Soluções para MS550, Métodos de Matemática Aplicada I, e F520, Métodos Matemáticos da Física I

Lista 7 - Função de Legendre

Equações eventualmente útil:

(ST) $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(GE) $$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

(1) $$\Gamma(z+1) = z\Gamma(z),\ \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$$

(2) $$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z)$$

(BG) $$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

(BT) $$B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta d\theta$$

(BI) $$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$$

(SP) $$(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1),\ (\alpha)_0 = 1$$

(3) $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

(4) $$\frac{(\alpha)_n}{m!} = \binom{\alpha+n-1}{n},\ \frac{(-\alpha)_n}{n!} = (-1)^n\binom{\alpha}{n}$$

(EH) $$z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]\, y' - \alpha\beta y = 0$$

(SH) $$_2F_1(\alpha,\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$$

(5) $$_2F_1(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt$$

(EL) $$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$

(FL) $$P_\nu(x) = {}_2F_1(-\nu,\nu+1,1;(1-x)/2)$$

(FGL) $$g(-x,t) = (1-2xt+t^2)^{-1/2}$$

(FGLS) $$g(-x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$$

(6) $$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = 2\,(2n+1)^{-1}\,\delta_{mn}$$

(7) $$Q_n(x) = \frac{1}{2^{n+1}}\int_{-1}^{1} \frac{(1-y^2)^n}{(x-y)^{n+1}}\,\mathrm{d}y$$

(ELA) $$(1-x^2)y'' - 2xy' + \left(\nu(\nu+1) - m^2(1-x^2)^{-1}\right) y = 0$$

(FLA) $$P_\nu^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_\nu(x)$$

(8) $$\sum_{n=0}^{\infty} P_{n+m}(x)t^n = \frac{(2m)!}{2^n m!}\frac{(1-x^2)^{m/2}}{(1-2xt+t^2)^{m+1/2}}$$

(9) $$Q_\nu^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}Q_\nu(x)$$

(FR) $$F_n(x) = \frac{1}{\rho(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\rho(x) s^n(x)\right]$$

1. Mostre as relações de recorrência que seguem derivando a função geratriz.

Solução: *Seja $g(x,t)$ a função geratriz de Legendre. Para $g(1,t)$ temos*

$$\begin{aligned}
g(1,t) &= \left(1 - 2t + t^2\right)^{-1/2} && \textit{por } (\text{FGL}) \\
&= \left[(1-t)^2\right]^{-1/2} \\
&= (1-t)^{-1} \\
&= \sum_{n=0}^{\infty} t^n && \textit{pela expansão em Taylor,} \\
g(1,t) &= \sum_{n=0}^{\infty} P_n(1) t^n && \textit{por } (\text{FGLS}),
\end{aligned}$$

de onde segue que

(10) $$P_n(1) = 1.$$

Tomando agora $g(-x,t)$ temos

$$\begin{aligned}
g(-x,t) &= \left(1 - 2(-x)t + t^2\right)^{-1/2} && \textit{por } (\text{FGL}) \\
&= \left(1 - 2x(-t) + (-t)^2\right)^{-1/2} \\
&= \sum_{n=0}^{\infty} P_n(x)(-1)^n t^n, \\
g(-x,t) &= \sum_{n=0}^{\infty} P_n(-x) t^n && \textit{por } (\text{FGLS})
\end{aligned}$$

de onde segue que

(11) $$P_n(-x) = (-1)^n P_n(x).$$

Utilizando (10) *e* (11) *segue que*

$$P_n(-1) = (-1)^n.$$

Derivando $g(x,t)$ em relação a t temos

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial t} &= \frac{\partial}{\partial t} \left(1 - 2xt + t^2\right)^{-1/2} && \textit{por } (\text{FGL}) \\
&= (-2x + 2t)\,(-1/2) \left(1 - 2xt + t^2\right)^{-3/2} \\
&= (x - t) \left(1 - 2xt + t^2\right)^{-1/2} \left(1 - 2xt + t^2\right)^{-1} \\
&= (x - t)\, g(x,t) \left(1 - 2xt + t^2\right)^{-1},
\end{aligned}$$

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$$\left(1 - 2xt + t^2\right) \frac{\partial g(x,t)}{\partial t} = (x - t) g(x,t).$$

Usando (FGLS) *na equação anterior temos*

$$\begin{aligned}
\left(1 - 2xt + t^2\right) \frac{\partial}{\partial t} \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] &= (x - t) \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] \\
\left(1 - 2xt + t^2\right) \left[\sum_{n=0}^{\infty} P_n(x) n t^{n-1}\right] &= (x - t) \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] \\
\sum_{n=0}^{\infty} P_n(x) n \left(t^{n-1} - 2xt^n + t^{n+1}\right) &= \sum_{n=0}^{\infty} P_n(x) (xt^n - t^{n+1}).
\end{aligned}$$

Agrupando as potências de t segue que

$$(2n + 1) x P_n(x) = (n + 1) P_{n+1}(x) + n P_{n-1}(x). \tag{12}$$

Derivando $g(x,t)$ em relação a x temos

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial x} &= \frac{\partial}{\partial x} \left(1 - 2xt + t^2\right)^{-1/2} && \textit{por (FGL)} \\
&= (-2t)(-1/2) \left(1 - 2xt + t^2\right)^{-3/2} \\
&= t \left(1 - 2xt + t^2\right)^{-1/2} \left(1 - 2xt + t^2\right)^{-1} \\
&= t g(x,t) \left(1 - 2xt + t^2\right)^{-1},
\end{aligned}$$

$$\left(1 - 2xt + t^2\right) \frac{\partial g(x,t)}{\partial t} = -t g(x,t).$$

Usando (FGLS) *na equação anterior temos*

$$\begin{aligned}
\left(1 - 2xt + t^2\right) \frac{\partial}{\partial x} \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] &= t \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] \\
\left(1 - 2xt + t^2\right) \left[\sum_{n=0}^{\infty} P_n'(x) t^n\right] &= t \left[\sum_{n=0}^{\infty} P_n(x) t^n\right] \\
\sum_{n=0}^{\infty} P_n'(x) \left(t^n - 2xt^{n+1} + t^{n+2}\right) &= \sum_{n=0}^{\infty} P_n(x) t^{n+1}.
\end{aligned}$$

Agrupando as potências de t segue que

$$P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x).$$

Derivando (12) *temos*

$$(2n + 1)(P_n(x) + x P_n'(x)) = (n + 1) P_{n+1}'(x) + n P_{n-1}'(x)$$

e usando o resultado na expressão anterior segue que

$$(2n + 1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x). \tag{13}$$

TODO

$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x). \tag{14}$$

Substituindo (14) *em* (13) *segue que*

$$\begin{aligned}
(2n+1)P_n(x) &= P'_{n+1}(x) - [-nP_n(x) + xP'_n(x)] \\
&= P'_{n+1}(x) + nP_n(x) - xP'_n(x), \\
(n+1)P_n(x) &= P'_{n+1}(x) - xP'_n(x).
\end{aligned}$$

Tomando $n \to n-1$ *na equação anterior temos*

$$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x)$$

e somando com (14) *multiplicado por* $(-x)$ *segue que*

$$\begin{aligned}
nP_{n-1}(x) - xP'_{n-1} &= P'_n(x) - xP'_{n-1}(x) - \left[-nxP_n(x) + x^2P'_n(x)\right], \\
nP_{n-1}(x) &= P'_n(x) + nxP_n(x) - x^2P'_n(x), \\
nP_{n-1}(x) - nxP_n(x) &= (1-x^2)P'_n(x).
\end{aligned}$$

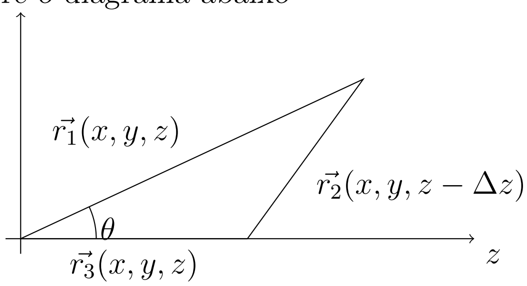
Outras relações:

$$\begin{aligned}
&(1-x^2)P'_n(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x), \\
&P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x).
\end{aligned}$$

2. (Exercício 12.1.7 do Arfken) Mostre que

$$P_n(\cos\theta) = (-1)^n \frac{r^{n+1}}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r}\right).$$

Solução: *Considere o diagrama abaixo*



Então

$$\begin{aligned}
|\vec{r_1} - \vec{r_3}| &= \sqrt{r_1^2 + r_3^2 - 2r_1r_3\cos(\theta)}, \\
|\vec{r_1} - \vec{r_3}|^{-1} &= (r_1^2 + r_2^3 - 2r_1r_3\cos(\theta))^{-1/2}
\end{aligned}$$

$$= r_1^{-1}\sum_{n=0}^{\infty} P_n(\cos(\theta))(r_3 r_1^{-1})^n \qquad \textit{por (FGL) e (FGLS)}$$

$$= \sum_{n=0}^{\infty} P_n(\cos(\theta))(\Delta z^n(-1)^n)r^{-n+1}.$$

Por outro lado

$$[r(z-\Delta z)]^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{1}{r}\right)(\Delta z)^n.$$

Portanto,

$$P_n(\cos(\theta)) = \frac{(-1)^n r^{n+1}}{n!}\frac{\mathrm{d}}{\mathrm{d}z^n}\left(\frac{1}{r}\right).$$

3. Mostre, usando explicitamente coordenadas esféricas (r,θ,ϕ), que

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}}.$$

Solução: *Como*

$$P_n(\cos\theta) = (-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

já está em coordenadas esféricas, temos:

$$\begin{aligned}
\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] &= \frac{\partial}{\partial z}\left[\frac{(-1)^n r^{n+1}}{n! r^{n+1}}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)\right] \\
&= \frac{\partial}{\partial z}\left[\frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)\right] \\
&= \frac{(-1)^n}{n!}\frac{\partial^{n+1}}{\partial z^{n+1}}\left(\frac{1}{r}\right).
\end{aligned}$$

Utilizando a expressão para $P_{n+1}(\cos\theta)$ de um exercício anterior temos

$$\begin{aligned}
P_{n+1}(\cos\theta) &= \frac{r^{n+2}(-1)^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial z^{n+1}}\left(\frac{1}{r}\right), \\
\frac{\partial^{n+1}}{\partial z^{n+1}}\left(\frac{1}{r}\right) &= \frac{(n+1)!P_{n+1}(\cos\theta)}{(-1)^{n+1}r^{n+2}}.
\end{aligned}$$

E substituindo na expressão anterior,

$$\begin{aligned}
\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] &= \frac{(-1)}{n!}\frac{(n+1)!}{(-1)^{n+1}}\frac{P_{n+1}(\cos\theta)}{r^{n+2}} \\
&= \frac{-(n+1)P_{n+1}(\cos\theta)}{r^{n+2}}.
\end{aligned}$$

4. Mostre que

(a) (T6 de 2010) $\int_{-1}^{1} P_n(x)\,\mathrm{d}x = 0, n \geq 1,$

Solução: *Utilizando* (6) *e fazendo* $P_m(x) = 1 = P_0(x)$ *temos*

$$\int_{-1}^{1} P_n(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{0n} = \begin{cases} 2, & \text{se } n = 0, \\ 0, & \text{se } n \geq 1. \end{cases}$$

(b) $\int_{-1}^{1} x^{2n+1} P_{2m}(x)\,\mathrm{d}x = 0, m \neq n,$

Solução: *Sabemos que* x^{2n+1} *é uma função ímpar,* $(-x)^{2n+1} = (-1)x^{2n+1}$*, e que* $P_{2m}(x)$ *é uma função par,* $P_{2m}(-x) = P_{2m}(x)$*. Logo,* $f(x) = x^{2n+1}P_{2m}(x)$ *é uma função ímpar,* $m \neq n$*, e portanto*

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0, m \neq n.$$

(c) $\int_{-1}^{1} x^m P_n(x)\,\mathrm{d}x = 0, m < n,$

Solução: *Temos que*

$$\begin{aligned}
\int_{-1}^{1} x^m P_n(x)\,\mathrm{d}x &= \int_{-1}^{1} \sum_{k=0}^{m} a_k P_k(x) P_n(x)\,\mathrm{d}x \\
&= \sum_{k=0}^{m} a_k \int_{-1}^{1} P_k(x) P_n(x)\,\mathrm{d}x \\
&= \sum_{k=0}^{m} a_k \left[\frac{2}{2n+1}\delta_{mn}\right] && \text{por } (6) \\
&= \sum_{k=0}^{m} a_k\,[0] \\
&= 0.
\end{aligned}$$

(d) $\int_{-1}^{1} x^n P_n(x)\,\mathrm{d}x = \left[2^{n+1}(n!)^2\right]/(2n+1)!.$

Solução: *Por* (FR) *temos que*

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right]$$

e portanto o n*-ésimo termo de* P_n *corresponde a*

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{2n}) &= (2n)(2n-1)\ldots(2n-n+1)x^n \\
&= (2n)\ldots(n+1)x^n \\
&= \frac{(2n)!}{n!}x^n.
\end{aligned}$$

Utilizando a expressão acima temos que

$$\begin{aligned} P_n &= \frac{1}{2^n n!}\frac{(2n)!}{n!}x^n, \\ x^n &= \frac{2^n (n!)^2}{(2n)!}P_n. \end{aligned}$$

Logo,

$$\begin{aligned} \int_{-1}^{1} x^n P_n(x)\,\mathrm{d}x &= \int_{-1}^{1} \frac{2^n (n!)^2}{(2n)!} P_n P_n \,\mathrm{d}x \\ &= \frac{2^n (n!)^2}{(2n)!} \int_{-1}^{1} P_n(x) P_n(x)\,\mathrm{d}x \\ &= \frac{2^n (n!)^2}{(2n)!} \frac{2}{2n+1}\delta_{nm} && \text{por (6)} \\ &= \frac{2^n (n!)^2}{(2n)!} \frac{2}{2n+1} \\ &= \frac{2^{n+1} (n!)}{(2n+1)!} \end{aligned}$$

5. Mostre que
   (a) (P3 de 2006, T6 de 2010, T6 de 2011) $P_n'(1) = n(n+1)/2$,
   Solução: Método 1
   Pela função geratriz temos que

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Derivando em relação a x obtemos

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) t^n.$$

Para $x = 1$ verifica-se que

$$\begin{aligned} \frac{t}{(1-2t+t^2)^{3/2}} &= \frac{1}{\left[(1-t)^2\right]^{3/2}} \\ &= \frac{t}{(1-t)^3} \\ &= \sum_{n=0}^{\infty} P_n'(1) t^n. \end{aligned}$$

Sabendo que

$$(1-t)^{-3} = \sum_{k=0}^{\infty} \frac{(3)_k}{k!} t^k$$

$$= \sum_{k=0}^{\infty} \frac{(k+2)!}{2!k!} t^k$$
$$= \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} t^k$$

temos então que

$$\frac{t}{(1-t)^3} = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} t^{k+1}$$
$$= \sum_{n=1}^{\infty} \frac{(n+1)n}{2} t^n$$
$$= \sum_{n=0}^{\infty} P_n'(1) t^n.$$

Concluimos então que $P_0'(1) = 0$ *e* $P_n'(1) = n\,(n+1)\,/2$, $n = 1, 2, \ldots$, *que equivale a* $P_n'(1) = n\,(n+1)\,/2$, $n = 0, 1, 2, \ldots$.

Método 2

Temos que

$$\begin{aligned}
P_n(x) &= {}_2F_1(-n, n+1, 1; (1-x)/2) \\
P_n'(x) &= {}_2F_1'(-n, n+1, 1; (1-x)/2)\,(-1/2) \\
&= (-1/2)\,(-n)(n+1)(1)^{-1}\,{}_2F_1(-n+1, n+2m, 2; (1-x)/2) \\
P_n'(1) &= n(n+1)(2)^{-1}\,{}_2F_1(-n+1, n+2, 2; 0) \\
&= n(n+1)/2.
\end{aligned}$$

(b) (E de 2006) $P_n'(-1) = (-1)^{n+1} n(n+1)/2$.

Solução: *Temos que*

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

derivando em relação a x *obtemos*

$$\frac{t}{[1-2xt+t^2]^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) t^n$$

e fazendo $x = -1$

$$\frac{t}{[1+2t+t^2]^{3/2}} = \frac{t}{\left[(1+t)^2\right]^{3/2}}$$
$$= \frac{t}{(1+t)^3}$$
$$= \sum_{n=0}^{\infty} P_n'(-1) t^n.$$

Sabendo que

$$\begin{aligned}(1+t)^{-3} &= \sum_{k=0}^{\infty}\frac{(3)_k}{k!}(-t)^k\\ &= \sum_{k=0}^{\infty}\frac{(k+2)!}{2!k!}(-1)^k t^k\\ &= \sum_{k=0}^{\infty}\frac{(k+2)(k+1)}{2}(-1)^k t^k\end{aligned}$$

temos então que

$$\begin{aligned}\frac{t}{(1-t)^3} &= \sum_{k=0}^{\infty}\frac{(k+2)(k+1)}{2}(-1)^k t^{k+1}\\ &= \sum_{n=1}^{\infty}\frac{(n+1)n}{2}(-1)^{n-1}t^n\\ &= \sum_{n=0}^{\infty}P_n'(-1)t^n.\end{aligned}$$

Concluimos então que $P_0'(-1)=0$ *e* $P_n'(-1)=(-1)^{n-1}n\,(n+1)\,/2=(-1)^{n+1}n(n+1)/2$, $n=1,2,\ldots$, *que equivale a* $P_n'(1)=(-1)^{n+1}n\,(n+1)\,/2$, $n=0,1,2,\ldots$.

6. Construa (a menos de uma normalização) os polinômios de Legendre $P_n(x)$ utilizando o processo de ortogonalização de Gram-Schmidt aplicado à base $\{1,x,x^2,x^3,\ldots\}$ do espaço dos polinômios no intervalo $(-1,1)$ equipado com o produto escalar $(f,g)=\int_{-1}^{1}f(x)g(x)\,\mathrm{d}x$.

Solução: *Seja* $\{v_0,v_1,v_2,\ldots\}=\{1,x,x^2,\ldots\}$. *Então pelo processo de ortogonalização de Gram-Schmidt temos que*

$$\begin{aligned}u_0 &= v_0 = 1 = P_0(x)\\ u_1 &= v_1 - \frac{v_1^t u_0}{\|u_0\|\|^2}u_0\\ &= x - \frac{1\int_{-1}^{1}x\,\mathrm{d}x}{\int_{-1}^{1}\mathrm{d}x}\\ &= x - 0/2 = x = P_1(x)\\ u_2 &= v_2 - \frac{v_2^t u_0}{\|u_0\|^2}u_0 - \frac{v_2^t u_1}{\|u_1\|^2}u_1\\ &= x^2 - \frac{1\int_{-1}^{1}x^2\,\mathrm{d}x}{\int_{-1}^{1}\mathrm{d}x} - \frac{x\int_{-1}^{1}x^2\,\mathrm{d}x}{\int_{-1}^{1}\mathrm{d}x}\\ &= x^2 - 1/3 = (2/3)P_2(x)\\ &\vdots\end{aligned}$$

7. Mostre que as funções de Legentre de segunda espécie $Q_n(x)$ satisfazem as seguintes relações de recorrência:

   (a) $(2n+1)xQ_n(x) = (n+1)Q_{n+1}(x) + nQ_{n-1}(x)$,

   Solução: *A partir de* (7) *temos que*

$$\begin{aligned}(n+1)Q_{n+1} + nQ_{n-1}(x) &= (n+1)\frac{1}{2}\int_{-1}^{1}\frac{P_{n+1}(y)}{x-1y}\,\mathrm{d}y + n\frac{1}{2}\int_{-1}^{1}\frac{P_{n-1}(y)}{x-y}\\ &= \frac{1}{2}\int_{-1}^{1}\frac{(n+1)P_{n+1}+nP_{n-1}}{x-y}\,\mathrm{d}y\\ &= (2n+1)xQ_n(x),\end{aligned}$$

   onde a última passagem decorre de $(2n+1)P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$ *(ver notas de aula).*

   (b) $(2n+1)Q_n(x) = Q'_{n+1}(x) - Q'_{n-1}(x)$.

   Solução: *A partir de* (7) *temos que*

$$\begin{aligned}Q'_{n+1}(x) - Q'_{n-1}(x) &= \frac{1}{2}\int_{-1}^{1}\frac{P'_{n+1}(y)}{x-y}\,\mathrm{d}y - \frac{1}{2}\int_{-1}^{1}\frac{P'_{n-1}(y)}{x-y}\,\mathrm{d}y\\ &= \frac{1}{2}\int_{-1}^{1}\frac{P1_{n+1}-P'_{n-1}}{x-y}\,\mathrm{d}y\\ &= \frac{1}{2}\int_{-1}^{1}\frac{(2n+1)P_n(y)}{x-y}\,\mathrm{d}y\\ &= (2n+1)Q_n(x).\end{aligned}$$

8. Mostre que $Q_n(-x) = (-1)^{n+1}Q_n(x)$.

   Solução: *Temos que*

$$\begin{aligned}Q_n(-x) &= \frac{1}{2}\int_{-1}^{1}\frac{P_n(y)}{-x-y}\,\mathrm{d}y\\ &= \frac{1}{2}\int_{-1}^{1}\frac{P_n(-u)}{-(x-u)}\,\mathrm{d}u \qquad\qquad u=-y\\ &= \frac{1}{2}\int_{-1}^{1}\frac{(-1)^nP_n(u)}{-(x-u)}\,\mathrm{d}u\\ &= (-1)^{n+1}Q_n(x).\end{aligned}$$

9. Mostre que

$$n\left[P_n(x)Q_{n-1}(x) - Q_n(x)Q_{n-1}(x)\right] = P_1(x)Q_0(x) - P_0(x)Q_1(x) = 1.$$

Solução: *Temos que*

$$\begin{aligned}
I &= n\left[P_n(x)Q_{n-1}(x) - Q_n(x)P_{n-1}(x)\right] \\
&= P_1(x)Q_0(x) - Q_1(x)P_0(x) && n=1 \\
&= P_1(x)\frac{1}{2}\int_{-1}^{1}\frac{P_0(y)}{x-y}\,\mathrm{d}y - \frac{1}{2}\int_{-1}^{1}\frac{P_1(y)}{x-y}\,\mathrm{d}yP_0(x) \\
&= \frac{x}{2}\int_{-1}^{1}\frac{1}{x-y}\,\mathrm{d}y - \frac{1}{2}\int_{-1}^{1}\frac{y}{x-y}\,\mathrm{d}y \\
&= \frac{1}{2}\left[x\left(-\log(x-y)\big|_{y=-1}^{1}\right) - \left(x\log(y-x)-y\right)\big|_{y=-11}^{1}\right] \\
&= 2^{-1}\left[-x\log(x-1) + x\log(x+1) - x\log(1-x) + 1 + x\log(-1-x) + 1\right] \\
&= 1.
\end{aligned}$$

10. Mostre que os polinômios de Legendre associados $P_n^m(x)$ satisfazem as seguintes relações de recorrência:

   (a) $P_n^{m+1}(x) - 2mx\left(1-x^2\right)^{-1/2}P_n^m(x) + \left[n(n_1) - m(m-1)\right]P_n^{m-1}(x) = 0,$

   Solução: *Sabemos que os polinômios de Legendre associados $P_n^m(x)$ satisfazem* (ELA) *e portanto*

$$\begin{aligned}
0 &= (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^m(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_n^m(x) + \left(n(n+1) - \frac{m^2}{1-x^2}\right)P_n^m(x) \\
&= (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[(1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_n(x)\right] - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_n^m(x) \\
&\quad + \left(n(n+1) - \frac{m^2}{1-x^2}\right)P_n^m(x) \\
&= -mP_n^m + m(m-2)x^2(1-x^2)^{-1}P_n^m - 2mx(1-x^2)^{-1/2}P_n^{m+1} + \\
&\quad + P_n^{m+2} + 2x^2m(1-x^2)^{-1}P_n^m - 2x(1-x^2)^{-1/2}P_n^{m+1} \\
&\quad + \left(n + (n+1) - m^2(1-x^2)^{-1}\right)P_n^m \\
&= P_n^{m+2} + \left[-2x(m+1)(1-x^2)^{-1/2}\right]P_n^{m+1} \\
&\quad + \left[-m(1-x^2) + m(m+2)x^2 + 2x^2m + n(n+1)(1-x^2) - m^2\right](1-x)^{-1}P_n^m \\
&= P_n^{m+2} + \left[-2x(m+1)(1-x^2)^{-1/2}\right]P_n^{m+1} \\
&\quad + \left[-m + mx^2 + m^2x - 2mx^2 + 2mx^2 + n(n+1)(1-x^2) - m^2\right](1-x)^{-1}P_n^m \\
&= P_n^{m+2} + \left[-2x(m+1)(1-x^2)^{-1/2}\right]P_n^{m+1} \\
&\quad + \left[-m(1-x^2) - m^2(1-x^2) + n(n+1)(1-x^2)\right](1-x)^{-1}P_n^m \\
&= P_n^{m+2} + \left[-2x(m+1)(1-x^2)^{-1/2}\right]P_n^{m+1} \\
&\quad + \left[m(m+1) + n(n_1)\right]P_n^m.
\end{aligned}$$

   Fazendo $m \to m-1$ segue que

$$0 = P_n^{m+1} - 2xm(1-x^2)^{-1/2}P_n^m + \left[n(n+1) - (m-1)m\right]P_n^{m-1}.$$

(b) $(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x) = 0,$

Solução: *Derivando (8) em relação a t temos que*

$$\begin{aligned}
\frac{(2m)!}{2^m m!}\frac{(2m+1)(x-t)}{(1-2xt+t^2)^{m+3/2}} &= \sum_{n=0}^{\infty} P_{n+m}^m(x) n t^{n-1}, \\
(2m+1)(x-t)\sum_{n=0}^{\infty} P_{n+m}^m(x) t^n &= \sum_{n=0}^{\infty} P_{n+m}^m(x) n t^{n-1}(1-2xt+t^2) \\
(2m+1)xP_{n+m}^m - (2m+1)P_{n+m-1}^m &= (n+1)P_{n+m+1}^m \\
&\quad - 2nxP_{n+m}^m + (n+1)P_{n+m-1}^n \\
\left[(2m+1)x + 2nx\right] P_{n+m}^m &= \left[(2m+1) + (n-1)\right] P_{n+m-1}^m + (n+1)P_{n+m+1}^m.
\end{aligned}$$

Fazendo $n = k - m$ segue que

$$\left[x(2k+1)\right] P_k^m = (m+k)\, P_{k=1}^m + (k-m+1)\, P_{k+1}.$$

(c) $(2n+1)(1-x^2)^{1/2}P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x),$

Solução: *Por (13) segue que*

$$(2n+1)P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)$$

TODO

(d) $(1-x^2)^{1/2}P_n^{m\prime}(x) = 2^{-1}P_n^{m+1}(x) - 2^{-1}(n+m)(n-m+1)P_n^{m-1}(x).$

Solução: *Segue de (ELA) ao efetuar a mudança de variável representada por $y(x) = (1-x^2)^{m/2}w(x)$ que*

$$(1-x^2)w'' - 2(m+1)xw' + (\nu - m)(\nu + m + 1)w = 0.$$

Como $P_n^{m-1}(x)$ satisfaz a equação acima temos que

$$\begin{aligned}
0 &= (1-x^2)P_n^{m+1}(x) - 2mxP_n^m(x) + (n-m+1)(m+m)P_n^{m-1}(x) \\
&= (1-x^2)^{m/2}P_n^{m+1}(x) - 2mx(1-x^2)^{m/2-1}P_n^m(x) \\
&\quad + (n+m)(n-m+1)(1-x^2)^{m/2-1}P_n^{m-1} \\
&= (1-x^2)^{-1/2}P_n^{m+1}(x) - 2\left[-P_n^{m\prime}(x) + (1-x^2)^{-1/2}P_n^{m+1}(x)\right] \\
&\quad + (n+m)(n-m+1)(1-x^2)^{-1/2}P_n^{m-1}(x) \\
&= (1-x^2)^{1/2}2P_n^{m\prime}(x) - P_n^{m+1}(x) + (n+m)(n-m+1)P_n^{m-1}(x).
\end{aligned}$$

Da equação acima segue que

$$\sqrt{1-x^2}P_n^{m\prime} = 2^{-1}P_n^{m+1}(x) - 2^{-1}(n+m)(n-m+1)P_n^{m-1}(x).$$

11. Definia $P_n^{-m}(x)$ $(m \geq 0)$ usando a fórmula de Rodrigues, ou seja,

$$P_n^{-m}(x) = \frac{(1-x^2)^{-m/2}}{2^n n!} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \left(x^2-1\right)^n.$$

Mostre que

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x).$$

Solução: *Temos que*

$$\begin{aligned}
P_n^{-m}(x) &= \frac{(1-x^2)^{-m/2}}{2^n n!} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} (x^2-1)^n \\
&= \frac{(1-x^2)^{-m/2}}{2^n n!} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \left[(x-1)(x+1)\right]^n \\
&= \frac{(1-x^2)^{-m/2}}{2^n n!} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \left[(x-1)^n(x+1)^n\right] \\
&= \frac{(1-x^2)^{-m/2}}{2^n n!} \sum_{k=0}^{n-m} \frac{(n-m)!}{(n-m-k)!k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x-1)^n \frac{\mathrm{d}^{n-m-k}}{\mathrm{d}x^{m-n-k}}(x+1)^n \\
&= \frac{(1-x^2)^{-m/2}}{2^n n!} \sum_{k=0}^{n-m} \frac{(n-m)!}{(n-m-k)!k!} \left[\frac{n!}{(n-k)!}(x-1)^{n-k} \frac{n!}{(m-k)!}(x+1)^{m+k}\right], \\
P_n^m(x) &= \frac{(1-x^2)^{m/2}}{2^n n!} \sum_{k=m}^{n} \frac{(n+m)!}{(n-m-k)!k!} \left[\frac{n!}{(n-k)!}(x-1)^{n-k} \frac{n!}{(k-m)!}(x+1)^{k-m}\right] \\
&= \frac{(1-x^2)^{-m/2}}{2^n n!}(1-x^2)^m \sum_{j=0}^{n-m} \frac{(n+m)!}{(n-j)!(m+j)!} \left[\frac{n!}{(n-m-j)!}(x-1)^{n-m-j} \frac{n!}{j!}(x+1)^j\right] \\
&= \frac{(-1)^m(1-x^2)^{-m/2}}{2^n n!} \sum_{j=0}^{n-m} \frac{(n+m)!}{(n-m-j)!j!} \left[\frac{n!}{(n-j)!}(x-1)^{n-j} \frac{n!}{(m+j)!}(x+1)^{m+j}\right] \\
&= \frac{(n+m)!}{(n-m)!}(-1)^m P_n^{-m}.
\end{aligned}$$

Logo, segue que

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x).$$

12. Mostre que

$$P_n^m(0) = \begin{cases} (-1)^{(n-m)/2}(n+m-1)!! \left[(n-m)!!\right]^{-1}, & n+m \text{ é par}, \\ 0, & n+m \text{ é ímpar}. \end{cases}$$

Solução: *Para $x = 0$ temos que (8) assume a forma*

$$\begin{aligned}
\frac{(2m)!}{2^m m!}\frac{1}{(1+t^2)^{m+1/2}} &= \sum_{n=0}^{\infty} P_{n+m}^m(0)t^n \\
&= \sum_{k=m}^{\infty} P_k^m(0)t^{k-m} \\
&= \frac{(2m)!}{2^m m!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(m+1/2)_n t^{2n} \\
&= \frac{(2m)!}{2^m m!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(2m+2n-1)!!}{(2m-1)!!}\frac{t^{2n}}{2^n} \\
&= \frac{(2m)!}{2^m m!}\sum_{k=0}^{\infty}\frac{(-1)^{(k-m)/2}}{((k-m)/2)!}\frac{(m+k-1)!!}{(2m-1)!!}\frac{t^{k+m}}{2^{k-m/2}} \\
&= \frac{(2m)!}{2^m m!}\sum_{k=0}^{\infty}\frac{(-1)^{(k-m)/2}}{(k-m)!!}\frac{2^{(k-m)/2}}{2^{k-m/2}}\frac{(m+k-1)!!}{(2m-1)!!}t^{k-m} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{(k-m)/2}}{(k-m)!}(m+k-1)!!t^{k-m}.
\end{aligned}$$

Logo, segue que

$$P_k^m(0) = (-1)^{(k-m)/2}\frac{(m+k-1)!!}{(k-m)!!}.$$

13. Sejam $P_n^m(x)$ as funções associadas de Legendere de primeira esécie. Mostre que

$$P_n^n(\cos\theta) = (2n-1)!!\sin^n\theta, \qquad n = 0,1,2,\ldots.$$

Solução: *Temos que*

$$\begin{aligned}
P_n^n(x) &= (1-x^2)^{n/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}P_n(x) \\
&= \frac{(1-x^2)^{n/2}}{2^n n!}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n.
\end{aligned}$$

Mas

$$\begin{aligned}
(x^2-1)^n &= \sum_{k=0}^{n}\binom{n}{k}(x^2)^{n-k}(-1)^k \\
&= x^{2n}\underbrace{-\binom{n}{1}x^{2n-2}+\binom{n}{2}x^{2n-4}-\ldots+(-1)^n}_{\textit{nulo ao calcular } \mathrm{d}^2n/\mathrm{d}x^{2n}}
\end{aligned}$$

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[\left(x^2-1\right)^n\right] = \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(x^{2n}\right)$$

$$
\begin{aligned}
&= 2n(2n-1)\ldots 1 \\
&= (2n)!
\end{aligned}
$$

e portanto $P_n^n(x) = (1-x^2)^{n/2}\,(2n)!\,(2^n n!)^{-1}$. *Como*

$$
\begin{aligned}
\frac{(2n)!}{2^n n!} &= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 2\cdot 1}{2^n n(n-1)\ldots 1} \\
&= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 2\cdot 1}{(2n-2)\ldots 2} \\
&= (2n-1)(2n-3)\ldots 1 \\
&= (2n-1)!!
\end{aligned}
$$

temos que $P_n^n(x) = (1-x^2)^{n/2}\,(2n-1)!!$.
Fazendo $x = \cos\theta$ *temos que* $(1-x^2)^{1/2} = \sin\theta$, $\theta \in [0,\pi]$. *Se* $x \in [-1,1]$ *então* $\theta \in [0,\pi]$ *e portanto* $P_n^n(\cos\theta) = (2n-1)!!\sin^n\theta$.

14. Seja P_n o n-ésimo polinômio de Legendre ($n = 0, 1, 2, \ldots$). Calcule $P_n(0)$.

    Solução:

15. (E de 2010) Seja P_n o n-ésimo polinômio de Legendre. Mostre que

$$
\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) P_n(x') = \delta(x-x'),
$$

$x, x' \in (-1,1)$. Dica: lembre-se que os polinômios de Legendere forma um conjunto ortogonal completo, com $\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = 2(2n+1)^{-1}\delta_{nm}$.

Solução:

16. (T6 de 2011) Seja $P_n(x)$ ($n = 0, 1, 2, \ldots$) os polinômios de Legendere. Mostre que

$$
P_n'(x) = (2n-1)P_{n+1}(x) + (2n-5)P_{n-3} + (2n-9)P_{n-5}(x) + \ldots + \Delta_n,
$$

onde $\Delta_n = 3P_1(x)$ se n for par e $\Delta_n = P_0(x)$ se n for ímpar.

Solução: *Temos* (13),

$$
(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x),
$$

e fazendo $n \to n-1$ *obtemos*

$$
\begin{aligned}
P_n'(x) &= (2n-1)P_{n-1}(x) + P_{n-2}'(x) \\
P_{n-2}'(x) &= (2n-5)P_{n-3}(x) + P_{n-4}'(x) && n \to n-2 \\
P_{n-4}'(x) &= (2n-9)P_{n-5}(x) + P_{n-6}'(x) && n \to n-2
\end{aligned}
$$

$$\vdots$$

Somando as equações acima obtemos

$$P_n'(x) = (2n-1)P_{n+1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \dots.$$

Se n é par, a última relação é

$$P_2(x) = 3P_1(x) + P_0(x) = 3P_1(x)$$

e se n é ímpar, a última relação é

$$P_1(x) = P_0(x) + 0 = P_0(x).$$

Logo,

$$\Delta_n = \begin{cases} 3P_1(x), & n \text{ par}, \\ P_0(x), & n \text{ ímpar}. \end{cases}$$

17. (P2 de 2011) Sejam $P_n(x)$ os polinômios de Legendre ($n = 0, 1, 2, \dots$). Mostre que

(a) $P_n(1) = 1$,

Solução: *Temos* (FGL) *e* (FGLS). *Logo para $x = 1$ temos*

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(1)t^n &= \frac{1}{\sqrt{1-2t+t^2}} \\ &= \frac{1}{1-t} \\ &= \sum_{n=0}^{\infty} t^n \\ &= 1. \end{aligned}$$

(b) $\int_0^1 P_{2n}(x)\,\mathrm{d}x = 0$ $(n \neq 0)$.

Solução: *Utilizando* (13) *temos que*

$$\begin{aligned} \int_0^1 P_{2n}(x)\,\mathrm{d}x &= \int_0^1 \left[\frac{P_{2n+1}'(x) - P_{2n-1}'(x)}{2(2n)+1}\right] \mathrm{d}x \\ &= \frac{1}{4n+1}\left[P_{2n+1}(x)\big|_0^1 - P_{2n-1}(x)\big|_0^1\right] \\ &= \frac{1}{(4n+1)}\left[P_{2n+1}(1) - P_{2n-1}(0) - P_{2n-1}(1) + P_{2n-1}(0)\right] \\ &= \frac{1}{(4n+1)}\left[-P_{2n-1}(0) + P_{2n-1}(0)\right] \\ &= \frac{1}{(4n+1)}\left[-0+0\right] \end{aligned}$$

★

$$= 0,$$

onde $\bigstar$ *decorre de*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(0)t^n &= \frac{1}{\sqrt{1+t^2}} \\
&= \sum_{m=0}^{\infty} \frac{(1/2)_m(-1)^m(t^2)^m}{m!} \\
P_{2n+1}(0) &= 0.
\end{aligned}$$

18. (E de 2011) Sejam $P_n(x)$ os polinômios de Legendre ($n = 0, 1, 2, \ldots$). Mostre que

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

Solução: *Temos que*

$$\begin{aligned}
g(x,t) &= \frac{1}{\sqrt{1-2xt+t^2}} \\
&= \sum_{n=0}^{\infty} P_n(x)t^n.
\end{aligned}$$

Derivando em relação a t

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial t} &= \left(\frac{-1}{2}\right)\frac{-2x+2t}{[1-2xt+t^2]^{3/2}} \\
&= \frac{x-t}{1-2xt+t^2} g(x,t) \\
(x-t)g(x,t) &= (1-2xt+t^2)\frac{\partial g(x,t)}{\partial t} \\
(x-t)\sum_{n=0}^{\infty} P_n(x)t^n &= (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} \\
\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} &= \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1} \\
\sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n &= \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n + \sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^n \\
xP_0(x) + \sum_{n=1}^{\infty}(2n+1)xP_n(x)t^n &= P_1(x) + \sum_{n=1}^{\infty}(n+1)P_{n+1}(x)t^n + \sum_{n=1}^{\infty} nP_{n-1}(x)t^n.
\end{aligned}$$

Por fim, temos que

$$\begin{cases} xP_0(x) = P_1(x), \\ (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 1, 2, \ldots. \end{cases}$$

Referências

[1] G.B. Arfken and H.J. Weber. *Mathematical Methods For Physicists*. Elsevier, 2005.