EE 400 — Exame
10/12/08 — 2º sem/08
Solução:

1) Utiliza-se o teorema de Stokes:

$$I = \oint_C \vec{F} \cdot d\vec{r} = \int_S \text{rot}\, \vec{F} \cdot \vec{n} \, dA$$

Cálculo do rot $\vec{F}$:

$$\begin{array}{ccccc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{array}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}$$

$$= (1 - 3)\, \vec{i} + (2 - 1)\, \vec{j} + (1 - 2)\, \vec{k}$$

$$= -2\vec{i} + \vec{j} - \vec{k}$$

Portanto

$$\text{rot}\, \vec{F} \cdot \vec{n} = (-2\vec{i} + \vec{j} - \vec{k}) \cdot \frac{1}{3}(\vec{i} + 2\vec{j} + 2\vec{k})$$

$$= \frac{1}{3}(-2 + 2 - 2) = -\frac{2}{3}$$

Finalmente:

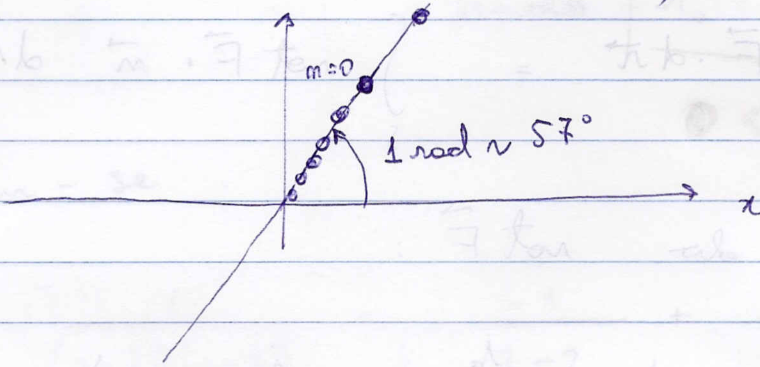
$$I = -\frac{2}{3} \int dA = -\frac{2}{3} \cdot \pi \cdot 2^2 = \boxed{-\frac{8\pi}{3}}$$

raio do disco = 2

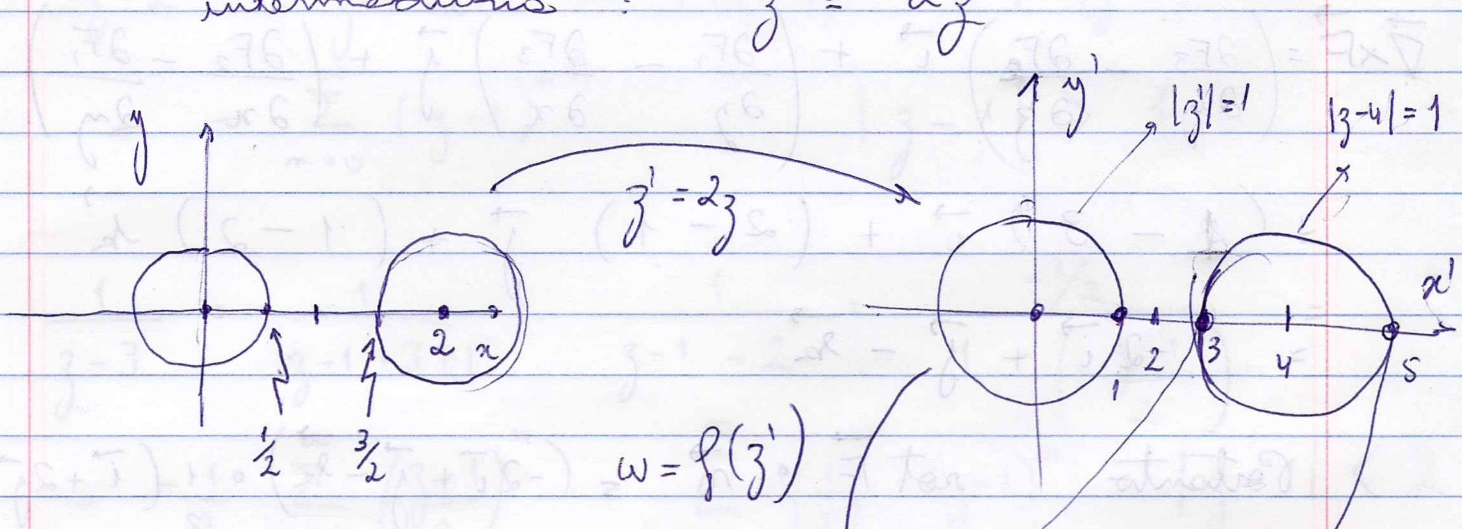
2) $e^{i} = e^{i \log e} = e^{i(\log e + i\,2n\pi)}$

$= e^{i(1 + i2n\pi)} = e^{-2n\pi + i} =$

$= e^{-2n\pi}\left(\cos 1 + i \,\text{sen}\, 1\right)$

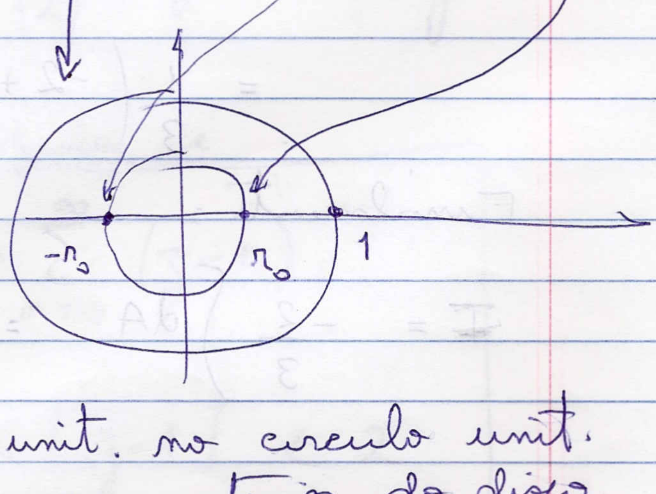


3) Vamos considerar uma transformação intermediária: $z' = 2z$



$$w = f(z') = e^{i\theta_0}\,\frac{z' - z_0}{z'\bar{z}_0 - 1}$$

$|z_0| > 1$



Esta função mapeia o círculo unit. no círculo unit. e mapeia o interior do disco no <u>exterior</u> do disco

Vamos considerar $z_0 = \overline{z_0}$ (z_0 é real)

e ainda:

$$\begin{cases} f(3) = r_0 \\ f(5) = -r_0 \end{cases} \Longrightarrow \begin{cases} r_0 e^{-i\theta_0} = \dfrac{3 - z_0}{3z_0 - 1} \\ -r_0 e^{-i\theta_0} = \dfrac{5 - z_0}{5z_0 - 1} \end{cases}$$

$$\Longrightarrow \frac{3-z_0}{3z_0-1} = -\frac{5-z_0}{5z_0-1} \Longrightarrow (3-z_0)(5z_0-1) = -(5-z_0)(3z_0-1)$$

$$\Longrightarrow -5z_0^2 + 16z_0 - 3 = -\left(-3z_0^2 + 16z_0 - 5\right)$$

$$\Longrightarrow 8z_0^2 - 32z_0 + 8 = 0 \Longrightarrow z_0^2 - 4z_0 + 1 = 0$$

$$z_0 = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$|z_0| > 1 \Longrightarrow \boxed{z_0 = 2 + \sqrt{3}}$$

$$r_0 e^{i\theta_0} = \frac{3 - 2 - \sqrt{3}}{3(2+\sqrt{3}) - 1} = \frac{5-\sqrt{3}}{5+3\sqrt{3}} =$$

$$= \frac{5-\sqrt{3}}{(5+3\sqrt{3})} \frac{(5-3\sqrt{3})}{(5-3\sqrt{3})} = \frac{25 - 20\sqrt{3} + 9}{25 - 27} =$$

$$= \frac{34 - 20\sqrt{3}}{-2} = -17 + 10\sqrt{3} \begin{array}{l} > 0 \\ < 1 \end{array} \quad \boxed{\text{logo: } e^{i\theta_0} = 1}$$

(obs: $1{,}7 < \sqrt{3} < 1{,}8$)

Em conclusão:

$$w = \frac{2z - z_0}{2z z_0 - 1} \quad \text{com} \quad \boxed{z_0 = 2 + \sqrt{3}}$$

$$\text{sendo} \quad \boxed{r_0 = 10\sqrt{3} - 17}$$

$$0 < r_0 < 1$$

4) Tem-se:

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$\frac{-1}{z-2} = \frac{-1}{z-1-2+1} = \frac{-1}{-1+(z-1)} = \frac{1}{1-(z-1)}$$

$$= -\sum_{n=0}^{\infty} (z-1)^{-n-1} \qquad |z-1| > 1$$

$$\frac{1}{z-3} = \frac{1}{z-1-3+1} = \frac{1}{z-1-2} = \frac{-1/2}{1-\left(\frac{z-1}{2}\right)} =$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n+1}(z-1)^n \qquad |z-1| < 2$$

Portanto a série desejada é:

$$f(z) = \sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n+1}(z-1)^n - \sum_{n=0}^{\infty}(z-1)^{-n-1}$$

$$1 < |z-1| < 2$$

5) $I = \int_{-\infty}^{\infty} \frac{dx}{(4+x^2)^2}$

Esta integral satisfaz: | grau (den) $\geq$ grau (num) + 2
| não tem polos reais

Portanto $I = \oint_C \frac{dz}{(4+z^2)^2}$

Sendo C um caminho que envolve todo o semi-plano superior.

Portanto $I = 2\pi i \, K$

onde K é o resíduo de

$f(z) = \frac{1}{(4+z^2)^2}$ em $z_0 = 2i$

$f(z) = \frac{1}{(z+2i)^2 (z-2i)^2}$

Claramente $z = 2i$ é um polo de 2ª ordem sendo a única singularidade no semi-plano superior.

$K = \phi'(2i)$ sendo $\phi(z) = \frac{1}{(z+2i)^2}$

Logo: $$\phi'(z) = \frac{-2}{(z+2i)^3}$$

$$\Longrightarrow \phi'(2i) = \frac{-2}{(4i)^3} = -\frac{i}{32} = K$$

Finalmente:

$$\boxed{I = 2\pi i\, K = \frac{\pi}{16}}$$