Soluções para MS550, Métodos de Matemática Aplicada I, e F520, Métodos Matemáticos da Física I

Lista 6 - Função de Bessel

Equações eventualmente útil:

(ST) $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(GE) $$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

(1) $$\Gamma(z+1) = z\Gamma(z),\ \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$$

(2) $$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z)$$

(BG) $$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

(BT) $$B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta d\theta$$

(BI) $$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$$

(SP) $$(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1),\ (\alpha)_0 = 1$$

(3) $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

(4) $$\frac{(\alpha)_n}{m!} = \binom{\alpha+n-1}{n},\ \frac{(-\alpha)_n}{n!} = (-1)^n \binom{\alpha}{n}$$

(EB) $$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

(5) $$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

(6) $$J_\nu(x) = \frac{x^\nu \exp(-ix)}{2^\nu \Gamma(\nu+1)} {}_1F_1(\nu+1/2, 2\nu+1; 2ix)$$

(7) $$J_{-\nu}(x) = (-1)^\nu J_\nu(x)$$

(GFB) $$\exp(x(t-t^{-1})/2) = \sum_{k=-\infty}^{+\infty} t^k J_k(x)$$

(EBM) $$x^2y'' + xy' - (x^2 + v^2)y = 0$$

(8) $$I_\nu(x) = i^{-\nu} J_\nu(x)$$

1. Mostre, diretamente a partir da definição, que
   (a) (Equação 18.92 do Riley) $\mathrm{d}[x^\nu J_\nu(x)]/\mathrm{d}x = x^\nu J_{\nu-1}(x)$

   Solução: *Temos que*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x} x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \frac{x^{2k+\nu}}{2^{2k+\nu}} && \textit{por } (5)\\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} x^\nu \frac{(-1)^k x^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}\\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(-1)^k(2k+2\nu)x^{2k+2\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(k+\nu)x^{2k+2\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu-1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^k(k+\nu)x^{2k+2\nu-1}}{k!(k+\nu)\Gamma(k+\nu)2^{2k+\nu-1}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2\nu-1}}{k!\Gamma(k+\nu)2^{2k+\nu-1}} \\
&= x^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+(\nu-1)}}{k!\Gamma(k+(\nu-1)+1))2^{2k+(\nu-1)}} \\
&= x^{\nu} J_{\nu-1}(x) && \textit{por } (5).
\end{aligned}
$$

(b) $\mathrm{d}[x^{-\nu}J_\nu(x)]/\mathrm{d}x = -x^{-\nu}J_{\nu+1}(x)$

Solução: *Temos que*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\nu}J_\nu(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x} x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \frac{x^{2k+\nu}}{2^{2k+\nu}} && \textit{por } (5) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} x^{-\nu} \frac{(-1)^k x^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k) x^{-\nu} x^{2k+\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k (2k) x^{2k+\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k k x^{2k+\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu-1}} \\
&= x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu-1}}{(k-1)!\Gamma(k+\nu+1)2^{2k+\nu-1}} \\
&= -x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k-1} x^{2(k-1)+\nu+1}}{(k-1)!\Gamma((k-1)+(\nu+1)+1)2^{2(k-1)+\nu+1}} \\
&= x^{-\nu} J_{\nu+1}(x) && \textit{por } (5).
\end{aligned}
$$

(c) $J_{\nu-1}(x) + J_{\nu+1}(x) = (2\nu/x)J_\nu(x)$

Solução: *Temos que*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\nu}J_{\nu}(x)\right] = \nu x^{\nu-1}J_{\nu}(x) + x^{\nu}J_{\nu}'(x) = x^{\nu}J_{\nu-1}(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}J_{\nu}(x)\right] = -\nu x^{-\nu-1}J_{\nu}(x) + x^{-\nu}J_{\nu}'(x) = -x^{-\nu}J_{\nu+1}(x).$$

Então, para $x \neq 0$*, temos que*

$$\nu J_{\nu}(x) + xJ_{\nu}'(x) = xJ_{\nu-1}(x),$$
$$-\nu J_{\nu}(x) + xJ_{\nu}'(x) = -xJ_{\nu+1}(x).$$

Logo,

$$\begin{aligned}\nu J_{\nu}(x) + xJ_{\nu}'(x) - (-\nu J_{\nu}(x) + xJ_{\nu}'(x)) &= xJ_{\nu-1}(x) - (-xJ_{\nu+1}(x))\\ 2\nu J_{\nu}(x) &= xJ_{\nu-1}(x) + xJ_{\nu+1}(x)\\ 2\nu x^{-1}J_{\nu}(x) &= J_{\nu-1}(x) + J_{\nu+1}(x).\end{aligned}$$

Também é possível utilizar (GFB) *para mostrar a igualdade pedida (ver detalhes em Riley).*

(d) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$

Solução: *Temos que*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\nu}J_{\nu}(x)\right] = \nu x^{\nu-1}J_{\nu}(x) + x^{\nu}J_{\nu}'(x) = x^{\nu}J_{\nu-1}(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}J_{\nu}(x)\right] = -\nu x^{-\nu-1}J_{\nu}(x) + x^{-\nu}J_{\nu}'(x) = -x^{-\nu}J_{\nu+1}(x).$$

Então, para $x \neq 0$*, temos que*

$$\nu J_{\nu}(x) + xJ_{\nu}'(x) = xJ_{\nu-1}(x),$$
$$-\nu J_{\nu}(x) + xJ_{\nu}'(x) = -xJ_{\nu+1}(x).$$

Logo,

$$\begin{aligned}\nu J_{\nu}(x) + xJ_{\nu}'(x) + (-\nu J_{\nu}(x) + xJ_{\nu}'(x)) &= xJ_{\nu-1}(x) + (-xJ_{\nu+1}(x))\\ 2xJ_{\nu}'(x) &= xJ_{\nu-1}(x) - xJ_{\nu+1}(x)\\ 2J_{\nu}'(x) &= J_{\nu-1}(x) - J_{\nu+1}(x).\end{aligned}$$

2. (Exercício 11.1.4 do Arfken) Mostre que vale a expressão de Jacobi-Anger

$$\exp(ix\cos\theta) = \sum_{k=-\infty}^{+\infty} i^{k}J_{k}(x)\exp(ik\theta).$$

Solução: *Fazendo que* $t = i\exp(i\theta)$, *temos que*

$$\begin{aligned}\frac{t-t^{-1}}{2} &= \frac{i\exp(i\theta)}{2} - \frac{\exp(-i\theta)}{2}\\ &= \frac{i\exp(i\theta)}{2} + \frac{i\exp(-i\theta)}{2}\\ &= i\left(\frac{\exp(i\theta)+\exp(-i\theta)}{2}\right)\\ &= i\cos\theta.\end{aligned}$$

Logo, por (GFB) *para* $t = i\exp(i\theta)$ *temos*

$$\begin{aligned}\exp(ix\cos\theta) &= \sum_{k=-\infty}^{+\infty} (i\exp(i\theta))^k J_k(x)\\ &= \sum_{k=-\infty}^{+\infty} i^k \exp(ik\theta) J_k(x).\end{aligned}$$

3. (Exercício 11.1.5 do Arfken) Mostre que

   (a) $\cos x = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)$,

   Solução: *Fazendo* $(t-t^{-1})2^{-1} = i$ *temos que*

   $$\exp(ix) = \sum_{k=-\infty}^{+\infty} i^k J_k(x) \qquad\qquad \textit{por (GFB)},$$

   e fazendo $(t-t^{-1})2^{-1} = -i$ *temos que*

   $$\exp(-ix) = \sum_{k=-\infty}^{+\infty} (-i)^k J_k(z). \qquad\qquad \textit{por (GFB)}$$

   Então,

   $$\begin{aligned}\cos(x) &= (\exp(ix) + \exp(-ix))\, 2^{-1}\\ &= \left[\sum_{k=-\infty}^{+\infty} J_k(x)(-1)^k i^k + \sum_{k=-\infty}^{+\infty} J_k(x) i^k\right] 2^{-1}\\ &= \left[\sum_{k=-\infty}^{+\infty} J_{-k}(x) i^k + \sum_{k=-\infty}^{+\infty} J_k(x) i^k\right] 2^{-1}\\ &= \left[\sum_{k=-\infty}^{+\infty} (J_{-k}(x) + J_k(x))\, i^k\right] 2^{-1} && \textit{por } \bigstar\\ &= \left[2\sum_{k=-\infty}^{+\infty} J_k(x) i^k\right] 2^{-1}\end{aligned}$$

$$= \sum_{k=-\infty}^{+\infty} J_k(x) i^k$$
$$= J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x),$$

onde ★ corresponde a soma de k's simétricos.
Para k ímpar, i.e., $k = 2\bar{k}$, temos

$$\sum_{k=0}^{\infty} J_{2k+1}(x) i^{-2k+1} + \sum_{k=0}^{\infty} J_{2k+1}(x) i^{2k+1} = 0$$

e para k par, i.e., $k = 2\bar{k}$, temos

$$\sum_{k=1}^{\infty} J_{2k}(x) i^{-2k} + \sum_{n=1}^{\infty} J_{2k}(x) i^{2k} = 2\sum_{n=1}^{\infty} J_{2k}(z) i^{2k}.$$

Logo,

$$\cos(x) = J_0(x) + 2\sum_{k=0}^{\infty}(-1)^n J_{2n}(z).$$

(b) $\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$.
Solução: *Fazendo $t = i$ em (GFB) temos que*

$$\exp(ix^m) = \sum_{k=-\infty}^{+\infty} J_k(x) i^k,$$

e fazendo $t = -i$ temos que

$$\exp(-ix) = \sum_{k=-\infty}^{+\infty} J_k(x)(-1)^k i^k.$$

Então temos que

$$\begin{aligned}
\sin(x) &= (\exp(ix) - \exp(-ix))\,(2i)^{-1} \\
&= (-i)2^{-1}\,(\exp(ix) - \exp(-ix)) \\
&= (-i)2^{-1}\left[\sum_{k=-\infty}^{+\infty} J_k(x) i^k - \sum_{k=-\infty}^{+\infty} J_k(x)(-1)^k i^k\right] \\
&= (-i)2^{-1}\left[J_0(x) + 2\sum_{k=1}^{+\infty} J_k(x) i^k - J_0(x) - 2\sum_{k=1}^{+\infty} J_k(x)(-1)^k i^k\right] \\
&= (-i)2^{-1}\left[2\sum_{k=1}^{+\infty} J_k(x) i^k - 2\sum_{k=1}^{+\infty} J_k(x)(-1)^k i^k\right]
\end{aligned}$$

$$= -\sum_{k=1}^{+\infty} J_k(x) i^{k+1} + \sum_{k=1}^{+\infty} J_k(x)(-1)^k i^{k+1}.$$

Para k par, i.e., $k = 2\bar{k}$, temos

$$-\sum_{k=1}^{\infty} J_{2k}(x) i^{2k+1} + \sum_{k=1}^{\infty} J_{2k}(x) i^{2k+1} = 0$$

e para k ímpar, i.e., $k = 2\bar{k} + 1$, temos

$$\begin{aligned} -\sum_{k=0}^{\infty} J_{2k+1}(x) i^{2k+2} + \sum_{k=0}^{\infty} J_{2k+1}(x)(-1)^{2k+1} i^{2k+2} &= -\sum_{n=0}^{\infty} J_{2k+1}(-1)^k(-1)^k \\ &\quad + \sum_{k=0}^{\infty} J_{2k+1}(x)(-1)(-1)^k(-1) \\ &= 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(x). \end{aligned}$$

Logo,

$$\sin(x) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(x).$$

4. (Exercício 11.1.2 do Arfken) Mostre, a partir da função geratriz, que vale a fórumla de adição para as funções de Bessel: $J_n(u+v) = \sum_{m=-\infty}^{+\infty} J_m(u) J_{n-m}(v)$.

Solução: *Temos que*

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} t^n J_n(u+v) &= \exp\big((u+v)(t-t^{-1})/2\big) && \textit{por } (\text{GFB}) \\ &= \exp(u(t-t^{-1})/2)\exp(v(t-t^{-1})/2) \\ &= \sum_{m=-\infty}^{+\infty} t^m J_m(u) \sum_{l=-\infty}^{+\infty} t^l J_l(v) && \textit{por } (\text{GFB}) \\ &= \sum_{m=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} t^{m+l} J_m(u) J_l(v) \\ &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} t^n J_m(u) J_{n-m}(v) && n = l+m \\ &= \sum_{n=-\infty}^{+\infty} t^n \sum_{m=-\infty}^{+\infty} J_m(u) J_{n-m}(v) \end{aligned}$$

Por comparação termo a termos, concluimos que

$$J_n(u+v) = \sum_{m=-\infty}^{+\infty} J_m(u) J_{n-m}(v).$$

5. (Exercício 11.1.10 do Arfken) Mostre que

$$J_n(x) = (-1)^n z^n \left(x^{-1}\,\mathrm{d}/\mathrm{d}x\right)^n J_0(x).$$

Solução: *Iremos mostrar por indução finita.*

Para $n = 0$ temos que

$$\begin{aligned} J_0(x) &= (-1)^0 (x)^0 \left(x^{-1}\,\mathrm{d}/\mathrm{d}x\right)^0 J_0(x) \\ &= J_0(x). \end{aligned}$$

Como hipótese de indução temos que

$$J_n(x) = (-1)^n x^n \left(x^{-1}\,\mathrm{d}/\mathrm{d}x\right)^n J_0(x).$$

E como tese de indução

$$J_{n+1}(x) = (-1)^{n+1} x^{n+1} \left(x^{-1}\,\mathrm{d}/\mathrm{d}x\right)^{n+1} J_0(x).$$

Mas pelo exercício 1(b) desta lista temos que

$$(\mathrm{d}/\mathrm{d}x)\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \qquad \nu = n$$

e manipulando a expressão acima temos que

$$\begin{aligned} J_{n+1}(x) &= -x^n\,(\mathrm{d}/\mathrm{d}x)\left(x^{-n} J_n(x)\right) \\ &= -x^n\,(\mathrm{d}/\mathrm{d}x)\left[x^{-n}(-1)^n x^n \left(x^{-1}\mathrm{d}/\mathrm{d}x\right)^n J_0(x)\right] && \text{por } \bigstar \\ &= (-1)^{n+1} x^n\,(\mathrm{d}/\mathrm{d}x)\left[\left(x^{-1}\mathrm{d}/\mathrm{d}x\right)^n J_0(x)\right] \\ &= (-1)^{n+1} x^{n+1} z^{-1}\,(\mathrm{d}/\mathrm{d}z)\left[\left(z^{-1}\mathrm{d}/\mathrm{d}z\right)^n J_0(z)\right] \\ &= (-1)^{n+1} x^{n+1} \left(z^{-1}\mathrm{d}/\mathrm{d}z\right)^{n+1} J_0(z), \end{aligned}$$

onde a última expressão corresponde a nossa tese de indução e $\bigstar$ a hipótese de indução.

6. (E de 2006, Exercício 11.1.9 do Arfken) Seja $J_0(z)$ a função de Bessel de primeira espécie e ordem zero. Mostre que

$$J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

Solução: *Temos que*

$$
\begin{aligned}
\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{(1-t^2)^{1/2}}\,\mathrm{d}t &= \frac{2}{\pi}\int_0^1 (1-t^2)^{-1/2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(zt)^{2n}}{(2n)!}\right]\mathrm{d}t && \textit{expansão em série}\\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\int_0^1 (1-t^2)^{-1/2}(t^2)^n\,\mathrm{d}t\\
&= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\int_0^1 (1-y)^{-1/2}y^{n-1/2}\,\mathrm{d}y && t^2=y\\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}B(n+1/2,1/2) && \textit{por (BI)}\\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)} && \textit{por (BG)}\\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\frac{\sqrt{\pi}\sqrt{\pi}\Gamma(2n)}{n!\Gamma(n)2^{2n-1}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\frac{\Gamma(2n)}{n!\Gamma(n)2^{2n-1}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!}\frac{2n\Gamma(2n)}{(2n)!n\Gamma(n)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!}\frac{\Gamma(2n+1)}{(2n)!n\Gamma(n)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!}\frac{\Gamma(2n+1)}{(2n)!\Gamma(n+1)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!}\frac{1}{\Gamma(n+1)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!n!}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+0+1)}\frac{z^{2n}}{2^{2n}}\\
&= J_0(z) && \textit{por (5).}
\end{aligned}
$$

7. Mostre que

$$
J_v(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,\mathrm{d}\theta.
$$

Solução: *Temos que*

$$
\begin{aligned}
\int_0^{\pi/2} \cos(x\sin\theta)\cos^{2\nu}\theta\,\mathrm{d}\theta &= \int_0^{\pi/2}\sum_{k=0}^{\infty}\frac{(-1)^k(x\sin\theta)^{2k}}{(2k)!}\cos^{2\nu}\theta\,\mathrm{d}\theta && \textit{série de Taylor}\\
&= \sum_{k=0}^{\infty}\int_0^{\pi/2}\frac{(-1)^k x^{2k}\sin^{2k}\theta}{(2k)!}\cos^{2\nu}\theta\,\mathrm{d}\theta\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\int_0^{\pi/2}\sin^{2k}\theta\cos^{2\nu}\theta\,\mathrm{d}\theta\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}2^{-1}B(k+1/2,\nu+1/2) && \textit{por } (\mathrm{BT})\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}2^{-1}\frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)} && \textit{por } (\mathrm{BG})\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}2^{-1}\frac{\sqrt{\pi}\Gamma(2k)}{2^{2k-1}\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)} && \textit{por } (2)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\frac{\sqrt{\pi}\Gamma(2k)}{2^{2k}\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)} && \textit{por } (2)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{(2k)!}\frac{\sqrt{\pi}\Gamma(2k)}{\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{\Gamma(2k+1)}\frac{\sqrt{\pi}\Gamma(2k)}{\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{2k\Gamma(2k)}\frac{\sqrt{\pi}\Gamma(2k)}{\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{2k}\frac{\sqrt{\pi}}{\Gamma(k)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{2}\frac{\sqrt{\pi}}{\Gamma(k+1)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k}}{2}\frac{\sqrt{\pi}}{\Gamma(k+1)}\frac{\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}
\end{aligned}
$$

Logo,

$$
\begin{aligned}
\frac{2(x/2)^{\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\ \mathrm{d}\theta &= \frac{2(x/2)^{\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}\sqrt{\pi}\Gamma(\nu+1/2)}{2\Gamma(k+1)\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{2(x/2)^{\nu}(-1)^k(x/2)^{2k}\sqrt{\pi}\Gamma(\nu+1/2)}{\sqrt{\pi}\Gamma(\nu+1/2)2\Gamma(k+1)\Gamma(k+\nu+1)}\\
&= \sum_{k=0}^{\infty}\frac{2(x/2)^{\nu}(-1)^k(x/2)^{2k}\sqrt{\pi}}{\sqrt{\pi}2\Gamma(k+1)\Gamma(k+\nu+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(x/2)^{\nu}(-1)^k (x/2)^{2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \\
&= J_{\nu}(x)
\end{aligned}
$$

8. (Exercício V-33 do Farrell) Mostre que $\int_0^{\infty} J_0(x)\,\mathrm{d}x = 1$, assumindo que a integral $\int_0^{\infty} J_n(x)\,\mathrm{d}x$, $n = 0, 1, 2, \ldots$ é convergente e dado que

$$
(\bigstar) \qquad \int_0^{\pi} \frac{\cos(n\phi)\,\mathrm{d}\phi}{a - ib\cos\phi} = \frac{\pi i^n}{\sqrt{a^2+b^2}} \left[\frac{\sqrt{a^2+b^2} - a}{b}\right]^n, \qquad b \neq 0, i = \sqrt{-1},
$$

$$
(\bigstar\bigstar) \qquad J_n(x) = \frac{(-i)^n}{\pi} \int_0^{\pi} \exp(ix\cos\phi)\cos(n\phi)\,\mathrm{d}\phi, \quad i = \sqrt{-1}, n = 0, 1, 2, \ldots.
$$

Solução: *Se a é qualquer constante maior que zero, então a integral $\int_0^{\infty} \exp(-ax)J_n(bx)\,\mathrm{d}x$ é convergente. Então, utilizando ($\bigstar\bigstar$) temos*

$$
\begin{aligned}
\int_0^{\infty} \exp(-ax)J_n(bx)\,\mathrm{d}x &= \int_0^{\infty} \exp(-ax)\left[\frac{(-i)^n}{n}\int_0^{\pi} \exp(ibx\cos\phi)\cos(n\phi)\,\mathrm{d}\phi\right]\mathrm{d}x \\
&= \frac{(-i)^n}{n}\int_0^{\infty} \exp(-ax)\left[\int_0^{\pi} \exp(ibx\cos\phi)\cos(n\phi)\,\mathrm{d}\phi\right]\mathrm{d}x
\end{aligned}
$$

Se considerarmos x e ϕ como um plano de coordenadas retangulares, então podemos considerar a integral multipla anterior equivalente a integral dupla na região R do plano $x\phi$ correspondente aos pontos (x, ϕ) tal que $x \geq 0$ e $0 \leq \phi \leq \phi$. E assim a integral dupla na região R pode ser igual a integral múltipla na qual a ordem de integração encontra-se invertida. Então

$$
\begin{aligned}
\int_0^{\infty} \exp(-ax)J_n(bx)\,\mathrm{d}x &= \frac{(-i)^n}{n}\int_0^{\pi}\left[\int_0^{\infty} \exp(-ax)\exp(ibx\cos\phi)\cos(n\phi)\,\mathrm{d}x\right]\mathrm{d}\phi \\
&= \frac{(-i)^n}{n}\int_0^{\pi}\left[\int_0^{\infty} \exp(-ax)\exp(ibx\cos\phi)\,\mathrm{d}x\right]\cos(n\phi)\,\mathrm{d}\phi \\
&= \frac{(-i)^n}{n}\int_0^{\pi}\left[\int_0^{\infty} \exp(-(a-ib\cos\phi)x)\,\mathrm{d}x\right]\cos(n\phi)\,\mathrm{d}\phi \\
&= \frac{(-i)^n}{n}\int_0^{\pi}\left[\left.\frac{\exp(-(a-ib\cos\phi)x)}{-(a-ib\cos\phi)}\right|_{x=0}^{x=\infty}\right]\cos(n\phi)\,\mathrm{d}\phi \\
&= \frac{(-i)^n}{n}\int_0^{\pi} \frac{1}{a-ib\cos\phi}\cos(n\phi)\,\mathrm{d}\phi \\
&= \frac{1}{\sqrt{a^2+b^2}}\left[\frac{\sqrt{a^2+b^2}-a}{b}\right]^n,
\end{aligned}
$$

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onde a última passagem corresponde a aplicação de (★).

A expressão

$$\int_0^\infty \exp(-ax) J_n(bx)\,\mathrm{d}x = \frac{1}{\sqrt{a^2+b^2}}\left[\frac{\sqrt{a^2+b^2}-a}{b}\right]^n \tag{9}$$

foi obtida sob a hipótese de que $a > 0$, mas a integral do lado esquerdo é convergente para $a \geq 0$ e fazendo $a = 0$ verificamos a validade da expressão. Tomando $a = 0$, $b = 1$ e $n = 0$ obtemos a igualdade desejada.

9. Mostre que

$$\mathcal{L}[J_0(x)](p) = \int_0^\infty \exp(-px) J_0(x) dx = (1+p^2)^{-1/2}.$$

Solução: *A relação desejada decorre de (9) ao tomar $a = p$, $b = 0$ e $n = 0$.*

Também temos que

$$\begin{aligned}
\mathcal{L}[J_0(x)](p) &= \int_0^\infty \exp(-px) J_0(x), \mathrm{d}x \\
&= \int_0^\infty \exp(-px) \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} \mathrm{d}x && \textit{por } (5) \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}} \int_0^\infty x^{2k} \exp(-px)\,\mathrm{d}x && \textit{por } (1) \\
&= \frac{p^{2k+1}}{p^{2k+1}} \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}} \int_0^\infty x^{2k} \exp(-px)\,\mathrm{d}x \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}p^{2k+1}} \int_0^\infty (px)^{(2k+1)-1} \exp(-px)\,\mathrm{d}(px) \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}p^{2k+1}} \Gamma(2k+1) && \textit{por } (\mathrm{GE}) \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}p^{2k+1}} \frac{1}{\sqrt{\pi}} 2^{2k}\Gamma(k+1/2)\Gamma(k+1) && \textit{por } (2) \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}p^{2k+1}} \frac{1}{\sqrt{\pi}} 2^{2k}(1/2)_k\Gamma(1/2)k! \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!k!2^{2k}p^{2k+1}} 2^{2k}(1/2)_k k! \\
&= \frac{1}{p} \sum_{k=0}^\infty \frac{(-1)^k (1/2)_k}{K!} \left(\frac{1}{p}\right)^{2k}
\end{aligned}$$

$$= \frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k[(2k-1)!!]^2}{(2k)!}\left(\frac{1}{p}\right)^{2k}$$
$$= \frac{1}{p}\sum_{k=0}^{\infty}(-1)^k[(2k-1)!!]^2\frac{(1/p)^{2k}}{(2k)!}$$
$$= \frac{1}{\sqrt{1+p^2}}$$

onde o último passo é justificado pela expansão de $(1+z^2)^{-1/2}$, $z = 1/p$, *em série de Taylor.*

10. Mostre que

$$\int_0^{\infty} J_n(ax)dx = a^{-1}, n = 0, 1, 2, \ldots$$

Solução: *A relação desejada decorre de* (9) *ao tomar* $a = 0$, $b = a$ *e* $n = 0$.

Também temos que

$$\int_0^{\infty} J_n(ax)\,\mathrm{d}x = a^{-1}\int_0^{\infty} J_n(z)\,\mathrm{d}z.$$

Vamos mostrar que $\int_0^{\infty} J_n(z)\,\mathrm{d}z = 1$, *então*

$$\begin{aligned}\int_0^{\infty} J_n(z)\,\mathrm{d}z &= \int_0^{\infty}\left[J_{n-2}(z) - 2J'_{n-1}(z)\right]\,\mathrm{d}z\\ &= \int_0^{\infty} J_{n-2}(z)\,\mathrm{d}z - 2J_{n-1}(z)\big|_0^{\infty}\\ &= \int_0^{\infty} J_{n-2}(z)\,\mathrm{d}z.\end{aligned}$$

A expressão acima nos fornece uma relação de recorrência de modo que precisamos mostrar apenas que $\int_0^{\infty} J_0(z)\,\mathrm{d}z = 1$, *ver gráfico, e que* $\int_0^{\infty} J_1(z)\,\mathrm{d}z = 1$, *usar* $J_1(x) = -J_0'(x)$.

11. Mostre que a equação $x^4y'' + (\exp(2/x) - v^2)y = 0$ é satisfeita por $y = xJ_v(\exp(1/x))$.

Solução: *Temos que* $J_\nu(z)$ *satisfaz* (EB). *Escrevendo* $y(x) = xJ_\nu(z)$, *onde* $z = \exp(1/x)$, *temos*

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= J_\nu(z)\frac{\mathrm{d}x}{\mathrm{d}x} + xJ'\nu(z)\frac{\mathrm{d}z}{\mathrm{d}x}\\ &= J_\nu(z) + zx^{-1}J'_\nu(z)\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= J'_\nu(z)\frac{\mathrm{d}z)}{\mathrm{d}x} - \frac{1}{x}J'_\nu(z)\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{z}{x^2}J'_\nu(z) - \frac{z}{x}J''_v(z)\frac{\mathrm{d}z}{\mathrm{d}x}\\ &= \frac{z^2}{x^3}J''_\nu(z) + \frac{z}{x^3}J'_\nu(z).\end{aligned}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-\exp(1/x)}{x^2} = \frac{-z}{x^2},$$

Substituindo na equação desejada temos

$$\begin{aligned} x^4 y'' + (\exp(2/x) - \nu^2) y &= xz^2 J_\nu''(z) + x\ J_\nu'(z) + (z^2 - \nu^2) x J_\nu(z) \\ &= x \left[z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) \right] \\ &= x\,[0] \qquad \text{(EB)} \\ &= 0. \end{aligned}$$

12. (Exercício VI-2 do Farrell) Mostre que a solução geral de

$$x^2 \frac{d^2 y}{dy^2} + (1 - 2\alpha) x \frac{dy}{dx} + \left[\beta^2 \gamma^2 x^{2\gamma} + (\alpha^2 - n^2 \gamma^2) \right] y = 0$$

é dada por $Ax^\alpha J_n(\beta x^\gamma) + Bx^\alpha Y_n(\beta x^\gamma)$.

Solução: *Devemos mostrarmos que $x^\alpha J_n(\beta x^\gamma)$ e $x^\alpha Y_n(\beta x^\gamma)$ são soluções da equação desejada. Comecemos com $y = x^\alpha J_n(\beta x^\gamma)$. Então*

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x} \left[x^\alpha J_n \right] \\ &= x^\alpha \frac{\mathrm{d}}{\mathrm{d}x} J_n + J_n \frac{\mathrm{d}}{\mathrm{d}x} x^\alpha \\ &= x^\alpha (\beta \gamma x^{\gamma - 1}) J_n' + J_n \alpha x^{\alpha - 1} \\ &= \beta \gamma x^{\alpha + \gamma - 1} J_n' + \alpha x^{\alpha - 1} J_n \\ y'' &= \beta^2 \gamma^2 x^{\alpha + 2\gamma - 2} J_n'' + \beta \gamma \left[(\alpha + \gamma - 1) x^{\alpha + \gamma - 2} + \alpha x^{\alpha + \gamma - 2} \right] J_n' + \alpha (\alpha - 1) x^{\alpha - 2} J_n \end{aligned}$$

Substituindo na equação temos

$$\begin{aligned} \gamma^2 x^\alpha \left[\beta^2 x^{2\gamma} J_n'' + \beta x^\gamma J_n' + (\beta^2 x^{2\gamma} - n^2) J_n \right] &= 0, \\ \gamma^2 x^\alpha \left[(\beta^2 x^\gamma)^2 J_n'' + \beta x^\gamma J_n' + ((\beta x^\gamma)^2 - n^2) J_n \right] &= 0. \end{aligned}$$

Notamos que a expressão entre colchetes é (EB) *e assim concluimos que $y = x^\alpha J_n(\beta x^\gamma)$ satisfaz a equação desejada.*
Para $y = x^\alpha Y_n(\beta x^\gamma)$ o processo é semelhante.

As funções de Bessel esféricas $j_n(x)$ e $y_n(x)$ são definidas como

$$j_n(x) = \left(\frac{\pi}{2x} \right)^{-1/2} J_{n+1/2}(x), \tag{10}$$

$$y_n(x) = \left(\frac{\pi}{2x} \right)^{-1/2} Y_{n+1/2}(x) = (-1)^{n+1} \left(\frac{\pi}{2x} \right)^{-1/2} J_{-n-1/2}(x). \tag{11}$$

13. Mostre que a n-ésima função de Bessel esférica é dado por

$$f_l(x) = (-1)^l x^l \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^l f_0(x),$$

onde $f_l(x)$ denota tanto $j_l(x)$ como $y_n(x)$.

Solução: *Pelo exercício 1(a) desta lista temos que*

$$J_{\nu+1}(x) = -x^{\nu}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}J_{\nu}(x)\right].$$

Tomando $\nu = l + 1/2$ *na expressão acima temos*

$$\begin{aligned}
J_{l+1/2+1}(x) &= -x^{l+1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-(l+1/2)}J_{l+1/2}(x)\right] \\
J_{l+3/2}(x) &= -x^{l+1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l-1/2)}J_{l+1/2}(x)\right] \\
J_{l+3/2}(x) &= -x^{l+1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{-1/2}J_{l+1/2}(x)}{x^{l}}\right] \\
x^{-1/2}J_{l+3/2}(x) &= -x^{l}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{-1/2}J_{l+1/2}(x)}{x^{l}}\right] \\
j_{l+1}(x) &= -x^{l}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l}j_{l}(x)\right] && \textit{por } (10) \\
j_{l}(x) &= -x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+1}j_{l-1}(x)\right] && l+1 \to l
\end{aligned}$$

Aplicando a expressão acima recursivamente temos

$$\begin{aligned}
j_{l}(x) &= -x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+1}j_{l-1}(x)\right] \\
&= -x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+1}(-1)x^{l-2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+2}j_{l-2}(x)\right]\right] \\
&= (-1)^{2}x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+1}x^{l-2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+2}j_{l-2}(x)\right]\right] \\
&= (-1)^{2}x^{l-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+2}j_{l-2}(x)\right]\right] \\
&= (-1)^{2}\frac{x^{l}}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-l+2}j_{l-2}(x)\right]\right] \\
&= \dots \\
&= (-1)^{l}x^{l}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}j_{0}(x).
\end{aligned}$$

Para $y_l(x)$ *o processo é análogo tomando no início* $\nu = l - 1/2$ *e utilizando que* $Y_{l+1/2}(x) = (-1)^{l+1}J_{-l-1/2}(x)$.

14. Mostre que

 (a) $j_n(x) = (-1)^n x^n \left(x^{-1}d/dx\right)^n x^{-1}\sin x$,

 Solução: *Sabendo que* $j_0(x) = \sin(x)x^{-1}$ *e pelo exercício anterior utilizamos uma simples substituição para concluir que* $j_n(x) = (-1)^n x^n \left(x^{-1}d/dx\right)^n x^{-1}\sin x$.

 (b) $y_n(x) = (-1)^{n+1} x^n \left(x^{-1}d/dx\right)^n x^{-1}\cos x$.

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Solução: *Sabendo que* $y_0(x) = -\cos(x)x^{-1}$ *e pelo exercício anterior utilizamos uma simples substituição para concluir que* $y_n(x) = (-1)^{n+1}x^n\left(x^{-1}d/dx\right)^n x^{-1}\cos x$.

15. Mostre que
    (a) $j_{n-1}(x) + j_{n+1}(x) = x^{-1}(2n+1)j_n(x)$,
    Solução:
    (b) $nj_{n-1}(x) - (n_1)j_{n+1}(x) = (2n+1)j_n'(x)$,
    Solução:
    (c) $y_{n-1}(x) + y_{n+1}(x) = x^{-1}(2n+1)y_n(x)$,
    Solução:
    (d) $ny_{n-1}(x) - (n+1)y_{n+1}(x) = (2n+1)y_n'(x)$.
    Solução:

As funções de Bessel esféricas modificadas $i_n(x)$ e $k_n(x)$ são definidas como

$$i_n(x) = \sqrt{\frac{\pi}{2x}}I_{n+1/2}(x), \tag{12}$$

$$k_n(x) = \sqrt{\frac{2}{\pi x}}K_{n+1/2}(x), \tag{13}$$

onde $I_\nu(x)$ e $K_\nu(x)$ são as funções de Bessel modificadas de primeira e segunda espécie.

16. Mostre que
    (a) $i_{n+1}(x) = x^n\,\mathrm{d}(x^{-n}i_n(x))/\mathrm{d}x$,
    Solução:
    (b) $k_{n+1}(x) = -x^n\,\mathrm{d}(x^{-n}k_n(x))/\mathrm{d}x$.
    Solução:

17. Mostre que
    (a) $i_n(x) = x^n(x^{-1}\,\mathrm{d}/\mathrm{d}x)^n(\sinh(x)x^{-1})$,
    Solução:
    (b) $k_n(x) = (-1)^n x^n(x^{-1}\,\mathrm{d}/\mathrm{d}x)^n(\exp(-x)x^{-1})$.
    Solução:

18. Mostre que
    (a) $i_{n-1}(x) - i_{n+1}(x) = (2n+1)x^{-1}i_n(x)$,
    Solução:
    (b) $ni_{n-1}(x) + (n+1)i_{n+1}(x) = (2n+1)i_n'(x)$,

Solução:

(c) $k_{n-1}(x) - k_{n+1}(x) = -(2n+1)x^{-1}k_n(x)$,

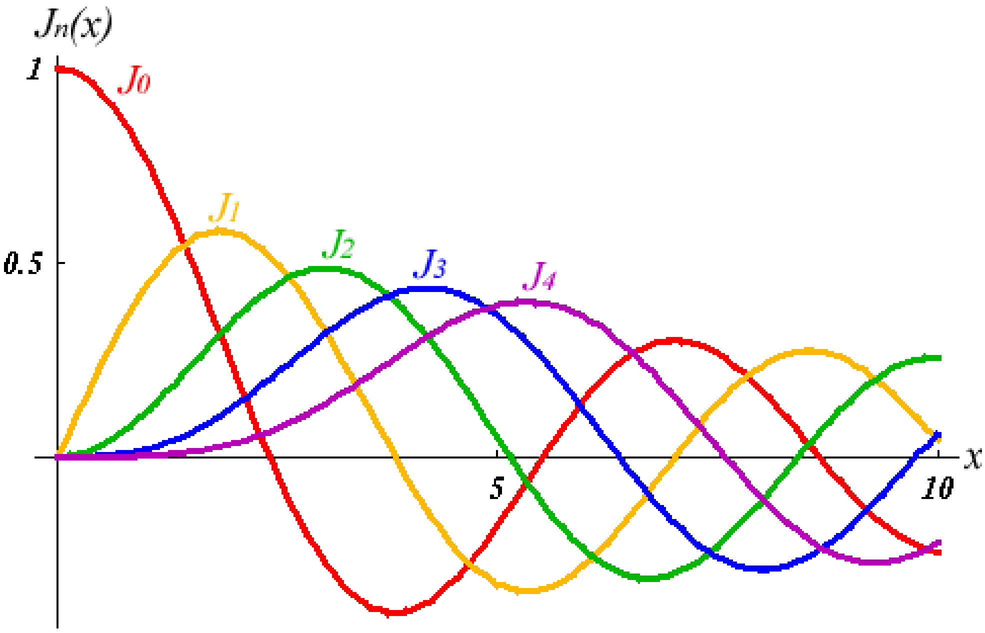
Solução:

(d) $nk_{n-1}(x) + (n+1)k_{n+1}(x) = -(2n+1)k_n'(x)$.

Solução:

19. (P2 de 2006) Faça um esboço do gráfico das funções de Bessel de primeira espécie e de ordem zero, um e dois.

Solução: *Como apresentado nas notas de aula, as funções de Bessel de primeira espécie apresentam comportamento oscilatório e que apenas $J_0(x)$ não se anula para $x = 0$.*



20. (P3 de 2006) As funções de Bessel esféricas $j_n(x)$ são definidas através de

$$j_n(x) = (\pi/(2x))^{1/2} J_{n+1/2}(x),$$

onde $J_\nu(x)$ são as funções de Bessel de primeira espécie e de ordem ν. Mostre que

$$j_0(x) = \sin(x)/x.$$

Solução: *Temos que*

$$\begin{aligned} j_n(x) &= (\pi/(2x))^{1/2} J_{n+1/2}(x) \\ &= (\pi/(2x))^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1/2}}{2^{2k+1/2} k! \Gamma(k+1/2+1)} \qquad \textit{por } (5) \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1}\Gamma(k+1)\Gamma(k+1+1/2)} \\
&= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\sqrt{\pi}\Gamma(2k+2)} && \textit{por } (2) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(2k+2)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} \\
&= x^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
&= x^{-1} \sin(x) && \textit{pela série de Taylor}
\end{aligned}
$$

21. (P2 de 2010) Considere a equação de Bessel $y''(x) + x^{-1}y'(x) + (1 - v^2x^{-2})y(x) = 0$ e suas soluções de primeira espécie $J_\nu, \nu \geq 0$.

    (a) Mostre que J_0 possui um número infinito de zeros no intervalo $(0, \infty)$ (não vale usar a expressão assintótica de J_0 sem deduzí-la). Dica: transformação de variáveis $y(x) = x^{-1/2}u(x)$.

    Solução:

    (b) Deduza uma fóruma para $J_{1/2}$ em termos de funções elementares. Faça o mesmo para $J_{3/2}$.

    Solução:

22. (T5 de 2011) Mostre que

$$
J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_{-1}^{1} (1 - t^2)^{\nu - 1/2} \exp(izt)\, \mathrm{d}t.
$$

Solução: *Temos que*

$$
\begin{aligned}
I &= \int_{-1}^{1} (1 - t^2)^{\nu-1/2} \exp(izt)\, \mathrm{d}t \\
&= \int_{-1}^{1} (1 - t^2)^{\nu-1/2} \left(\cos(zt) + i\sin(zt)\right)\, \mathrm{d}t \\
&= \int_{-1}^{1} \underbrace{(1 - t^2)^{\nu-1/2} \cos(zt)}_{\textit{função par}}\, \mathrm{d}t + i \int_{-1}^{1} \underbrace{(1 - t^2)^{\nu-1/2} \sin(zt)}_{=0,\ \textit{função ímpar}}\, \mathrm{d}t \\
&= 2\int_{0}^{1} (1 - t^2)^{\nu-1/2} \cos(zt)\, \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \int_0^1 (1-t^2)^{\nu-1/2} t^{2n}\, \mathrm{d}t && \textit{expansão em série} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \int_0^1 (1-y)^{\nu-1/2} y^{n-1/2}\, \mathrm{d}t && t^2 = y \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} B(\nu+1/2, n+1/2) && \textit{por (BI)} \\
&= \sum_{n=0} \infty \frac{(-1)^n z^{2n}}{(2n)!} \frac{\Gamma(\nu+1/2)\Gamma(n+1/2)}{\Gamma(\nu+n+1)} && \textit{por (BG)} \\
&= \sqrt{\pi}\Gamma(\nu+1/2) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \frac{\Gamma(2n)}{\Gamma(n) 2^{2n-1} \Gamma(\nu+n+1)} && \textit{por (2)} \\
&= \sqrt{\pi}\Gamma(\nu+1/2) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)\Gamma(2n)} \frac{\Gamma(2n)}{\Gamma(n) 2^{n-1} \Gamma(\nu+n+1)} \\
&= \sqrt{\pi}\Gamma(\nu+1/2) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)} \frac{1}{\Gamma(n) 2^{n-1} \Gamma(\nu+n+1)} \\
&= \sqrt{\pi}\Gamma(\nu+1/2) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n} \\
&= \sqrt{\pi}\Gamma(\nu+1/2) \left(\frac{z}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu} \\
&= \sqrt{\pi}\Gamma(\nu+1/2)(z/2)^{-\nu} J_\nu(z).
\end{aligned}
$$

23. (P2 de 2011, E de 2011) Seja $J_\nu(x)$ as funções de Bessel de primeira espécie e ordem n $(n = 0, 1, 2, \ldots)$. Mostre que

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} J_n(x) = J_0(\sqrt{x^2 - 2xt}).$$

Solução: *Temos que*

$$
\begin{aligned}
J_0(\sqrt{x^2-2xt}) &= \sum_{m=0}^{\infty} \frac{(-1)^m \left(2^{-1}\sqrt{x^2-2xt}\right)^{2m}}{(m!)^2} && \textit{por (5)} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} (x^2-2xt)^m \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} \sum_{n=0}^{m} \binom{m}{n} (x^2)^{m-n}(-2xt)^n && \textit{expansão binomial} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} (x^2)^{m-n}(-2xt)^n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} x^{2m-2n}(-1)^n 2^n x^n t^n \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^m}{2^{2m}(m!)^2} \frac{m!}{n!(m-n)!} x^{2m-2n}(-1)^n 2^n x^n t^n \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^m m! 2^n x^{2m-2n}(-1)^n x^n t^n}{2^{2m}(m!)^2 n!(m-n)!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{m+n} x^{2m-n} t^n}{2^{2m-n} m! n!(m-n)!} \\
&= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(-1)^{m+n} x^{2m-n} t^n}{2^{2m-n} m! n!(m-n)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+2n} x^{2(n+k)-n} t^n}{2^{2(n+k)-n}(n+k)! n! k!} && m-n=k \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}(n+k)! k!} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} J_n(x) && \textit{por } (5).
\end{aligned}
$$

24. A função geratriz das funções de Bessel é $g(x,t) = \exp(x(t-t^{-1})/2)$.
   (a) Use-a para mostrar que $J_n(u+v) = \sum_{m=-\infty}^{\infty} J_m(u) J_{n-m}(v)$.
   Solução: *Temos que*

$$
\begin{aligned}
\exp\big((u+v)(t+t^{-1})/2\big) &= \exp\big(u(t+t^{-1})/2\big) \exp\big(v(t+t^{-1})/2\big) \\
\sum_{n=-\infty}^{\infty} J_n(u+v) t^n &= \sum_{m=-\infty}^{\infty} J_m(u) t^m \sum_{k=-\infty}^{\infty} J_k(v) t^k \\
\sum_{n=-\infty}^{\infty} J_n(u+v) t^n &= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_m(u) J_k(v) t^{m+k} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} J_m(u) J_{n-m}(v) \right) t^n && m+k=n.
\end{aligned}
$$

   Pela útima equação concluímos que$J_n(u+v) = \sum_{m=-\infty}^{\infty} J_m(u) J_{n-m}(v)$.

   (b) Mostre que $|J_0(x)| \leq 1$ e que $|J_n(x)| \leq 2^{-1/2}$ para $n = 1, 2, 3, \ldots$.
   Solução: *Tomando* $n = 0$, $u = -v = x$ *temos que*

$$
\begin{aligned}
J_0(0) &= \sum_{m=-\times}^{\infty} J_m(x) J_{-m}(-x) \\
&= \sum_{m=-\infty}^{\infty} J_m(x)(-1)^m J_m(-x) && \textit{por } (7)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=-\infty}^{\infty} J_m(x)(-1)^m(-1)^m J_m(x) \\
&= \sum_{m=-\infty}^{\infty} J_m^2(x) \\
&= J_0^2(x) + \sum_{m=-\infty}^{-1} J_m^2(x) + \sum_{m=1}^{\infty} J_m^2(x) \\
&= J_0^2(x) + 2\sum_{m=1}^{\infty} J_m^2(x).
\end{aligned}
$$

Pela definição de $J_m(x)$ temos que $J_0(0) = 1$ e portanto

$$
1 = J_0^2(x) + 2\sum_{m=1}^{\infty} J_m^2(x)
$$

que implica em

$$
\begin{cases}
J_0^2(x) \leq 1, \\
2J_m^2(x) \leq 1, \quad m = 1, 2, \ldots.
\end{cases}
$$

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