# Soluções para MS550, Métodos de Matemática Aplicada I, e F520, Métodos Matemáticos da Física I

## Lista 5 - Função Hipergeométrica

Equações eventualmente útil:

(ST) $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(GE) $$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

(1) $$\Gamma(z+1) = z\Gamma(z),\ \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$$

(2) $$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z)$$

(BG) $$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

(BT) $$B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta d\theta$$

(BI) $$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$$

(SP) $$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1),\ (\alpha)_0 = 1$$

(3) $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

(4) $$\frac{(\alpha)_n}{m!} = \binom{\alpha+n-1}{n},\ \frac{(-\alpha)_n}{n!} = (-1)^n\binom{\alpha}{n}$$

(EH) $$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]\, y' - \alpha\beta y = 0$$

(SH) $${}_2F_1(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$$

(5) $${}_2F_1(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt$$

(6) $${}_2F_1(\alpha,\beta,\gamma;z) = \frac{\alpha\beta}{\gamma}\,{}_2F_1(\alpha+1,\beta+1,\gamma+1;z)$$

(EHC) $$zy'' + (\gamma - z)y' - \alpha y = 0$$

(SHC) $${}_1F_1(\alpha,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!}$$

1. Mostre as relações de Kumer listadas abaixo.

   Dica: Como indicado nas notas de aula a demonstração pode ser realizada utilizando a representação integram da função hipergeométrica, (5), com as mudanças de variável $t \to 1-t$, $t \to (1-z-tz)^{-1}$ e $t \to (1-t)/(1-tz)$, conhecidas como transformações hipergeométricas de Euler.

   (a) ${}_2F_1(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;z)$,

   ***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \quad \textit{por (5)} \\
&= \frac{1}{B(\gamma-\beta,\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \quad B(z,w)=B(w,z)
\end{aligned}$$

$$= \frac{1}{B(\gamma-\beta,\beta)}\bigstar,$$

*onde*

$$\begin{aligned}
\bigstar &= \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \\
&= \int_0^1 \left(\frac{1-v}{1-vz}\right)^{\beta-1}\left(1-\frac{1-v}{1-vz}\right)^{\gamma-\beta-1}\left(1-\frac{z(1-v)}{1-vz}\right)^{-\alpha}\frac{1-z}{(1-vz)^2}dv \quad t=\frac{1-v}{1-vz} \\
&= \int_0^1 \frac{(1-v)^{\beta-1}(1-vz-1+v)^{\gamma-\beta-1}(1-vz-z+zv)^{-\alpha}(1-z)}{(1-vz)^{\beta-1+\gamma-\beta-1-\alpha+2}}dz \\
&= \int_0^1 \frac{(1-v)^{\beta-1}v^{\gamma-\beta-1}(1-z)^{\gamma-\beta-1}(1-z)^{-\alpha}(1-z)}{(1-vz)^{\gamma-\alpha}}dz \\
&= \int_0^1 (1-vz)^{-\gamma+\alpha}(1-v)^{\beta-1}v^{\gamma-\beta-1}(1-z)^{\gamma-\beta-\alpha}dv \\
&= (1-z)^{\gamma-\alpha-\beta}\int_0^1 v^{\gamma-\beta-1}(1-v)^{\beta-1}(1-vz)^{-\gamma+\alpha}dv.
\end{aligned}$$

*Portanto,*

$$\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \frac{1}{B(\gamma-\beta,\beta)}(1-z)^{\gamma-\alpha-\beta}\int_0^1 v^{\gamma-\beta-1}(1-v)^{\beta-1}(1-vz)^{-\gamma+\alpha}dv \\
&= (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;z).
\end{aligned}$$

(b) ${}_2F_1(\alpha,\beta,\gamma;z) = (1-z)^{-\alpha}{}_2F_1(\alpha,\gamma-\beta,\gamma;z/(z-1))$,

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \quad \text{por (5)} \\
&= \frac{1}{B(\gamma-\beta,\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \quad B(z,w)=B(w,z) \\
&= \frac{1}{B(\gamma-\beta,\beta)}\bigstar,
\end{aligned}$$

*onde*

$$\begin{aligned}
\bigstar &= \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt \\
&= \int_1^0 (1-v)^{\beta-1}v^{\gamma-\beta-1}(1-(1-v)z)^{-\alpha}(-dv) \qquad t=1-v \\
&= \int_0^1 (1-v)^{\beta-1}v^{\gamma-\beta-1}(1-(1-v)z)^{-\alpha}dv \\
&= \int_0^1 (1-v)^{\beta-1}v^{\gamma-\beta-1}(1-z+vz)^{-\alpha}dv \\
&= \int_0^1 (1-v)^{\beta-1}v^{\gamma-\beta-1}\left[\frac{(1-z)^{-\alpha}}{(1-z)^{-\alpha}}\left((1-z)+vz\right)^{-\alpha}\right]dv
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} (1-z)^{-\alpha} \left(1+\frac{vz}{1-z}\right)^{-\alpha} dv \\
&= (1-z)^{-\alpha} \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} \left(1+\frac{vz}{1-z}\right)^{-\alpha} dv \\
&= (1-z)^{-\alpha} \int_0^1 v^{\gamma-\beta-1} (1-v)^{\beta-1} \left(1-\frac{vz}{z-1}\right)^{-\alpha} dv.
\end{aligned}
$$

*Portanto,*

$$
\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \frac{1}{B(\gamma-\beta,\beta)} (1-z)^{-\alpha} \int_0^1 v^{\gamma-\beta-1} (1-v)^{\beta-1} \left(1-\frac{vz}{z-1}\right)^{-\alpha} dv \\
&= (1-z)^{-\alpha}\, {}_2F_1(\alpha,\gamma-\beta,\gamma;z/(z-1)).
\end{aligned}
$$

(c) ${}_2F_1(\alpha,\beta,\gamma;z) = (1-z)^{-\beta}{}_2F_1(\gamma-\alpha,\beta,\gamma;z/(z-1))$.
***Solução:*** *Temos que*

$$
\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} && \textit{por (SH)} \\
&= \sum_{n=0}^{\infty} \frac{(\beta)_n(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!} \\
&= {}_2F_1(\beta,\alpha,\gamma;z) && \textit{por (SH)} \\
&= \frac{1}{B(\alpha,\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta} dt && \textit{por (5)} \\
&= \frac{1}{B(\alpha,\gamma-\alpha)} \bigstar,
\end{aligned}
$$

*onde*

$$
\begin{aligned}
\bigstar &= \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta} dt \\
&= \int_1^0 (1-v)^{\alpha-1} v^{\gamma-\alpha-1} \left(1-(1-v)z\right)^{-\beta} (-dv) && t = 1-v \\
&= \int_0^1 (1-v)^{\alpha-1} v^{\gamma-\alpha-1} \left(1-(1-v)z\right)^{-\beta} dv \\
&= \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} (1-z+vz)^{-\alpha} dv \\
&= \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} \left[\frac{(1-z)^{-\alpha}}{(1-z)^{-\alpha}} \left((1-z)+vz\right)^{-\alpha}\right] dv \\
&= \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} (1-z)^{-\alpha} \left(1+\frac{vz}{1-z}\right)^{-\alpha} dv \\
&= (1-z)^{-\alpha} \int_0^1 (1-v)^{\beta-1} v^{\gamma-\beta-1} \left(1+\frac{vz}{1-z}\right)^{-\alpha} dv
\end{aligned}
$$

$$= (1-z)^{-\alpha}\int_0^1 v^{\gamma-\beta-1}(1-v)^{\beta-1}\left(1-\frac{vz}{z-1}\right)^{-\alpha}dv.$$

*Portanto,*

$$\begin{aligned}{}_2F_1(\alpha,\beta,\gamma;z) &= \frac{1}{B(\alpha,\gamma-\alpha)}(1-z)^{-\alpha}\int_0^1 v^{\gamma-\beta-1}(1-v)^{\beta-1}\left(1-\frac{vz}{z-1}\right)^{-\alpha}dv \\ &= (1-z)^{-\beta}\,{}_2F_1(\gamma-\alpha,\beta,\gamma;z/(z-1)).\end{aligned}$$

2. (Exercício 13.4.8 do Arfken) Mostre que

$${}_2F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \gamma \neq 0,-1,-2,\dots,\gamma > \alpha+\beta.$$

***Solução:*** *Temos que*

$$\begin{aligned}{}_2F_1(\alpha,\beta,\gamma;1) &= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}dt && \textit{por } (5) \\ &= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}dt \\ &= \frac{1}{B(\beta,\gamma-\beta)}B(\alpha-\beta-\alpha,\beta) && \textit{por } (\mathrm{BI}) \\ &= \frac{\Gamma(\beta+\gamma-\beta)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\frac{\Gamma(\beta)\Gamma(\gamma-\beta-\gamma)}{\Gamma(\beta+\gamma-\beta-\alpha)} && \textit{por } (\mathrm{BG}) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\frac{\Gamma(\beta)\Gamma(-\beta)}{\Gamma(\gamma-\alpha)} \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}.\end{aligned}$$

3. (Exame de 2006) Mostre que

$$\int_0^\infty e^{-st}{}_1F_1(\alpha,\gamma;t)dt = s^{-1}{}_2F_1(\alpha,1,\gamma,s^{-1}).$$

***Solução:*** *Temos que*

$$\begin{aligned}\int_0^\infty e^{-st}\,{}_1F_1(\alpha,\gamma,t)dt &= \int_0^\infty e^{-st}\sum_{n=0}^\infty \frac{(\alpha)_n}{(\gamma)_n}\frac{t^n}{n!}dt && \textit{por } (\mathrm{SH}) \\ &= \sum_{n=0}^\infty \frac{(\alpha)_n}{(\gamma)_n}\frac{1}{n!}\int_0^\infty e^{-st}t^n dt\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{1}{n!} \int_0^{\infty} e^{-u} \frac{u^n}{s^n} \frac{du}{s} && st = u \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{1}{n!} \frac{1}{s^{n+1}} \int_0^{\infty} e^{-u} u^n du \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{1}{n!} \frac{1}{s^{n+1}} \Gamma(n+1) && \textit{por } (\text{GE}) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{1}{n!} \frac{1}{s^{n+1}} n! \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{1}{s^{n+1}} \\
&= s^{-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} s^{-n} \\
&= s^{-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{(1)_n}{n!} s^{-n} \\
&= s^{-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (1)_n}{(\gamma)_n} \frac{s^{-n}}{n!} \\
&= s^{-1} \, {}_2F_1(\alpha, 1, \gamma; s^{-1}) && \textit{por } (\text{SH}).
\end{aligned}
$$

4. Mostre que a série hipergeométrica ${}_2F_1(\alpha, \beta, \gamma; z)$ reduz-se a um polinômio quando $\alpha$ ou $\beta$ é um número inteiro negativo.

***Solução:*** *Quando $\alpha$ é inteiro negativo, i.e., $\alpha = -k$, $k \in \mathbb{N}$ e $k < n$ temos que*

$$(\alpha)_n = (-k)_n = (-k)(-k+1)\dots\underbrace{(-k+k)}_{=0}(-k+k+1)\dots(-k+n+1).$$

*Verificamos então que a série é não nula até $n = k-1$. Então,*

$$
\begin{aligned}
{}_2F_1(-k, \beta, \gamma; z) &= \sum_{n=0}^{k-1} \frac{(-k)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} && \textit{por } (\text{SH}) \\
&= \sum_{n=0}^{k-1} \frac{(-k)_n (\beta)_n}{(\gamma)_n n!} z^n
\end{aligned}
$$

*que é um polinômio de grau $k-1$.*

*Para o caso de $\beta$ ser um número inteiro negativo basta substituir $\alpha$ por $\beta$ no racioncínio acima.*

5. (Ver exercício 18.11 do Riley) Mostre que:

(a) $(1-z)^{-\alpha} = {}_2F_1(\alpha, \beta, \beta; z)$;

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(\alpha, \beta, \beta; z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta)_n} \frac{z^n}{n!} && \textit{por } \text{(SH)} \\
&= \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!} \\
&= 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!} z^3 + \dots \\
&= (1-z)^{-\alpha}
\end{aligned}$$

*onde o último passo decorre da expansão pela série de Taylor da função* $(1-z)^{-\alpha}$ *em* $z = 0$.

(b) $z^n = {}_2F_1(-n, 1, 1; 1-z)$, para $n = 0, 1, 2, \dots$;

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(-k, 1, 1; 1-z) &= \sum_{n=0}^{\infty} \frac{(-k)_n (1)_n}{(1)_n} \frac{(1-z)^n}{n!} && \textit{por } \text{(SH)} \\
&= \sum_{n=0}^{\infty} (-k)_n \frac{(1-z)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-k)_n}{n!} (1-z)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{k}{n} (1-z)^n && \textit{por } (4) \\
&= \sum_{n=0}^{\infty} \binom{k}{n} (-1+z)^n \\
&= z^n
\end{aligned}$$

*onde o último passo decorre da expansão pela série de Taylor da função* $z^n$ *em* $z = 0$.

(c) $1 + \binom{a}{1} z + \binom{a}{2} z^2 + \dots \binom{a}{m} z^m = \binom{a}{m} z^m \, {}_2F_1(-m, 1, a-m+1; -z^{-1})$;

***Solução:*** *Temos que*

$$\begin{aligned}
\binom{a}{m} z^m \, {}_1F_1(-m, 1, a-m+1; -z^{-1}) &= \binom{a}{m} z^m \sum_{n=0}^{\infty} \frac{(-m)_n (1)_n}{(a-m+1)_n} \frac{(-z^{-1})^n}{n!} && \textit{por } \text{(SH)} \\
&= \binom{a}{m} z^m \sum_{n=0}^{\infty} \frac{(-m)_n}{(a-m+1)_n} \frac{(1)_n}{n!} (-z^{-1})^n \\
&= \binom{a}{m} z^m \sum_{n=0}^{\infty} \frac{(-m)_n}{(a-m+1)_n} (-z^{-1})^n && (1)_n = n! \\
&= \binom{a}{m} z^m \sum_{n=0}^{\infty} \frac{(-m)_n (-1)^n}{(a-m+1)_n} z^{-n}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{a}{m} z^m \sum_{n=0}^{m} \frac{m!}{(m-n)!(a-m+1)_n} z^{-n} && \textit{por } \bigstar \\
&= \binom{a}{m} \sum_{n=0}^{m} \frac{m!}{(m-n)!(a-m+1)_n} z^{m-n} \\
&= \frac{a!}{(a-m)!m!} \sum_{n=0}^{m} \frac{m!}{(m-n)!(a-m+1)_n} z^{m-n} \\
&= \sum_{n=0}^{m} \frac{a!m!}{(a-m)!m!(m-n)!(a-m+1)_n} z^{m-n} \\
&= \sum_{n=0}^{m} \frac{a!}{(a-m)!(m-n)!(a-m+1)_n} z^{m-n} \\
&= \sum_{n=0}^{m} \binom{a}{m-n} z^{m-n} && \textit{por } \bigstar\bigstar
\end{aligned}
$$

*onde* $\bigstar$ *corresponde a*

$$
\begin{aligned}
(-m)_n(-1)^n &= (-m)(-m+1)(-m+2)\dots(-m+n-1)(-1)^n \\
&= (-m)(-1)(-m+1)(-1)(-m+2)(-1)\dots(-m+n-1)(-1) \\
&= (m)(m-1)(m-2)\dots(m-n+1) = m!
\end{aligned}
$$

e $\bigstar\bigstar$ a

$$
\begin{aligned}
(a-m)!(a-m+1)_n &= (a-m)!(a-m+1)(a-m+2)\dots(a-m+n) \\
&= (a-(m-n))!.
\end{aligned}
$$

(d) (Exemplo4.6 das notas de aula) $\ln(1-z) = -z\,{}_2F_1(1,1,2;z)$;

***Solução:*** *Temos que*

$$
\begin{aligned}
{}_2F_1(1,1,2;z) &= \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n} \frac{z^n}{n!} && \textit{por } \text{(SH)} \\
&= \sum_{n=0}^{\infty} \frac{n!n!}{(n+1)!} \frac{z^n}{n!} && (2)_n = (n+1)! \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n+1} \\
&= z^{-1} \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \\
&= -z^{-1} \ln(1-z)
\end{aligned}
$$

*onde o último passo decorre da expansão pela série de Taylor da função* $\ln(1-z)$ *em* $z = 0$.

(e) $\ln\left((1+z)/(1-z)\right) = 2z\,{}_2F_1(1/2, 1, 3/2; z^2)$;

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(1/2,1,3/2;z^2) &= \sum_{n=0}^{\infty} \frac{(1/2)_n (1)_n}{(3/2)_n} \frac{(z^2)^n}{n!} && \textit{por (SH)}\\
&= \sum_{n=0}^{\infty} \frac{(1/2)_n n!}{(3/2)_n} \frac{(z^2)^n}{n!}\\
&= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n} \frac{n!}{n!} (z^2)^n\\
&= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n} \frac{(z^2)^n}{n!}\\
&= 1 + \sum_{n=1}^{\infty} \frac{(1/2)(3/2)(5/2)\ldots((2n-1)/2)}{(3/2)(5/2)\ldots((2n+1)/2)} (z^2)^n\\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} (z^2)^n\\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} z^{2n}\\
&= \left(\frac{1}{2z}\right)\left(\ln\left(\frac{1+z}{1-z}\right)\right)
\end{aligned}$$

*onde o último passo decorre da expansão pela série de Taylor da função* $\ln\left((1+z)/(1-z)\right)$ *em* $z = 0$.

(f) $\arcsin(z) = z\,{}_2F_1(1/2, 1/2, 3/2; z^2)$;

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(1/2,1/2,3/2;z^2) &= \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{(3/2)_n} \frac{(z^2)^n}{n!} && \textit{por (SH)}\\
&= 1 + \sum_{n=1}^{\infty} \frac{((1/2)(3/2)(5/2)\ldots((2n-1)/2))^2}{(3/2)(5/2)\ldots((2n+1)/2)} (z^2)^n\\
&= 1 + \sum_{n=1}^{\infty} \frac{(1/2)(3/2)(5/2)\ldots((2n-1)/2)}{2n+1} (z^2)^n\\
&= 1 + \sum_{n=1}^{\infty} \frac{(1)(3)\ldots(2n-1)}{2^n(2n+1)} (z^2)^n\\
&= z^{-1} \arcsin(z)
\end{aligned}$$

*onde o último passo decorre da expansão pela série de Taylor da função* $\arcsin(z)$ *no ponto* $z = 0$.

(g) (Exemplo 4.7 das notas de aula) $\cos(az) = {}_2F_1(a/2, -a/2, 1/2; \sin^2 z)$;

***Solução:*** *Para $z \to 0$ temos que $z \approx \sin z$ e portanto podemos dizer que ${}_2F_1(a/2, -a/2, 1/2; \sin^2 z)$ ${}_2F_1(a/2, -a/2, 1/2; z^2)$. Então temos*

$$
\begin{aligned}
{}_2F_1(a/2, -a/2, 1/2; \sin^2 z) &= \sum_{n=0}^{\infty} \frac{(a/2)_n(-a/2)_n}{(1/2)_n} \frac{(z^2)^n}{n!} && \textit{por (SH)} \\
&= \sum_{n=0}^{\infty} \frac{(a/2)_n(-a/2)_n}{(1/2)_n} \frac{z^{2n}}{n!} \\
&= 1 + \frac{(a/2)(-a/2)}{(1/2)} z^2 z \\
&\quad + \frac{(a/2)(-a/2)(a/2+1)(-a/2+1)}{2!(1/2)(3/2)} z^4 z + \dots \\
&= 1 - \frac{a^2}{2} z^2 + \frac{a^2(a^2-4)}{2(3)} z^4 + \dots \\
&\approx 1 - \frac{a^2}{2} z^2 + \frac{a^4}{2^3(3)} z^4 + \dots \\
&= 1 - \frac{1}{2}(az)^2 + \frac{1}{24} + (az)^4 + \dots \\
&= \cos(az)
\end{aligned}
$$

*onde o último passo decorre da expansão pela série de Taylor da função $\cos(az)$ no ponto $z = 0$.*

*Nota: Uma demonstração mais formal encontra-se no Riley onde é determinado a constante do termo $z^6$ e um teste que consiste em transformar a equação hipergeométrica original utilizando a transformação caracterizada por $z = \sin^2 z$.*

(h) $B(x,y)x^{-1}\,{}_2F_1(x, 1-y, x+1; 1)$.

***Solução:*** *Temos que*

$$
\begin{aligned}
{}_2F_1(x, 1-y, x+1; 1) &= \frac{1}{B(1-y, x+y)} \int_0^1 t^{-y}(1-t)^{x+y-1}(1-t)^{-x} dt && \textit{por (5)} \\
&= \frac{1}{B(1-y, x+y)} \int_0^1 t^{-y}(1-t)^{y-1} dt \\
&= \frac{1}{B(1-y, x+y)} B(-y+1, y) && \textit{por (BI)} \\
&= \frac{\Gamma(-y+1)\Gamma(y)}{\Gamma(1)} \frac{\Gamma(x+1)}{\Gamma(1-y)\Gamma(x+y)} \\
&= \frac{\Gamma(1-y)\Gamma(y)\Gamma(x+1)}{\Gamma(1)\Gamma(1-y)\Gamma(x+y)} && \textit{por (BG)} \\
&= \frac{\Gamma(y)\Gamma(x+1)}{\Gamma(x+y)} \\
&= \frac{\Gamma(y)x\Gamma(x)}{\Gamma(x+y)} && \textit{por (1)} \\
&= xB(x,y) && \textit{por (BG)}
\end{aligned}
$$

(i) $K(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2 \phi\right)^{-1/2} d\phi = (\pi/2)\, {}_2F_1(1/2, 1/2, 1; k^2)$.

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(1/2,1/2,1;k^2) &= \frac{1}{B(1/2,1-1/2)} \int_0^1 t^{1/2-1}(1-t)^{1-1/2-1}(1-tk^2)^{-1/2} dt \quad \textit{por } (5)\\
&= \frac{1}{B(1/2,1-1/2)} \int_0^1 t^{-1/2}(1-t)^{-1/2}(1-tk^2)^{-1/2} dt\\
&= \frac{1}{B(1/2,-1/2)} \bigstar,
\end{aligned}$$

*onde*

$$\begin{aligned}
\bigstar &= \int_0^{\pi/2} \frac{1}{\sin\phi}\frac{1}{\cos\phi}(1-k^2\sin^2\phi)^{-1/2} 2\sin\phi\cos\phi d\phi \qquad t = \sin^2\phi\\
&= 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi.
\end{aligned}$$

*Portanto*

$$\begin{aligned}
{}_2F_1(1/2,1/2,1;k^2) &= \frac{1}{B(1/2,-1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi\\
&= \frac{\Gamma(1/2+1/2)}{\Gamma(1/2)\Gamma(1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi \qquad \textit{por } (\mathrm{BG})\\
&= \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi\\
&= \frac{1}{\sqrt{\pi}\sqrt{\pi}} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi \qquad \textit{por } (1)\\
&= \frac{2}{\pi}\int_0^{\pi/2} (1-k^2\sin^2\phi)^{-1/2} d\phi.
\end{aligned}$$

(j) $E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{1/2} d\phi = (\pi/2)\ {}_2F_1(-1/2, 1/2, 1; k^2)$.

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(-1/2,1/2,1;k^2) &= \frac{1}{B(1/2,1-1/2)} \int_0^1 t^{1/2-1}(1-t)^{1-1/2}(1-k^2t)^{1/2} dt \quad \textit{por } (5)\\
&= \frac{1}{B(1/2,1-1/2)} \int_0^1 t^{-1/2}(1-t)^{1/2}(1-k^2t)^{1/2} dt\\
&= \frac{1}{B(1/2,1/2)} \bigstar,
\end{aligned}$$

*onde*

$$\begin{aligned}
\bigstar &= \int_0^{\pi/2} \frac{1}{\sin\phi}\frac{1}{\cos\phi}(1-k^2\sin^2\phi)^{1/2} 2\sin\phi\cos\phi d\phi \qquad t = \sin^2\phi\\
&= 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi.
\end{aligned}$$

*Portanto*

$$\begin{aligned}
{}_2F_1(-1/2, 1/2, 1; k^2) &= \frac{1}{B(1/2,1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi \\
&= \frac{\Gamma(1/2+1/2)}{\Gamma(1/2)\Gamma(1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi && \textit{por (BG)} \\
&= \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi \\
&= \frac{1}{\sqrt{\pi}\sqrt{\pi}} 2\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi && \textit{por (1)} \\
&= \frac{2}{\pi}\int_0^{\pi/2} (1-k^2\sin^2\phi)^{1/2} d\phi.
\end{aligned}$$

(k) $\exp(z)\,(1 + z/(a-1)) = {}_1F_1(a, a-1; z)$

***Solução:*** *Temos que*

$$\begin{aligned}
{}_1F_1(a, a-1; z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(a-1)_n} \frac{z^n}{n!} && \textit{por (SHC)} \\
&= \sum_{n=0}^{\infty} \frac{a(a+1)\dots(a+n-1)}{(a-1)a\dots(a+n-2)} \frac{z^n}{n!} && \textit{por (SP)} \\
&= \sum_{n=0}^{\infty} \frac{a+n-1}{a-1} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(1 + \frac{n}{a-1}\right) \frac{z^n}{n!} \\
&= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) + \frac{1}{a-1}\left(\sum_{n=0}^{\infty} \frac{nz^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) + \frac{1}{a-1}\left(\sum_{n=0}^{\infty} \frac{z^n}{(n-1)!}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) + \frac{z}{a-1}\left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!}\right) \\
&= \exp(z) + \frac{z}{a-1}\exp(z)
\end{aligned}$$

*onde o último passo decorre da expansão pela série de Taylor da função* $\exp(z)$ *no ponto* $z = 0$.

6. (Ver exemplo 4.7 das notas de aula) Os polinômios de Jacobi $P_n^{(\alpha,\beta)}(z)$ estão relacionados com a função hipergeométrica por

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\,{}_2F_1(-n, \alpha+\beta+n+1, \alpha+1; (1-z)/2).$$

Mostre que esses polinômios satisfacem a equação

$$(1-z^2)y'' + (\beta - \alpha - (\alpha+\beta+2)z)y' + n(n+\alpha+\beta+1)y = 0.$$

***Solução:*** *Temos que* $f(y) = {}_2F_1(-n, \alpha+\beta+n+1, \alpha+1; y)$ *satisfaz a equação hipergeométrica dada por*

$$y(1-y)f''(y) + [(\alpha+1) - ((-n) + (\alpha+\beta+n+1) + 1)y]\, f'(y) - (-n)(\alpha+\beta+n+1)f(y) = 0 \quad \textit{por}$$

*que pode ser simplificada para*

$$y(1-y)f''(y) + [(\alpha+1) - (\alpha+\beta+2)y]\, f'(y) + n(\alpha+\beta+n+1)f(y) = 0.$$

*Para* $y = (1-z)/2$ *temos que*

$$\frac{d}{dy} = \left(\frac{-1}{2}\right)\frac{d}{dz}, \tag{7}$$

$$\frac{d^2}{dy^2} = \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\frac{d^2}{dz^2} = \frac{1}{4}\frac{d^2}{dz^2}. \tag{8}$$

*Então para* $\phi = f((1-z)/2)$ *temos a seguinte equação hipergeométrica*

$$\frac{1-z}{2}\left(1 - \frac{1-z}{2}\right)\phi'' + \left[(\alpha+1) - (\alpha+\beta+2)\frac{1-z}{2}\right]\phi' + n(\alpha+\beta+n+1)\phi = 0$$

*que pode ser simplificada para*

$$\begin{aligned}
\frac{1-z}{2}\left(\frac{1+z}{2}\right)\phi'' + \left[\frac{2(\alpha+1) - (\alpha+\beta+2)(1-z)}{2}\right]\phi' + n(\alpha+\beta+n+1)\phi &= 0\\
\frac{1-z}{2}\left(\frac{1+z}{2}\right)\phi'' + \left[\frac{2(\alpha+1) - (\alpha+\beta+2)(1-z)}{2}\right]\phi' + n(\alpha+\beta+n+1)\phi &= 0\\
\frac{1-z^2}{4}\phi'' + \left[\frac{2\alpha+2-\alpha-\beta-2+(\alpha+\beta+2)z)}{2}\right]\phi' + n(\alpha+\beta+n+1)\phi &= 0\\
\frac{1-z^2}{4}\phi'' + \left[\frac{\alpha-\beta+(\alpha+\beta+2)z)}{2}\right]\phi' + n(\alpha+\beta+n+1)\phi &= 0\\
(1-z^2)\frac{\phi''}{4}\phi'' + [-\alpha+\beta-(\alpha+\beta+2)z)]\,\frac{-\phi'}{2} + n(\alpha+\beta+n+1)\phi &= 0\\
(1-z^2)y'' + [-\alpha+\beta-(\alpha+\beta+2)z)]\, y' + n(\alpha+\beta+n+1)y &= 0.
\end{aligned}$$

7. (Ver exemplo 4.7 das notas de aula) Os polinômios de Hermite $H_n(z)$ podem ser escritos em termos da função hipergeométrica confluente como

$$H_n(z) = 2^n U(-n/2, 1/2, z^2).$$

Mostre que esses polinômios satisfazem a equação

$$y'' - 2zy' + 2ny = 0.$$

***Solução:*** *Temos que* $f(y) = {}_1F_1(-n/2, 1/2; y)$ *satisfaz a equação hipergeométrica confluente dada por*

$$yf''(y) + (1/2 - y)\, f'(y) - (-n/2)f(y) = 0 \qquad \text{por (EHC)}$$

*que poder ser simplificada para*

$$yf''(y) + ((1 - 2y)/2)\, f'(y) + (n/2)f(y) = 0.$$

*Para* $y = z^2$ *temos que*

$$\begin{aligned} \frac{d}{dy} &= 2z\frac{d}{dz}, \\ \frac{d^2}{dy^2} &= 2\frac{df}{dz} + 4z^2\frac{d}{dz}. \end{aligned}$$

*Então para* $\phi = f(z^2)$ *temos a seguinte equação hipergeométrica confluente*

$$z^2\phi'' + \big((1 - 2z^2)/2\big)\, \phi' + (n/2)\phi = 0$$

*que pode ser simplificada para*

$$\begin{aligned} 4z^2\phi'' + \left(2 - 4z^2\right)\phi' + 2n\phi &= 0 \\ 4z^2\phi'' + 2\phi' - 4z^2\phi' + 2n\phi &= 0 \\ (4z^2\phi'' + 2\phi') - 2z(2z\phi') + 2n\phi &= 0 \\ y'' - 2zy' + 2ny &= 0. \end{aligned}$$

8. (Ver exemplo 4.7 das notas de aula) Mostre que

$$M_{k,m}(z) = \exp(-z/2)z^{m+1/2}\,{}_1F_1(1/2 + m - k, 1 + 2m; z)$$

satisfaz a equação de Whittaker,

$$w'' + \left[-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{z^2}\right] w = 0.$$

***Solução:*** *Temos que* $f(z) = {}_1F_1(1/2 + m - k, 1 + 2m; z) = e^{z/2}z^{-m-1/2}M_{k,m}(z)$ *satisfaz a equação hipergeométrica confluente dada por*

$$zf''(z) + (1 + 2m - z)f'(z) - (1/2 + m - k)f(z) = 0 \qquad \text{por (EHC).}$$

*Para* $f(z) = e^{z/2}z^{-m-1/2}M$ *temos que*

$$
\begin{aligned}
f'(z) &= \frac{1}{2}e^{z/2}z^{-m-1/2}M + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M + e^{z/2}z^{-m-1/2}M', \\
f''(z) &= \frac{1}{4}e^{z/2}z^{-m-1/2}M + \frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M \\
&\quad + \frac{1}{2}e^{z/2}z^{-m-1/2}M' + \frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M \\
&\quad + e^{z/2}\left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-m-5/2}M + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M' \\
&\quad + \frac{1}{2}e^{z/2}z^{-m-1/2}M' + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M' + e^{z/2}z^{-m-1/2}M''.
\end{aligned}
$$

*Substituido na equação hipergeométrica confluente temos que*

$$
\begin{aligned}
0 &= z\frac{1}{4}e^{z/2}z^{-m-1/2}M + z\frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M \\
&\quad + z\frac{1}{2}e^{z/2}z^{-m-1/2}M' + z\frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M \\
&\quad + ze^{z/2}\left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-m-5/2}M + ze^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M' \\
&\quad + z\frac{1}{2}e^{z/2}z^{-m-1/2}M' + ze^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M' + ze^{z/2}z^{-m-1/2}M'' \\
&\quad + (1+2m-z)\left[\frac{1}{2}e^{z/2}z^{-m-1/2}M + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M + e^{z/2}z^{-m-1/2}M'\right] \\
&\quad - (1/2+m-k)e^{z/2}z^{-m-1/2}M \\
0 &= \frac{1}{4}e^{z/2}z^{-m+1/2}M + \frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M \\
&\quad + \frac{1}{2}e^{z/2}z^{-m+1/2}M' + \frac{1}{2}e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M \\
&\quad + e^{z/2}\left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-m-3/2}M + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M' \\
&\quad + \frac{1}{2}e^{z/2}z^{-m+1/2}M' + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M' + e^{z/2}z^{-m+1/2}M'' \\
&\quad + (1+2m-z)\left[\frac{1}{2}e^{z/2}z^{-m-1/2}M + e^{z/2}\left(-m-\frac{1}{2}\right)z^{-m-3/2}M + e^{z/2}z^{-m-1/2}M'\right] \\
&\quad - (1/2+m-k)e^{z/2}z^{-m-1/2}M \\
0 &= \frac{1}{4}z^{-m+1/2}M + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M \\
&\quad + \frac{1}{2}z^{-m+1/2}M' + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-m-1/2}M \\
&\quad + \left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-m-3/2}M + \left(-m-\frac{1}{2}\right)z^{-m-1/2}M' \qquad e^{z/2} \neq 0 \\
&\quad + \frac{1}{2}z^{-m+1/2}M' + \left(-m-\frac{1}{2}\right)z^{-m-1/2}M' + z^{-m+1/2}M'' \\
&\quad + (1+2m-z)\left[\frac{1}{2}z^{-m-1/2}M + \left(-m-\frac{1}{2}\right)z^{-m-3/2}M + z^{-m-1/2}M'\right] \\
&\quad - (1/2+m-k)z^{-m-1/2}M
\end{aligned}
$$

$$
\begin{aligned}
0 = &\frac{1}{4}M + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-1}M \\
&+ \frac{1}{2}M' + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-1}M \\
&+ \left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-2}M + \left(-m-\frac{1}{2}\right)z^{-1}M' \\
&+ \frac{1}{2}M' + \left(-m-\frac{1}{2}\right)z^{-1}M' + M'' \\
&+ (1+2m-z)\left[\frac{1}{2}z^{-1}M + \left(-m-\frac{1}{2}\right)z^{-2}M + z^{-1}M'\right] \\
&- (1/2+m-k)z^{-1}M
\end{aligned}
\qquad z^{-m+1/2} \neq 0
$$

$$
\begin{aligned}
0 = &\frac{1}{4}M + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-1}M + \frac{1}{2}\left(-m-\frac{1}{2}\right)z^{-1}M \\
&+ \left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-2}M \\
&+ (1+2m-z)\left[\frac{1}{2}z^{-1}M + \left(-m-\frac{1}{2}\right)z^{-2}M\right] - (1/2+m-k)z^{-1}M \\
&+ \frac{1}{2}M' + (1+2m-z)z^{-1}M' + \frac{1}{2}M' + \left(-m-\frac{1}{2}\right)z^{-1}M' \\
&+ \left(-m-\frac{1}{2}\right)z^{-1}M' + M''
\end{aligned}
$$

$$
\begin{aligned}
0 = &\frac{1}{4}M + \left[\frac{1}{2}\left(-m-\frac{1}{2}\right) + \frac{1}{2}\left(-m-\frac{1}{2}\right)\right]z^{-1}M \\
&+ \left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-2}M \\
&+ (1+2m-z)\left[\frac{1}{2}z^{-1}M + \left(-m-\frac{1}{2}\right)z^{-2}M\right] - (1/2+m-k)z^{-1}M \\
&+ \left[\frac{z}{2} + (1+2m-z) + \frac{z}{2} + \left(-m-\frac{1}{2}\right) + \left(-m-\frac{1}{2}\right)\right]z^{-1}M' + M''
\end{aligned}
$$

$$
\begin{aligned}
0 = &\frac{1}{4}M + \left[-m-\frac{1}{2}\right]z^{-1}M + \left(-m-\frac{1}{2}\right)\left(-m-\frac{3}{2}\right)z^{-2}M \\
&+ (1+2m-z)\left[\frac{1}{2}z - m - \frac{1}{2}\right]z^{-2}M - (1/2+m-k)z^{-1}M \\
&+ [z+1+2m-z-2m-1]\,z^{-1}M' + M''
\end{aligned}
$$

$$
\begin{aligned}
0 = &\frac{1}{4}M + \left[-m-\frac{1}{2}\right]\left(z-m-\frac{3}{2}\right)z^{-2}M \\
&+ (1+2m-z)\left[\frac{1}{2}z - m - \frac{1}{2}\right]z^{-2}M - (1/2+m-k)z^{-1}M + M''
\end{aligned}
$$

$$
0 = \left[-\frac{1}{4} + \frac{k}{z} + \left(-m^2+\frac{1}{4}\right)\frac{1}{z^2}\right]M + M''
$$

$$0 = \left[ -\frac{1}{4} + \frac{k}{z} + \frac{(1/4 - m^2)}{z^2} \right] M + M''$$

9. (P2 de 2006) Seja ${}_2F_1(a, b, c; z)$ a função hipergeométrica. Mostre que

$${}_2F_1(-n, b, b; z) = (1-z)^n,$$

com $|z| < 1$ e $n \in \mathbb{N}$.

***Solução:*** *Temos que*

$$\begin{aligned}
{}_2F_1(-n, b, b; z) &= \sum_{k=0}^{\infty} \frac{(-n)_k (b)_k}{(b)_k} \frac{z^k}{k!} && \text{(SH)} \\
&= \sum_{k=0}^{\infty} (-n)_k \frac{z^k}{k!} \\
&= \sum_{k=0}^{n} (-n)_k \frac{z^k}{k!} && (-n)_k = 0, k = n+1, n+2, \dots \\
&= \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{z^k}{k!} && \text{(SP)} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-z)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-z)^k \\
&= (1-z)^n.
\end{aligned}$$

10. (P2 de 2011) Seja ${}_2F_1(\alpha, \beta, \gamma; x)$ a função hipergeométrica. Motre que

$${}_2F_1(\alpha, \beta, \beta - \alpha + 1; -1) = \frac{\Gamma(1 + \beta - \alpha)\Gamma(1 + \beta/2)}{\Gamma(1 + \beta)\Gamma(1 + \beta/2 - \alpha}.$$

***Solução:*** *Por* (5) *temos que*

$${}_2F_1(\alpha, \beta, \beta - \alpha + 1; -1) = \frac{1}{B(\beta, (\beta - \alpha + 1) - \beta)}$$
$$\int_0^1 t^{\beta-1} (1-t)^{(\beta-\alpha+1)-\beta-1} (1 - t(-1))^{-\alpha} dt$$
$$= \frac{1}{B(\beta, 1 - \alpha} \int_0^1 t^{\beta-1} (1-t)^{-\alpha} (1+t)^{-\alpha} dt$$

$$
\begin{aligned}
&= \frac{1}{B(\beta, 1-\alpha} \int_0^1 t^{\beta-1}(1-t^2)^{-\alpha} dt && (1-t)(1+t) = 1-t^2 \\
&= \frac{1}{B(\beta, 1-\alpha} \int_0^1 (y^{1/2})^{\beta-1}(1-y)^{-\alpha}(1/2)y^{-1/2} dy && t = y^{1/2} \\
&= \frac{1}{2B(\beta, 1-\alpha} \int_0^1 y^{\beta/2-1}(1-y)^{-\alpha} dy \\
&= \frac{B(\beta/2, -\alpha+1)}{2B(\beta, 1-\alpha)} && \textit{por } (\mathrm{BI}) \\
&= \frac{1}{2}\frac{\Gamma(\beta/2)\Gamma(1-\alpha)\Gamma(1-\alpha+\beta)}{\Gamma(1-\alpha+\beta/2)\Gamma(\beta)\Gamma(1-\alpha)} && \textit{por } (\mathrm{BG}) \\
&= \frac{\beta}{2}\frac{\Gamma(\beta/2)\Gamma(1+\beta-\alpha)}{\Gamma(\beta)\Gamma(1+\beta/2-\alpha)} \\
&= \frac{\Gamma(1+\beta/2)\Gamma(1+\beta-\alpha)}{\Gamma(1+\beta)\Gamma(1+\beta/2-\alpha)}.
\end{aligned}
$$

11. (Exame de 2011) Seja ${}_2F_1(\alpha, \beta, \gamma; x)$ a função hipergeométrica. Mostre que

$$ {}_2F_1(\alpha, \alpha/2+1, \alpha/2; x) = (1+x)(1-x)^{-\alpha-1}. $$

***Solução:*** *Temos que*

$$
\begin{aligned}
{}_2F_1(\alpha, \alpha/2+1, \alpha/2; x) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha/2+1)_n}{(\alpha/2)_n}\frac{x^n}{n!} && (\mathrm{SH}) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n(1+2n/\alpha)(\alpha/2)_n}{(\alpha/2)_n}\frac{x^n}{n!} && \star \\
&= \sum_{n=0}^{\infty} (\alpha)_n \frac{x^2}{n!} \\
&= \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + \frac{2}{\alpha}\sum_{n=1}^{\infty} n(\alpha)_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + \frac{2}{\alpha}\sum_{n=0}^{\infty} (\alpha)_{n+1} \frac{x^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} + 2x\sum_{n=0}^{\infty} (\alpha+1)_n \frac{x^n}{n!} && (\alpha)_{n+1} = \alpha(\alpha+1)_n \\
&= (1-x)^{-\alpha} + 2x(1-x)^{-\alpha-1} \\
&= (1-x)^{-\alpha-1}(1-x+2x) \\
&= (1+x)(1-x)^{-\alpha-1},
\end{aligned}
$$

*onde*

$$ \star = \left(\frac{\alpha}{2}+1\right)_n = \frac{\Gamma(\alpha/2+1+n)}{\Gamma(\alpha/2+1)} = \frac{(\alpha/2+n)\Gamma(\alpha/2+n)}{(\alpha/2)\Gamma(\alpha/2)} = (1+2n/\alpha)(\alpha/2)_n. $$

# Referências

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[2] K.F. Riley, M.P. Hobson, and S.J. Bence. *Mathematical Methods for Physics and Engineering*. Cambridge University Press, 2006.