STAT 471: Introduction to R Programming Lecture

Lecture 3: Bisection & Newton-Raphson Algorithms

Cory Suzuki

Department of Mathematics & Statistics
California State University, Long Beach

8 September 2025

Overview

1. Bisection Algorithm

2. Newton-Raphson Algorithm

What is the Bisection Algorithm?

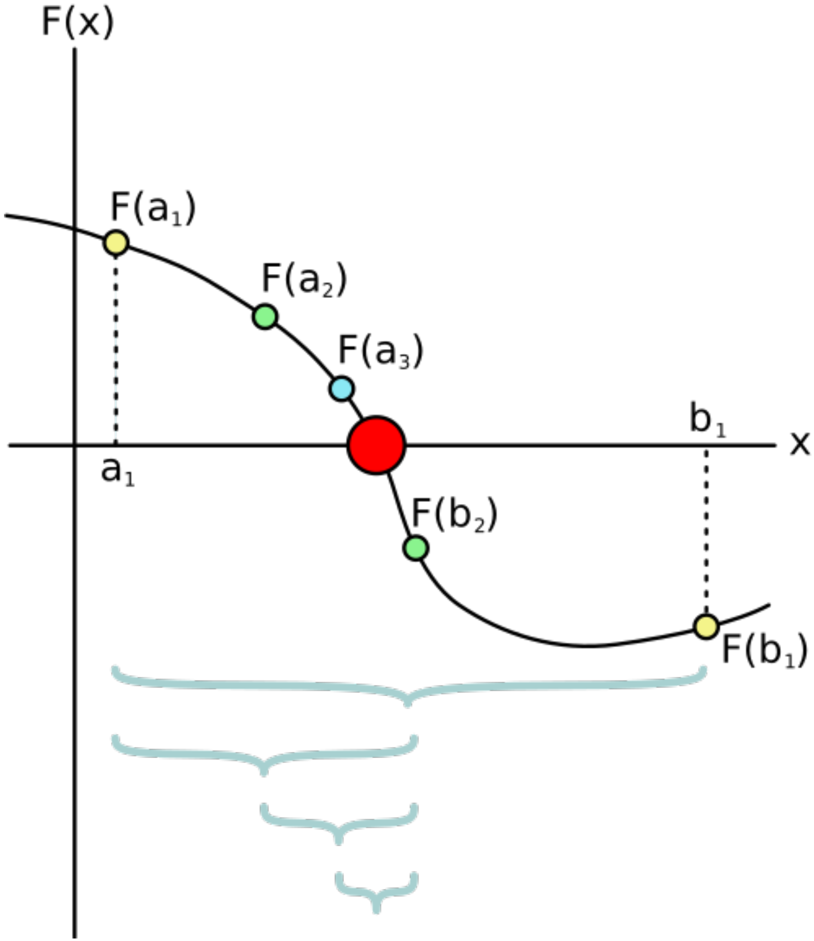
- Iteratively finds the roots of a function within a given interval
- The interval is then iteratively split in half (bisection) until the root is found

Bisection Algorithm

Step 1. Calculate the midpoint $c = \frac{(a+b)}{2}$
Step 2. Calculate f(c)
Step 3. If $|f(c)|$ is small (getting closer to 0), return c and stop the algorithm
Step 4. Otherwise, if the sign of f(c) is negative, replace it with a. If the sign of f(c) is positive, replace it with b.

Note: Intermediate Value Theorem must hold!

How the Algorithm Works



Example

Let $f(x) = x^2 - 6$ for $[2,3]$. Assume IVT holds.
Find p_0 and p_1.

$f(2) = (2)^2 - 6 = -2$ ✓ $\rightarrow p_0 = \frac{2+3}{2} = \boxed{\frac{5}{2}} \rightarrow f(\frac{5}{2}) = (\frac{5}{2})^2 - 6 = \frac{1}{4}$

$f(3) = (3)^2 - 6 = 3$

Since $f(\frac{5}{2}) = \frac{1}{4} > 0$, new interval is $[2, \frac{5}{2}]$.

$f(2) = -2$ ✓ $\rightarrow p_1 = \frac{2 + \frac{5}{2}}{2} = \frac{9}{2} \cdot \frac{1}{2} = \boxed{\frac{9}{4}} \approx 2.25$

$f(\frac{5}{2}) = \frac{1}{4}$

Hence $p_0 = \frac{5}{2}$, $p_1 = \frac{9}{4}$.

More iterations $\Rightarrow$ More accurate approx.

Newton-Raphson Algorithm Background

Sometimes it can be hard to find the roots of functions, so we may need to resort to numerical methods.

- Newton's method helps to iteratively find roots of functions using the "best linear approximation"
- Requires knowledge of the first and second derivatives of the function

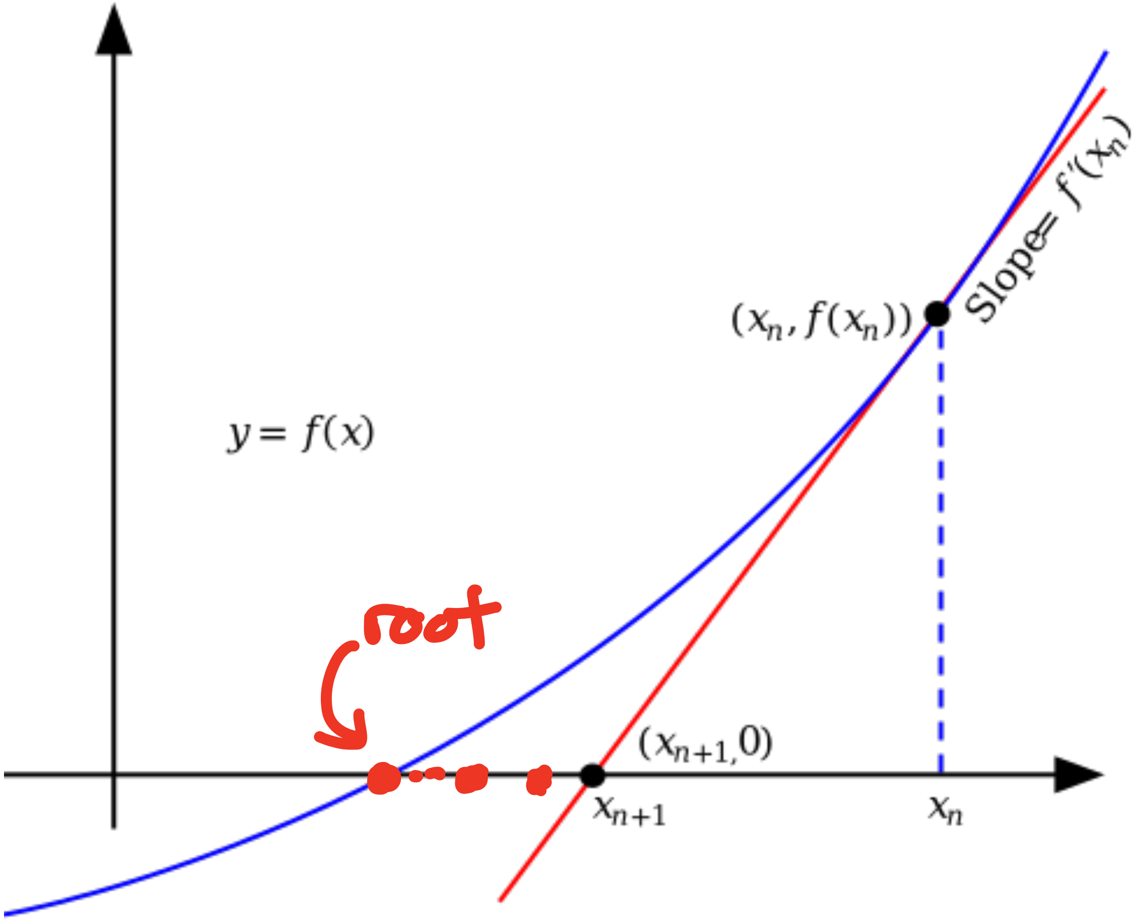
Newton-Raphson Algorithm

Let x_0 be the initial guess of the root. Then,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where x_{i+1} will be the approximation of the root.

Visualization of the Newton-Raphson Algorithm



Tangent line is the "best linear approx" to f.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Example of Newton's Method by Hand

Consider $f(x) = x^2 - 4$ with initial guess $x_0 = 1$

Find x_1, x_2.

$f'(x) = 2x$

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(1)^2-4}{2(1)} = 1 - \frac{-3}{2} = 1 + \frac{3}{2} = \frac{5}{2} = \boxed{2.5}$$

$$x_2 = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{(2.5)^2-4}{2(2.5)} = \boxed{2.05}$$

Closer to root $\rightarrow 2.05$

$x = \pm 2$ ← "truth"

Memory Lane: Maximum Likelihood Estimators (MLE's) and the STAT 381 Counterattack

Recall from STAT 381 that Maximum Likelihood Estimators can help us estimate the parameters of a probability distribution. This is done by finding:

- Using the known distribution's probability density function (pdf), $f_X(x)$ to create the Likelihood Function $L(x;\theta)$
- Taking the log of the Likelihood function to get the Log-Likelihood Function $\ell(x;\theta)$
- Maximize the Log-Likelihood Function by taking the derivative of $\ell(x;\theta)$ with respect to the parameter θ and finding the root (solving the equation equaling to 0)

MLE Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Ber}(p)$ with pmf $p(x) = p^x (1-p)^{1-x}$, $x = 1, 2, \ldots$

$$L(x;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} \cdots$$

$$= p^{\sum_{i=1}^{n} x_i}(1-p)^{1-\sum_{i=1}^{n} x_i}$$

$$\ell(x;p) = \ln(L(x;p)) = \ln\left(p^{\sum_{i=1}^{n} x_i}(1-p)^{1-\sum_{i=1}^{n} x_i}\right)$$

$$= \ln(p) \sum_{i=1}^{n} x_i + \ln(1-p) \cdot \sum_{i=1}^{n}(1-x_i)$$

MLE Example 1 Continued

$$\ell'(x;p) = \frac{\partial \ell(x;p)}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p}$$

Let $\ell'(x;p) = 0$ and solve for p to get $\hat{p}$.

$$\frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} = 0 \Rightarrow \frac{(1-p)\sum_{i=1}^{n} x_i}{p(1-p)} - \frac{np - p\sum_{i=1}^{n} x_i}{p(1-p)} = 0$$

$$\Rightarrow \frac{\sum x_i - \cancel{p\sum x_i} - np + \cancel{p\sum x_i}}{p(1-p)} = 0 \Rightarrow \sum_{i=1}^{n} x_i - \cancel{np} = 0$$

$$\qquad \cancel{+np} \quad +np$$

$$np = \sum_{i=1}^{n} x_i \Rightarrow \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\bar{x}$

MLE Example 2

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Find MLE for μ.

$$L(x;\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma)^{-n/2} e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}}$$

$$\ell(x;\theta) = \ln(L(x;\mu)) = \ln\left((2\pi\sigma)^{-n/2} e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}}\right)$$

$$= -\frac{n}{2}\ln(2\pi\sigma) - \frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}$$

MLE Example 2 Continued

$$\ell'(x;m) = \frac{\partial \ell}{\partial m} = \frac{\partial}{\partial m}\left[-\frac{n}{2}\ln(2\pi\sigma) - \frac{\sum_{i=1}^{n}(x_i-m)^2}{2\sigma^2}\right]$$

$$= 0 + \frac{\cancel{2}\sum_{i=1}^{n}(x_i-m)}{\cancel{2}\sigma^2} = \frac{\sum_{i=1}^{n}(x_i-m)}{\sigma^2}$$

$$\hat{m} = \frac{\sum_{i=1}^{n}x_i}{n} = \boxed{\bar{x}}$$

Let $\ell'(x;m) = 0$

$$\Rightarrow \frac{\sum_{i=1}^{n}(x_i-m)}{\sigma^2} = 0 \Rightarrow \sum_{i=1}^{n}x_i - \cancel{nm} = 0$$

$$+\cancel{nm} \quad +nm$$

$$\frac{\sum_{i=1}^{n}x_i}{n} = \frac{\cancel{n}m}{\cancel{n}}$$

Good example of this is for the weibull dist.

Sometimes this process fails when no closed-form solution exists. For this class, we will explore how to find closed-form MLE's using the Newton-Raphson algorithm. We can slightly modify the Newton-Raphson algorithm to find MLE's that may be difficult to obtain!

We will learn how to use this powerful algorithm to help us find MLE's using R.

In the above examples, the MLE's we found are our theoretical results. These results will be used to verify the Newton-Raphson MLE algorithm!

Next Time...

Importing Data and Data Cleaning

Announcements

Homework 1 is due this Saturday at 11:59pm!