T$_6$ ( ) F520 ( ) MS550 · Nome: ______________________ RA: _________

Sejam $P_n(x)$ os polinômios de Legendre de ordem n. Mostre que:

(i) $P'_{2n}(0) = 0; \qquad P'_{2n+1}(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}, \qquad (n = 0, 1, 2, \ldots);$

(ii) $\displaystyle\int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)}, \qquad (n = 1, 2, \ldots).$

(i) $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{k=0}^{\infty} P_k(x) t^k \xRightarrow{\frac{d}{dx}} \frac{t}{[1-2xt+t^2]^{3/2}} = \sum_{k=0}^{\infty} P'_k(x) t^k$$

$\boxed{x=0}$

$$t(1+t^2)^{-3/2} = t\sum_{n=0}^{\infty} \frac{(3/2)_n}{n!}(-t^2)^n = \sum_{n=0}^{\infty} \frac{(3/2)_n}{n!}(-1)^n t^{2n+1} = \sum_{k=0}^{\infty} P'_k(0) t^k$$

$$\therefore \quad P'_{2n}(0) = 0, \quad P'_{2n+1}(0) = (-1)^n \frac{(3/2)_n}{n!}$$

Mas: $$\left(\tfrac{3}{2}\right)_n = \frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{3}{2}+n-1\right) = \frac{1}{2^n}\cdot\frac{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}{2\cdot 4\cdots 2n} = \frac{1}{2^{2n}}\frac{(2n+1)!}{n!}$$

$$\therefore \quad P'_{2n+1}(0) = \frac{(-1)^n(2n+1)!}{2^{2n}(n!)^2}$$

+1,0

(ii) Vamos usar (★):

$$\int_{-1}^{+1} (xP_{n+1}(x))(xP_{n-1}(x))\,dx = \int_{-1}^{+1}\left[\frac{(n+2)}{(2n+3)}P_{n+2}(x) + \frac{(n+1)}{(2n+3)}P_n(x)\right]\left[\frac{n}{(2n-1)}P_n(x) + \frac{(n-1)}{(2n-1)}P_{n-2}(x)\right]dx =$$

$$= \frac{(n+2)n}{(2n+3)(2n-1)}\underbrace{\langle P_{n+2},P_n\rangle}_{=0} + \frac{(n+2)(n-1)}{(2n+3)(2n-1)}\underbrace{\langle P_{n+2},P_{n-2}\rangle}_{=0} + \frac{n(n+1)}{(2n+3)(2n-1)}\underbrace{\langle P_n,P_n\rangle}_{\frac{2}{2n+1}} + \frac{(n+1)(n-1)}{(2n+3)(2n-1)}\underbrace{\langle P_n,P_{n-2}\rangle}_{=0} =$$

$$= \frac{2n(n+1)}{(2n+3)(4n^2-1)}$$

+1,0

FORMULÁRIO EVENTUALMENTE ÚTIL

$$P_n(x) = {}_2F_1(-n, n+1, 1; \tfrac{1-x}{2}), \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n], \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x), \quad (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad (★)$$

$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x), \quad (1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x),$$

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x), \quad (1-x)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}x^n, \quad \int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}.$$