Chaotic Behavior of the Triple Pendulum
A Computational Approach

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1 Introduction

Appendix A Equations of Motion

A.1 Full Solution

We can describe the position of a given mass by its Cartesian coordinates (x_i, y_i). The position of a given mass is the sum of its position relative to its pivot point and the position of its pivot point:

$$
\begin{aligned}
x_1 &= l\sin\phi_1 & y_1 &= l(1-\cos\phi_1)\\
x_2 &= x_1 + l\sin\phi_2 & y_2 &= y_1 + l(1-\cos\phi_2)\\
x_3 &= x_2 + l\sin\phi_3 & y_3 &= y_2 + l(1-\cos\phi_3).
\end{aligned}
$$

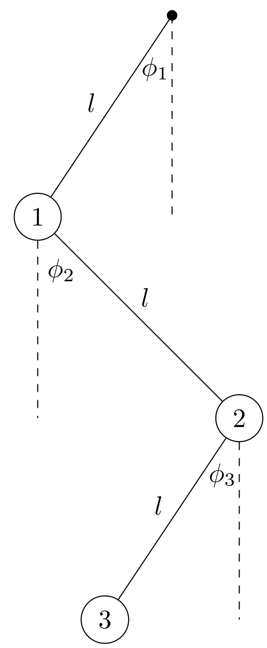

Figure 1: The triple pendulum.

Making the appropriate substitutions, we have

$$
\begin{aligned}
x_1 &= l\sin\phi_1\\
y_1 &= l(1-\cos\phi_1)\\
x_2 &= l(\sin\phi_1 + \sin\phi_2)\\
y_2 &= l((1-\cos\phi_1) + (1-\cos\phi_2))\\
x_3 &= l(\sin\phi_1 + \sin\phi_2 + \sin\phi_3)\\
y_3 &= l((1-\cos\phi_1) + (1-\cos\phi_2) + (1-\cos\phi_3)).
\end{aligned}
$$

Taking time derivatives to find velocity, we have

$$
\begin{aligned}
\dot{x}_1 &= l\dot{\phi}_1\cos\phi_1\\
\dot{y}_1 &= l\dot{\phi}_1\sin\phi_1\\
\dot{x}_2 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2)\\
\dot{y}_2 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2)\\
\dot{x}_3 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3)\\
\dot{y}_3 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3).
\end{aligned}
$$

For the kinetic energy, we will need to find $v^2 = \dot{x}^2\dot{y}^2$ for each mass.

$$
\begin{aligned}
v_1^2 &= \dot{x}_1^2 + \dot{y}_1^2\\
&= (l\dot{\phi}_1\cos\phi_1)^2 + (l\dot{\phi}_1\sin\phi_1)^2\\
&= l^2(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1)\\
&= l^2\dot{\phi}_1^2
\end{aligned}
$$

$$
\begin{aligned}
v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2\\
&= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2))^2\\
&= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_1^2\sin^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + \dot{\phi}_2^2\sin^2\phi_2\right)\\
&= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_2^2\sin^2\phi_2 + 2\dot{\phi}_1\dot{\phi}_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\right)\\
&= l^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
v_3^2 &= \dot{x}_3^2 + \dot{y}_3^2\\
&= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3))^2
\end{aligned}
$$

$$
\begin{aligned}
&= l^2\big(\dot\phi_1^2\sin^2\phi_1 + \dot\phi_2^2\sin^2\phi_2 + \dot\phi_3^2\sin^2\phi_3 + 2\dot\phi_1\dot\phi_2\sin\phi_1\sin\phi_2 + 2\dot\phi_1\dot\phi_3\sin\phi_1\sin\phi_3 + 2\dot\phi_2\dot\phi_3\sin\phi_2\sin\phi_3 \\
&\quad + \dot\phi_1^2\cos^2\phi_1 + \dot\phi_2^2\cos^2\phi_2 + \dot\phi_3^2\cos^2\phi_3 + 2\dot\phi_1\dot\phi_2\cos\phi_1\cos\phi_2 + 2\dot\phi_1\dot\phi_3\cos\phi_1\cos\phi_3 + 2\dot\phi_2\dot\phi_3\cos\phi_2\cos\phi_3\big)
\end{aligned}
$$

$$
= l^2\left(\dot\phi_1^2 + \dot\phi_2^2 + \dot\phi_3^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\right)
$$

We can now write the kinetic and potential energy for the system by summing up $-mgy_i$ and $\frac{1}{2}mv_i^2$ for each mass i. We will drop the constant terms from the potential energy expression since they will be lost when we take the partial derivatives of the Lagrangian:

$$
\begin{aligned}
U &= U_1 + U_2 + U_3 \\
&= -mgl\,(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3)
\end{aligned}
$$

$$
\begin{aligned}
T &= T_1 + T_2 + T_3 \\
&= \frac{1}{2}ml^2\left(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\right)
\end{aligned}
$$

So the Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= T + U \\
&= -mgl\,(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3) + \frac{1}{2}ml^2\big(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) \\
&\qquad + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\big)
\end{aligned}
$$

Since our generalized coordinates are ϕ_1, ϕ_2, and ϕ_3, we must now take partial derivatives of the Lagrangian with respect to each ϕ_i and $\dot\phi_i$:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi_1} &= 3mgl\sin\phi_1 - 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - ml^2\dot\phi_1\dot\phi_3\sin(\phi_1-\phi_3) \\
\frac{\partial\mathcal{L}}{\partial\phi_2} &= 2mgl\sin\phi_2 + 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - ml^2\dot\phi_2\dot\phi_3\sin(\phi_2-\phi_3) \\
\frac{\partial\mathcal{L}}{\partial\phi_3} &= mgl\sin\phi_3 + ml^2\dot\phi_1\dot\phi_3\sin(\phi_1-\phi_3) + ml^2\dot\phi_2\dot\phi_3\sin(\phi_2-\phi_3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\dot\phi_1 + 2ml^2\dot\phi_2\cos(\phi_1-\phi_2) + ml^2\dot\phi_3\cos(\phi_1-\phi_3) \\
\frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\dot\phi_2 + 2ml^2\dot\phi_1\cos(\phi_1-\phi_2) + ml^2\dot\phi_3\cos(\phi_2-\phi_3) \\
\frac{\partial\mathcal{L}}{\partial\dot\phi_3} &= ml^2\dot\phi_3 + ml^2\dot\phi_1\cos(\phi_1-\phi_3) + ml^2\dot\phi_2\cos(\phi_2-\phi_3)
\end{aligned}
$$

Next, we find $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_i}$ for each i:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\ddot\phi_1 + 2ml^2\ddot\phi_2\cos(\phi_1-\phi_2) - 2ml^2\dot\phi_2\sin(\phi_1-\phi_2)(\dot\phi_1-\dot\phi_2) \\
&\quad + ml^2\ddot\phi_3\cos(\phi_1-\phi_3) - ml^2\dot\phi_3\sin(\phi_1-\phi_3)(\dot\phi_1-\dot\phi_3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\ddot\phi_2 + 2ml^2\ddot\phi_1\cos(\phi_1-\phi_2) - 2ml^2\dot\phi_1\sin(\phi_1-\phi_2)(\dot\phi_1-\dot\phi_2) \\
&\quad + ml^2\ddot\phi_3\cos(\phi_2-\phi_3) - ml^2\dot\phi_3\sin(\phi_2-\phi_3)(\dot\phi_2-\dot\phi_3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_3} = &ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3) - ml^2\dot{\phi}_1\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\\
&+ ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3) - ml^2\dot{\phi}_2\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)
\end{aligned}
$$

We can then write the equations of motion for each of our generalized coordinates, according to the Euler-Lagrange condition $\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$. For ϕ_1 we have

$$
\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}
$$

$$
\begin{aligned}
3mgl\sin\phi_1 - 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) = &3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2)\\
&- 2ml^2\dot{\phi}_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\
&+ ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\
&- ml^2\dot{\phi}_3\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)
\end{aligned}
$$

$$
\begin{aligned}
3mgl\sin\phi_1 = &3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\
&+ 2ml^2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_3^2\sin(\phi_1-\phi_3)
\end{aligned}
$$

$$
\frac{3g}{l}\sin\phi_1 = 3\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_1-\phi_3) + 2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) \tag{1}
$$

For ϕ_2 we have

$$
\frac{\partial \mathcal{L}}{\partial \phi_2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}
$$

$$
\begin{aligned}
2mgl\sin\phi_2 + 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = &2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2)\\
&- 2ml^2\dot{\phi}_1\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\
&+ ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\
&- ml^2\dot{\phi}_3\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)
\end{aligned}
$$

$$
\begin{aligned}
2mgl\sin\phi_2 = &2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\
&+ 2ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_3^2\sin(\phi_2-\phi_3)
\end{aligned}
$$

$$
\frac{2g}{l}\sin\phi_2 = 2\ddot{\phi}_2 + 2\ddot{\phi}_1\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_2-\phi_3) + 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) \tag{2}
$$

For ϕ_3 we have

$$
\frac{\partial \mathcal{L}}{\partial \phi_3} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_3}
$$

$$
\begin{aligned}
mgl\sin\phi_3 + ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) + ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = &ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3)\\
&- ml^2\dot{\phi}_1\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\\
&+ ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3)
\end{aligned}
$$

$$- ml^2\dot{\phi}_2 \sin(\phi_2 - \phi_3)(\dot{\phi}_2 - \dot{\phi}_3)$$

$$\begin{aligned} mgl \sin\phi_3 = ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1 \cos(\phi_1 - \phi_3) + ml^2\ddot{\phi}_2 \cos(\phi_2 - \phi_3) \\ + ml^2\dot{\phi}_1^2 \sin(\phi_1 - \phi_3) - ml^2\dot{\phi}_2^2 \sin(\phi_2 - \phi_3) \end{aligned}$$

$$\frac{g}{l}\sin\phi_3 = \ddot{\phi}_3 + \ddot{\phi}_1 \cos(\phi_1 - \phi_3) + \ddot{\phi}_2 \cos(\phi_2 - \phi_3) + \dot{\phi}_1^2 \sin(\phi_1 - \phi_3) - \dot{\phi}_2^2 \sin(\phi_2 - \phi_3) \tag{3}$$

Rearranging (1), (2), and (3) by moving the second derivatives to one side, we have

$$\begin{aligned} 3\ddot{\phi}_1 + 2\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \ddot{\phi}_3 \cos(\phi_1 - \phi_3) + &= -2\dot{\phi}_2^2 \sin(\phi_1 - \phi_2) + \dot{\phi}_3^2 \sin(\phi_1 - \phi_3) + \frac{3g}{l}\sin\phi_1 \\ 2\ddot{\phi}_1 \cos(\phi_1 - \phi_2) + 2\ddot{\phi}_2 + \ddot{\phi}_3 \cos(\phi_2 - \phi_3) &= -2\dot{\phi}_1^2 \sin(\phi_1 - \phi_2) + \dot{\phi}_3^2 \sin(\phi_2 - \phi_3) + \frac{2g}{l}\sin\phi_2 \\ \ddot{\phi}_1 \cos(\phi_1 - \phi_3 + \ddot{\phi}_2 \cos(\phi_2 - \phi_3)) + \ddot{\phi}_3 &= -\dot{\phi}_1^2 \sin(\phi_1 - \phi_3) + \dot{\phi}_2^2 \sin(\phi_2 - \phi_3) + \frac{g}{l}\sin\phi_3 \end{aligned}$$

We can rewrite this system equations as a single matrix equation:

$$\begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3) \\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3) \\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_3^2 \sin(\phi_1-\phi_3) - 2\dot{\phi}_2^2 \sin(\phi_1-\phi_2) + \frac{3g}{l}\sin\phi_1 \\ \dot{\phi}_3^2 \sin(\phi_2-\phi_3) - 2\dot{\phi}_1^2 \sin(\phi_1-\phi_2) + \frac{2g}{l}\sin\phi_2 \\ \dot{\phi}_2^2 \sin(\phi_2-\phi_3) - \dot{\phi}_1^2 \sin(\phi_1-\phi_3) + \frac{g}{l}\sin\phi_3 \end{pmatrix}$$

We can then solve for our vector of second derivatives by multiplying on the left by the inverse of the first matrix:

$$\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3) \\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3) \\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}_3^2 \sin(\phi_1-\phi_3) - 2\dot{\phi}_2^2 \sin(\phi_1-\phi_2) + \frac{3g}{l}\sin\phi_1 \\ \dot{\phi}_3^2 \sin(\phi_2-\phi_3) - 2\dot{\phi}_1^2 \sin(\phi_1-\phi_2) + \frac{2g}{l}\sin\phi_2 \\ \dot{\phi}_2^2 \sin(\phi_2-\phi_3) - \dot{\phi}_1^2 \sin(\phi_1-\phi_3) + \frac{g}{l}\sin\phi_3 \end{pmatrix} \tag{4}$$

The matrix inverse in the above equation is very tedious to calculate. We used a computer algebra system[1] to do the inverse and the matrix multiplication afterwards. The result is

$$\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) - \frac{3g\sin(\phi_1)}{l}\right)\left(\frac{2\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)^2}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} + \frac{2\cos(\phi_1-\phi_2)^2}{2\cos(\phi_1-\phi_2)^2-3} - 1\right) - \left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) - \frac{2g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} + \frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right) + \frac{2\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} \\ -\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) - \frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} + \frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right) + \frac{3}{2}\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) - \frac{2g\sin(\phi_1)}{l}\right)\left(\frac{1}{2\cos(\phi_1-\phi_2)^2-3} + \frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{(2\cos(\phi_1-\phi_2)^2-3)^2\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}\right) - \frac{3\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} \\ \frac{2\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} + \frac{6\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} - \frac{3\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} \end{pmatrix},$$

which is far too messy to use computationally. Instead, we chose to construct the matrices in (4) in `numpy` and numerically calculate the inverse and multiplication. Since many of the terms are repeated (especially the trigonometric functions of angle differences), this has the added advantage of allowing us to precompute these functions.

A.2 Small-angle Approximation

To check the validity of our numerical solution, we would like an analytical solution. Clearly a full analytical solution is impossible. However, by assuming that ϕ_1, ϕ_2, and ϕ_3 remain small, we can radically simplify the equations of motion to reach a closed-form expression for ϕ_1, ϕ_2, and phi_3 as a function of time.

We begin by substituting $\cos\phi_i = 1 - \frac{1}{2}\phi_i^2$, the second order Maclaurin approximation for cosine, into our expression for potential energy. We also substitute $\dot{\phi}_i\dot{\phi}_j\cos(\phi_i - \phi_j) = \dot{\phi}_i\dot{\phi}_j$, since for the small product

[1]SageMath, the Sage Mathematics Software System (Version 7.6), The Sage Developers, 2017, `http://www.sagemath.org`.

$\dot{\phi}_i\dot{\phi}_j$, the cosine term is negligible. This allows us to simplify our expression for kinetic energy as well. We find

$$\begin{aligned}
U &= -mgl\left(3(1-\tfrac{1}{2}\phi_1^2) + 2(1-\tfrac{1}{2}\phi_2^2 + (1-\tfrac{1}{2}\phi_3^2))\right) \\
&\simeq mgl\left(\frac{3}{2}\phi_1^2 + \frac{2}{2}\phi_2^2 + \frac{1}{2}\phi_3^2\right) \quad \text{(since the constant terms will be dropped when the derivative is taken)}
\end{aligned}$$

$$T = \frac{1}{2}ml^2(3\dot{\phi}_1^2 + 2\dot{\phi}_2^2 + \dot{\phi}_1^2 + 4\dot{\phi}_1\phi_2 + 2\dot{\phi}_1\phi_3 + 2\dot{\phi}_2\phi_3),$$

so the Lagrangian is

$$\begin{aligned}
\mathcal{L} &= T + U \\
&= \frac{1}{2}ml^2(3\dot{\phi}_1^2 + 2\dot{\phi}_2^2 + \dot{\phi}_1^2 + 4\dot{\phi}_1\phi_2 + 2\dot{\phi}_1\phi_3 + 2\dot{\phi}_2\phi_3) - mgl\left(\frac{3}{2}\phi_1^2 + \frac{2}{2}\phi_2^2 + \frac{1}{2}\phi_3^2\right)
\end{aligned}$$

Taking derivatives as required, we have

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi_1} &= -3mgl\phi_1, & \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} &= ml^2(3\dot{\phi}_1 + 4\dot{\phi}_2 + 2\dot{\phi}_3) & &\Rightarrow & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} &= ml^2(3\ddot{\phi}_1 + 4\ddot{\phi}_2 + 2\ddot{\phi}_3) \\
\frac{\partial\mathcal{L}}{\partial\phi_2} &= -2mgl\phi_2, & \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} &= ml^2(4\dot{\phi}_1 + 2\dot{\phi}_2 + 2\dot{\phi}_3) & &\Rightarrow & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_3} &= ml^2(4\ddot{\phi}_1 + 2\ddot{\phi}_2 + 2\ddot{\phi}_3) \\
\frac{\partial\mathcal{L}}{\partial\phi_3} &= -mgl\phi_3, & \frac{\partial\mathcal{L}}{\partial\dot{\phi}_3} &= ml^2(2\dot{\phi}_1 + 2\dot{\phi}_2 + \dot{\phi}_3) & &\Rightarrow & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_3} &= ml^2(2\ddot{\phi}_1 + 2\ddot{\phi}_2 + \ddot{\phi}_3)
\end{aligned}$$

Applying our equilibrium condition $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_i} = \frac{\partial\mathcal{L}}{\partial\phi_i}$, we have our three equations of motion:

$$\begin{aligned}
ml^2(3\ddot{\phi}_1 + 4\ddot{\phi}_2 + 2\ddot{\phi}_3) &= -3mgl\phi_1 \\
ml^2(4\ddot{\phi}_1 + 2\ddot{\phi}_2 + 2\ddot{\phi}_3) &= -2mgl\phi_2 \\
ml^2(2\ddot{\phi}_1 + 2\ddot{\phi}_2 + \ddot{\phi}_3) &= -mgl\phi_3
\end{aligned}$$

Cancelling the common term of ml^2, we have

$$\begin{aligned}
3\ddot{\phi}_1 + 4\ddot{\phi}_2 + 2\ddot{\phi}_3 &= -\frac{3g}{l}\phi_1 \\
4\ddot{\phi}_1 + 2\ddot{\phi}_2 + 2\ddot{\phi}_3 &= -\frac{2g}{l}\phi_2 \\
2\ddot{\phi}_1 + 2\ddot{\phi}_2 + \ddot{\phi}_3 &= -\frac{g}{l}\phi_3
\end{aligned}$$

We can write this system of equations as a single matrix equation

$$M\ddot{\vec{\phi}} = -G\vec{\phi}$$

where

$$M = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \qquad G = \frac{g}{l}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We assume a periodic solution of the form

$$\vec{\phi} = \mathrm{Re}\ \vec{z}, \quad \text{where} \quad \vec{z} = \vec{a}\,\mathrm{e}^{\mathrm{i}\omega t}$$

Taking the second derivative of $\vec{z}$, we find

$$\ddot{\vec{z}} = -\omega^2 \vec{a}\, \mathrm{e}^{\mathrm{i}\omega t} = -omega^2 \vec{z}$$

So our differential equation becomes

$$-\omega^2 M \vec{a} \mathrm{e}^{\mathrm{i}\omega t} = -G \vec{a} \mathrm{e}^{\mathrm{i}\omega t},$$

which by rearranging and cancelling the common exponential term yields

$$(G - \omega^2 M)\vec{a} = 0$$

This equation will have nontrivial solutions when $\det(G - \omega^2 M) = 0$. Calculating the determinant (again with a computer algebra system), we find

$$2w^6 + \frac{18g}{l} w^4 + \frac{18g^2}{l^2} w^2 - \frac{6g^3}{l^3} = 0$$