Chaotic Behavior of the Triple Pendulum
A Computational Approach

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1 Introduction

In this paper, we will report on our findings regarding the behavior and motion of an undamped triple pendulum made of three equal masses attached by three massless rods of equal length. We used the Lagrangian formulation to create equations of motion for the three masses, then examined the system using Python. We used Fourier transformations as a tool to help us determine if the pendulum's motion was chaotic or periodic. **Did we examine how changing the masses affects the motion or are we giving up on that?** Our findings helped us find some order about a system that initially appeared to be purely chaotic.

2 Conservation of Energy

After finding our equations of motion for the three masses, we first wanted to test that they were correct. Our plots of the masses' angles versus time seemed reasonable, but we wanted to confirm our suspicions before delving deeper into the problem. We plotted kinetic, potential and total energy versus time on the same set of axes and saw that mechanical energy of the system is conserved (Fig. 1).

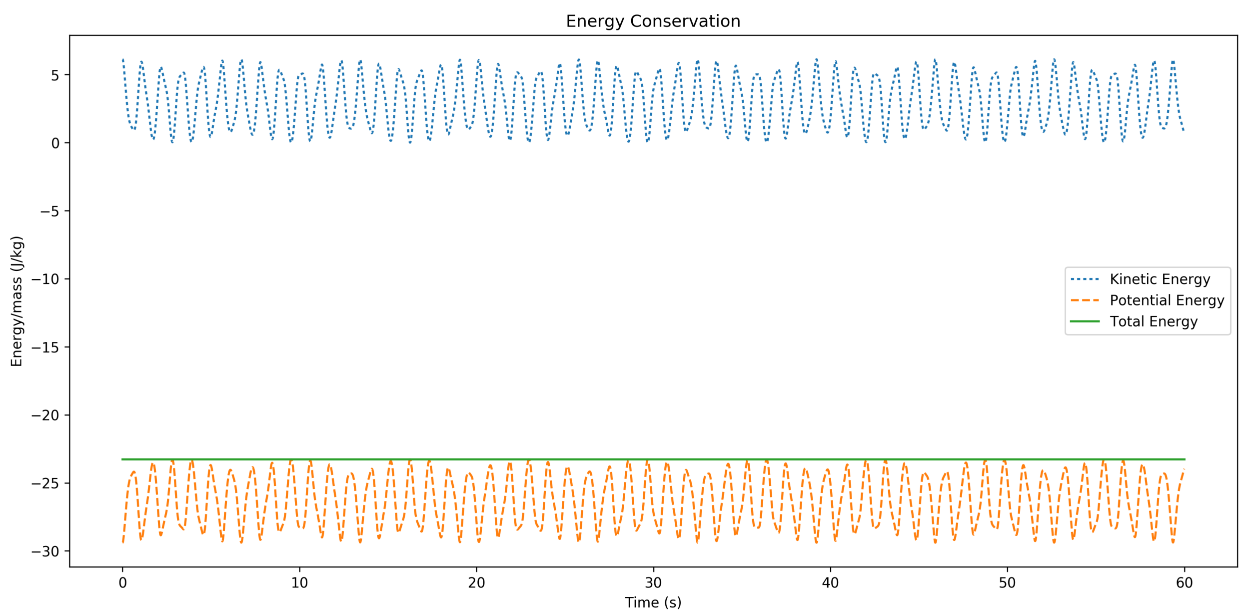

Figure 1: Kinetic, potential, and total energy over time for $\dot{\phi}_3(0) = 7\,\mathrm{s}^{-1}$.

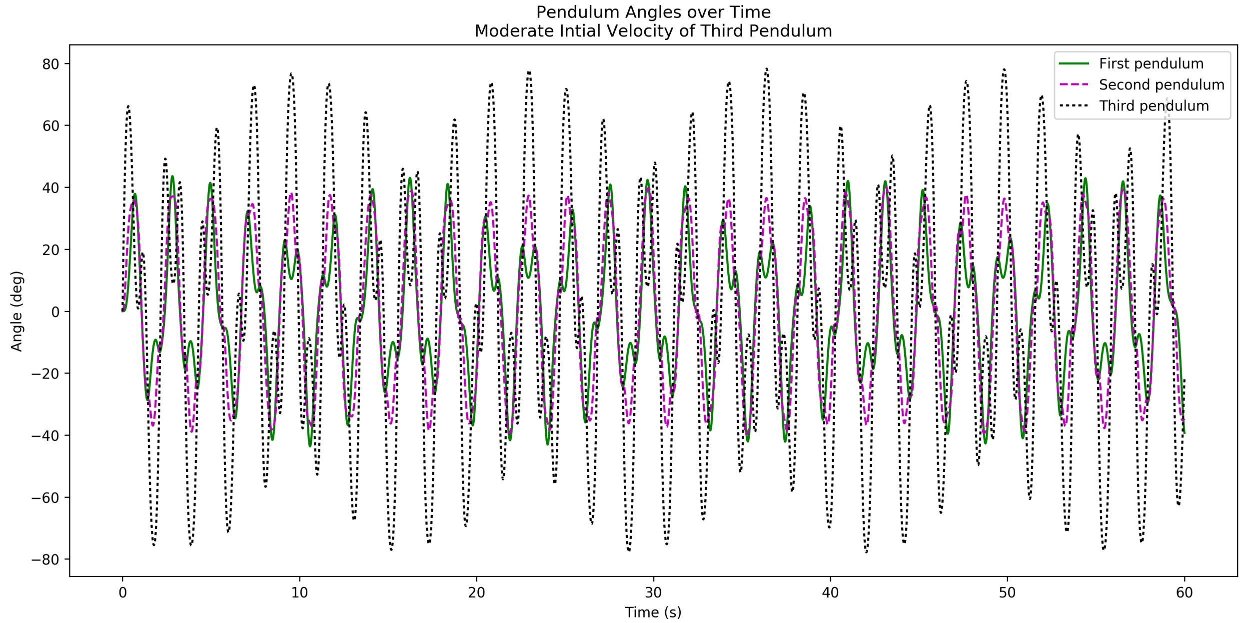

Figure 2: ϕ_1, ϕ_2, and ϕ_3 over time for $\dot{\phi}_3(0) = 7\,\mathrm{s}^{-1}$

3 Chaotic or Periodic?

As we first began to examine the system, we kept all initial angles and angular velocities save for the lowest hanging mass's at zero. For short time frames and an initial angular velocity of $\dot{\phi}_3 = 7\,\mathrm{s}^{-1}$, the motion of the system appears to be chaotic. However, increasing the time window to 1,000 seconds showed something different. In Figure 2, we can see that there appears to be some sort of regularity to the pendulum's motion. Looking at such a plot is not sufficient to declare periodic victory, so we needed to use some other tools to help us describe the system's motion.

4 Results

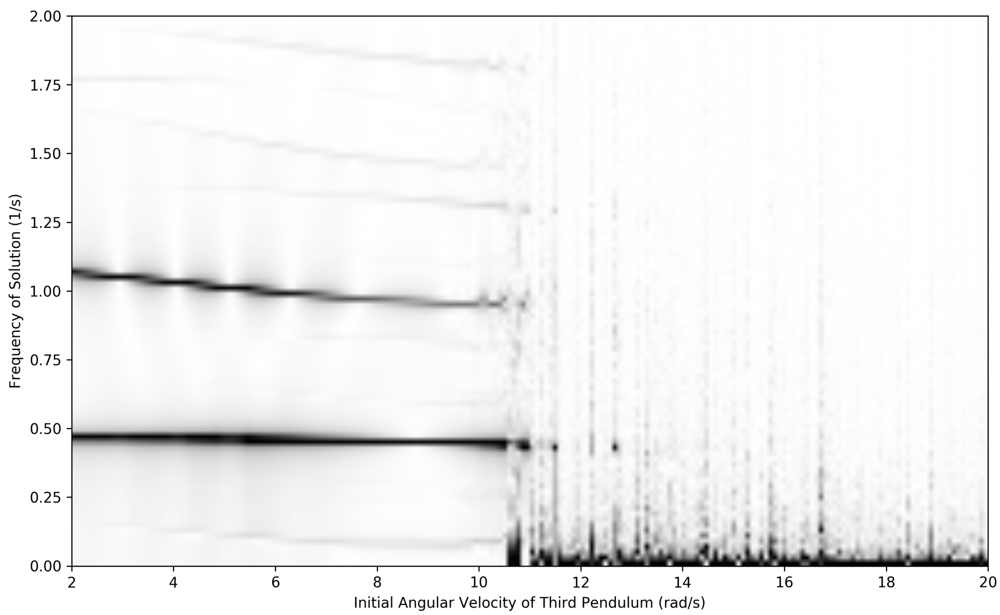

Figure 3: "Bifurcation Diagram" for $2 \leq \dot{\phi}_3(0) \leq 20$

Appendix A Equations of Motion

We can describe the position of a given mass by its Cartesian coordinates (x_i, y_i). The position of a given mass is the sum of its position relative to its pivot point and the position of its pivot point:

$$\begin{aligned} x_1 &= l\sin\phi_1 & y_1 &= -l\cos\phi_1 \\ x_2 &= x_1 + l\sin\phi_2 & y_2 &= y_1 + -l\cos\phi_2 \\ x_3 &= x_2 + l\sin\phi_3 & y_3 &= y_2 + -l\cos\phi_3. \end{aligned}$$

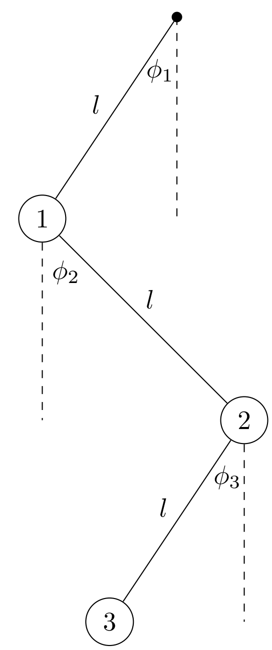

Figure 4: The triple pendulum.

Making the appropriate substitutions, we have

$$\begin{aligned} x_1 &= l\sin\phi_1 \\ y_1 &= -l\cos\phi_1 \\ x_2 &= l(\sin\phi_1 + \sin\phi_2) \\ y_2 &= -l(\cos\phi_1 + \cos\phi_2) \\ x_3 &= l(\sin\phi_1 + \sin\phi_2 + \sin\phi_3) \\ y_3 &= -l(\cos\phi_1 + \cos\phi_2 + \cos\phi_3). \end{aligned}$$

Taking time derivatives to find velocity, we have

$$\begin{aligned} \dot{x}_1 &= l\dot{\phi}_1\cos\phi_1 \\ \dot{y}_1 &= l\dot{\phi}_1\sin\phi_1 \\ \dot{x}_2 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2) \\ \dot{y}_2 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2) \\ \dot{x}_3 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3) \\ \dot{y}_3 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3). \end{aligned}$$

For the kinetic energy, we will need to find $v^2 = \dot{x}^2\dot{y}^2$ for each mass.

$$\begin{aligned} v_1^2 &= \dot{x}_1^2 + \dot{y}_1^2 \\ &= (l\dot{\phi}_1\cos\phi_1)^2 + (l\dot{\phi}_1\sin\phi_1)^2 \\ &= l^2(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1) \\ &= l^2\dot{\phi}_1^2 \end{aligned}$$

$$\begin{aligned} v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2 \\ &= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2))^2 \\ &= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_1^2\sin^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + \dot{\phi}_2^2\sin^2\phi_2\right) \\ &= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_2^2\sin^2\phi_2 + 2\dot{\phi}_1\dot{\phi}_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\right) \\ &= l^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right) \end{aligned}$$

$$\begin{aligned} v_3^2 &= \dot{x}_3^2 + \dot{y}_3^2 \\ &= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3))^2 \\ &= l^2\big(\dot{\phi}_1^2\sin^2\phi_1 + \dot{\phi}_2^2\sin^2\phi_2 + \dot{\phi}_3^2\sin^2\phi_3 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\sin\phi_1\sin\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\sin\phi_2\sin\phi_3 \\ &\quad + \dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_3^2\cos^2\phi_3 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\cos\phi_1\cos\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\cos\phi_2\cos\phi_3\big) \end{aligned}$$

$$= l^2 \left(\dot\phi_1^2 + \dot\phi_2^2 + \dot\phi_3^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1 - \phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1 - \phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2 - \phi_3)\right)$$

We can now write the kinetic and potential energy for the system by summing up $-mgy_i$ and $\frac{1}{2}mv_i^2$ for each mass i:

$$\begin{aligned} U &= U_1 + U_2 + U_3 \\ &= -mgl\left(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3\right) \end{aligned}$$

$$\begin{aligned} T &= T_1 + T_2 + T_3 \\ &= \frac{1}{2}ml^2\left(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1 - \phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1 - \phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2 - \phi_3)\right) \end{aligned}$$

So the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= mgl\left(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3\right) + \frac{1}{2}ml^2\big(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1 - \phi_2) \\ &\qquad + 2\dot\phi_1\dot\phi_3\cos(\phi_1 - \phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2 - \phi_3)\big) \end{aligned}$$

Since our generalized coordinates are ϕ_1, ϕ_2, and ϕ_3, we must now take partial derivatives of the Lagrangian with respect to each ϕ_i and $\dot\phi_i$:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi_1} &= -3mgl\sin\phi_1 - 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1 - \phi_2) - ml^2\dot\phi_1\dot\phi_3\sin(\phi_1 - \phi_3) \\ \frac{\partial\mathcal{L}}{\partial\phi_2} &= -2mgl\sin\phi_2 + 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1 - \phi_2) - ml^2\dot\phi_2\dot\phi_3\sin(\phi_2 - \phi_3) \\ \frac{\partial\mathcal{L}}{\partial\phi_3} &= -mgl\sin\phi_3 + ml^2\dot\phi_1\dot\phi_3\sin(\phi_1 - \phi_3) + ml^2\dot\phi_2\dot\phi_3\sin(\phi_2 - \phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\dot\phi_1 + 2ml^2\dot\phi_2\cos(\phi_1 - \phi_2) + ml^2\dot\phi_3\cos(\phi_1 - \phi_3) \\ \frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\dot\phi_2 + 2ml^2\dot\phi_1\cos(\phi_1 - \phi_2) + ml^2\dot\phi_3\cos(\phi_2 - \phi_3) \\ \frac{\partial\mathcal{L}}{\partial\dot\phi_3} &= ml^2\dot\phi_3 + ml^2\dot\phi_1\cos(\phi_1 - \phi_3) + ml^2\dot\phi_2\cos(\phi_2 - \phi_3) \end{aligned}$$

Next, we find $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_i}$ for each i:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\ddot\phi_1 + 2ml^2\ddot\phi_2\cos(\phi_1 - \phi_2) - 2ml^2\dot\phi_2\sin(\phi_1 - \phi_2)(\dot\phi_1 - \dot\phi_2) \\ &\quad + ml^2\ddot\phi_3\cos(\phi_1 - \phi_3) - ml^2\dot\phi_3\sin(\phi_1 - \phi_3)(\dot\phi_1 - \dot\phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\ddot\phi_2 + 2ml^2\ddot\phi_1\cos(\phi_1 - \phi_2) - 2ml^2\dot\phi_1\sin(\phi_1 - \phi_2)(\dot\phi_1 - \dot\phi_2) \\ &\quad + ml^2\ddot\phi_3\cos(\phi_2 - \phi_3) - ml^2\dot\phi_3\sin(\phi_2 - \phi_3)(\dot\phi_2 - \dot\phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_3} &= ml^2\ddot\phi_3 + ml^2\ddot\phi_1\cos(\phi_1 - \phi_3) - ml^2\dot\phi_1\sin(\phi_1 - \phi_3)(\dot\phi_1 - \dot\phi_3) \\ &\quad + ml^2\ddot\phi_2\cos(\phi_2 - \phi_3) - ml^2\dot\phi_2\sin(\phi_2 - \phi_3)(\dot\phi_2 - \dot\phi_3) \end{aligned}$$

We can then write the equations of motion for each of our generalized coordinates, according to the Euler-Lagrange condition $\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$. For ϕ_1 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}$$

$$\begin{aligned}-3mgl\sin\phi_1 - 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) = {} & 3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2)\\ & - 2ml^2\dot{\phi}_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\ & + ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\ & - ml^2\dot{\phi}_3\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\end{aligned}$$

$$\begin{aligned}-3mgl\sin\phi_1 = {} & 3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\ & + 2ml^2\dot{\phi}_2^2\sin(\phi_1-\phi_2) + ml^2\dot{\phi}_3^2\sin(\phi_1-\phi_3)\end{aligned}$$

$$-\frac{3g}{l}\sin\phi_1 = 3\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_1-\phi_3) + 2\dot{\phi}_2^2\sin(\phi_1-\phi_2) + \dot{\phi}_3^2\sin(\phi_1-\phi_3) \tag{1}$$

For ϕ_2 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}$$

$$\begin{aligned}-2mgl\sin\phi_2 + 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = {} & 2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2)\\ & - 2ml^2\dot{\phi}_1\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\ & + ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\ & - ml^2\dot{\phi}_3\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)\end{aligned}$$

$$\begin{aligned}-2mgl\sin\phi_2 = {} & 2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\ & - 2ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_2) + ml^2\dot{\phi}_3^2\sin(\phi_2-\phi_3)\end{aligned}$$

$$-\frac{2g}{l}\sin\phi_2 = 2\ddot{\phi}_2 + 2\ddot{\phi}_1\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_2-\phi_3) - 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) + \dot{\phi}_3^2\sin(\phi_2-\phi_3) \tag{2}$$

For ϕ_3 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_3} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_3}$$

$$\begin{aligned}-mgl\sin\phi_3 + ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) + ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = {} & ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3)\\ & - ml^2\dot{\phi}_1\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\\ & + ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3)\\ & - ml^2\dot{\phi}_2\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)\end{aligned}$$

$$-mgl\sin\phi_3 = ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3) + ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3)$$

$$-ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_3)-ml^2\dot{\phi}_2^2\sin(\phi_2-\phi_3)$$

$$-\frac{g}{l}\sin\phi_3=\ddot{\phi}_3+\ddot{\phi}_1\cos(\phi_1-\phi_3)+\ddot{\phi}_2\cos(\phi_2-\phi_3)-\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3) \tag{3}$$

Rearranging (1), (2), and (3) by moving the second derivatives to one side, we have

$$\begin{aligned}
3\ddot{\phi}_1+2\ddot{\phi}_2\cos(\phi_1-\phi_2)+\ddot{\phi}_3\cos(\phi_1-\phi_3)+&=-2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g}{l}\sin\phi_1\\
2\ddot{\phi}_1\cos(\phi_1-\phi_2)+2\ddot{\phi}_2+\ddot{\phi}_3\cos(\phi_2-\phi_3)&=2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g}{l}\sin\phi_2\\
\ddot{\phi}_1\cos(\phi_1-\phi_3)+\ddot{\phi}_2\cos(\phi_2-\phi_3)+\ddot{\phi}_3&=+\dot{\phi}_1^2\sin(\phi_1-\phi_3)+\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g}{l}\sin\phi_3
\end{aligned}$$

We can rewrite this system equations as a single matrix equation:

$$\begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3)\\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3)\\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix}\begin{pmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\\ \ddot{\phi}_3\end{pmatrix}=\begin{pmatrix}-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\frac{3g}{l}\sin\phi_1\\ -\dot{\phi}_3^2\sin(\phi_2-\phi_3)+2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\frac{2g}{l}\sin\phi_2\\ \dot{\phi}_2^2\sin(\phi_2-\phi_3)+\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\frac{g}{l}\sin\phi_3\end{pmatrix}$$

We can then solve for our vector of second derivatives by multiplying on the left by the inverse of the first matrix:

$$\begin{pmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\\ \ddot{\phi}_3\end{pmatrix}=\begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3)\\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3)\\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix}^{-1}\begin{pmatrix}-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\frac{3g}{l}\sin\phi_1\\ -\dot{\phi}_3^2\sin(\phi_2-\phi_3)+2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\frac{2g}{l}\sin\phi_2\\ \dot{\phi}_2^2\sin(\phi_2-\phi_3)+\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\frac{g}{l}\sin\phi_3\end{pmatrix} \tag{4}$$

The matrix inverse in the above equation is very tedious to calculate. We used a computer algebra system[1] to do the inverse and the matrix multiplication afterwards. The result is

$$\begin{pmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\\ \ddot{\phi}_3\end{pmatrix}=\begin{pmatrix}
\frac{1}{3}\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g\sin(\phi_1)}{l}\right)\left(\frac{2\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)^2}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6}+\frac{2\cos(\phi_1-\phi_2)^2}{2\cos(\phi_1-\phi_2)^2-3}-1\right)-\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}+\frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right)+\frac{2\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6}\\
-\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}+\frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right)+\frac{3}{2}\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g\sin(\phi_1)}{l}\right)\left(\frac{1}{2\cos(\phi_1-\phi_2)^2-3}+\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{(2\cos(\phi_1-\phi_2)^2-3)^2\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}\right)-\frac{3\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}\\
\frac{2\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6}+\frac{6\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6}-\frac{3\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}
\end{pmatrix},$$

which is far too messy to use computationally. Instead, we chose to construct the matrices in (4) in `numpy` and numerically calculate the inverse and multiplication. Since many of the terms are repeated (especially the trigonometric functions of angle differences), this has the added advantage of allowing us to precompute these functions.

[1]SageMath, the Sage Mathematics Software System (Version 7.6), The Sage Developers, 2017, `http://www.sagemath.org`.