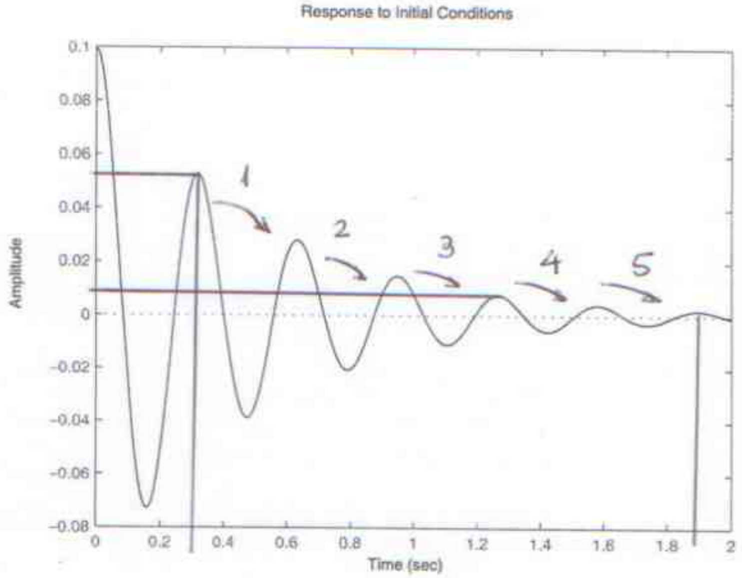
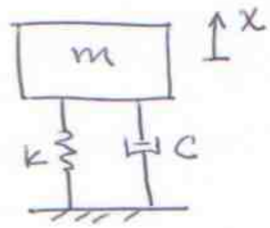
EM607 - Primeira Prova - 6/4/2010

Nome: Gabarito RA:

1. (valor 2.0) Uma mesa de $1000kg$ para ensaios de vibração é apoiada sobre molas e amortecedores. O deslocamento vertical da mesa quando esta é colocada em uma condição inicial de deslocamento vertical de 0,1m e é solta é a mostrada na figura a seguir. Determine a equação diferencial que governa o movimento da mesa quando esta é excitada por uma força vertical senoidal de amplitude A e frequência Ω.





$$5T = 1{,}9 - 0{,}3 \Rightarrow T = 0{,}32s \Rightarrow \omega_d = \frac{2\pi}{T} = 19{,}63 \text{ rad/s}$$

$$\delta = \frac{1}{3} \ln \frac{0{,}05}{0{,}01} = 0{,}54 \; ; \; \xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0{,}54}{\sqrt{4\pi^2 + 0{,}54^2}} \cong 0{,}1$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \Rightarrow \omega_n = \frac{19{,}63}{\sqrt{1 - 0{,}1^2}} \cong 20 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{1000}} = 20 \Rightarrow K = 400000 \text{ N/m}$$

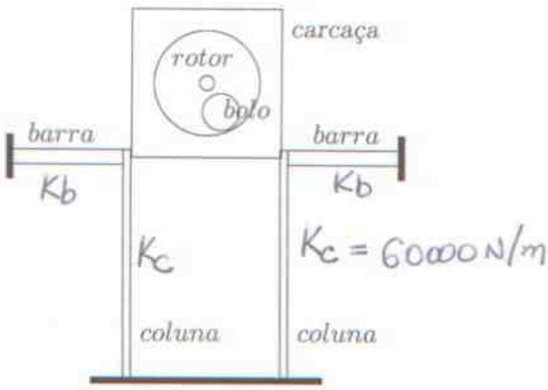
$$m\ddot{x} + c\dot{x} + kx = f(t) = A \operatorname{sen} \Omega t$$

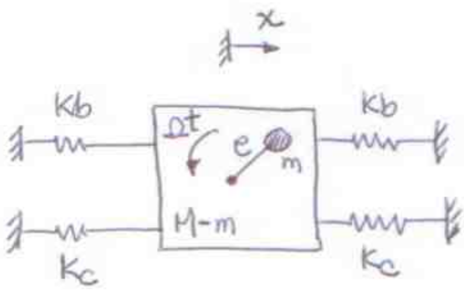
$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m} = \frac{A}{m} \operatorname{sen} \Omega t \; ; \; \frac{c}{m} = 2\xi\omega_n = 2 \times 0{,}1 \times 20 = 4$$

$$\boxed{\ddot{x} + 4\dot{x} + 400x = \frac{A}{1000} \operatorname{sen} \Omega t}$$

2. Um misturador industrial que opera a $3000rpm$ é suportado por duas colunas verticais e por duas barras horizontais conforme esquematizado na figura. A massa do misturador vazio é de $80kg$. A massa do bolo a ser misturado é de $30kg$ e se encontra deslocada do centro geométrico em $60mm$. A rigidez horizontal de cada coluna vertical é de $60000N/m$. Considere apenas o movimento horizontal do misturador.

   (a) (Valor 2.5) Determine a área das barras horizontais (comprimento de $1,0m$ cada e módulo de elasticidade de $206E9Pa$) para que a frequência de operação seja 25% superior à frequência natural do sistema. Nota: os valores numéricos são hipotéticos.





$\Omega = 3000\,rpm = 314,16\ rad/s$

$M - m = 80 \Rightarrow M = 110\ kg$

$m = 30\ kg\ ;\quad e = 0,06\ m.$

$E = 206 \times 10^{9}\ Pa\ ;\quad \ell = 1\ m.$

$$\Omega = 1,25\,\omega_n \Rightarrow r = \frac{\Omega}{\omega_n} = 1,25\ ;\qquad \omega_n = \frac{314,16}{1,25} = 251,33\ rad/s$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{2k_b + 2k_c}{M}} = 251,33 \Rightarrow \frac{2k_b + 120000}{110} = 63166,77 \Rightarrow$$

$$\Rightarrow K_b = 3414\,172,35 = \frac{EA}{\ell} \Rightarrow A = \frac{3414\,172,35 \times 1}{206 \times 10^{9}} = 16,574 \times 10^{-6}\ m^2$$

$$\boxed{A = 16,574\ mm^2}$$

(b) (Valor 1.5) Determine máxima força que atua em cada suporte das barras horizontais na condição de operação.

$$\frac{X}{me/M} = \frac{r^2}{\sqrt{(1-r^2)^2}} \Rightarrow X = \frac{\frac{me}{M} \quad r^2}{\sqrt{(1-r^2)^2}} \Rightarrow$$

$$X = \frac{\frac{30 \times 0{,}06}{110} \cdot 1{,}25^2}{\sqrt{(1-1{,}25^2)^2}} = 0{,}04545$$

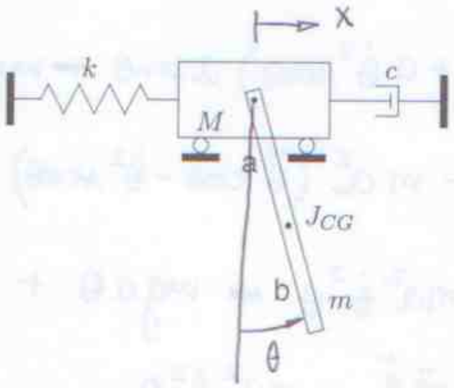
$$F_{barra} = K_b \cdot X = 155174{,}13 \text{ N}$$

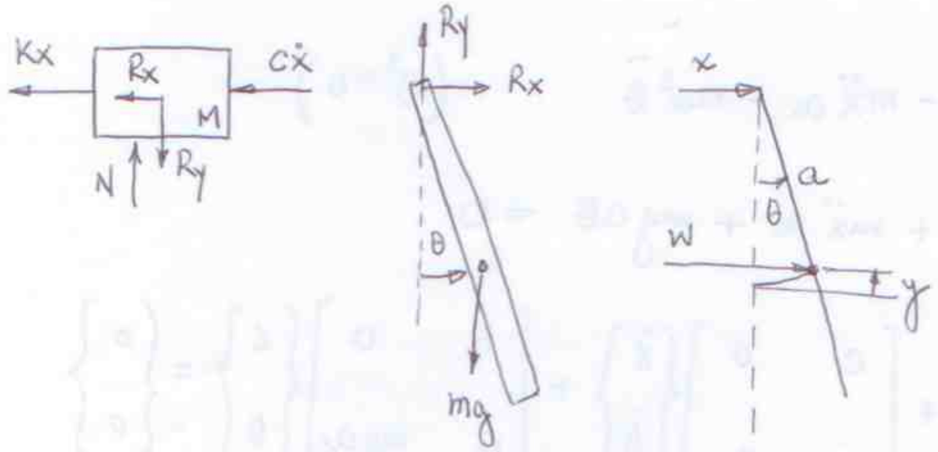
$$\boxed{F_{barra} = 155{,}2 \text{ KN}}$$

(suporte da barra)

3. Considere o sistema da figura com mola de rigidez k, constante de amortecimento c, bloco de massa M, barra de massa m e de momento de inércia J_{CG}.

   (a) (Valor 2.5) Escreva a equação matricial do movimento em termos da posição horizontal do bloco e do ângulo da barra em relação à vertical. Considerar pequenos deslocamentos / pequenas velocidades.





$$W = x + a\,\mathrm{sen}\theta$$

$$y = a\,(1 - \cos\theta)$$

$$M\ddot{x} = -Kx - c\dot{x} - R_x \; ; \quad m\ddot{y} = R_y - mg \; ; \quad m\ddot{w} = R_x \; ;$$

$$J_{cg}\,\ddot{\theta} = -R_y\, a\,\mathrm{sen}\theta - R_x\, a\cos\theta \; ;$$

$$\dot{w} = \dot{x} + a\,\dot{\theta}\cos\theta \; ; \quad \ddot{w} = \ddot{x} + a\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\,\mathrm{sen}\theta\right) \; ;$$

$$\dot{y} = a\,\dot{\theta}\,\mathrm{sen}\theta \; ; \quad \ddot{y} = a\left(\ddot{\theta}\,\mathrm{sen}\theta + \dot{\theta}^2\cos\theta\right) \; ;$$

$$M\ddot{x} + c\dot{x} + Kx + m\left[\ddot{x} + a\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\,\mathrm{sen}\theta\right)\right] = 0$$

$$(M+m)\,\ddot{x} + c\dot{x} + Kx + m\,a\,\ddot{\theta} = 0 \qquad \left(\cos\theta \simeq 1 \; ; \; \mathrm{sen}\theta \simeq \theta \; ; \; \dot{\theta}^2 \simeq 0\right)$$

$$J_{CG}\ddot{\theta} = -\left(m\ddot{y} + mg\right) a\,\text{sen}\theta - m\left(\ddot{x} + a\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\text{sen}\theta\right)\right) a\cos\theta$$

$$J_{CG}\ddot{\theta} = -m\left(a\ddot{\theta}\,\text{sen}\theta + a\dot{\theta}^2\cos\theta\right) a\,\text{sen}\theta - mg\,a\,\text{sen}\theta +$$

$$- m\ddot{x}\,a\cos\theta - m a^2\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\text{sen}\theta\right)\cos\theta$$

$$J_{CG}\ddot{\theta} = -m a^2\ddot{\theta}\theta^2 - ma^2\dot{\theta}^2\theta - mga\theta +$$

$$- m\ddot{x}a - ma^2\ddot{\theta} + ma^2\dot{\theta}^2\theta$$

$$J_{CG}\ddot{\theta} = -mga\theta - m\ddot{x}a - ma^2\ddot{\theta} \qquad \left(\theta^2 \simeq 0\right)$$

$$J_{CG}\ddot{\theta} + ma^2\ddot{\theta} + m\ddot{x}a + mga\theta = 0$$

$$\begin{bmatrix} M+m & ma \\ ma & J_{CG}+ma^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & mga \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Linearizando já no início:

$$W = x + a\theta \;;\quad \dot{W} = \dot{x} + a\dot{\theta} \;;\quad \ddot{W} = \ddot{x} + a\ddot{\theta} \;;$$

$$\dot{y} = a\dot{\theta}\theta \;;\quad \ddot{y} = a\ddot{\theta}\theta + a\dot{\theta}^2 \simeq a\ddot{\theta}\theta$$

$$M\ddot{x} + c\dot{x} + kx + m\left[\ddot{x} + a\ddot{\theta}\right] = 0$$

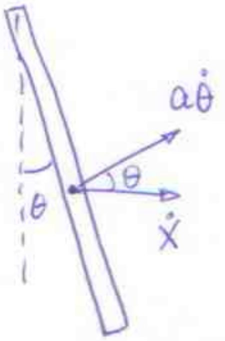
$$\boxed{(M+m)\ddot{x} + c\dot{x} + kx + ma\ddot{\theta} = 0}$$

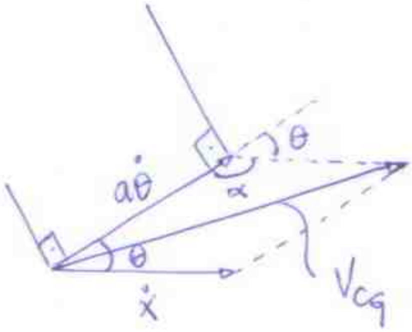
$$J_{CG}\ddot{\theta} = -\left(ma\ddot{\theta}\theta + mg\right)a\theta - m\left(\ddot{x} + a\ddot{\theta}\right)a$$

$$J_{CG}\ddot{\theta} + ma^2\ddot{\theta}\,\theta^2 + mga\theta + m\ddot{x}a + ma^2\ddot{\theta} = 0$$

$$\boxed{\left(J_{CG} + ma^2\right)\ddot{\theta} + m\ddot{x}a + mga\theta = 0}$$

Lagrange:



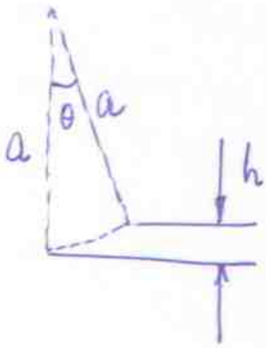


$\alpha + \theta = 180°$

$\cos\theta = -\cos\alpha$

$$V_{cg}^2 = \dot{x}^2 + (a\dot{\theta})^2 - 2\dot{x}\,a\dot{\theta}\cos\alpha = \dot{x}^2 + (a\dot{\theta})^2 + 2\dot{x}\,a\dot{\theta}\cos\theta$$

$$E_C = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[\dot{x}^2 + (a\dot{\theta})^2 + 2\dot{x}\,a\dot{\theta}\cos\theta\right] + \frac{1}{2} J_{cg}\,\dot{\theta}^2 = T$$



$h = a - a\cos\theta$

$$E_p = \frac{1}{2} k x^2 + mga(1 - \cos\theta) = V$$

$$D = \frac{1}{2} c\dot{x}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k \; ; \quad q = \begin{Bmatrix} x \\ \theta \end{Bmatrix}$$

$$Q_k = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{\partial T}{\partial \dot{x}} = M\dot{x} + m\dot{x} + ma\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ma\left(\ddot{\theta}\cos\theta + \dot{\theta}. -\operatorname{sen}\theta\,\dot{\theta}\right)$$

$$\frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta} + m\dot{x}a\cos\theta + J_{cq}\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ma^2\ddot{\theta} + ma\cos\theta\ddot{x} + ma\dot{x}. -\operatorname{sen}\theta\dot{\theta} + J_{cq}\ddot{\theta}$$

$$\frac{\partial T}{\partial x} = 0; \quad \frac{\partial T}{\partial \theta} = m\dot{x}a\dot{\theta}. -\operatorname{sen}\theta;$$

$$\frac{\partial V}{\partial x} = Kx; \quad \frac{\partial V}{\partial \theta} = -mga. -\operatorname{sen}\theta = mga\operatorname{sen}\theta$$

$$\frac{\partial D}{\partial \dot{x}} = c\dot{x}; \quad \frac{\partial D}{\partial \dot{\theta}} = 0;$$

$\operatorname{sen}\theta \simeq \theta$; $\cos\theta \simeq 1$;

$$(M+m)\ddot{x} + ma\ddot{\theta} - \cancel{\dot{\theta}^2}^{\simeq 0}\theta - 0 + Kx + c\dot{x} = 0$$

$$\boxed{(M+m)\ddot{x} + c\dot{x} + Kx + ma\ddot{\theta} = 0}$$

$$(J_{cq} + ma^2)\ddot{\theta} + ma\ddot{x} - \cancel{ma\dot{x}\dot{\theta}\theta} - (\cancel{-ma\dot{x}\dot{\theta}\theta}) + mga\theta = 0$$

$$\boxed{(J_{cq} + ma^2)\ddot{\theta} + ma\ddot{x} + mga\theta = 0}$$

(b) (Valor 1.5) Considere os seguintes valores hipotéticos para os parâmetros do sistema: $k = 20$, $c = 0$, $M = 2$, $m = 1$, $J_{CG} = 3$. Determine as frequências naturais do sistema e os modos de vibração. $g = 10$; $a = 0{,}6$; $b = 0{,}4$.

$$\underbrace{\begin{bmatrix} 3 & 0{,}6 \\ 0{,}6 & 3{,}36 \end{bmatrix}}_{[M]} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} 20 & 0 \\ 0 & 6 \end{bmatrix}}_{[K]} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left([M]r^2 + [K]\right) = 0 \Rightarrow \det \begin{bmatrix} 3r^2 + 20 & 0{,}6\,r^2 \\ 0{,}6\,r^2 & 3{,}36\,r^2 + 6 \end{bmatrix} = 0$$

$$(3r^2 + 20)(3{,}36\,r^2 + 6) - 0{,}36\,r^4 = 0$$

$$10{,}08\,r^4 + 18\,r^2 + 67{,}2\,r^2 + 120 - 0{,}36\,r^4 = 0$$

$$9{,}72\,r^4 + 85{,}2\,r^2 + 120 = 0 \Rightarrow r_{1,2} = \pm\, 2{,}646\,j$$

$$r_{3,4} = \pm\, 1{,}3278\,j$$

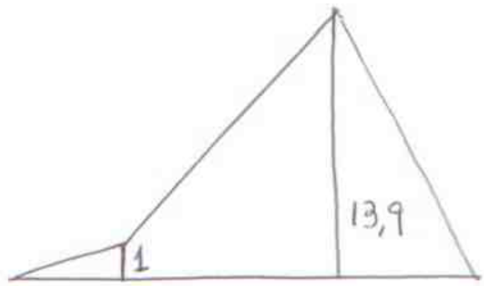
$$\boxed{\omega_1 = 1{,}3278 \; ; \; \omega_2 = 2{,}646 \; ;}$$ (frequências naturais)

$$r_{1,2}^2 = -1{,}7631 \; ; \; r_{3,4}^2 = -7{,}0024 \; ;$$

$$\begin{bmatrix} 3 \times (-1{,}7631) + 20 & 0{,}6 \times (-1{,}7631) \\ 0{,}6\,(-1{,}7631) & 3{,}36 \times (-1{,}7631) + 6 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$14{,}711\,A_1 - 1{,}058\,A_2 = 0 \Rightarrow \frac{A_1}{A_2} = 0{,}0719 \rightarrow \begin{Bmatrix} 1 \\ 13{,}906 \end{Bmatrix}$$

(primeiro modo)

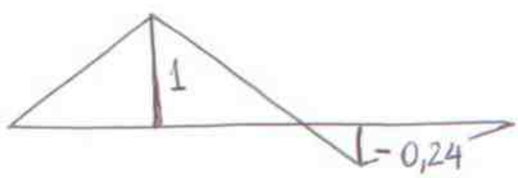


(representação do primeiro modo)

$$\begin{bmatrix} 3\times(-7{,}0024)+20 & 0{,}6\times(-7{,}0024) \\ 0{,}6\times(-7{,}0024) & 3{,}36\times(-7{,}0024)+6 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-1{,}0072\,A_1 - 4{,}2014\,A_2 = 0 \Rightarrow \frac{A_1}{A_2} = -4{,}1714 \Rightarrow \begin{Bmatrix} 1 \\ -0{,}2397 \end{Bmatrix}$$

(segundo modo)



(representação do segundo modo)

Algumas equações:

$$k = \frac{P}{\Delta} = \frac{3EI}{l^3}$$

$$k = \frac{P}{\Delta} = \frac{48EI}{l^3}$$

$$k = \frac{P}{\Delta} = \frac{AE}{l}$$

$$k = \sum_{i=1}^{n} k_i$$

$$\frac{1}{k} = \sum_{i=1}^{n} \frac{1}{k_i}$$

$$\delta = \ln \frac{x(t)}{x(t+T)}$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\delta = \frac{1}{n} \ln \frac{x(t)}{x(t+nT)}$$

$$m\ddot{x} + c\dot{x} + kx = F_0 sen(wt)$$

$$\ddot{x} + 2\xi w_n \dot{x} + w_n^2 x = f_0 sen(wt)$$

$$M\ddot{x} + c\dot{x} + kx = mew^2 sen(wt)$$

$$\frac{X}{(me/M)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\tan\theta = \frac{2\xi r}{1-r^2}$$

$$r_{pico} = 1/\sqrt{1-2\xi^2}$$

$$r_{pico} = \sqrt{1-2\xi^2}$$

$$r_{pico} = \frac{\sqrt{-1+\sqrt{1+8\xi^2}}}{2\xi}$$

$$\frac{F_T}{mew_n^2} = \frac{r^2\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$