Métodos I - 1S11 - Lista 5

(1) Mostre que

$$ {}_2F_1(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},\qquad \gamma\neq 0,-1,-2,\ldots,\qquad \gamma>\alpha+\beta. $$

(2) Mostre que

$$ \begin{aligned} {}_2F_1(\alpha,\beta,\gamma;z) &= (1-z)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;z),\\ {}_2F_1(\alpha,\beta,\gamma;z) &= (1-z)^{-\alpha}\ {}_2F_1(\alpha,\gamma-\beta,\gamma;z/(z-1))\\ {}_2F_1(\alpha,\beta,\gamma;z) &= (1-z)^{-\beta}\ {}_2F_1(\gamma-\alpha,\beta,\gamma;z/(z-1)). \end{aligned} $$

(3) Mostre que

$$ \int_0^\infty \mathrm{e}^{-st}\ {}_1F_1(\alpha,\gamma;t)\,dt = s^{-1}\,{}_2F_1(\alpha,1,\gamma;s^{-1}). $$

(4) Mostre que a série hipergeométrica ${}_2F_1(\alpha,\beta,\gamma;z)$ reduz-se a um polinômio quando α ou β é um número inteiro negativo.

(5) Mostre que

$$ \begin{aligned} & z^n = {}_2F_1(-n,1,1;1-z)\quad (n=0,1,2,\ldots),\\ & 1+\binom{a}{1}z+\binom{a}{2}z^2+\cdots+\binom{a}{m}z^m\\ &\qquad = \binom{a}{m}z^m\,{}_2F_1(-m,1,a-m+1;-z^{-1}),\\ & \ln(1-z) = -z\ {}_2F_1(1,1,2;z),\\ & \ln\left(\frac{1+z}{1-z}\right) = 2z\ {}_2F_1(1/2,1,3/2;z^2),\\ & \arcsin z = z\ {}_2F_1(1/2,1/2,3/2;z^2), \end{aligned} $$

$$ \begin{aligned} \cos az &= {}_2F_1(a/2,-a/2,1/2;(\sin z)^2),\\ B(x,y) &= x^{-1}\ {}_2F_1(x,y,x+1;1),\\ K(k) &= \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}\\ &= \frac{\pi}{2}\ {}_2F_1(1/2,1/2,1;k^2),\\ E(k) &= \int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi\\ &= \frac{\pi}{2}\ {}_2F_1(-1/2,1/2,1;k^2). \end{aligned} $$

(6) Os polinômios de Jacobi $P_n^{(\alpha,\beta)}(z)$ estão relacionados com a função hipergeométrica por

$$ P_n^{(\alpha,\beta)}(z)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\,{}_2F_1(-n,\alpha+\beta+n+1,\alpha+1;(1-z)/2). $$

Mostre que esses polinômios satisfazem a equação

$$ (1-z^2)y''+(\alpha-\beta-(\alpha+\beta+2)z]y'+n(n+\alpha+\beta+1)y=0. $$

(7) Os polinômios de Hermite $H_n(z)$ podem ser escritos em termos da função hipergeométrica confluente como

$$ H_n(z)=2^nU(-n/2,1/2;z^2). $$

Mostre que esses polinômios satisfazem a equação

$$ y''-2zy'+2ny=0. $$