Anomalous Fluctuations of Extremes in Many-Particle Diffusion

Jacob B Hass, Aileen Carroll-Godfrey, and Eric I Corwin
Department of Physics and Materials Science Institute, University of Oregon, Eugene, Oregon 97403, USA.

Ivan Z Corwin
Department of Mathematics, Columbia University, New York, New York 10027, USA.

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Over one hundred years ago Einstein created a remarkably simple and powerful theory describing the behavior of a single diffusing particle. That theory has since been applied countless times to successfully model widely disparate systems. However, this theory neglects the effects a shared environment has on the particles. As a consequence, the Einstein theory dramatically fails to predict the behavior of extreme diffusion, i.e. outlier particles which have moved the farthest from their starting points. We study particles undergoing a random walk in a beta distributed environment and provide theoretical predictions, which we confirm numerically, of the behavior of the maximally displaced particle. By introducing a shared environment, we find three scaling regimes relating to the KPZ equation for the variance of the maximally displaced particle, contrary to the Einstein diffusion model which predicts a single scaling regime. Understanding the behavior of outliers will have wide ranging applicability to physical, biological, epidemiological, economic, and social systems where outliers often determine behavior.

I. OUTLINE

1. Introduction
2. Background (Einstein, which in our language is a deterministic CDF calculation plus a Gumbel-ish process)
3. Discrete model for diffusion, how to compute the CDF from it, how to compute the maximum value from it
4. Computational methods to implement this model
5. What is already known about this particular discrete model (asymptotics in large and moderate deviations)
6. Convert known results in large and moderate deviations into $log(N)$, $log(N)^2$ language)
7. Matching the two regions together
8. Sample the CDF: computing the actual max by including the Gumbel (already described in Background). Explicitly compute it.
9. Results, comparisons of numerics to theory and to pure Einstein, across broad range of N

II. INTRODUCTION

Understanding outliers is the key to describing many important phenomena, including biological processes dependent on the interactions between strands of DNA, patient zero spreading a virus to a new area in a pandemic, many chemical processes, and how microscopic price fluctuations relate to long-term macroscopic trends in the stock market [1–4]. How do we predict the behavior of these outliers? In diffusive environments, where many particles spread outward from their originating source, we can use Einstein's theory of diffusion to describe the bulk behavior of the particles [5, 6]. Einstein described diffusion as particles undergoing independent random walks. The random-walk description ignores the effect the shared environment has on the particles, resulting in its potential failure to correctly describe the diffusion of the outliers. Particles near each other should be influenced by the medium in the same way, but in Einstein's theory of diffusion, their behavior is totally independent from one another. As a result, when analyzing the system as a whole, the distributions of outlier particle locations and first passage times will differ from that predicted by Einstein's theory. The objective of this work is to show this discrepancy and demonstrate the need for a new theory of extreme diffusion.

We study the behavior of the maximally displaced particle undergoing a 1D random walk in a beta distributed environment. Einstein's model of diffusion predicts the variance of the maximally displaced particle to grow proportionally to time. Additional theories of diffusion have been introduced with variances that scale with t^α for $\alpha > 1$, super-diffusive, or $\alpha < 1$, sub-diffusive. By studying particles moving in beta distributed environments, we found three scaling regimes which depend on the underlying beta distribution.

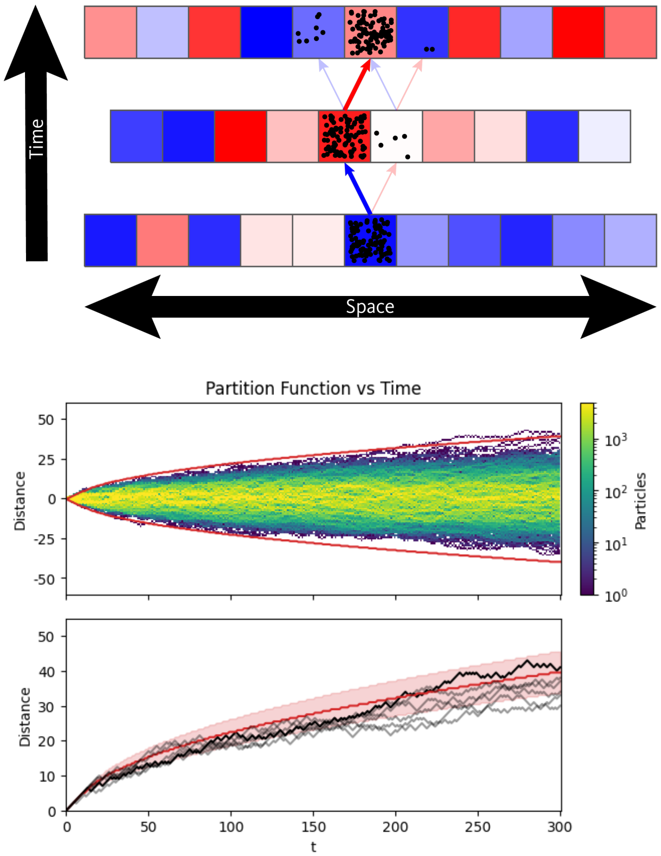

FIG. 1. System over time

III. BACKGROUND

IV. DISCRETE SYSTEMS

V. COMPUTATIONAL METHODS

We compute the PDF

VI. ASYMPTOTICS IN LARGE AND MODERATE DEVIATIONS

VII. THE $log(N)$ AND $log(N)^2$ REGIMES

VIII. MATCHING TWO REGIONS

IX. SAMPLING THE CDF

X. RESULTS

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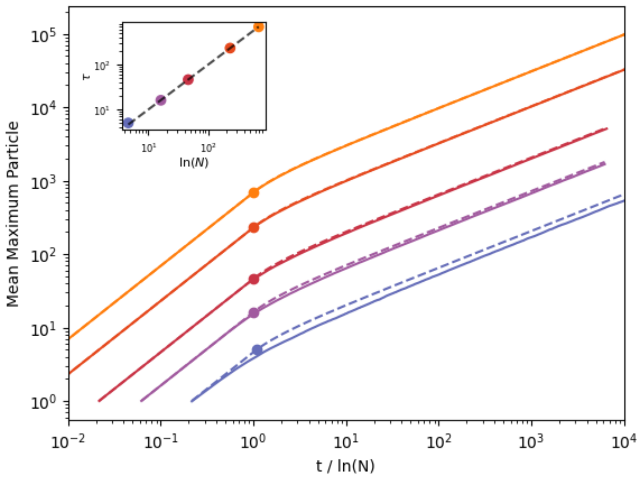

FIG. 2. Mean max particle over time

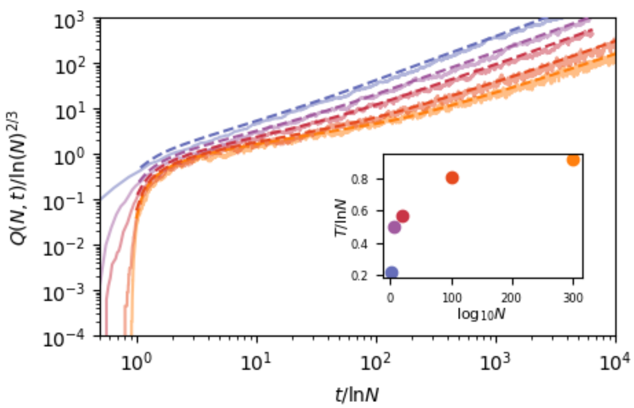

FIG. 3. Max particle variance over time

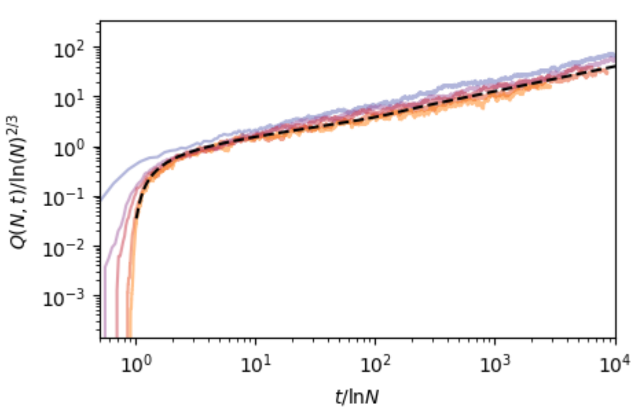

FIG. 4. Quantile variance over time