( )F520   ( )MS550 - Primeira Prova - 06/05/2013

RA: ________________   **Nome:** ____________________________________

(1) (i) Sejam as coordenadas esféricas (r, θ, ϕ) dadas por

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta,$$

onde $0 \leq r < \infty$, $0 \leq \theta \leq \phi$ e $0 \leq \phi < 2\pi$.
(i) Mostre que os vetores tangentes unitários $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ são dados por

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k}, \\
\mathbf{e}_\theta &= \cos\theta\cos\phi\boldsymbol{i} + \cos\theta\sin\phi\boldsymbol{j} - \sin\theta\boldsymbol{k}, \\
\mathbf{e}_\phi &= -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j},
\end{aligned}$$

onde $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ são os vetores unitários cartesianos tais que $\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
(ii) Seja o campo vetorial $\mathbf{V}$ dado por

$$\mathbf{V} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{yz}{x^2+y^2}\boldsymbol{i} - \frac{xz}{x^2+y^2}\boldsymbol{j}\right).$$

Mostre que em termos de coordenadas esféricas

$$\mathbf{V} = -\frac{\cot\theta}{r}\mathbf{e}_\phi.$$

(iii) Utilizando coordenadas esféricas, calcule o rotacional de $\mathbf{V}$.

(2) Seja a equação diferencial

$$x(1-x)y'' + (1-5x)y' - 4y = 0.$$

Ao utilizar o método de Frobenius para resolver essa equação diferencial encontramos que a equação indicial correspondente apresenta raízes iguais $r_1 = r_2 = 0$, e com isso obtemos que uma das soluções em forma de série dessa equação diferencial é

$$y_1(x) = \sum_{n=0}^{\infty}(n+1)^2 x^n.$$

Utilize o método de Frobenius para encontrar uma segunda solução $y_2(x)$ linearmente independente.

(3) Mostre que nenhuma solução não trivial da equação

$$z^2y'' + zy' + y = 0$$

que é real no semieixo real positivo do plano complexo pode ser real no semieixo real negativo.

(4) Mostre que

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx = \pi.$$

🛈 Valor das questões: (1) 3,0 (2) 3,0 (3) 2,5 (4) 1,5.

Formulário (eventualmente útil)

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{e}_{q_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{e}_{q_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{e}_{q_3},$$

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_3 h_1 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right],$$

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_{q_1} & h_2\mathbf{e}_{q_2} & h_3\mathbf{e}_{q_3} \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix},$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad \Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\, dt,$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z),$$

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \qquad B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt$$

$$B(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\theta \sin^{2w-1}\theta\, d\theta,$$

(1) $\vec{r} = (r\sin\theta\cos\phi)\vec{\imath} + (r\sin\theta\sin\phi)\vec{\jmath} + (r\cos\theta)\vec{k}$

(i)

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k} \qquad \therefore \vec{e}_r = \frac{\partial\vec{r}}{\partial r} \checkmark$$

$$h_r = \sqrt{\underbrace{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi}_{\sin^2\phi} + \cos^2\theta} = 1$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\vec{\imath} + r\cos\theta\sin\phi\,\vec{\jmath} - r\sin\theta\,\vec{k}$$

$$\therefore h_\theta = \sqrt{\underbrace{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi}_{r^2\cos^2\theta} + r^2\sin^2\theta} = r \qquad \therefore \vec{e}_\theta = \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta} \checkmark$$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{\imath} + r\sin\theta\cos\phi\,\vec{\jmath} \qquad \therefore \vec{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi} \checkmark$$

$$\therefore h_\phi = \sqrt{r^2\sin^2\theta\sin^2\phi + r\sin^2\theta\cos^2\phi} = r\sin\theta$$

+1,0

(ii) $\vec{\imath} = (\vec{\imath}\cdot\vec{e}_r)\vec{e}_r + (\vec{\imath}\cdot\vec{e}_\theta)\vec{e}_\theta + (\vec{\imath}\cdot\vec{e}_\phi)\vec{e}_\phi$

$$\therefore \vec{\imath} = \sin\theta\cos\phi\,\vec{e}_r + \cos\theta\cos\phi\,\vec{e}_\theta - \sin\phi\,\vec{e}_\phi$$

$$\vec{\jmath} = (\vec{\jmath}\cdot\vec{e}_r)\vec{e}_r + (\vec{\jmath}\cdot\vec{e}_\theta)\vec{e}_\theta + (\vec{\jmath}\cdot\vec{e}_\phi)\vec{e}_\phi$$

$$\therefore \vec{\jmath} = \sin\theta\sin\phi\,\vec{e}_r + \cos\theta\sin\phi\,\vec{e}_\theta + \cos\phi\,\vec{e}_\phi$$

$$\sqrt{x^2+y^2+z^2} = r$$

$$x^2+y^2 = r^2\sin^2\theta$$

$$\therefore \vec{V} = \frac{1}{r\cdot r^2\sin^2\theta}\Big[r^2\sin\theta\sin\phi\cos\theta\left(\sin\theta\cos\phi\,\vec{e}_r + \cos\theta\cos\phi\,\vec{e}_\theta - \sin\phi\,\vec{e}_\phi\right)$$
$$- r^2\sin\theta\cos\phi\cos\theta\left(\sin\theta\sin\phi\,\vec{e}_r + \cos\theta\sin\phi\,\vec{e}_\theta + \cos\phi\,\vec{e}_\phi\right)\Big]$$

$$\vec{V} = \frac{1}{r^3 \sin^2\theta}\Big[\big(\underbrace{r^2\sin^2\theta\cos\theta\sin\phi\cos\phi - r^2\sin^2\theta\cos\phi\sin\phi\cos\phi}_{=0}\big)\vec{e}_r$$

$$+\big(\underbrace{r^2\sin\theta\cos^2\theta\sin\phi\cos\phi - r^2\sin\theta\cos^2\theta\sin\phi\cos\phi}_{=0}\big)\vec{e}_\theta$$

$$-\big(\underbrace{r^2\sin\theta\sin^2\phi\cos\theta + r^2\sin\theta\cos^2\phi\cos\theta}_{r^2\sin\theta\cos\theta}\big)\vec{e}_\phi\Big]$$

$$= \frac{1}{r^3\sin^2\theta}\left(-r^2\sin\theta\cos\theta\right)\vec{e}_\phi \qquad \therefore \quad \vec{V} = -\frac{\cot\theta}{r}\vec{e}_\phi$$

+1,0

iii

$$\vec{\nabla}\times\vec{V} = \frac{1}{r \cdot r\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 1\cdot 0 & r\cdot 0 & \underbrace{r\sin\theta\left(-\frac{\cot\theta}{r}\right)}_{-\cos\theta} \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}(-1)\frac{\partial}{\partial\theta}(\cos\theta)\,\vec{e}_r = \frac{1}{r^2}\vec{e}_r$$

$$\therefore \quad \vec{\nabla}\times\vec{V} = \frac{1}{r^2}\vec{e}_r = \frac{\vec{r}}{r^3}$$

+1,0

2)

$$y_2(x) = y_1(x)\ln x + x^0 \sum_{n=1}^{\infty} a_n x^n = y_1(x)\ln x + \sum_{n=1}^{\infty} a_n x^n$$

$$y_2'(x) = y_1' \ln x + y_1 \frac{1}{x} + \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y_2''(x) = y_1'' \ln x + 2y_1' \frac{1}{x} - y_1 \frac{1}{x^2} + \sum_{n=1}^{\infty} n(n-1) a_n x^{n-1}$$

$$(*) \Rightarrow \underbrace{[x(1-x)y_1'' + (1-5x)y_1' - 4y_1]}_{=0}\ln x + 2y_1' - \cancel{y_1\frac{1}{x}} + \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1}$$

$$-2xy_1' + y_1 - \sum_{n=1}^{\infty} n(n-1)a_n x^n + \cancel{y_1 \frac{1}{x}} + \sum_{n=1}^{\infty} n a_n x^{n-1} - 5y_1 - 5\sum_{n=1}^{\infty} n a_n x^n - 4\sum_{n=1}^{\infty} a_n x^n = 0$$

usando a expressaõ de $y_1(x)$:

$$2\sum_{n=1}^{\infty} n(n+1)^2 x^{n-1} + \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} - 2\sum_{n=1}^{\infty} n(n+1)^2 x^n - \sum_{n=1}^{\infty} n(n-1)a_n x^n$$

$$+\sum_{n=1}^{\infty} n a_n x^{n-1} - 4\sum_{n=0}^{\infty} (n+1)^2 x^n - 5\sum_{n=1}^{\infty} n a_n x^n - 4\sum_{n=1}^{\infty} a_n x^n = 0$$

$$\therefore 2\cdot 1\cdot 2^2 x^0 + 2\sum_{n=1}^{\infty} (n+1)(n+2)^2 x^n + \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - 2\sum_{n=1}^{\infty} n(n+1)^2 x^n$$

$$-\sum_{n=1}^{\infty} n(n-1)a_n x^n + 1\cdot a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - 4 - 4\sum_{n=1}^{\infty} (n+1)^2 x^n$$

$$-\sum_{n=1}^{\infty} (5n+4) a_n x^n = 0$$

+1,0

$$\therefore \begin{cases} 8-4+a_1=0 \quad \Longrightarrow \quad \boxed{a_1=-4=-2\cdot 2} \\ 2(n+1)(n+2)^2+(n+1)n\,a_{n+1}-2n(n+1)^2-n(n-1)a_n+(n+1)a_{n+1} \\ \quad -4(n+1)^2-(5n+4)\,a_n=0 \,, \qquad n=1,2,3,\ldots \quad \circledast \end{cases}$$

(+0,5)

$$\circledast\ \underbrace{(n+1)\Big[\underbrace{2(n+2)^2-\underbrace{2n(n+1)-4(n+1)}_{-2(n+2)(n+1)}}_{\underbrace{(n+2)\left[2(n+2)-2(n+1)\right]}_{(n+2)\cdot 2}}\Big]}+(n+1)^2a_{n+1}-\Big[\underbrace{\underbrace{n(n-1)+5n+4}_{n^2+4n+4}}_{(n+2)^2}\Big]a_n=0$$

$$\therefore\ 2(n+1)(n+2)+(n+1)^2a_{n+1}-(n+2)^2a_n=0$$

$$\therefore\ \boxed{a_{n+1}=\left(\frac{n+2}{n+1}\right)\left[\left(\frac{n+2}{n+1}\right)a_n-2\right]} \qquad n=1,2,3,\ldots$$

(+1,0)

$\underline{n=2}$

$$a_2=\frac{3}{2}\left[\frac{3}{2}\cdot a_1-2\right]=\frac{3}{2}(-6-2)=-3\cdot 4=-3\cdot 2\cdot 2$$

$\underline{n=3}$

$$a_3=\frac{4}{3}\left[\frac{4}{3}(-3\cdot 4)-2\right]=\frac{4}{3}(-16-2)=-4\cdot 3\cdot 2$$

$\underline{n=4}$

$$a_4=\frac{5}{4}\left[\frac{5}{4}(-4\cdot 6)-2\right]=\frac{5}{4}(-32)=-5\cdot 4\cdot 2$$

$$\therefore\ \boxed{a_n=-2n(n+1)} \qquad n=1,2,3\ldots$$

(+0,5)

$$\therefore\ y_2(x)=y_1(x)\ln x-2\sum_{n=1}^{\infty}n(n+1)x^n$$

onde: $y_1(x)=\sum_{n=0}^{\infty}(n+1)^2x^n$

(3) $\left.\begin{array}{l} z^2y'' + zy' + y = 0 \\ y = z^r \end{array}\right\} \Rightarrow r(r-1) + r + 1 = 0 \Rightarrow r^2+1=0 \therefore r = \pm i$

$$\therefore y = c_1 z^{i} + c_2 z^{-i} = c_1 e^{i\ln z} + c_2 e^{-i\ln z} \quad \text{, onde } c_1, c_2 \in \mathbb{C}$$

+0,5

Escrevendo $z = re^{i\theta}$, $c_1 = \alpha_1 + i\beta_1$, $c_2 = \alpha_2 + i\beta_2$, temos

$$y = (\alpha_1 + i\beta_1)\, e^{i\ln r} e^{-\theta} + (\alpha_2 + i\beta_2)\, e^{-i\ln r} e^{\theta}$$

$$= (\alpha_1 + i\beta_1) e^{-\theta} [\cos(\ln r) + i\sin(\ln r)] + (\alpha_2 + i\beta_2) e^{\theta} [\cos(\ln r) - i\sin(\ln r)]$$

$$= [(\alpha_1 e^{-\theta} + \alpha_2 e^{\theta}) \cos(\ln r) + (-\beta_1 e^{-\theta} + \beta_2 e^{\theta}) \sin(\ln r)]$$

$$+ i\,[(\alpha_1 e^{-\theta} - \alpha_2 e^{\theta}) \sin(\ln r) + (\beta_1 e^{-\theta} + \beta_2 e^{\theta}) \cos(\ln r)] \quad (*)$$

semieixo real positivo $\Rightarrow \theta = 0$

$$(*) \Rightarrow y_+ = (\alpha_1+\alpha_2)\cos(\ln r) + (\beta_2-\beta_1)\sin(\ln r) + i\left[(\alpha_1-\alpha_2)\sin(\ln r) + (\beta_1+\beta_2)\cos(\ln r)\right]$$

$$\therefore y_+ \in \mathbb{R} \iff \boxed{\alpha_1 = \alpha_2,\ \beta_1 = -\beta_2} \quad (\mathrm{I})$$

semieixo real negativo $\Rightarrow \theta = \pi$

$$(*) \Rightarrow y_- = (\alpha_1 e^{-\pi} + \alpha_2 e^{\pi})\cos(\ln r) + (-\beta_1 e^{-\pi} + \beta_2 e^{\pi})\sin(\ln r) + i\left[(\alpha_1 e^{-\pi} - \alpha_2 e^{\pi})\sin(\ln r) + (\beta_1 e^{-\pi} + \beta_2 e^{\pi})\cos(\ln r)\right]$$

$$\therefore y_- \in \mathbb{R} \iff \boxed{\alpha_1 e^{-\pi} = \alpha_2 e^{\pi},\ \beta_1 e^{-\pi} = -\beta_2 e^{\pi}} \quad (\mathrm{II})$$

+1,5

MAS (I) e (II) NÃO podem ser satisfeitas ao mesmo tempo!

+0,5

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(4) Escrevendo $\frac{x+1}{2} = t \quad \therefore x = 2t-1$, temos

$$I = \int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^1 \sqrt{\frac{2t}{2(1-t)}}\,2\,dt = 2\int_0^1 t^{1/2}(1-t)^{-1/2}\,dt$$

$$= 2B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{2\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = 2\cdot\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1\cdot\Gamma(1)} =$$

$$= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

+1,5