( )F520 ( )MS550 - Segunda Prova - 25/06/2012

RA: __________________ **Nome:** ____

(1) Encontre os autovalores e autofunções do problema de Sturm-Liouville

$$\begin{cases} xy'' + y' + \lambda xy = 0, \quad 0 < x < 1, \\ y(1) = 0, \quad \lim\limits_{x \to 0^+} |y(x)| < \infty. \end{cases}$$

(2) Seja ${}_2F_1(\alpha, \beta, \gamma; x)$ a função hipergeométrica. Mostre que

$${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{\gamma-\alpha-\beta} \; {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma; x).$$

(3) Sejam $P_n(x)$ os polinômios de Legendre de ordem n. Mostre que

$$\begin{array}{ll} \text{(i)} & P_n(-x) = (-1)^n P_n(x), \\ \text{(ii)} & (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x). \end{array}$$

(4) Sejam $J_m(x)$ as funções de Bessel de primeira espécie e ordem m. Mostre que

$$\begin{array}{ll} \text{(i)} & \displaystyle\int_0^\infty J_{2n+1}(ax)\,\mathrm{d}x = \frac{1}{a}, \quad n = 0, 1, \ldots \\ \text{(ii)} & \displaystyle\int_0^\infty \frac{J_{2n}(x)}{x}\,\mathrm{d}x = \frac{1}{2n}, \quad n = 1, 2, \ldots. \end{array}$$

(5) Em um mesmo gráfico faça um esboço do comportamento das funções de Bessel de primeira espécie de ordem zero, um e dois, identificando claramente os limites $x \to 0$ e $x \to \infty$ e o(s) zero(s) dessas funções. Faça o mesmo para as funções de Bessel modificadas de primeira espécie de ordem zero, um e dois.

ℹ Valor das questões: (1) 2,5 (2) 1,5 (3) 1,0 + 1,5 (4) 1,0 + 1,0 (5) 1,5.

Formulário Eventualmente Útil

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right], \qquad \nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \mathbf{e}_{q_1} & h_2 \mathbf{e}_{q_2} & h_3 \mathbf{e}_{q_3} \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix},$$

$$\nabla \cdot (f\mathbf{V}) = \mathbf{V} \cdot \nabla f + f \nabla \cdot \mathbf{V}, \qquad \nabla \times (f\mathbf{V}) = f \nabla \times \mathbf{V} + \nabla f \times \mathbf{V}, \qquad \nabla(fg) = f\nabla g + g \nabla f,$$

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{e}_{q_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{e}_{q_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{e}_{q_3}, \qquad h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta, \qquad \Gamma(1/2) = \sqrt{\pi},$$

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z),$$

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta\, d\theta, \qquad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt$$

$${}_2F_1(\alpha,\beta,\gamma;z) = \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \quad {}_2F_1(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt$$

$$U(a,b;z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt} t^{a-1}(1+t)^{-a+b-1}\,\mathrm{d}t, \quad \frac{\mathrm{d}^n U(a,b;z)}{\mathrm{d}z^n} = (-1)^n (a)_n U(a+n, b+n; z),$$

$$U(a,b;z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\, {}_1F_1(a-b+1, 2-b; z) + \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\, {}_1F_1(a,b;z), \quad {}_1F_1(a,b;z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}, \quad J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x), \quad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x),$$

$$\frac{d}{dx}(x^{-\nu}J_\nu(x)) = -x^{-\nu}J_{\nu+1}(x), \quad \frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x), \quad \mathrm{e}^{x(t-t^{-1})/2} = \sum_{k=-\infty}^{+\infty} t^k J_k(x)$$

$$J_n(u+v) = \sum_{m=-\infty}^{+\infty} J_m(u)J_{n-m}(v) \quad J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n J_0(x) \quad J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\, dt$$

$$J_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\, d\theta, \quad I_\nu(x) = i^{-\nu}J_\nu(ix)$$

$$P_n(x) = {}_2F_1(-n, n+1, 1; \tfrac{1-x}{2}), \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n], \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^\infty P_n(x)t^n,$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x), \quad (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x),$$

$$P_{n-1}'(x) = -nP_n(x) + xP_n'(x), \quad (1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x),$$

$$P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x), \quad (1-x)^{-\alpha} = \sum_{n=0}^\infty \frac{(\alpha)_n}{n!}x^n, \quad \int_{-1}^1 P_n(x)P_m(x)\, dx = \frac{2}{2n+1}\delta_{mn}.$$

(1) $xy'' + y' + \lambda xy = 0$

$\therefore x^2y'' + xy' + \lambda x^2 y = 0$

(i) $\boxed{\lambda < 0}$ $\quad \lambda = -k^2 \quad (k>0)$

$\therefore x^2y'' + xy' - k^2x^2y = 0 \Rightarrow y = A\,I_0(kx) + B\,K_0(kx)$

I_0: Bessel modificada 1ª espécie, ordem 0 $\quad$ K_0: Bessel modificada 2ª espécie, ordem 0

$\lim_{x\to 0^+} |y(x)| < \infty \Rightarrow \boxed{B=0}$ pois $\lim_{x\to 0^+} K_0(x) \to \infty$

$y(1) = 0 \Rightarrow A\,\underbrace{I_0(k)}_{\neq 0} = 0 \Rightarrow \boxed{A=0}$ $\therefore$ apenas solução trivial

+0,5

(ii) $\boxed{\lambda = 0}$

$xy'' + y' = (xy')' = 0 \Rightarrow xy' = c_1 \Rightarrow y = c_1 \ln x + c_2$

$\lim_{x\to 0^+} |y(x)| < \infty \Rightarrow \boxed{c_1 = 0}$ $\quad \therefore$ apenas solução trivial

$y(1) = c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

+0,5

(iii) $\boxed{\lambda > 0}$ $\quad \lambda = k^2 \quad (k>0)$

$x^2y'' + xy' + k^2x^2y = 0 \Rightarrow y = A\,J_0(kx) + B\,Y_0(kx)$

J_0: Bessel 1ª espécie ordem 0 $\quad$ Y_0: Bessel 2ª espécie ordem 0

$\lim_{x\to 0^+} |y(x)| < \infty \Rightarrow \boxed{B=0}$ pois $\lim_{x\to 0^+} Y(x) \to -\infty$

$y(1) = 0 = A\,J_0(k) = 0 \Rightarrow J_0(k) = 0 \Rightarrow k = \alpha_{0n}$ ← n-ésimo zero de $J_0(x)$ $(n = 1, 2, 3, \ldots)$

$\therefore$ autovalores: $\lambda_n = \alpha_{0n}^2$

autofunções: $y_n(x) = J_0(\alpha_{0n} x)$

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(2) $${}_2F_1(\alpha,\beta,\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt$$

Considere a mudança de variável de integração: $t \to z = \dfrac{1-t}{1-tx}$ (notas de aula pg. 130)

$$\therefore\ z - ztx = 1-t \Rightarrow t = \frac{1-z}{1-zx}$$

$$1-tx = 1-\frac{x(1-z)}{1-zx} = \frac{1-zx-x+zx}{1-zx} = \frac{1-x}{1-zx}$$

$t=0 \Rightarrow z=1$

$t=1 \Rightarrow z=0$

$$1-t = \frac{1-zx-(1-z)}{1-zx} = \frac{z(1-x)}{1-zx}$$

$$\frac{dt}{dz} = \frac{(-1)(1-zx)-(1-z)(-x)}{(1-zx)^2} = \frac{-(1-x)}{(1-zx)^2}$$

$$\therefore\ {}_2F_1(\alpha,\beta,\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_1^0 \frac{(1-z)^{\beta-1}}{(1-zx)^{\beta-1}}\, z^{\gamma-\beta-1}\frac{(1-x)^{\gamma-\beta-1}}{(1-zx)^{\gamma-\beta-1}}\frac{(1-x)^{-\alpha}}{(1-zx)^{-\alpha}}\frac{(1-x)(-dz)}{(1-zx)^2}$$

$$= \frac{(1-x)^{\gamma-\beta-1-\alpha+1}}{B(\beta,\gamma-\beta)}\int_0^1 z^{\gamma-\beta-1}(1-z)^{\beta-1}(1-zx)^{\overbrace{1-\beta-\gamma+\beta+1+\alpha-2}^{}}_{-(\gamma-\alpha)}\,dz$$

$\beta' = \gamma-\beta$ $\qquad \gamma' = \gamma$ $\qquad \alpha' = \gamma-\alpha$

$$= (1-x)^{\gamma-\beta-\alpha}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;x)$$

+1,5

(3) Veja Exemplo 4.11, pg. 166 das notas de aula

(i) (eq. 4.202) +1,0 ; (ii) (eq. 4.203) +1,5

④ (i) $\boxed{J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)}$

Para $\nu \neq 0 \Rightarrow \int_0^\infty J_{\nu-1}(x)dx - \int_0^\infty J_{\nu+1}(x)dx = 2\int_0^\infty J_\nu'(x)dx = 2\left[J_\nu(x)\right]_0^\infty$

$$= 0 - 0 = 0 \quad (\nu \neq 0)$$

$$\therefore \int_0^\infty J_{\nu+1}(x)dx = \int_0^\infty J_{\nu-1}(x)dx \qquad (\nu\neq 0)$$

$$\therefore \int_0^\infty J_{2n+1}(x)dx = \int_0^\infty J_{2n-1}(x)dx = \dots = \int_0^\infty J_1(x)dx$$

Mas: $\frac{d}{dx}\left(x^{-\nu}J_\nu(x)\right) = -x^{-\nu}J_{\nu+1}(x) \Rightarrow \frac{d}{dx}J_0(x) = -J_1(x)$

$$\therefore \int_0^\infty J_1(x)dx = -\int_0^\infty \frac{d}{dx}J_0(x)dx = -J_0(x)\Big|_0^\infty = 0-(-J_0(0)) = 1$$

$$\therefore \int_0^\infty J_{2n+1}(x)dx = 1, \quad n = 0,1,2,\dots$$

+1,0

(ii) $\boxed{\frac{2\nu}{x}J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x)}$

$\nu = 2n \quad 2(2n)\int_0^\infty \frac{J_{2n}(x)}{x}dx = \underbrace{\int_0^\infty J_{2n+1}(x)dx}_{1} + \underbrace{\int_0^\infty J_{2n-1}(x)dx}_{1} = 2$

$$\therefore \int_0^\infty \frac{J_{2n}(x)}{x}dx = \frac{1}{2n} \qquad n = 1,2\dots$$

+1,0

⑤ Veja notas de aula, pg <u>142</u> e pg <u>151</u>.

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