Segunda Prova de MA311, Turma A

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Questão 1 (2,0 pontos). Resolva por transformada de Laplace o seguinte problema de valor inicial:

$$2y'' + y' + 4y = \delta\left(t - \frac{\pi}{6}\right) \operatorname{sen} t, \quad y(0) = y'(0) = 0.$$

Questão 2 (2,0 pontos). Calcular a seguinte transformada inversa de Laplace:

$$\mathcal{L}^{-1}\left\{\ln\left(x^3 + 3x^2 - 4\right)\right\}(t).$$

Questão 3. (2,0 pontos). Usando autovalores, autovetores e o método de variação de parâmetros, determine a solução geral do sistema:

$$\mathrm{x}'(t) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \mathrm{x}(t) + \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}.$$

Questão 4 (2,0 pontos). Determine a solução geral real do sistema:

$$t\mathrm{x}'(t) = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} \mathrm{x}(t).$$

Questão 5 (a) (1,0 ponto). Estude a convergência da sequência $(a_n)_{n\in\mathbb{N}}$, $a_n = \frac{2n^2+3}{n^2-n+1}\cos\frac{n\pi}{4}$, usando somente definição ou teorema dado em aula.

(b) (1,0 ponto) Calcule a soma da série $\sum_{n=2}^{\infty} \frac{3}{(4n-3)(4n+1)}$.

Observação: Justificar convenientemente todas as passagens e argumentos usados nas resoluções das questões acima.

1- $2y'' + y' + 4y = \delta(t-\pi/6)\,\text{sen}\,t\;, \quad y(0) = y'(0) = 0$

$$\mathcal{L}\{\delta(t-\pi/6)\,\text{sen}\,t\}(x) = e^{-x\pi/6}\,\text{sen}\,\pi/6 = \frac{1}{2}e^{-x\pi/6}$$

$$\mathcal{L}\{2y''+y'+4y\}(x) = 2\left(x^2Y - xy(0) - y'(0)\right) + \left(xY - y(0)\right) + 4Y$$

$$= Y(2x^2+x+4) \quad \text{onde } Y = \mathcal{L}\{y(t)\}(x)$$

$$Y = \frac{1}{2}\,\frac{e^{-x\pi/6}}{2x^2+x+4}$$

0,8

$$= \frac{1}{4}e^{-x\pi/6}\,\frac{1}{\left(x^2+\frac{x}{2}+\frac{1}{16}\right)+\frac{31}{16}}$$

$$= \frac{1}{\sqrt{31}}e^{-x\pi/6}\,\frac{\sqrt{31}/4}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{31}}{4}\right)^2}$$

0,4

$$= \frac{1}{\sqrt{31}}e^{-x\pi/6}\,\mathcal{L}\left\{\text{sen}\,\frac{\sqrt{31}}{4}t\right\}\left(x+\frac{1}{4}\right)$$

$$= \frac{1}{\sqrt{31}}e^{-x\pi/6}\,\mathcal{L}\left\{e^{-t/4}\,\text{sen}\,\frac{\sqrt{31}}{4}t\right\}(x)$$

0,4

$$= \mathcal{L}\left\{\frac{1}{\sqrt{31}}\,u_{\pi/6}(t)\,e^{-(t-\pi/6)/4}\,\text{sen}\,\frac{\sqrt{31}}{4}(t-\pi/6)\right\}(x)$$

Resposta:

$$y(t) = \frac{1}{\sqrt{31}}\,u_{\pi/6}(t)\,e^{-(t-\pi/6)/4}\,\text{sen}\,\frac{\sqrt{31}}{4}(t-\pi/6).$$

0,4

2- $\mathcal{L}^{-1}\left\{ \ln(x^3+3x^2-4)\right\}(t) = f(t)$

$$\ln(x^3+3x^2-4) = \mathcal{L}\{f(t)\}(x)$$

$$\Rightarrow \frac{d}{dx}\left[\ln(x^3+3x^2-4)\right] = \frac{d}{dx}\mathcal{L}\{f(t)\}(x)$$

$$\Rightarrow \frac{3x^2+6x}{x^3+3x^2-4} = \mathcal{L}\{-t f(t)\}(x)$$

0,7

$$\frac{3x^2+6x}{x^3+3x^2-4} = \frac{3x(x+2)}{(x+2)^2(x-1)} = \frac{3x}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$$

$$\frac{3x}{(x+2)(x-1)} = \frac{3x}{x^2+x-2}$$

$$\begin{cases} x=1 \Rightarrow B=1 \\ x=-2 \Rightarrow A=2 \end{cases}$$

$$\frac{3x^2+6x}{x^3+3x^2-4} = \frac{2}{x+2}+\frac{1}{x-1}$$

0,6

$$= 2\,\mathcal{L}\{e^{-2t}\}(x) + \mathcal{L}\{e^{t}\}(x)$$

$$= \mathcal{L}\{2e^{-2t}+e^{t}\}(x)$$

0,4

$$-t f(t) = 2e^{-2t}+e^{t}$$

$$f(t) = \frac{-2e^{-2t}-e^{t}}{t}$$

0,3

3- $X'(t) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} X(t) + \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$

$$p(\lambda) = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$$

$\underline{\underline{\lambda_1 = 4}}$: $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow 3x - y = 4x \Rightarrow y = -x$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v_1}, \ x \neq 0$ (autovetores associados ao autovalor $\lambda_1 = 4$)

0,2

$\underline{\underline{\lambda_2 = 2}}$: $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow 3x - y = 2x \Rightarrow y = x$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ x \neq 0$ (autovetores associados ao autovalor $\lambda_2 = 2$)

0,2

$X_H(t) = C_1 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}}_{y_1(t)} + C_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}_{y_2(t)}$ (solução geral do sistema homogêneo associado)

0,4

$\Phi(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{bmatrix}$ <u>matriz fundamental</u>

$X_p(t) = \Phi(t)\, U(t)$ solução particular do sistema dado

$\Phi(t)\, U'(t) = B(t)$

$B(t) = \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix} \qquad U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$

$$W = W\left(y_1(t), y_2(t)\right) = \begin{vmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{vmatrix} \neq 2e^{6t}$$

$$\begin{bmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix} \Longleftrightarrow \begin{cases} e^{4t} u_1' + e^{2t} u_2' = 4e^{2t} \\ -e^{4t} u_1' + e^{2t} u_2' = 4e^{4t} \end{cases}$$

0,3

$$u_1' = \frac{\begin{vmatrix} 4e^{2t} & e^{2t} \\ 4e^{4t} & e^{2t} \end{vmatrix}}{W} = \frac{1}{2} e^{-6t}\left(4e^{4t} - 4e^{6t}\right) = 2e^{-2t} - 2$$

$$u_1 = \int \left(2e^{-2t} - 2\right) dt = -e^{-2t} - 2t$$

0,3

$$u_2' = \frac{\begin{vmatrix} e^{4t} & 4e^{2t} \\ -e^{4t} & 4e^{4t} \end{vmatrix}}{W} = \frac{1}{2} e^{-6t}\left(4e^{8t} + 4e^{6t}\right) = 2e^{2t} + 2$$

$$u_2 = \int \left(2e^{2t} + 2\right) dt = e^{2t} + 2t$$

0,3

$$X_p(t) = \begin{bmatrix} e^{4t} & e^{2t} \\ -e^{4t} & e^{2t} \end{bmatrix} \begin{bmatrix} -e^{-2t} - 2t \\ e^{2t} + 2t \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -1 + 2t \\ 1 + 2t \end{bmatrix} + e^{4t} \begin{bmatrix} 1 - 2t \\ 1 + 2t \end{bmatrix}$$

<u>Resposta</u> : $X(t) = X_H(t) + X_p(t)$

0,3

4- (*) $t X'(t) = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} X(t)$

$$x = \ln t \Rightarrow \frac{dX}{dt} = \frac{dX}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dX}{dx}$$

(**) $X'(x) = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} X(x)$

0,4

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 2 = \lambda^2 - 4\lambda + 5$$

$\lambda_1 = 2+i$ e $\lambda_2 = \overline{\lambda}_1 = 2-i$ são os autovalores

$\underline{\underline{\lambda_1 = 2+i}}$: $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (2+i)\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x = y(-1-i)$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$, $y \neq 0$, $v_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

0,5

$$\phi(x) = v_1 e^{\lambda_1 x} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} e^{2x}(\cos x + i \operatorname{sen} x)$$

$$= \underbrace{e^{2x}\begin{bmatrix} -\cos x + \operatorname{sen} x \\ \cos x \end{bmatrix}}_{u_1(x)} + i\,\underbrace{e^{2x}\begin{bmatrix} -\operatorname{sen} x - \cos x \\ \operatorname{sen} x \end{bmatrix}}_{u_2(x)}$$

0,5

$$X(x) = C_1 e^{2x}\begin{bmatrix} -\cos x + \operatorname{sen} x \\ \cos x \end{bmatrix} + C_2 e^{2x}\begin{bmatrix} -\operatorname{sen} x - \cos x \\ \operatorname{sen} x \end{bmatrix}$$ (solução de (**))

0,2

$$X(t) = C_1 t^2\begin{bmatrix} -\cos \ln t + \operatorname{sen} \ln t \\ \cos \ln t \end{bmatrix} + C_2 t^2\begin{bmatrix} -\operatorname{sen} \ln t - \cos \ln t \\ \operatorname{sen} \ln t \end{bmatrix}$$ (solução de (*))

0,4

5a) $$a_n = \frac{2n^2+3}{n^2-n+1}\cos\left(\frac{n\pi}{4}\right)$$

$$\frac{n\pi}{4} = 2\pi p \implies n = 8p$$

$$b_p = a_{8p} = \frac{128p^2+3}{64p^2-8p+1}\cos(2\pi p) = \frac{128p^2+3}{64p^2-8p+1}$$

$$\lim_{p\to\infty} b_p = \lim_{p\to\infty} \frac{128+\frac{3}{p^2}}{64-\frac{8}{p}+\frac{1}{p^2}} = \frac{128+0}{64-0+0} = 2$$

$$\frac{n\pi}{4} = 2\pi p + \frac{\pi}{2} \implies n = 8p+2$$

$$c_p = a_{8p+2} = \frac{2(8p+2)^2+3}{(8p+2)^2-(8p+2)+1}\cos\left(2\pi p + \frac{\pi}{2}\right) = 0$$

$$\lim_{p\to\infty} c_p = \lim_{p\to\infty} 0 = 0.$$

$$\begin{cases} (c_p) \text{ e } (b_p) \text{ são duas subsequências convergentes } (a_n) \\ \lim\limits_{p\to\infty} c_p = 0 \neq 2 = \lim\limits_{p\to\infty} b_p \end{cases}$$

$\implies$ a sequência (a_n) diverge por teorema dado em aula

1,0

5b) $$\sum_{m=2}^{\infty} \underbrace{\frac{3}{(4m-3)(4m+1)}}_{a_m}$$

$$\frac{1}{4m-3} - \frac{1}{4m+1} = \frac{4}{(4m-3)(4m+1)}$$

$$a_m = \frac{3}{4}\left[\frac{1}{4m-3} - \frac{1}{4m+1}\right]$$

0,2

$$S_m = a_2 + a_3 + \cdots + a_{m-1} + a_m$$

$$= \frac{3}{4}\left[\left(\frac{1}{5} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{13}}\right) + \cdots + \left(\cancel{\frac{1}{4m-7}} - \cancel{\frac{1}{4m-3}}\right) + \left(\cancel{\frac{1}{4m-3}} - \frac{1}{4m+1}\right)\right]$$

$$= \frac{3}{4}\left(\frac{1}{5} - \frac{1}{4m+1}\right)$$

0,4

$$S = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{3}{4}\left(\frac{1}{5} - \frac{1}{4m+1}\right)$$

$$= \frac{3}{20} - \lim_{m\to\infty} \frac{3/4}{4m+1} =$$

$$= \frac{3}{20} - \frac{3/4}{\infty} = \frac{3}{20} - 0 = \frac{3}{20}$$

é a soma da série dada

0,4