T_4 ( ) F520 ( ) MS550 · Nome: ______________________ RA: __________

Os polinômios de Hermite $H_n(z)$ podem ser escritos em termos da função hipergeométrica confluente de segunda espécie como

$$H_n(z) = 2^n\, U(-n/2, 1/2; z^2).$$

Mostre que esses polinômios satisfazem a equação

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}y}{\mathrm{d}z} + 2ny = 0.$$

$\mathcal{U}(\alpha, \gamma; x)$ satisfaz

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0$$

$$x = z^2 \Rightarrow \begin{cases} \dfrac{dy}{dx} = \dfrac{1}{2z}\dfrac{dy}{dz} \\ \dfrac{d^2y}{dx^2} = \dfrac{1}{2z}\dfrac{d}{dz}\left(\dfrac{1}{2z}\dfrac{dy}{dz}\right) = -\dfrac{1}{4z^3}\dfrac{dy}{dz} + \dfrac{1}{4z^2}\dfrac{d^2y}{dz^2} \end{cases}$$

$\therefore \mathcal{U}(\alpha, \gamma; z^2)$ satisfaz

$$z^2\left[-\frac{1}{4z^3}\frac{dy}{dz} + \frac{1}{4z^2}\frac{d^2y}{dz^2}\right] + [\gamma - z^2]\frac{1}{2z}\frac{dy}{dz} - \alpha y = 0$$

$$\therefore \frac{d^2y}{dz^2} + (2\gamma - 1)\frac{1}{z}\frac{dy}{dz} - 2z\frac{dy}{dz} - 4\alpha y = 0$$

Tomando $\alpha = -n/2$, $\gamma = 1/2$, temos que $\mathcal{U}(-n/2, 1/2; z^2)$ satisfaz

$$\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + 2ny = 0 \quad (*)$$

Como 2^n é cte, $H_n(z) = 2^n\,\mathcal{U}(-n/2, 1/2, z^2)$ satisfaz $(*)$.

Formulário eventualmente útil

$${}_2F_1(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \quad {}_2F_1(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

$$U(a,b;z) = \frac{1}{\Gamma(a)}\int_0^{\infty} \mathrm{e}^{-zt}t^{a-1}(1+t)^{-a+b-1}\,\mathrm{d}t, \quad \frac{\mathrm{d}^n U(a,b;z)}{\mathrm{d}z^n} = (-1)^n(a)_n U(a+n, b+n; z),$$

$$U(a,b;z) = \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}\,{}_1F_1(a-b+1, 2-b; z) + \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a,b;z), \quad {}_1F_1(a,b;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$