T_6 ( ) F520 ( ) MS550 · Nome: ____________________ **RA:** __________

As funções de Legendre de segunda espécie $Q_n(x)$ $(n = 0, 1, 2, \ldots)$ podem ser definidas como

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(y)}{x-y}\, \mathrm{d}y,$$

onde $P_n(x)$ são os polinômios de Legendre. Mostre que eles satisfazem as seguintes relações de recorrência:

$$\text{(i)} \qquad (2n+1)xQ_n(x) = (n+1)Q_{n+1}(x) + nQ_{n-1}(x),$$
$$\text{(ii)} \qquad (2n+1)Q_n(x) = Q'_{n+1}(x) - Q'_{n-1}(x).$$

$$(i)\ (2n+1)\, x Q_n(x) = \frac{(2n+1)}{2}\int_{-1}^{+1} \frac{x P_n(y)}{x-y}\,dy = \frac{(2n+1)}{2}\int_{-1}^{+1} \frac{(x-y+y)P_n(y)}{x-y}\,dy =$$

$$= \frac{(2n+1)}{2}\left[\int_{-1}^{+1} P_n(y)\,dy + \int_{-1}^{+1}\frac{y P_n(y)}{x-y}\,dy\right] \overset{\overset{(\star)}{\downarrow}}{\underset{\underset{(\star\star)}{\uparrow}}{=}} \frac{1}{2}\left[\int_{-1}^{+1}[P'_{n+1}(y) - P'_{n-1}(y)]\,dy + \int_{-1}^{+1}\frac{[(n+1)P_{n+1}(y) + nP_{n-1}(y)]}{x-y}\,dy\right] =$$

$$= \frac{1}{2}\left[P_{n+1}(y)\Big|_{-1}^{+1} - P_{n-1}(y)\Big|_{-1}^{+1}\right] + \frac{(n+1)}{2}\int_{-1}^{+1}\frac{P_{n+1}(y)}{x-y}\,dy + \frac{n}{2}\int_{-1}^{+1}\frac{P_{n-1}(y)}{x-y}\,dy =$$

$$= \frac{1}{2}\Big[\underbrace{1 - (-1)^{n+1} - 1 + (-1)^{n-1}}_{=0}\Big] + (n+1)Q_{n+1}(x) + nQ_{n-1}(x) \quad ✓$$

+1,0

$$(ii)\ (2n+1)Q_n(x) = \frac{1}{2}\int_{-1}^{+1}\frac{(2n+1)P_n(y)}{x-y}\,dy \overset{\overset{(\star)}{\downarrow}}{=} \frac{1}{2}\int_{-1}^{+1}\frac{P'_{n+1}(y)}{x-y}\,dy - \frac{1}{2}\int_{-1}^{+1}\frac{P'_{n-1}(y)}{x-y}\,dy =$$

$$= \frac{1}{2}\left[\frac{P_{n+1}(y)}{x-y}\Big|_{-1}^{+1} - \int_{-1}^{+1}\frac{P_{n+1}(y)}{(x-y)^2}\,dy - \frac{P_{n-1}(y)}{x-y}\Big|_{-1}^{+1} + \int_{-1}^{+1}\frac{P_{n-1}(y)}{(x-y)^2}\,dy\right] =$$

$$= \frac{1}{2}\left[\cancel{\frac{1}{x-1}} - \cancel{\frac{(-1)^{n+1}}{x+1}} - \cancel{\frac{1}{x-1}} + \cancel{\frac{(-1)^{n-1}}{x+1}}\right] + \frac{1}{2}\frac{d}{dx}\int_{-1}^{+1}\frac{P_{n+1}(y)}{x-y}\,dy - \frac{1}{2}\frac{d}{dx}\int_{-1}^{+1}\frac{P_{n-1}(y)}{x-y}\,dy =$$

$$= \frac{d}{dx}Q_{n+1}(x) - \frac{d}{dx}Q_{n-1}(x) \quad ✓$$

+1,0

Formulário eventualmente útil

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \quad (n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x), \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

$$(\star)\ (2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad (1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0,$$

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x), \quad P_n(-x) = (-1)^nP_n(x), \quad P_n(1) = 1,$$

$$(\star\star)\ (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad (1-x^2)P'_n(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x).$$