Chaotic Behavior of the Triple Pendulum

A Computational Approach

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1 Introduction

Appendix A Equations of Motion

We can describe the position of a given mass by its Cartesian coordinates (x_i, y_i). The position of a given mass is the sum of its position relative to its pivot point and the position of its pivot point:

$$
\begin{aligned}
x_1 &= l \sin\phi_1 & y_1 &= -l\cos\phi_1 \\
x_2 &= x_1 + l\sin\phi_2 & y_2 &= y_1 + -l\cos\phi_2 \\
x_3 &= x_2 + l\sin\phi_3 & y_3 &= y_2 + -l\cos\phi_3.
\end{aligned}
$$

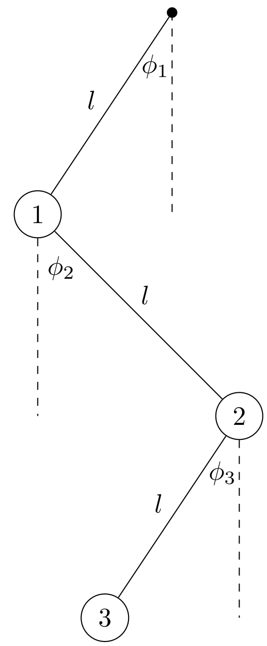

Figure 1: The triple pendulum.

Making the appropriate substitutions, we have

$$
\begin{aligned}
x_1 &= l\sin\phi_1 \\
y_1 &= -l\cos\phi_1 \\
x_2 &= l(\sin\phi_1 + \sin\phi_2) \\
y_2 &= -l(\cos\phi_1 + \cos\phi_2) \\
x_3 &= l(\sin\phi_1 + \sin\phi_2 + \sin\phi_3) \\
y_3 &= -l(\cos\phi_1 + \cos\phi_2 + \cos\phi_3).
\end{aligned}
$$

Taking time derivatives to find velocity, we have

$$
\begin{aligned}
\dot{x}_1 &= l\dot{\phi}_1\cos\phi_1 \\
\dot{y}_1 &= l\dot{\phi}_1\sin\phi_1 \\
\dot{x}_2 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2) \\
\dot{y}_2 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2) \\
\dot{x}_3 &= l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3) \\
\dot{y}_3 &= l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3).
\end{aligned}
$$

For the kinetic energy, we will need to find $v^2 = \dot{x}^2\dot{y}^2$ for each mass.

$$
\begin{aligned}
v_1^2 &= \dot{x}_1^2 + \dot{y}_1^2 \\
&= (l\dot{\phi}_1\cos\phi_1)^2 + (l\dot{\phi}_1\sin\phi_1)^2 \\
&= l^2(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1) \\
&= l^2\dot{\phi}_1^2
\end{aligned}
$$

$$
\begin{aligned}
v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2 \\
&= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2))^2 \\
&= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_1^2\sin^2\phi_1 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + \dot{\phi}_2^2\sin^2\phi_2\right) \\
&= l^2\left(\dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_1^2\sin^2\phi_1 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_2^2\sin^2\phi_2 + 2\dot{\phi}_1\dot{\phi}_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\right) \\
&= l^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
v_3^2 &= \dot{x}_3^2 + \dot{y}_3^2 \\
&= (l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2 + \dot{\phi}_3\cos\phi_3))^2 + (l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2 + \dot{\phi}_3\sin\phi_3))^2 \\
&= l^2\big(\dot{\phi}_1^2\sin^2\phi_1 + \dot{\phi}_2^2\sin^2\phi_2 + \dot{\phi}_3^2\sin^2\phi_3 + 2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\sin\phi_1\sin\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\sin\phi_2\sin\phi_3 \\
&\quad + \dot{\phi}_1^2\cos^2\phi_1 + \dot{\phi}_2^2\cos^2\phi_2 + \dot{\phi}_3^2\cos^2\phi_3 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2 + 2\dot{\phi}_1\dot{\phi}_3\cos\phi_1\cos\phi_3 + 2\dot{\phi}_2\dot{\phi}_3\cos\phi_2\cos\phi_3\big)
\end{aligned}
$$

$$= l^2\left(\dot\phi_1^2 + \dot\phi_2^2 + \dot\phi_3^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\right)$$

We can now write the kinetic and potential energy for the system by summing up $-mgy_i$ and $\frac{1}{2}mv_i^2$ for each mass i:

$$\begin{aligned} U &= U_1 + U_2 + U_3 \\ &= -mgl\left(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3\right) \end{aligned}$$

$$\begin{aligned} T &= T_1 + T_2 + T_3 \\ &= \frac{1}{2}ml^2\left(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\right) \end{aligned}$$

So the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= mgl\left(3\cos\phi_1 + 2\cos\phi_2 + \cos\phi_3\right) + \frac{1}{2}ml^2\big(3\dot\phi_1^2 + 2\dot\phi_2^2 + \dot\phi_3^2 + 4\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) \\ &\qquad + 2\dot\phi_1\dot\phi_3\cos(\phi_1-\phi_3) + 2\dot\phi_2\dot\phi_3\cos(\phi_2-\phi_3)\big) \end{aligned}$$

Since our generalized coordinates are ϕ_1, ϕ_2, and ϕ_3, we must now take partial derivatives of the Lagrangian with respect to each ϕ_i and $\dot\phi_i$:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi_1} &= -3mgl\sin\phi_1 - 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - ml^2\dot\phi_1\dot\phi_3\sin(\phi_1-\phi_3) \\ \frac{\partial\mathcal{L}}{\partial\phi_2} &= -2mgl\sin\phi_2 + 2ml^2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - ml^2\dot\phi_2\dot\phi_3\sin(\phi_2-\phi_3) \\ \frac{\partial\mathcal{L}}{\partial\phi_3} &= -mgl\sin\phi_3 + ml^2\dot\phi_1\dot\phi_3\sin(\phi_1-\phi_3) + ml^2\dot\phi_2\dot\phi_3\sin(\phi_2-\phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\dot\phi_1 + 2ml^2\dot\phi_2\cos(\phi_1-\phi_2) + ml^2\dot\phi_3\cos(\phi_1-\phi_3) \\ \frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\dot\phi_2 + 2ml^2\dot\phi_1\cos(\phi_1-\phi_2) + ml^2\dot\phi_3\cos(\phi_2-\phi_3) \\ \frac{\partial\mathcal{L}}{\partial\dot\phi_3} &= ml^2\dot\phi_3 + ml^2\dot\phi_1\cos(\phi_1-\phi_3) + ml^2\dot\phi_2\cos(\phi_2-\phi_3) \end{aligned}$$

Next, we find $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_i}$ for each i:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_1} &= 3ml^2\ddot\phi_1 + 2ml^2\ddot\phi_2\cos(\phi_1-\phi_2) - 2ml^2\dot\phi_2\sin(\phi_1-\phi_2)(\dot\phi_1-\dot\phi_2) \\ &\quad + ml^2\ddot\phi_3\cos(\phi_1-\phi_3) - ml^2\dot\phi_3\sin(\phi_1-\phi_3)(\dot\phi_1-\dot\phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_2} &= 2ml^2\ddot\phi_2 + 2ml^2\ddot\phi_1\cos(\phi_1-\phi_2) - 2ml^2\dot\phi_1\sin(\phi_1-\phi_2)(\dot\phi_1-\dot\phi_2) \\ &\quad + ml^2\ddot\phi_3\cos(\phi_2-\phi_3) - ml^2\dot\phi_3\sin(\phi_2-\phi_3)(\dot\phi_2-\dot\phi_3) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\phi_3} &= ml^2\ddot\phi_3 + ml^2\ddot\phi_1\cos(\phi_1-\phi_3) - ml^2\dot\phi_1\sin(\phi_1-\phi_3)(\dot\phi_1-\dot\phi_3) \\ &\quad + ml^2\ddot\phi_2\cos(\phi_2-\phi_3) - ml^2\dot\phi_2\sin(\phi_2-\phi_3)(\dot\phi_2-\dot\phi_3) \end{aligned}$$

We can then write the equations of motion for each of our generalized coordinates, according to the Euler-Lagrange condition $\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$. For ϕ_1 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}$$

$$\begin{aligned}-3mgl\sin\phi_1 - 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) = {} & 3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2)\\ & - 2ml^2\dot{\phi}_2\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\ & + ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\ & - ml^2\dot{\phi}_3\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\end{aligned}$$

$$\begin{aligned}-3mgl\sin\phi_1 = {} & 3ml^2\ddot{\phi}_1 + 2ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_1-\phi_3)\\ & + 2ml^2\dot{\phi}_2^2\sin(\phi_1-\phi_2) + ml^2\dot{\phi}_3^2\sin(\phi_1-\phi_3)\end{aligned}$$

$$-\frac{3g}{l}\sin\phi_1 = 3\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_1-\phi_3) + 2\dot{\phi}_2^2\sin(\phi_1-\phi_2) + \dot{\phi}_3^2\sin(\phi_1-\phi_3) \tag{1}$$

For ϕ_2 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}$$

$$\begin{aligned}-2mgl\sin\phi_2 + 2ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = {} & 2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2)\\ & - 2ml^2\dot{\phi}_1\sin(\phi_1-\phi_2)(\dot{\phi}_1-\dot{\phi}_2)\\ & + ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\ & - ml^2\dot{\phi}_3\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)\end{aligned}$$

$$\begin{aligned}-2mgl\sin\phi_2 = {} & 2ml^2\ddot{\phi}_2 + 2ml^2\ddot{\phi}_1\cos(\phi_1-\phi_2) + ml^2\ddot{\phi}_3\cos(\phi_2-\phi_3)\\ & - 2ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_2) + ml^2\dot{\phi}_3^2\sin(\phi_2-\phi_3)\end{aligned}$$

$$-\frac{2g}{l}\sin\phi_2 = 2\ddot{\phi}_2 + 2\ddot{\phi}_1\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_2-\phi_3) - 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) + \dot{\phi}_3^2\sin(\phi_2-\phi_3) \tag{2}$$

For ϕ_3 we have

$$\frac{\partial \mathcal{L}}{\partial \phi_3} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_3}$$

$$\begin{aligned}-mgl\sin\phi_3 + ml^2\dot{\phi}_1\dot{\phi}_3\sin(\phi_1-\phi_3) + ml^2\dot{\phi}_2\dot{\phi}_3\sin(\phi_2-\phi_3) = {} & ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3)\\ & - ml^2\dot{\phi}_1\sin(\phi_1-\phi_3)(\dot{\phi}_1-\dot{\phi}_3)\\ & + ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3)\\ & - ml^2\dot{\phi}_2\sin(\phi_2-\phi_3)(\dot{\phi}_2-\dot{\phi}_3)\end{aligned}$$

$$-mgl\sin\phi_3 = ml^2\ddot{\phi}_3 + ml^2\ddot{\phi}_1\cos(\phi_1-\phi_3) + ml^2\ddot{\phi}_2\cos(\phi_2-\phi_3)$$

$$- ml^2\dot{\phi}_1^2\sin(\phi_1-\phi_3) - ml^2\dot{\phi}_2^2\sin(\phi_2-\phi_3)$$

$$-\frac{g}{l}\sin\phi_3 = \ddot{\phi}_3 + \ddot{\phi}_1\cos(\phi_1-\phi_3) + \ddot{\phi}_2\cos(\phi_2-\phi_3) - \dot{\phi}_1^2\sin(\phi_1-\phi_3) - \dot{\phi}_2^2\sin(\phi_2-\phi_3) \tag{3}$$

Rearranging (1), (2), and (3) by moving the second derivatives to one side, we have

$$\begin{aligned}
3\ddot{\phi}_1 + 2\ddot{\phi}_2\cos(\phi_1-\phi_2) + \ddot{\phi}_3\cos(\phi_1-\phi_3) + &= -2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) - \frac{3g}{l}\sin\phi_1 \\
2\ddot{\phi}_1\cos(\phi_1-\phi_2) + 2\ddot{\phi}_2 + \ddot{\phi}_3\cos(\phi_2-\phi_3) &= 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) - \frac{2g}{l}\sin\phi_2 \\
\ddot{\phi}_1\cos(\phi_1-\phi_3) + \ddot{\phi}_2\cos(\phi_2-\phi_3) + \ddot{\phi}_3 &= +\dot{\phi}_1^2\sin(\phi_1-\phi_3) + \dot{\phi}_2^2\sin(\phi_2-\phi_3) - \frac{g}{l}\sin\phi_3
\end{aligned}$$

We can rewrite this system equations as a single matrix equation:

$$\begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3) \\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3) \\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} -\dot{\phi}_3^2\sin(\phi_1-\phi_3) - 2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \frac{3g}{l}\sin\phi_1 \\ -\dot{\phi}_3^2\sin(\phi_2-\phi_3) + 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \frac{2g}{l}\sin\phi_2 \\ \dot{\phi}_2^2\sin(\phi_2-\phi_3) + \dot{\phi}_1^2\sin(\phi_1-\phi_3) - \frac{g}{l}\sin\phi_3 \end{pmatrix}$$

We can then solve for our vector of second derivatives by multiplying on the left by the inverse of the first matrix:

$$\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} 3 & 2\cos(\phi_1-\phi_2) & \cos(\phi_1-\phi_3) \\ 2\cos(\phi_1-\phi_2) & 2 & \cos(\phi_2-\phi_3) \\ \cos(\phi_1-\phi_3) & \cos(\phi_2-\phi_3) & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}_3^2\sin(\phi_1-\phi_3) - 2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \frac{3g}{l}\sin\phi_1 \\ -\dot{\phi}_3^2\sin(\phi_2-\phi_3) + 2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \frac{2g}{l}\sin\phi_2 \\ \dot{\phi}_2^2\sin(\phi_2-\phi_3) + \dot{\phi}_1^2\sin(\phi_1-\phi_3) - \frac{g}{l}\sin\phi_3 \end{pmatrix} \tag{4}$$

The matrix inverse in the above equation is very tedious to calculate. We used a computer algebra system[1] to do the inverse and the matrix multiplication afterwards. The result is

$$\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) - \frac{3g\sin(\phi_1)}{l}\right)\left(\frac{2\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)^2}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} + \frac{2\cos(\phi_1-\phi_2)^2}{2\cos(\phi_1-\phi_2)^2-3} - 1\right) - \left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) - \frac{2g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} + \frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right) + \frac{2\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} \\ -\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_1-\phi_3) - \frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} + \frac{\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}\right) + \frac{3}{2}\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \dot{\phi}_3^2\sin(\phi_2-\phi_3) - \frac{2g\sin(\phi_1)}{l}\right)\left(\frac{1}{2\cos(\phi_1-\phi_2)^2-3} + \frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{(2\cos(\phi_1-\phi_2)^2-3)^2\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)}\right) - \frac{3\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} \\ \frac{2\left(2\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_1-\phi_3)-\frac{3g\sin(\phi_1)}{l}\right)\left(\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))\cos(\phi_1-\phi_2)}{2\cos(\phi_1-\phi_2)^2-3}-\cos(\phi_1-\phi_3)\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} + \frac{6\left(\dot{\phi}_1^2\sin(\phi_1-\phi_3)-\dot{\phi}_2^2\sin(\phi_2-\phi_3)-\frac{g\sin(\phi_1)}{l}\right)}{2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6} - \frac{3\left(2\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\dot{\phi}_3^2\sin(\phi_2-\phi_3)-\frac{2g\sin(\phi_1)}{l}\right)(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))}{(2\cos(\phi_1-\phi_2)^2-3)\left(2\cos(\phi_1-\phi_3)^2-\frac{(2\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_3)-3\cos(\phi_2-\phi_3))^2}{2\cos(\phi_1-\phi_2)^2-3}-6\right)} \end{pmatrix},$$

which is far too messy to use computationally. Instead, we chose to construct the matrices in (4) in `numpy` and numerically calculate the inverse and multiplication. Since many of the terms are repeated (especially the trigonometric functions of angle differences), this has the added advantage of allowing us to precompute these functions.

[1] SageMath, the Sage Mathematics Software System (Version 7.6), The Sage Developers, 2017, `http://www.sagemath.org`.