T_5 ( ) F520 ( ) MS550 · Nome: ____________________ RA: __________

Mostre que

$$\int_0^{\pi/2} J_1(x\cos\theta)\,\mathrm{d}\theta = \frac{1-\cos x}{x},$$

onde $J_1(\cdot)$ denota a função de Bessel de primeira espécie e ordem um.

$$I = \int_0^{\pi/2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x\cos\theta}{2}\right)^{2n+1}}{n!\,(n+1)!}\right]d\theta = \sum_{n=0}^{\infty}\frac{(-1)^n\left(x/2\right)^{2n+1}}{n!\,(n+1)!}\underbrace{\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta}_{\frac{1}{2}B(n+1,\,1/2)} =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+2}\,\cancel{n!}\,(n+1)!}\,\frac{\cancel{\Gamma(n+1)}\,\overbrace{\Gamma(1/2)}^{\sqrt{\pi}}}{\Gamma(n+1+\frac{1}{2})} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\sqrt{\pi}}{2^{2n+2}\,\Gamma(n+2)\,\Gamma(n+3/2)} = (*)$$

+0,8

Duplicação com $z=n+\frac{3}{2}$ $\Big\} \Rightarrow 2^{2(z+\frac{3}{2})-1}\,\Gamma(n+\frac{3}{2})\,\Gamma(n+2) = \sqrt{\pi}\,\Gamma\left(2\left(n+\frac{3}{2}\right)\right) = \sqrt{\pi}\,(2n+2)!$

$$\therefore (*) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}\,(2n+2)!} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!} =$$

+0,7

$$= \frac{1}{x}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}\right] = \frac{1}{x}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^{n-1}x^{2n}}{(2n)!}\right]$$

$$= \frac{1}{x}\left[1-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right] = \frac{1}{x}\left(1-\cos x\right)$$

+0,5

Formulário (eventualmente útil)

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},\quad J_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta,$$

$$e^{x(t-t^{-1})/2}=\sum_{k=-\infty}^{+\infty}t^kJ_k(x),\qquad \Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt,\qquad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\qquad 2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z),$$

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},\qquad B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt\qquad B(z,w)=2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta.$$