqrt{u^2+v^2}\vec{e}_u & \sqrt{u^2+v^2}\vec{e}_v & 1\vec{e}_z \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ -\frac{v}{4}\left(u^2+v^2\right)) & \frac{u}{4}\left(u^2+v^2\right)) & 0 \end{vmatrix} \\
&= \frac{1}{u^2+v^2}\left(\vec{e}_z\left(\frac{1}{4}\left(3u^2+v^2\right)\right)-\vec{e}_z\left(-\frac{1}{4}\left(u^2+3v^2\right)\right)\right) \\
&= \frac{1}{u^2+v^2}\left(\frac{\vec{e}_z}{4}\left(3u^2+v^2+u^2+3v^2\right)\right) \\
&= \frac{1}{u^2+v^2}\left(\frac{\vec{e}_z}{4}\left(4u^2+4v^2\right)\right) \\
&= \vec{e}_z.
\end{aligned}
$$

8. Sejam as coordenadas (u, v, z) definidas como

$$x = \frac{1}{2}(u^2+v^2),\ y = uv,\ z = z.$$

Mostre que esse sistema de coordenadas não é ortogonal.

Solução: *Mostrar que esse sistema de coordendas não é ortogonal equivale a mostrar que existe pelo menos um par (i, j), $i, j = 1, 2, 3$, tal que $\vec{e}_{q_i} \cdot \vec{e}_{q_j} \neq 0$.*

Começamos calculando os vetores tangentes

$$
\begin{aligned}
\frac{\partial \vec{r}}{\partial u} &= u\vec{i} + v\vec{j}, \\
\frac{\partial \vec{r}}{\partial v} &= v\vec{i} + u\vec{j}, \\
\frac{\partial \vec{r}}{\partial z} &= \vec{k}.
\end{aligned}
$$

Agora tentamos descobrir um par (i, j), $i, j = 1, 2, 3$, tal que $\vec{e}_{q_i} \cdot \vec{e}_{q_j} \neq 0$:

$$\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = uv + uv = 2uv \neq 0.$$

9. Calcule $\frac{\partial \vec{e}_{q_i}}{\partial q_j}$ com $i, j = 1, 2, 3$ quando q_i são as coordenadas esféricas e mostre que as únicas derivadas não nulas são

$$
\frac{\partial \vec{e}_r}{\partial \phi} = \sin\theta\vec{e}_\phi, \qquad \frac{\partial \vec{e}_\phi}{\partial \phi} = -\cos\theta\vec{e}_\theta - \sin\theta\vec{e}_r,
$$

$$
\frac{\partial \vec{e}_\theta}{\partial \phi} = \cos\theta\vec{e}_\phi, \qquad \frac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta, \qquad \frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r.
$$

Solução: *Seja* $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ *e* x, y *e* z *dados por* (10). *Primeiramente vamos calcular os fatores de escala dados por* (2)*:*

$$
\begin{aligned}
h_r &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = \sqrt{\sin^2 + \cos^2\theta} = 1,\\
h_\theta &= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin\phi + r^2\sin^2\theta} = \sqrt{r} = r,\\
h_\phi &= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta.
\end{aligned}
$$

Depois calculamos os vetores tangentes unitários dados por (3)*:*

$$
\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k},\\
\vec{e}_\theta &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k},\\
\vec{e}_\phi &= -\sin\phi\vec{i} + \cos\phi\vec{j}.
\end{aligned}
$$

Agora podemos calcular as derivadas dos vetores tangentes unitários:

$$
\begin{aligned}
\frac{\partial\vec{e}_r}{\partial\theta} &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k} = \vec{e}_\theta, & \frac{\partial\vec{e}_r}{\partial r} &= 0,\\
\frac{\partial\vec{e}_r}{\partial\phi} &= -\sin\theta\sin\phi\vec{i} + \sin\theta\cos\phi\vec{j} = \sin\theta\vec{e}_\phi, & \frac{\partial\vec{e}_\theta}{\partial r} &= 0\\
\frac{\partial\vec{e}_\theta}{\partial\theta} &= -\sin\theta\cos\phi\vec{i} - \sin\theta\sin\phi\vec{j} - \cos\theta\vec{k} = -\vec{e}_r, & \frac{\partial\vec{e}_\phi}{\partial r} &= 0,\\
\frac{\partial\vec{e}_\theta}{\partial\phi} &= -\cos\theta\sin\phi\vec{i} + \cos\theta\cos\phi\vec{j} = \cos\theta\vec{e}_\phi, & \frac{\partial\vec{e}_\phi}{\partial\theta} &= 0,\\
\frac{\partial\vec{e}_\phi}{\partial\phi} &= -\cos\phi\vec{i} - \sin\phi\vec{j} = -\cos\theta\vec{e}_\theta - \sin\theta\vec{e}_r.
\end{aligned}
$$

10. Seja o campo vetorial

$$\vec{V} = \frac{yz}{r(x^2+y^2)}\vec{i} - \frac{xz}{r(x^2+y^2)}\vec{j},$$

onde $r = \sqrt{x^2+y^2+z^2}$.

(a) Usando coordenadas cartesianas, mostre que

$$\nabla \times \vec{V} = \frac{\vec{r}}{r^3}.$$

Solução: *Utilizando* (7) *temos*

$$
\begin{aligned}
\nabla \times \vec{V} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{r(x^2+y^2)} & -\frac{xz}{r(x^2+y^2)} & 0 \end{vmatrix}\\
&= \frac{\partial\frac{yz}{r(x^2+y^2)}}{\partial z}\vec{j} + \frac{\partial\frac{xz}{r(x^2+y^2)}}{\partial x}\vec{k} - \frac{\partial\frac{yz}{r(x^2+y^2)}}{\partial y}\vec{k} - \frac{\partial\frac{xz}{r(x^2+y^2)}}{\partial z}\vec{i}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{-xz^2}{r^3\left(x^2+y^2\right)} + \frac{x}{r\left(x^2+y^2\right)}\right)\vec{i} + \left(\frac{-yz^2}{r^3\left(x^2+y^2\right)} + \frac{y}{r\left(x^2+y^2\right)}\right)\vec{j} \\
&\quad + \left(xz\left(\frac{-2x}{r\left(x^2+y^2\right)^3} - \frac{x}{r^3\left(x^2+y^2\right)}\right) + \frac{z}{r}\frac{1}{x^2+y^2}\right. \\
&\quad \left.-\left(yz\left(\frac{-2y}{r\left(x^2+y^2\right)^3} - \frac{y}{r^3\left(x^2+y^2\right)}\right) + \frac{z}{r}\frac{1}{x^2+y^2}\right)\right)\vec{k} \\
&= \frac{1}{r^3}\left(\left(\frac{-xz^2 + x\left(x^2+y^2+z^2\right)}{x^2+y^2}\right)\vec{i} + \left(\frac{-yz^2 + y\left(x^2+y^2+z^2\right)}{x^2+y^2}\right)\vec{j}\right. \\
&\quad \left. + \left(\frac{-2x^2z\left(x^2+y^2+z^2\right) - x\left(x^2+y^2\right)}{\left(x^2+y^2\right)^3} - \left(\frac{2y^2z\left(x^2+y^2+z^2\right) - y\left(x^2+y^2\right)^2}{\left(x^2+y^2\right)^3}\right)\right)\vec{k}\right) \\
&= \frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \\
&= \frac{\vec{r}}{r^3}.
\end{aligned}
$$

(b) Mostre que em termos de coordenadas esféricas

$$\vec{V} = -\frac{\cot\theta}{r}\vec{e}_\phi.$$

Solução: *Utilizando* (10), *escrevemos* $r = x\vec{i} + y\vec{j} + z\vec{k}$ *em coordenadas esféricas:*

$$\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\sin\phi\vec{j} + r\cos\theta\vec{k}.$$

Calculando o fator de escala utilizando (2) *temos*

$$
\begin{aligned}
h_r &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1, \\
h_\theta &= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta} = r, \\
h_\phi &= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta.
\end{aligned}
$$

E calculando os vetores tangentes unitários utilizando (3) *temos*

$$
\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\vec{k}, \\
\vec{e}_\theta &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \\
\vec{e}_\phi &= -\sin\phi\vec{i} + \cos\phi\vec{j}.
\end{aligned}
$$

Agora, por meio de (4), *escrevemos* $\vec{i}$, $\vec{j}$ e $\vec{k}$ *em função de* $\vec{e}_r$, $\vec{e}_\theta$ e $\vec{e}_\phi$*:*

$$
\begin{aligned}
\vec{i} &= \sin\theta\cos\phi\vec{e}_r + \cos\theta\cos\phi\vec{e}_\theta - \sin\vec{e}_\phi, \\
\vec{j} &= \sin\theta\sin\phi\vec{e}_r + \cos\theta\sin\phi\vec{e}_\theta + \cos\phi\vec{e}_\phi, \\
\vec{k} &= \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta.
\end{aligned}
$$

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r.gaia.cs@gmail.com

Por fim, substituimos x, y, z, $\vec{i}$, $\vec{j}$ e $\vec{k}$ em $\vec{V}$:

$$\begin{aligned}
\vec{V} &= \frac{(r\sin\theta\sin\phi)\,r\cos\theta}{r^3\sin\theta}\left(\sin\theta\cos\phi\vec{e}_r+\cos\theta\cos\phi\vec{e}_\theta-\sin\phi\vec{e}_\phi\right)\\
&\quad-\frac{(r\sin\theta\cos\phi)\,r\cos\theta\left(\sin\theta\sin\phi\vec{e}_r+\cos\theta\sin\phi\vec{e}_\theta+\cos\phi\vec{e}_\phi\right)}{r^3\sin\theta}\\
&=\frac{1}{r\sin\theta}\left(\vec{e}_r\left(\sin\phi\cos\theta\sin\theta\cos\phi-\sin\theta\cos\phi\sin\phi\cos\theta\right)\right.\\
&\quad\left.+\vec{e}_\theta\left(\sin\phi\cos^2\theta\cos\phi-\cos^2\theta\sin\phi\cos\phi\right)+\vec{e}_\phi\left(-\sin^2\phi\cos\theta-\cos^2\phi\cos\theta\right)\right)\\
&=-\frac{\cot\theta}{r}\vec{e}_\phi.
\end{aligned}$$

(c) Obtenha o resultado acima para $\nabla\times\vec{V}$ usando coordenadas esféricas.

Solução: *Utilizando (7) temos que*

$$\begin{aligned}
rot\ \vec{V} &= \frac{1}{r^2\sin\theta}\begin{bmatrix}\vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & r\sin\theta\frac{-\cot\theta}{r}\end{bmatrix}\\
&=\frac{1}{r^2\sin\theta}\left(\frac{\partial-\cos\theta}{\partial\theta}\vec{e}_r+\frac{\partial\cos\theta}{\partial r}r\vec{e}_\theta\right)\\
&=\frac{\vec{e}_r}{r^2}\\
&=\frac{\vec{v}}{r^3}.
\end{aligned}$$

11. Seja $f(r)$ uma função de $r=\sqrt{x^2+y^2+z^2}$. Mostre que

$$\begin{aligned}
\nabla f(r) &= f'(r)\vec{e}_r,\\
\nabla\cdot(\vec{e}_r f(r)) &= \frac{2}{r}f(r)+f'(r),\\
\nabla\times(\vec{e}_r f(r)) &= 0
\end{aligned}$$

usando:

(a) coordenadas cartesianas,

Solução:

(b) coordenads esféricas.

Solução:

12. Sejam

$$\begin{aligned}
\nabla &= \vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi},\\
\vec{V} &= V_r\vec{e}_r+V_\theta\vec{e}_\theta+V_\phi\vec{e}_\phi,
\end{aligned}$$

e $\vec{V} \cdot \nabla$ o operador dado por

$$\vec{V} \cdot \nabla = V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + \frac{V_\phi}{r \sin\theta}\frac{\partial}{\partial \phi}.$$

Mostre que $(\vec{V} \cdot \nabla) r = \vec{V}$.

Solução: *Utilizando* (10), *escrevemos* $r = x\vec{i} + y\vec{j} + z\vec{k}$ *em coordenadas esféricas:*

$$\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\sin\phi\vec{j} + r\cos\theta\vec{k}.$$

Calculando o fator de escala utilizando (2) *temos*

$$\begin{aligned}
h_r &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1,\\
h_\theta &= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta} = r,\\
h_\phi &= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta.
\end{aligned}$$

E calculando os vetores tangentes unitários utilizando (3) *temos*

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\vec{k},\\
\vec{e}_\theta &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k},\\
\vec{e}_\phi &= -\sin\phi\vec{i} + \cos\phi\vec{j}.
\end{aligned}$$

Agora, por meio de (4), *escrevemos* $\vec{i}$, $\vec{j}$ e $\vec{k}$ *em função de* $\vec{e}_r$, $\vec{e}_\theta$ e $\vec{e}_\phi$*:*

$$\begin{aligned}
\vec{i} &= \sin\theta\cos\phi\vec{e}_r + \cos\theta\cos\phi\vec{e}_\theta - \sin\vec{e}_\phi,\\
\vec{j} &= \sin\theta\sin\phi\vec{e}_r + \cos\theta\sin\phi\vec{e}_\theta + \cos\phi\vec{e}_\phi,\\
\vec{k} &= \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta.
\end{aligned}$$

Substituindo x, y, z, $\vec{i}$, $\vec{j}$ e $\vec{k}$ *em* $\vec{r}$ *temos*

$$\begin{aligned}
\vec{r} = {}& r\sin^2\theta\cos^2\phi\vec{e}_r + r\sin\theta\cos^2\phi\cos\theta\vec{e}_\theta - r\sin^2\theta\cos\phi\vec{e}_\phi\\
&+ r\sin^2\theta\sin^2\phi\vec{e}_r + r\sin\theta\sin^2\phi\cos\theta\vec{e}_\theta\\
&+ r\sin\theta\cos\phi\sin\phi\vec{e}_\phi + r\cos^2\theta\vec{e}_r - r\sin\theta\cos\theta\vec{e}_\theta\\
= {}& \left(r\sin^2\theta + r\cos^2\theta\right)\vec{e}_r + (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\vec{e}_\theta\\
= {}& r\vec{e}_r.
\end{aligned}$$

Por fim,

$$\begin{aligned}
\left(\vec{V} \cdot \nabla\right)\vec{r} &= \left(V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) r\vec{e}_r\\
&= V_r\vec{e}_r + \frac{V_\theta}{r}\frac{\partial r\vec{e}_r}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial r\vec{e}_r}{\partial\phi}\\
&= V_r\vec{e}_r + V_\theta\vec{e}_\theta + \frac{V_\phi\sin\theta\vec{e}_\phi}{\sin\theta}\\
&= V_r\vec{e}_r + V_\theta\vec{e}_\theta + V_\phi\vec{e}_\phi.
\end{aligned}$$

13. (T1 de 2011) Sejam (u, v, z) as coordenadas cilíndricas parabólicas, definidas como

$$x = (1/2)(u^2 - v^2), \qquad y = uv, \qquad z = z,$$

com $-\infty < u < +\infty$, $v \geq 0$, $-\infty < z < +\infty$.

(a) Mostre que esse sistema de coordenadas é ortogonal.

Solução: *Temos que*

$$r = (1/2)(u^2 - v^2)\vec{i} + uv\vec{j} + z\vec{k}$$

e portanto

$$\frac{\partial \vec{r}}{\partial u} = u\vec{i} + v\vec{j},$$
$$\frac{\partial \vec{r}}{\partial v} = -v\vec{i} + v\vec{j},$$
$$\frac{\partial \vec{r}}{\partial v} = k.$$

Logo,

$$h_u = \sqrt{u^2 + v^2},$$
$$h_v = \sqrt{u^2 + v^2},$$
$$h_z = 1$$

e

$$\vec{e}_u = (u\vec{i} + v\vec{j})/\sqrt{u^2 + v^2},$$
$$\vec{e}_v = (-v\vec{i} + u\vec{j})/\sqrt{u^2 + v^2},$$
$$\vec{e}_z = \vec{k}.$$

Então

$$\vec{e}_u\vec{e}_v = -uv + vu = 0,$$
$$\vec{e}_u\vec{e}_z = 0,$$
$$\vec{e}_v\vec{e}_z = 0$$

e assim concluimos que o sistema de coordenadas é ortogonal.

(b) Seja r o vetor posição, $r = x\vec{i} + y\vec{j} + z\vec{k}$. Mostre que em coordenadas cilíndricas parabólicas temos

$$r = (1/2)\sqrt{u^2 + v^2}(u\vec{e}_u + v\vec{e}_v) + z\vec{e}_z.$$

Solução: *Utilizando* $i(4)$ *temos que*

$$\begin{aligned}
\vec{i} &= \left(\vec{i}\cdot\vec{e}_u\right)\vec{e}_u + \left(\vec{i}\cdot\vec{e}_v\right)\vec{e}v + \left(\vec{i}\cdot\vec{e}_z\right)\vec{e}_z \\
&= \frac{u}{\sqrt{u^2+v^2}}\vec{e}_u - \frac{u}{\sqrt{u^2+v^2}}\vec{e}_v, \\
\vec{j} &= \left(\vec{j}\cdot\vec{e}_u\right)\vec{e}_u + \left(\vec{j}\cdot\vec{e}_v\right)\vec{e}_v + \left(\vec{j}+\vec{e}_z\right)\vec{e}_z \\
&= \frac{v}{\sqrt{u^2+v^2}}\vec{e}_u + \frac{u}{\sqrt{u^2+v^2}}\vec{e}_v, \\
\vec{k} &= \vec{e}_z.
\end{aligned}$$

Portanto,

$$\begin{aligned}
\vec{r} &= \left((u^2-v^2)/2\right)\left(\frac{u}{\sqrt{u^2+v^2}}\vec{e}_u - \frac{v}{\sqrt{u^2+v^2}}\vec{e}_v\right) \\
&\quad + uv\left(\frac{v}{\sqrt{u^2+v^2}}\vec{e}_u + \frac{u}{\sqrt{u^2+v^2}}\vec{e}_v\right) + z\vec{e}_z \\
&= \frac{u(u^2-v^2}{2\sqrt{u^2+v^2}}\vec{e}_u - \frac{v(u^2-v^2}{2\sqrt{u^2+v^2}}\vec{e}_v + \frac{uv^2}{\sqrt{u^2+v^2}}\vec{e}_u + \frac{u^2v}{\sqrt{u^2+v^2}}\vec{e}_v + z\vec{e}_z \\
&= \frac{u(u^2+v^2)}{2\sqrt{u^2+v^2}}\vec{e}_u + \frac{u(u^2+v^2)}{2\sqrt{u^2+v^2}}\vec{e}_v + z\vec{e}_z \\
&= \frac{\sqrt{u^2+v^2}}{2}(u\vec{e}_u + v\vec{e}_v) + z\vec{e}_z.
\end{aligned}$$

(c) Usando as coordenas, e sabendo que em um sistema de coordenadas curvilíneas ortogonal temos

$$\nabla\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3V_1) + \frac{\partial}{\partial q_2}(h_3h_1V_2) + \frac{\partial}{\partial q_3}(h_1h_2V_3)\right],$$

mostre que $\nabla\cdot\vec{r} = 3$.

Solução: *Pelo item anterior temos que* $r_u = (1/2)u\sqrt{u^2+v^2}$, $r_v = (1/2)v\sqrt{u^2+v^2}$ *e* $r_z = z$. *Logo,*

$$\begin{aligned}
\nabla\cdot\vec{r} &= \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\left(\frac{(u^2+v^2)u}{2}\right) + \frac{\partial}{\partial v}\left(\frac{(u^2+v^2)v}{2} + \frac{\partial}{\partial z}\left((u^2+v^2)z\right)\right)\right] \\
&= \frac{1}{u^2+v^2}\left[(3/2)u^2 + (1/2)v^2 + (1/2)u^2 + (3/2)v^2 + u^2 + v^2\right] \\
&= \frac{1}{u^2+v^2}\frac{6}{2}(v^2+u^2) = 3.
\end{aligned}$$

14. (P1 de 2011) (a) Sejam as coordenadas esféricas (r,θ,ϕ) dadas por

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\cos\phi, \qquad z = r\cos\theta,$$

onde $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$ e $0 \leq \phi \leq 2\pi$. Mostre que os vetores tangentes unitários $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\}$ são dados por

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \\
\vec{e}_\theta &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \\
\vec{e}_\phi &= -\sin\phi\vec{i} + \cos\phi\vec{j}.
\end{aligned}$$

onde $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$ são os vetores unitários cartesianos tais que $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solução: *Temos que* $\vec{r} = r\sin\theta\cos\phi\vec{i} + r\sin\theta\cos\phi\vec{j} + r\cos\theta\vec{k}$ *então utilizando* (1) *obtemos*

$$\begin{aligned}
\frac{\partial\vec{r}}{\partial r} &= \sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k}, \\
\frac{\partial\vec{r}}{\partial\theta} &= r\cos\theta\cos\phi\vec{i} + r\cos\theta\sin\phi\vec{j} - r\sin\theta\vec{k}, \\
\frac{\partial\vec{r}}{\partial\phi} &= -r\sin\theta\sin\phi\vec{i} + r\sin\theta\cos\phi\vec{j}.
\end{aligned}$$

Utilizando (2) *obtemos*

$$\begin{aligned}
h_r^2 &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \\
h_\theta^2 &= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2, \\
h_\phi^2 &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta.
\end{aligned}$$

Utilizando (3) *obtemos*

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{i} + \sin\theta\cos\phi\vec{j} + \cos\theta\vec{k}, \\
\vec{e}_\theta &= \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}, \\
\vec{e}_\phi &= -\sin\phi\vec{i} + \cos\phi\vec{j}.
\end{aligned}$$

(b) Sejam ∇ e $\vec{V}$ dados por

$$\nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \qquad \vec{V} = V_r\vec{e}_r + V_\theta\vec{e}_\theta + V_\phi\vec{e}_\phi,$$

e $\vec{V}\cdot\nabla$ o operador dado por

$$\vec{V}\cdot\nabla = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

Mostre que $(\vec{V}\cdot\nabla)\vec{r} = \vec{V}$.

Solução: *Tomemos que* $\vec{r} = r\vec{e}_r$ *e*

$$(\vec{V}\cdot\nabla)\vec{r} = V_r\frac{\partial r\vec{e}_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial r\vec{e}_r}{\partial\theta} + \frac{V\phi}{r\sin\theta}\frac{\partial r\vec{e}_r}{\partial\phi}.$$

Mas

$$
\begin{aligned}
\frac{\partial r\vec{e}_r}{\partial r} &= \left(\frac{\partial r}{\partial r}\right)\vec{e}_r + r\left(\frac{\partial \vec{e}_r}{\partial r}\right) && = \vec{e}_r, \\
\frac{\partial r\vec{e}_r}{\partial \theta} &= \left(\frac{\partial r}{\partial \theta}\right)\vec{e}_r + r\left(\frac{\partial \vec{e}_r}{\partial \theta}\right) \\
&= r\left(\cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}\right) = r\vec{e}_\theta, \\
\frac{\partial r\vec{e}_r}{\partial \phi} &= \left(\frac{\partial r}{\partial \phi}\right)\vec{e}_r + r\left(\frac{\partial \vec{e}_r}{\partial \phi}\right) \\
&= r\left(-\sin\theta\sin\phi\vec{i} + \sin\theta\cos\phi\vec{j}\right) = r\sin\theta\vec{e}_\phi.
\end{aligned}
$$

Logo,

$$
\begin{aligned}
\left(\vec{V}\cdot\nabla\right)\vec{r} &= V_r\vec{e}_r + \frac{V_\theta}{r}r\vec{e}_\theta + \frac{V_\phi}{r\sin\theta}r\sin\theta\vec{e}_\phi \\
&= V_r\vec{e}_r + V_\theta\vec{e}_\theta + V_\phi\vec{e}_\phi = \vec{V}.
\end{aligned}
$$

15. (T1 de 2012) Sejam as coordenads esféricas (r, θ, ϕ) dadas por

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\cos\phi, \qquad z = r\cos\theta,$$

onde $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$ e $0 \leq \phi \leq 2\pi$.

(a) Mostre que os fatores de escala são dados por

$$h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta.$$

Solução: *Utilizando* (2) *temos que*

$$
\begin{aligned}
h_r^2 &= (\sin\theta\cos\phi)^2 + (\sin\theta\cos\phi)^2 + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta = 1, \\
h_\theta^2 &= (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 \\
&= r^2\cos^2\theta + r^2\sin\theta = r^2, \\
h_\phi^2 &= (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 \\
&= r^2\sin^2\theta.
\end{aligned}
$$

(b) Seja $\vec{A}$ o campo vetorial dado por

$$\vec{A} = (r/3)\sin\theta\vec{e}_\phi.$$

Usando coordenadas esféricas, calcule $\vec{B} = \nabla\times\vec{A}$ e $\rho = \nabla\cdot\vec{B}$.

Solução: *Se* $\vec{A} = (r/3)\sin\theta\vec{e}_\phi$ *então* $A_r = 0$, $A_\theta = 0$ e $A_\phi = (r/3)\sin\theta$. *Portanto*

$$\begin{aligned}
\nabla\times\vec{A} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & (r^2\sin^2\theta)/3 \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left(\vec{e}_r(2r^2/3)\sin\theta\cos\theta - r\vec{e}_\theta(2r/3)\sin^2\theta\right) \\
&= (2/3)\cos\theta\vec{e}_r - (2/3)\sin\theta\vec{e}_\theta.
\end{aligned}$$

Então $\vec{B} = (2/3)\cos\theta\vec{e}_r - (2/3)\sin\theta\vec{e}_\theta$ *e portanto* $B_r = (2/3)\cos\theta$, $B_\theta = (-2/3)\sin\theta$ *e* $B_\phi = 0$. *Logo,*

$$\begin{aligned}
\nabla\cdot\vec{B} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta(2/3)\cos\theta) + \frac{\partial}{\partial\theta}(r\sin\theta(-2/3)\sin\theta)\right] \\
&= \frac{1}{r^2\sin\theta}\left[(4/3)r\sin\theta\cos\theta - (4/3)r\sin\theta\cos\theta\right] = 0.
\end{aligned}$$

16. (P1 de 2012) Sejam as coordenadas esféricas (r,θ,ϕ) dadas por

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\cos\phi, \qquad z = r\cos\theta,$$

onde $0 \le r \le \infty$, $0 \le \theta \le \pi$ e $0 \le \phi \le 2\pi$. Mostre que

$$\nabla\times(\cos\theta\nabla\phi) = \nabla(1/r).$$

Solução: *Temos que*

$$\nabla\times(\cos\theta\nabla\phi) = \nabla(\cos\theta)\times\nabla\phi + \cos\theta\underbrace{\nabla\times(\nabla\phi)}_{=0}.$$

Como

$$\begin{aligned}
\nabla(\cos\theta) &= \frac{1}{h_r}\underbrace{\frac{\partial\cos\theta}{\partial r}}_{=0}\vec{e}_r + \frac{1}{h_\theta}\frac{\partial\cos\theta}{\partial\theta}\vec{e}_\theta + \frac{1}{h_\phi}\underbrace{\frac{\partial\cos\theta}{\partial\phi}}_{=0}\vec{e}_\phi \\
&= \frac{1}{r}(-\sin\theta)\vec{e}_\theta, \\
\nabla\phi &= \frac{1}{h_r}\underbrace{\frac{\partial\phi}{\partial r}}_{=0}\vec{e}_r + \frac{1}{h_\theta}\underbrace{\frac{\partial\phi}{\partial\theta}}_{=0}\vec{e}_\theta + \frac{1}{h_\phi}\frac{\partial\phi}{\partial\phi}\vec{e}_\phi \\
&= \frac{1}{r\sin\theta}\vec{e}_\phi
\end{aligned}$$

então

$$\nabla\times(\cos\theta\nabla\phi) = -\frac{\sin\theta}{r}\vec{e}_\theta\times\frac{1}{r\sin\theta}\vec{e}_\phi = -\frac{1}{r^2}\vec{e}_r.$$

Por outro lado

$$\nabla\left(\frac{1}{r}\right)=\frac{1}{h_r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\vec{e}_r+\frac{1}{h_\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\right)}_{=0}\vec{e}_\theta+\frac{1}{h_\phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)}_{=0}\vec{e}_\phi=-\frac{1}{r^2}\vec{e}_r.$$

Portanto, concluimos que $\nabla\times(\cos\theta\nabla\phi)=\nabla(1/r)$.