Collocation Method for Nonlocal Neumann Problems

Here we use a quadrature method to solve nonlocal Neumann problems of the type discussed in [1].

I. INTRODUCTION

We are interested in solving the following nonlocal Neumann boundary value problem numerically:

$$\begin{cases} \mathcal{L}u(x) = f(x), & x \in (-L, L) \\ \mathcal{N}u(x) = g(x), & x \in [-L, L]^c \end{cases} \tag{1}$$

where the nonlocal operators $\mathcal{L}, \mathcal{N}$ are defined as

$$\begin{cases} \mathcal{L}u(x) = \int_{-\infty}^{\infty} [u(x) - u(y)]\nu(x-y)\ dy \\ \mathcal{N}u(x) = \int_{-L}^{L} [u(x) - u(y)]\nu(x-y)\ dy \end{cases} \tag{2}$$

and $\nu(y)$ is the kernel; the only difference between the two are the integration bounds. Further, $f(x), g(x)$ are a given forcing function and boundary condition, respectively.

We'll assume the following about ν:

(1) $\nu(x)$ is an even function

(2) ν is locally integrable everywhere (can't blow up at the origin)

(3) $\int_{-\infty}^{\infty} \nu(y)\ dy = 1$

We will also assume that u is C^2 except possibly at $x = \pm L$.

Like [**?**], the idea is to split the integral into multiple parts and then derive a discretization scheme for each piece. However, because we do not know the exact form u takes outside of $(-L, L)$ (e.g., a Dirichlet condition), we can't use the same trick as [**?**] to get rid of the integrals defined outside $(-L, L)$. Regardless, it turns out you can use the Neumann BC to do something similar.

First, we decompose the nonlocal operator $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L}u(x) &= u(x)\int_{-\infty}^{\infty} \nu(x-y)\ dy - \int_{-L}^{L} u(y)\nu(x-y)\ dy - \int_{|y|\geq L} u(y)\nu(x-y)\ dy \\ &= u(x) - \int_{-L}^{L} u(y)\nu(x-y)\ dy - \int_{|y|\geq L} u(y)\nu(x-y)\ dy \end{aligned}$$

Note that the first and second terms only depend on u inside $(-L, L)$ while the third third term requires we know u outside $(-L, L)$. As such, this form is problematic as any direct discretization would contain infinitely many $u(y_i)$ terms (from the last integral). In the next section we get around this by using the Neumann condition.

II. THE NEUMANN CONDITION

We can rearrange the nonlocal Neumann condition as follows:

$$\begin{aligned} g(z) &= \mathcal{N}u(z) \\ &= \int_{-L}^{L} [u(z) - u(y)]\nu(z-y)\ dy \\ &= u(z)\int_{-L}^{L} \nu(z-y)\ dy - \int_{-L}^{L} u(y)\nu(z-y)\ dy \end{aligned}$$

Solving for $u(z)$ then gives

$$u(z) = \frac{g(z) + \int_{-L}^{L} u(y)\nu(z-y)\ dy}{\int_{-L}^{L} \nu(z-y)\ dy} \tag{3}$$

Note that this defines u outside $(-L, L)$ strictly in terms of u inside $(-L, L)$. We can use this to calculate the third integral mentioned in the previous section. First, let us rewrite this with dummy variables as,

$$u(z) = g(z)h(z) + h(z)\int_{-L}^{L} u(\omega)\nu(z-\omega)\ d\omega$$

where

$$h(z) = \int_{-L}^{L} \nu(z-\omega)\ d\omega$$

We then have:

$$\begin{aligned} \int_{|y|\geq L} u(y)\nu(x-y)\ dy &= \int_{|y|\geq L} \left[g(y)h(y) + h(y)\int_{-L}^{L} u(\omega)\nu(y-\omega)\ d\omega\right]\nu(x-y)\ dy \\ &= \int_{|y|\geq L} g(y)h(y)\nu(x-y)\ dy + \int_{|y|\geq L} h(y)\bigg(\int_{-L}^{L} u(\omega)\nu(y-\omega)\ d\omega\bigg)\nu(x-y)\ dy \\ &= \int_{|y|\geq L} g(y)h(y)\nu(x-y)\ dy + \int_{-L}^{L} u(\omega)\bigg(\int_{|y|\geq L} h(y)\nu(y-\omega)\nu(x-y)\ dy\bigg)d\omega \end{aligned}$$

The last equality follows be switching the order of integration. Note that the first integral doesn't depend on u and can be calculated analytically (by hand). Likewise, the term in parenthesis inside the second integral doesn't depend on u and can also be calculated analytically. Provided both terms can be calculated, we see that what remains only depends on u inside $(-L, L)$!

III. THE REFORMULATED NONLOCAL OPERATOR

Letting

$$f_1(x,\omega) = \int_{|y|\geq L} h(y)\nu(y-\omega)\nu(x-y)\ dy \quad , \quad f_2(x) = \int_{|y|\geq L} g(y)h(y)\nu(x-y)\ dy$$

and collecting results, we have

$$\begin{aligned} \mathcal{L}u(x) &= u(x) - \int_{-L}^{L} u(y)\nu(x-y)\ dy - \int_{|y|\geq L} u(y)\nu(x-y)\ dy \\ &= u(x) - \int_{-L}^{L} u(y)\nu(x-y)\ dy - f_2(x) - \int_{-L}^{L} u(\omega)f_1(x,\omega)d\omega \end{aligned}$$

Because ω is just a dummy variable, we may write

$$\mathcal{L}u(x) = u(x) - \int_{-L}^{L} u(y)\nu(x-y)\,dy - f_2(x) - \int_{-L}^{L} u(y) f_1(x,y) dy$$

or

$$\boxed{\mathcal{L}u(x) = u(x) - f_2(x) - \int_{-L}^{L} u(y)\Big(\nu(x-y) + f_1(x,y)\Big)dy}. \tag{4}$$

In this form, it's clear that the nonlocal operator $\mathcal{L}$ can be calculated using data **only** on $(-L, L)$, reducing the original problem to one on a bounded domain.

The new boundary value problem is thus

$$u(x) - f_2(x) - \int_{-L}^{L} u(y)\Big(\nu(x-y) + f_1(x,y)\Big)dy = f(x)\ , \quad x \in (-L, L)$$

IV. DISCRETIZATION SCHEME

Let $x_i = ih$, for $-M \leq i \leq M$, such that $L = Mh$. For simplicity, we assume $u, \nu, f_1 \in C^2(-L, L)$ so that we may use the Trapezoidal Rule as our discretization. We thus have

$$\begin{aligned}
\mathcal{L}u(x_i) &= u(x_i) - f_2(x_i) - \int_{-L}^{L} u(y)\Big(\nu(x_i - y) + f_1(x_i, y)\Big)dy \\
&= u(x_i) - f_2(x_i) - h\Big[u(x_{-M})\frac{\nu(x_i - x_{-M}) + f_1(x_i, x_{-M})}{2} \\
&\quad + \sum_{j=-M+1}^{M-1} u(x_j)\big[\nu(x_i - x_j) + f_1(x_i, x_j)\big] \\
&\quad + u(x_M)\frac{\nu(x_i - x_M) + f_1(x_i, x_M)}{2}\Big] + O\big(h^2\big)
\end{aligned}$$

Again, note that y was just a dummy variable, i.e. $y_j = x_j$, and we replaced them accordingly in the trapezoidal rule.

It's worth noting that both $u(L)$ and $u(-L)$ appear in the discretization (as $u(x_M)$ and $u(x_{-M})$ respectively) even though our BVP is only defined on $(-L, L)$. That being said, these terms are artificial as the value of the integral doesn't change if we remove two points. Hence, for this discretization scheme, it should be understood that $u(L)$ and $u(-L)$ are defined as those values which make u continuous on $[-L, L]$. Alternatively, this problem could have been avoided from the start had we let the nonlocal operator be defined on $[-L, L]$.

Lastly, if we drop the $O\big(h^2\big)$ term and plug this into the boundary value problem then we will have a discretization of our system.

A. Matrix Form

Letting $\omega_{i,j} = \nu(x_i - x_j) + f_1(x_i, x_j)$ and $\hat{u}$ denote the vector

$$\begin{bmatrix} u_{-M} \\ \vdots \\ u_M \end{bmatrix}$$

, we may rewrite the operator in matrix form as

$$\widehat{\mathcal{L}u} = \hat{u} - \hat{f}_2 - h \begin{bmatrix} \frac{1}{2}\omega_{-M,-M} & \omega_{-M,-M+1} & \cdots & \omega_{-M,M-1} & \frac{1}{2}\omega_{-M,M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}\omega_{M,-M} & \omega_{M,-M+1} & \cdots & \omega_{M,M-1} & \frac{1}{2}\omega_{M,M} \end{bmatrix} \hat{u}.$$

If we let $\hat{\omega}$ be the previous matrix of $\omega_{i,j}$ terms, then we may write

$$\widehat{\mathcal{L}u} = \hat{u} - \hat{f}_2 - h\,\hat{\omega}\,\hat{u}.$$

Further, notice that $\hat{\omega}$ is a symmetric matrix since the evenness of ν implies $\omega_{i,j} = \omega_{j,i}$.

B. Discretized Neumann Problem

Collecting results, the nonlocal Neumann problem becomes the finite, symmetric, linear equation

$$(I - h\,\hat{\omega})\,\hat{u} = \hat{f} + \hat{f}_2$$

V. EXAMPLE

Consider the Neumann problem

$$\begin{cases} \mathcal{L}u(x) = f(x), & x \in (-L, L) \\ \mathcal{N}u(x) = g(x), & x \in [-L, L]^c \end{cases} \tag{5}$$

with

$$\nu(y) = \frac{1}{2}e^{-|y|}.$$

Then by direct calculation, we can calculate both $h(z)$ and $f_1(x, y)$:

$$\begin{aligned} h(z) &= \int_{-L}^{L} \nu(z - \omega)\, d\omega \\ &= \begin{cases} \frac{1}{2}e^{-(z+L)}[e^{2L} - 1] & y \geq L \\ \frac{1}{2}e^{z-L}[e^{2L} - 1] & y \leq L \end{cases} \end{aligned}$$

$$f_1(x, w) = \frac{1}{24}\frac{e^{2L} - 1}{e^{4L}}[e^{w+x} + e^{-(w+x)}]$$

If we now give the boundary data as $g(x) = 0$ then we have that

$$f_2(x) = 0$$

.

The last ingredient we need is the forcing function $f(x)$. Since we ultimately want to test the accuracy of our numerical method, we will work backwards, giving the solution $u(x)$ and then, by plugging it directly into Eq. (4), calculating the required $f(x)$. We will do this for two cases.

First, we take $u = x$. Letting

$$\alpha = \frac{13}{24}e^{-L} - \frac{1}{12}e^{-3L} + \frac{1}{24}e^{-5L} + \frac{11}{24}Le^{-L} + \frac{1}{24}Le^{-5L}$$

we have that the required $f(x)$ is

$$f(x) = 2\,\alpha\,\sinh(x)$$

For the second case, we let $u(x) = \text{sech}(x)$. In this case too an exact formula for $f(x)$ can be found, though it's rather large. Let

$$\beta = \frac{1}{2}\log(\frac{e^{2L}+1}{e^{2L}}) - \frac{1}{12}Le^{-2L} + \frac{1}{12}Le^{-4L}$$

and

$$\gamma = \frac{1}{2}\log(e^{-2L}+1) - \frac{1}{24}e^{-2L}\log(e^{-2L}+1) + \frac{1}{24}e^{-2L}\log(e^{2L}+1)$$
$$+\frac{1}{24}e^{-4L}\log(e^{-2L}+1) - \frac{1}{24}e^{-4L}\log(e^{2L}+1) - \frac{1}{6}Le^{-2L} + \frac{1}{6}Le^{-4L}.$$

Then we have that

$$f(x) = \text{sech}(x) - \frac{1}{2}e^{-x}\log(1+e^{2x}) - \frac{1}{2}e^{x}\log(1+e^{2x}) + xe^{x} + \beta e^{x} + \gamma e^{-x}$$

Notice that as $L \to \infty$, we recover the same forcing function (and solution) as we did for the Dirichlet problem. This could have probably been guessed by noting that $f_1 \to 0$ as $L \to \infty$, thereby (at least formally) reducing Eq. (4) into the infinite Dirichlet equation.

[1] S. Dipierro, X. Ros-Oton, and E. Valdinoci, Arxiv:1407.3313v3 (2014).