**$T_1$ ( ) F520 ( ) MS550 · Nome:** ________________________ **RA:** __________

Sejam $(u, v, z)$ as coordenadas cilíndricas parabólicas, definidas como

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z,$$

com $-\infty < u < +\infty$, $v \geq 0$, $-\infty < z < +\infty$.

(i) Mostre que esse sistema de coordenadas é ortogonal.

(ii) Seja $\mathbf{r}$ o vetor posição, $\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Mostre que em coordenadas cilíndricas parabólicas temos

$$\mathbf{r} = \frac{1}{2}\sqrt{u^2 + v^2}\,(u\mathbf{e}_u + v\mathbf{e}_v) + z\mathbf{e}_z.$$

(iii) Usando essas coordenadas, e sabendo que em um sistema de coordenadas curvilíneas ortogonal temos

$$\operatorname{div}\mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right],$$

mostre que $\operatorname{div}\mathbf{r} = 3$.

(i) $\vec{r} = \frac{1}{2}(u^2 - v^2)\vec{i} + uv\vec{j} + z\vec{k}$

$\frac{\partial \vec{r}}{\partial u} = u\vec{i} + v\vec{j} \Rightarrow h_u = \sqrt{u^2+v^2} \Rightarrow \vec{e}_u = \frac{u\vec{i} + v\vec{j}}{\sqrt{u^2+v^2}}$

$\frac{\partial \vec{r}}{\partial v} = -v\vec{i} + u\vec{j} \Rightarrow h_v = \sqrt{u^2+v^2} \Rightarrow \vec{e}_v = \frac{-v\vec{i} + u\vec{j}}{\sqrt{u^2+v^2}}$

$\frac{\partial \vec{r}}{\partial z} = \vec{k} \Rightarrow h_z = 1 \Rightarrow \vec{e}_z = \vec{k}$

$\left.\begin{array}{l} \vec{e}_u \cdot \vec{e}_v = -uv + vu = 0 \\ \vec{e}_u \cdot \vec{e}_z = 0 \\ \vec{e}_v \cdot \vec{e}_z = 0 \end{array}\right\} \therefore$ ortogonais ✓

+3.5

(ii) $\vec{i} = (\vec{i}\cdot\vec{e}_u)\vec{e}_u + (\vec{i}\cdot\vec{e}_v)\vec{e}_v + \underbrace{(\vec{i}\cdot\vec{e}_z)}_{=0}\vec{e}_z$

$\therefore \vec{i} = \frac{u}{\sqrt{u^2+v^2}}\vec{e}_u - \frac{v}{\sqrt{u^2+v^2}}\vec{e}_v$

$\vec{j} = (\vec{j}\cdot\vec{e}_u)\vec{e}_u + (\vec{j}\cdot\vec{e}_v)\vec{e}_v + \underbrace{(\vec{j}\cdot\vec{e}_z)}_{=0}\vec{e}_z$

$\therefore \vec{j} = \frac{v}{\sqrt{u^2+v^2}}\vec{e}_u + \frac{u}{\sqrt{u^2+v^2}}\vec{e}_v$

$\vec{k} = \vec{e}_z$

$$\vec{r} = \frac{1}{2}(u^2-v^2)\cdot\frac{u}{\sqrt{u^2+v^2}}\vec{e_u} - \frac{1}{2}(u^2-v^2)\frac{v}{\sqrt{u^2+v^2}}\vec{e_v}$$

$$+\frac{uv^2}{\sqrt{u^2+v^2}}\vec{e_u} + \frac{u^2v}{\sqrt{u^2+v^2}}\vec{e_v} + z\vec{e_z}$$

$$= \frac{1}{2}\frac{(u^2+v^2)u}{\sqrt{u^2+v^2}}\vec{e_u} + \frac{1}{2}\frac{(u^2+v^2)v}{\sqrt{u^2+v^2}}\vec{e_v} + z\vec{e_z}$$

$$\therefore \vec{r} = \frac{1}{2}\sqrt{u^2+v^2}\left(u\vec{e_u} + v\vec{e_v}\right) + z\vec{e_z}$$ ✓

+3.5

(iii) $r_u = \frac{1}{2}\sqrt{u^2+v^2}\,u$

$r_v = \frac{1}{2}\sqrt{u^2+v^2}\,v$

$r_z = z$

$$\therefore \operatorname{div}\vec{r} = \frac{1}{(u^2+v^2)}\left[\frac{\partial}{\partial u}\left[\frac{1}{2}(u^2+v^2)u\right] + \frac{\partial}{\partial v}\left[\frac{1}{2}(u^2+v^2)v\right] + \frac{\partial}{\partial z}\left[(u^2+v^2)z\right]\right]$$

$$= \frac{1}{(u^2+v^2)}\left[\frac{3}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 + \frac{3}{2}v^2 + u^2 + v^2\right]$$

$$= \frac{1}{(u^2+v^2)}\frac{6}{2}(u^2+v^2) = 3$$ ✓

+3.0