ME430A - Técnicas de Amostragem
Segundo semestre de 2011
Prova 1 - Solução

Questão 1
Item (a)

Distribuição exata de $\hat{\mu}$ para o plano A:

$\hat{\mu}$	2	2,5	3,5
$P_A(\hat{\mu})$	1/3	1/3	1/3

Esperança de $\hat{\mu}$:

$E_A(\hat{\mu}) = 2 \cdot \frac{1}{3} + 2,5 \cdot \frac{1}{3} + 3,5 \cdot \frac{1}{3} = \frac{8}{3}$

Variância de $\hat{\mu}$:

$$\begin{aligned} Var_A(\hat{\mu}) &= E(\hat{\mu}^2) - [E(\hat{\mu})]^2 \\ &= 2^2 \cdot \frac{1}{3} + (2,5)^2 \cdot \frac{1}{3} + (3,5)^2 \cdot \frac{1}{3} - (\frac{8}{3})^2 \\ &= \frac{7}{18} \end{aligned}$$

Erro-padrão de $\hat{\mu}$:

$EP_A(\hat{\mu}) = \sqrt{Var_A(\hat{\mu})} = 0,623$

Erro quadrático médio de $\hat{\mu}$:

$$\begin{aligned} EQM_A[\hat{\mu}] &= Var_A(\hat{\mu}) + B^2(\hat{\mu}) \\ &= \frac{7}{18} + (\frac{8}{3} - \frac{8}{3})^2 \\ &= \frac{7}{18} \end{aligned}$$

Item (b)

Distribuição exata de $\hat{\mu}$ para o plano B:

$\hat{\mu}$	1	2	2,5	3	3,5	4
$\mathrm{P}_B(\hat{\mu})$	1/9	2/9	2/9	1/9	2/9	1/9

Esperança de $\hat{\mu}$:

$E_B(\hat{\mu}) = 1 \cdot \frac{1}{9} + 2 \cdot \frac{2}{9} + 2,5 \cdot \frac{2}{9} + 3 \cdot \frac{1}{9} + 3,5 \cdot \frac{2}{9} + 4 \cdot \frac{1}{9} = \frac{8}{3}$

Variância de $\hat{\mu}$:

$$\begin{aligned} Var_B(\hat{\mu}) &= E(\hat{\mu}^2) - [E(\hat{\mu})]^2 \\ &= 1^2 \cdot \tfrac{1}{9} + 2^2 \cdot \tfrac{2}{9} + (2,5)^2 \cdot \tfrac{2}{9} + 3^2 \cdot \tfrac{1}{9} + (3,5)^2 \cdot \tfrac{2}{9} + 4^2 \cdot \tfrac{1}{9} - (\tfrac{8}{3})^2 \\ &= \tfrac{7}{9} \end{aligned}$$

Erro-padrão de $\hat{\mu}$:

$EP_B(\hat{\mu}) = \sqrt{Var_B(\hat{\mu})} = 0,882$

$$\begin{aligned} EQM_B[\hat{\mu}] &= Var_B(\hat{\mu}) + B^2(\hat{\mu}) \\ &= \tfrac{7}{9} + (\tfrac{8}{3} - \tfrac{8}{3})^2 \\ &= \tfrac{7}{9} \end{aligned}$$

Item (c)

O plano B é o melhor porque tem menor variância e, consequentemente, o menor EQM.

Item (d)

$$
\begin{aligned}
\delta &= z\sqrt{\sigma^2} \\
&= z\sqrt{\mathcal{V}(\hat{p})} \\
\delta^2 &= z^2\mathcal{V}(\hat{p}) \\
&= z^2\frac{(N-n)p(1-p)}{n(N-1)} \\
n &= \frac{z^2}{\delta^2}\frac{(N-n)p(1-p)}{N-1} \\
&= \frac{z^2}{\delta^2}\left(\frac{Np(1-p)}{N-1} - \frac{np(1-p)}{N-1}\right) \\
&= \frac{z^2}{\delta^2}\frac{Np(1-p)}{N-1} - \frac{z^2}{\delta^2}\frac{np(1-p)}{N-1}
\end{aligned}
$$

$$
\begin{aligned}
n + \frac{z^2}{\delta^2}\frac{np(1-p)}{N-1} &= \frac{Np(1-p)}{N-1} \\
n\left(1 + \frac{z^2p(1-p)}{\delta^2(N-1)}\right) &= \frac{Np(1-p)}{N-1}
\end{aligned}
$$

$$
\begin{aligned}
n &= \frac{Np(1-p)/(N-1)}{1+\frac{z^2p(1-p)}{\delta^2(N-1)}} \\
&= \frac{Np(1-p)/(N-1)}{\frac{(N-1)\delta^2+z^2p(1-p)}{\delta^2(N-1)}} \\
&= \frac{Np(1-p)}{\frac{(N-1)\delta^2+z^2p(1-p)}{\delta^2}} \\
&= \frac{N}{\frac{N-1}{p(1-p)}+\frac{z^2}{\delta^2}} \\
n &= \frac{N}{\frac{\delta^2(N-1)}{z^2p(1-p)}+1}
\end{aligned}
$$

Item (e)

$$n = \frac{2000}{\frac{(0{,}02)^2(2000-1)}{(1/2)(1/2)(1{,}96)^2}+1} = 1.091,36 \approx 1.092$$

Questão 2

Item (a)

Utilizando a equação do formulário que obtém o tamanho da amostra, temos

$$n = \frac{1500(100,25)(1,64)^2}{3^2 \times 1.500 + (100,25)(1,64)^2} = 29,37 \approx 30$$

Item (b)

$$\left(\tilde{\mu} - z_\alpha\sqrt{(1-f)\frac{\tilde{s}^2}{n}}; \tilde{\mu} + z_\alpha\sqrt{(1-f)\frac{\tilde{s}^2}{n}}\right)$$

$$\left(13,55 - 2,58\sqrt{\left(1-\frac{50}{1500}\right)\frac{93,32}{50}}; 13,55 + 2,58\sqrt{\left(1-\frac{50}{1500}\right)\frac{93,32}{50}}\right)$$

$$(10,08; 17,02)$$

Item (c)

$$
\begin{aligned}
\frac{(N-n)(N-n-1)}{N(N-1)} + 2\alpha + \frac{n(n-1)}{N(N-1)} &= 1 \\
\frac{(N-n)(N-n-1) + n(n-1)}{N(N-1)} + 2\alpha &= 1 \\
\frac{N(N-n-1) - n(N-n-1) + n(n-1)}{N(N-1)} + 2\alpha &= 1 \\
\frac{N^2 - Nn - N - nN + n^2 + n + n^2 - n}{N(N-1)} + 2\alpha &= 1 \\
\frac{N^2 - 2Nn - N + 2n^2}{N(N-1)} + 2\alpha &= 1 \\
\frac{N(N-1) - N^2 + 2Nn + N - 2n^2}{N(N-1)} &= 2\alpha \\
\frac{2Nn - 2n^2}{N(N-1)} &= 2\alpha \\
\frac{n(N-n)}{N(N-1)} &= \alpha
\end{aligned}
$$

Questão 3

Item (a)

$$
\begin{aligned}
\mathcal{E}(\hat{\mu}_2) &= \mathcal{E}\left(\frac{\displaystyle\sum_{i=2}^{N-2} Y_i}{N-2}\right) \\
&= \mathcal{E}\left(\frac{(N-2)\mu}{N-2}\right) \\
&= \mu
\end{aligned}
$$

Item (b)

$$\begin{aligned}\mathcal{V}(\hat{\mu}_{st}^2) &= \left(\frac{N-2}{N}\right)^2 \mathcal{V}(\hat{\mu}_2)\\ &= \left(\frac{N-2}{N}\right)^2 \left(1-\frac{n-2}{N-2}\right)\frac{s_2^2}{n-2}\end{aligned}$$

em que $s_2^2 = \frac{1}{N-3}\sum_{i=2}^{N-2}(y_i-\mu)^2$.

Item (c)

Considerando $s_2^2 < s^2$ e, N e n muito grandes, temos que a razão

$$\frac{\mathcal{V}(\hat{\mu}_2)}{\mathcal{V}(\hat{\mu})} = \frac{s_2^2}{s^2} < 1$$

ou seja,
$s_2^2 < s^2$, o que mostra que o estimador s_2^2 é melhor.