T_2 ( ) F520 ( ) MS550 · Nome: ____________________ RA: __________

Utilizando o método de Frobenius com $x_0 = 0$, encontre duas soluções linearmente independentes da equação

$$x^2y'' + (x^2 - x)y' + y = 0. \quad (*)$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \Longrightarrow \text{em}(*): \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} +$$

$$+ \sum_{n=0}^{\infty} a_n(n+r)x^{n+r+1} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\therefore \underbrace{a_0[r(r-1) - r + 1]}_{=0 \quad (1)} x^r + \sum_{n=1}^{\infty} \underbrace{[a_n(n+r)(n+r-1) - a_n(n+r) + a_n - a_{n-1}(n+r-1)]}_{=0 \quad (2)} x^{n+r} = 0$$

$$(1) \Rightarrow r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow \boxed{r_1 = r_2 = 1}$$

$$(2) \Rightarrow a_n(n+r-1)^2 + a_n(n+r-1) = 0, \quad n = 1, 2, 3 \ldots$$

$$\therefore n+r-1 \neq 0 \Rightarrow \boxed{a_n = \frac{-a_n}{n+r-1}} \quad (n = 1, 2, \ldots)$$

$\boxed{1^a \text{ solução}}$

$$\boxed{r = r_1 = r_2 = 1} \quad \therefore a_n = -\frac{a_{n-1}}{n}$$

$$\therefore a_1 = -\frac{a_0}{1}; \quad a_2 = -\frac{a_1}{2} = \frac{a_0}{2}; \quad a_3 = -\frac{a_2}{3} = \frac{-a_0}{3!} \quad \therefore \boxed{a_n = \frac{(-1)^n a_0}{n!}}$$

$n = 0, 1, 2, \ldots$

$$\therefore y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{n!} x^{n+1} = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = a_0 x e^{-x}$$

$$\therefore \boxed{Y_1(x) = x e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}}$$

1,0

$\boxed{2^a \text{ solução}}$

$$y_2(x) = Y_1(x)\ln x + x\sum_{n=1}^{\infty} b_n x^n$$

$$\therefore y_2'(x) = Y_1'(x)\ln x + Y_1(x)\frac{1}{x} + \sum_{n=1}^{\infty}(n+1)b_n x^n$$

$$y_2''(x) = Y_1''(x)\ln x + 2Y_1'(x)\frac{1}{x} - Y_1(x)\frac{1}{x^2} + \sum_{n=1}^{\infty} n(n+1)b_n x^{n-1}$$

em $(*) \Rightarrow$ $x^2 Y_1'' \ln x + 2 Y_1' x - Y_1 + \sum_{n=1}^{\infty} n(n+1) b_n x^{n+1}$

$+ (x^2 - x) Y_1' \ln x + x Y_1 - Y_1 + \sum_{n=1}^{\infty} (n+1) b_n x^{n+2} - \sum_{n=1}^{\infty} (n+1) b_n x^{n+1}$

$+ Y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+1} = 0$

$\parallel$

0 pois Y_1 satisfaz $(*)$

usando a expressão para $Y_1(x)$:

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (n+1) x^{n+1} - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} + \sum_{n=1}^{\infty} (n+1) b_n x^{n+2} + \sum_{n=1}^{\infty} n^2 b_n x^{n+1} = 0$$

$\therefore$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} x^{n+1} + \sum_{n=2}^{\infty} n b_{n-1} x^{n+1} + \sum_{n=1}^{\infty} n^2 b_n x^{n+1} = 0$$

$\boxed{n=1}$ $\quad -1 + b_1 = 0 \quad \therefore \boxed{b_1 = 1}$

$\boxed{n=2,3,\ldots}$ $\quad \frac{(-1)^n}{(n-1)!} + n b_{n-1} + n^2 b_n = 0 \quad \therefore \boxed{b_n = -\frac{b_{n-1}}{n} - \frac{(-1)^n}{n \cdot n!}}$

$\therefore b_2 = -\frac{1}{2} - \frac{1}{2 \cdot 2!} = -\frac{1}{2!}\left(1 + \frac{1}{2}\right)$

$b_3 = -\frac{1}{3}\left[\left(-\frac{1}{2!}\right)\left(1 + \frac{1}{2}\right)\right] - \frac{(-1)^3}{3 \cdot 3!} = \frac{1}{3!}\left(1 + \frac{1}{2} + \frac{1}{3}\right)$

$b_4 = -\frac{1}{4}\left[\frac{1}{3!}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] - \frac{(-1)^4}{4 \cdot 4!} = -\frac{1}{4!}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

$\therefore \boxed{b_n = \frac{(-1)^{n+1}}{n!} H_n, \quad \text{onde } H_n = \sum_{k=1}^{n} \frac{1}{k}}$ $\quad (n = 1, 2, \ldots)$

$$\therefore \boxed{\begin{aligned} Y_2(x) &= Y_1(x) \ln x + x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} H_n x^n \\ \therefore Y_2(x) &= x e^{-x} \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} H_n x^{n+1} \end{aligned}}$$

+1,0