GABARITO

1. (*2.5 pontos*) Resolva por transformada de Laplace o seguinte PVI:

$$y'' + 2y' + 2y = 2\delta(t - \pi)$$

onde $y(0) = 0$ e $y'(0) = 2$.

2. (*2.5 pontos*) Calcule a transformada inversa $\mathcal{L}^{-1}$ de

$$F(s) = \frac{1}{s^2(s^2 + s + 1)}$$

3. (*2.0 pontos*)

Encontre a solução geral real do sistema linear homogêneo de e.d.o.'s usando o método de autovalores e autovetores:

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}(t)$$

4. (*2.0 pontos*) Encontre a solução geral do sistema linear não-homogêneo utilizando o método de variação de parâmetros (indicando claramente a matriz fundamental)

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} e^{3t} + 1 \\ e^{t} \end{pmatrix}$$

e dado que a solução do homogêneo associado é:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

5. (*1.0 pontos*) Calcule a soma da série:

$$\sum_{n=2}^{\infty} \frac{\sqrt{3}\ \ 5^{(n+1)} + \pi\ \ 7^{(n-2)}}{8^{(n-1)}}.$$

1) $$y'' + 2y' + 2y = 2\delta(t-\pi) \ , \ y(0)=0 \ , \ y'(0)=2$$

$$\mathcal{L}\{y''+2y'+2y\}(x) = x^2\mathcal{L}\{y(t)\}(x) - x\,y(0) - y'(0)$$

$$+ 2\left(x\mathcal{L}\{y(t)\}(x) - y(0)\right) + 2\mathcal{L}\{y(t)\}(x)$$

$$Y = \mathcal{L}\{y(t)\}(x) \qquad = Y(x^2+2x+2) - 2$$

0,4

$$\mathcal{L}\{2\delta(t-\pi)\}(x) = 2e^{-\pi x}$$

0,2

$$Y(x^2+2x+2) - 2 = 2e^{-\pi x}$$

$$Y = \frac{1}{x^2+2x+2}\left(2 + 2e^{-\pi x}\right)$$

0,2

$$= \frac{1}{(x+1)^2+1}\left(2+2e^{-\pi x}\right)$$

0,3

$$= 2\mathcal{L}\{\operatorname{sen} t\}(x+1)\left(1+e^{-\pi x}\right)$$

$$= \mathcal{L}\{2e^{-t}\operatorname{sen} t\}(x) + e^{-\pi x}\mathcal{L}\{2e^{-t}\operatorname{sen} t\}(x)$$

0,5

$$= \mathcal{L}\{2e^{-t}\operatorname{sen} t\}(x) + \mathcal{L}\{2e^{-(t-\pi)}\operatorname{sen}(t-\pi)\,u_\pi(t)\}(x)$$

0,7

$$y(t) = 2e^{-t}\operatorname{sen} t + 2e^{-(t-\pi)}\operatorname{sen}(t-\pi)\,u_\pi(t)$$ (RESPOSTA)

0,2

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2) $$F(s) = \frac{1}{s^2(s^2+s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+s+1}$$

0,2

$$1 = s^3(A+C) + s^2(A+B+D) + s(A+B) + B$$

$$\begin{cases} A+C=0 \Rightarrow \boxed{C=1} \\ A+B+D=0 \\ A+B=0 \Rightarrow \boxed{A=-1} \\ \boxed{B=1} \end{cases} \qquad \boxed{D} = -(A+B) = -0 = \boxed{0}$$

$$F(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{s}{s^2+s+1}$$

0,5

$$= -\mathcal{L}\{1\}(s) + \mathcal{L}\{t\}(s) + \frac{s}{(s+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

0,5

$$= \mathcal{L}\{-1+t\}(s) + \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

0,6

$$= \mathcal{L}\{t-1\}(s) + \mathcal{L}\left\{\cos\frac{\sqrt{3}}{2}t\right\}\left(s+\frac{1}{2}\right) - \frac{1}{\sqrt{3}}\mathcal{L}\left\{\operatorname{sen}\frac{\sqrt{3}}{2}t\right\}\left(s+\frac{1}{2}\right)$$

$$= \mathcal{L}\{t-1\}(s) + \mathcal{L}\left\{e^{-t/2}\cos\frac{\sqrt{3}}{2}t\right\}(s) - \mathcal{L}\left\{\frac{e^{-t/2}}{\sqrt{3}}\operatorname{sen}\frac{\sqrt{3}}{2}t\right\}(s)$$

0,5

$$= \mathcal{L}\left\{t-1+e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{e^{-t/2}}{\sqrt{3}}\operatorname{sen}\frac{\sqrt{3}}{2}t\right\}(s)$$

RESPOSTA: $\mathcal{L}^{-1}\{F(s)\}(t) = t-1+e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{e^{-t/2}}{\sqrt{3}}\operatorname{sen}\frac{\sqrt{3}}{2}t$

0,2

3) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, $p_A(\lambda) = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 16$

$= \lambda^2 - 6\lambda + 25$

$\Delta = 36 - 100 = -64 \Rightarrow \lambda = \dfrac{+6 \pm 8i}{2}$ $\nearrow 3+4i = \lambda_1$, $\searrow 3-4i = \lambda_2$

0,3

$A - \lambda_1 I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$, $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A - \lambda_1 I)v = 0 \iff \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} -4ix - 4y = 0 \\ 4x - 4iy = 0 \end{cases}$

$y = -ix \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$x = 1 \Rightarrow v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

0,7

SOLUÇÃO COMPLEXA:

$e^{(3+4i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t}(\cos 4t + i \,\text{sen}\, 4t) \begin{bmatrix} 1 \\ -i \end{bmatrix}$

0,3

$= \underbrace{e^{3t} \begin{bmatrix} \cos 4t \\ \text{sen}\, 4t \end{bmatrix}}_{y_1(t)} + i \underbrace{e^{3t} \begin{bmatrix} \text{sen}\, 4t \\ -\cos 4t \end{bmatrix}}_{y_2(t)}$

0,5

RESPOSTA:

$y(t) = C_1 y_1(t) + C_2 y_2(t)$

$= C_1 e^{3t} \begin{bmatrix} \cos 4t \\ \text{sen}\, 4t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \text{sen}\, 4t \\ -\cos 4t \end{bmatrix}$

0,2

1,0 ⟵ Se achou os autovalores de forma errada (autovalores reais) e resolveu certo

4) $\Phi(t) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & e^{2t} \end{bmatrix}$ matriz fundamental

0,2

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \Phi(t)\, u' = g(t), \quad g(t) = \begin{bmatrix} e^{3t} + 1 \\ e^{t} \end{bmatrix}$$

$$x_p(t) = \Phi(t)\, u(t)$$

$$\begin{cases} e^{-3t} u_1' + e^{2t} u_2' = e^{3t} + 1 \\ -e^{-3t} u_1' + e^{2t} u_2' = e^{t} \end{cases}$$

0,5

$$\det \Phi(t) = e^{-t} + e^{-t} = 2e^{-t}$$

$$u_1' = \frac{e^{t}}{2} \begin{vmatrix} e^{3t}+1 & e^{2t} \\ e^{t} & e^{2t} \end{vmatrix} = \frac{1}{2}\left(e^{6t} + e^{3t} - e^{4t}\right)$$

$$u_2' = \frac{e^{t}}{2} \begin{vmatrix} e^{-3t} & e^{3t}+1 \\ -e^{-3t} & e^{t} \end{vmatrix} = \frac{1}{2}\left(e^{-t} + e^{t} + e^{-2t}\right)$$

0,4

$$u_1 = \frac{1}{2}\int \left(e^{6t} + e^{3t} - e^{4t}\right) dt = \frac{e^{6t}}{12} + \frac{e^{3t}}{6} - \frac{e^{4t}}{8}$$

$$u_2 = \frac{1}{2}\int \left(e^{-t} + e^{t} + e^{-2t}\right) dt = -\frac{e^{-t}}{2} + \frac{e^{t}}{2} - \frac{e^{-2t}}{4}$$

0,4

$$x_p(t) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -e^{-3t} & e^{2t} \end{bmatrix} \begin{bmatrix} e^{6t}/12 + e^{3t}/6 - e^{4t}/8 \\ -e^{-t}/2 + e^{t}/2 - e^{-2t}/4 \end{bmatrix}$$

0,3

Resposta:

$$y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + x_p(t)$$

0,2

5)

$$\sum_{n=2}^{\infty} \frac{\sqrt{3}\, 5^{n+1} + \pi\, 7^{n-2}}{8^{n-1}} =$$

$$= \sum_{n=2}^{\infty} \left(\frac{5\sqrt{3}}{8^{-1}} \left(\frac{5}{8}\right)^n + \frac{\pi\, 7^{-2}}{8^{-1}} \left(\frac{7}{8}\right)^n \right)$$

$$= 5 \cdot 8 \cdot \sqrt{3} \left(\left(\frac{5}{8}\right)^2 + \left(\frac{5}{8}\right)^3 + \left(\frac{5}{8}\right)^4 + \cdots \right)$$

0,3

$$+ \frac{8\pi}{7^2} \left(\left(\frac{7}{8}\right)^2 + \left(\frac{7}{8}\right)^3 + \left(\frac{7}{8}\right)^4 + \cdots \right)$$

$$= 5 \cdot 8 \cdot \sqrt{3} \cdot \frac{5^2}{8^2} \left(1 + \frac{5}{8} + \left(\frac{5}{8}\right)^2 + \cdots \right)$$

0,3

$$+ \frac{8\pi}{\cancel{7^2}} \cdot \frac{\cancel{7^2}}{8^2} \left(1 + \frac{7}{8} + \left(\frac{7}{8}\right)^2 + \cdots \right)$$

$$= \frac{5^3 \sqrt{3}}{8} \frac{1}{1 - \frac{5}{8}} + \frac{\pi}{8} \cdot \frac{1}{1 - \frac{7}{8}}$$

$$= \frac{5^3 \sqrt{3}}{\cancel{8}} \cdot \frac{\cancel{8}}{3} + \frac{\pi}{8} \cdot 8$$

0,4

$$= \frac{5^3}{\sqrt{3}} + \pi = \frac{125}{\sqrt{3}} + \pi.$$

0,3 + 0,3 — identificou como soma de duas séries geométricas corretamente

0,2 — a soma de cada série geométrica

-0,1 — cada erro de conta