Probability

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Weeks: 2
Dates: Monday 6/1, Tuesday 7/1, Monday 13/1 (Cancelled because of Ess Work), Thursday 16/1

Monday 6/1

1st half module

- What is probability?
  - Represented numerically
  - Determine probabilities
  - Where is probability used?
- Experimental Probability
  - Trials, Outcomes, Frequency, Relative frequency.
  - Relative frequency = Experimental probability(TOK 1)
  - Example : Dice and Two way table
  - Questions

2nd half module

- Sample Space
  - Sample space, Events
  - Set notation, subsets, Venn diagram
  - complementary events
  - Example: Venn diagram, 2D dimensional grid and Tree diagram
- Theoretical Probability
  - Equally likely definition
  - $P(A) = \frac{n(A)}{n(U)}$
  - Complementary Events $P(A) + P(A') = 1$
  - Questions

Tuesday 7/1

1st half module

- Addition law of probability
  - Work through Investigation 4
  - Definition $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Mutually exclusive
  - Examples:
  - Questions

2nd half module

- Independent events
  - Work through Investigation 5
  - Definition $P(A \cap B) = P(A) \times P(B)$
  - Questions

Thursday 16/1

1st half module

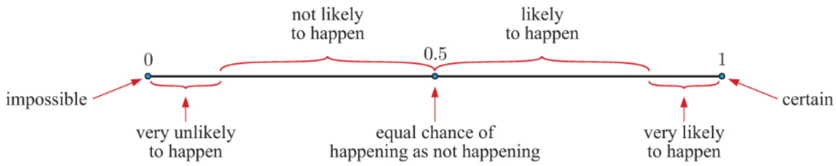
- Dependent events
  - Example:
  - Definition $P(A \cap B) = P(A) \times P(B \mid A)$
  - Experiments and Replacement
  - Questions

2nd half module

- Conditional probability
  - Example:
  - Definition $P(A \mid B) = \frac{n(A \cap B)}{n(B)}$
  - Equally likely $\implies$ conditional probability $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
  - Questions

1 Notes to Monday 6/1

Probability: In the real world we think of probability as a **chance or likelihood** of some event happening. We assign a number between 0 and 1 to the chance of a event occurring and we call this number the probability.



We can determine probabilities based on: **the results of an experiment** and/or **what we theoretically expect to happen**.

The theory of probability is used in a wide range of fields:

- Biology
- Economics
- Politics
- Sport
- Quality control
- Production planning
- Physics - Quantum mechanics and Statistical mechanics

Experimental Probability: When performing an experiment that involves chance the following information is needed.

- **Number of trials:** is the total number of times the experiment is performed.
- **Outcomes:** are the different results possible for one trial of the experiment.
- **Frequency of an outcome:** is the number of times the outcome is observed.
- **Relative frequency of an outcome:** is the frequency of the outcome expressed as a fraction or percentage of the total number of trials. $RF = \frac{Frequency}{Numberoftrials}$

Example: Dice probability experiment

- What is the theoretical probability of each face?
- How many trials will you do?
- What outcomes are possible? (Express in set notation)
- Express the frequency of an outcome in your experiment.
- Calculate the relative frequency of an outcome in your experiment.
- (Look at individual experiments vs. the sum of the trials)

In experiments, the relative frequency is the best estimate of the probability of that event occurring, **Relative frequency** = **Experimental probability**.

Example: Two way table

	Adult	Child	Total
Season ticket holder	1824	779	2603
Not a season ticket holder	3247	1660	4907
Total	5071	2439	7510

$$b_i : P(\text{A child}) = \frac{779 + 1660}{7510} = 0.32 \implies 32\% \tag{1}$$

$$b_{ii} : P(\text{Not a season ticket holder}) = \frac{3247 + 1660}{7510} = 0.65 \implies 65\% \tag{2}$$

$$b_{iii} : P(\text{An adult season ticket holder}) = \frac{1824}{7510} = 0.24 \implies 24\% \tag{3}$$

Questions:

- 10A - 1, 2, 3 (Page 243)
- 10A - 5 (Page 244)
- 10B - 3, 4 (Page 247)

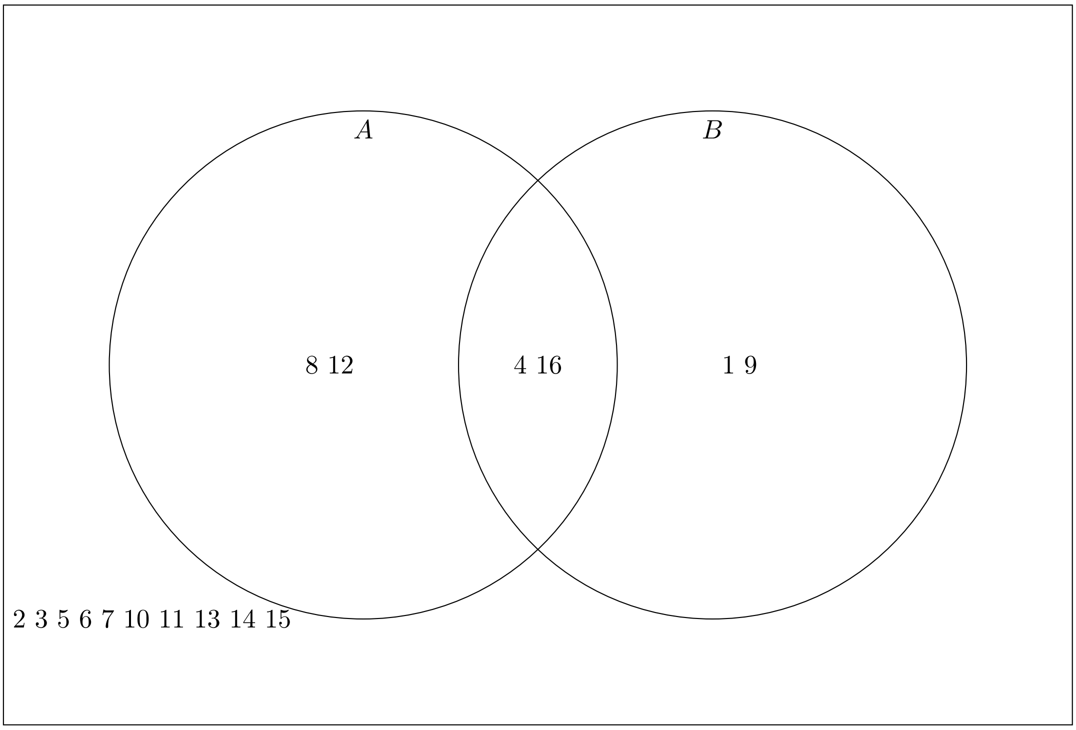
Sample space, events and complementary events

- The **sample space** U is the set of all possible outcomes of an experiment.
- An **event** is a set of outcomes in the sample space that have a particular property.
- The sample space is the **universal set** U.
- The outcomes are the **elements** of the sample space.
- Events are **subsets** of the sample space.
- We use set notation and Venn diagrams to solve probability problems.

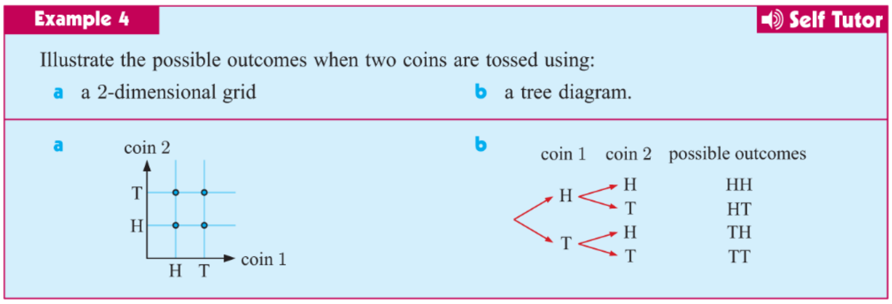
Complementary events: Two events are **complementary** if exactly one of the events must occur. If A is an event, then A' is the complementary event of A, or "not A".

Example: Venn diagram Q 10C.3

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \tag{4}$$
$$A = \{4, 8, 12, 16\} \tag{5}$$
$$B = \{1, 4, 9, 16\} \tag{6}$$



Example: 2D grid and Tree Diagram



Example 4 Self Tutor

Illustrate the possible outcomes when two coins are tossed using:

a a 2-dimensional grid **b** a tree diagram.

Notice in the Example that each outcome in the sample space {HH, HT, TH, TT} is represented by:

- a point on the grid
- a "branch" on the tree diagram.

Theoretical probability:

- **Equally Likely:** If a sample space has n outcomes which are equally likely to occur when the experiment is performed once, then each outcome has probability $\frac{1}{n}$ of occuring.
  - Example: Dice - 6 sides, each side is equally likely be rolled. Thus each side has a probability $\frac{1}{6}$ of occuring.
- When the outcome of an experiment are equally likely, the probability that an event A occurs is:

- $P(A) = \frac{\text{number of outcomes corresponding to A}}{\text{number of outcomes in the sample space}} = \frac{n(A)}{n(U)}$.
- Example: Dice - Probability of rolling either a 1 or a 5, Event $A =$ rolling either a 1 or a 5, $n(A) = 2$ and $n(U) = 6$.
- What is the complementary event A'?
- $P(A) + P(A') = 1$, A good sanity check!

Questions:

- 10C - 4, 5 (Page 249)
- 10D - 1, 7 (Page 252)
- 10D - 8, 10 (Page 253)
- 10D - 12(Use the example 9 above) (Page 254)
- 10D - 19 (Page 255)

2 Notes to Tuesday 7/1

Addition law of probability

- **Compound events:** More than one event in our sample space. This could be two or more processes in our experiment.
  - Two events A and B.
  - Both A and B occurs written as $A \cap B$, reads as "A intersection B".
  - A or B or Both occurs written as $A \cup B$, reads as "A union B".
- How are multiple probabilities added together?
- Investigation 4 - 2 Event Venn diagram (Page 258)
  - Suppose $U = \{x \mid \text{positive integers less than } 100\}$.
  - Let $A = \{\text{multiplies of 7 in U}\}$ and let $B = \{\text{multiples of 5 in U}\}$.
  - How many elements in:
    * U (99)
    * A (14)
    * B (19)
    * $A \cap B$ (2)
    * $A \cup B$ (31)
    *
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \tag{7}$$
$$31 = 14 + 19 - 2 \tag{8}$$
  - The probability of the events:
    * $P(A) = \frac{a+b}{a+b+c+d}$.
    * $P(B)$, $P(A \cap B)$, $P(A \cup B)$ and $P(A) + P(B) - P(A \cap B)$.

* What is the connection between $P(A \cup B)$ and $P(A) + P(B) - P(A \cap B)$.

- **Addition law of probability:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Mutually exclusive:** A and B are **disjointed** events $P(A \cap B) = 0$, the addition law becomes $P(A \cup B) = P(A) + P(B)$.

Example: (Page 259)

EXERCISE 10E

1 If $P(A) = 0.2$, $P(B) = 0.4$, and $P(A \cap B) = 0.05$, find $P(A \cup B)$.

2 If $P(A) = 0.4$, $P(A \cup B) = 0.9$, and $P(A \cap B) = 0.1$, find $P(B)$.

Questions:

- 10E - 3, 4 (Page 259)
- 10E - 5, 6, 7 (Page 260)

Independent Events

- Two events are **independent** if the occurrence of each event does not affect the occurrence of the other.
- How can we calculate $P(A \cap B)$ for two independent events?
  - Suppose a coin is tossed and a die is cast at the same time.
  - Does the outcome of the coin toss affect the die roll?
  - 2-Dimensional grid
  -

A	B	$P(A)$	$P(B)$	$P(A \cap B)$
Head	4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$
Head	Odd number	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
Tail	Number greater than 1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{5}{12}$
Tail	Number less than 3	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

  - Selecting balls from two boxes. 1st box - BBGG and 2nd box RRRW.
  - Does the outcome from one box affect the other?
  - 2-Dimensional Grid
  -

A	B	$P(A)$	$P(B)$	$P(A \cap B)$
Green from box X	Red from box Y	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$
Green from box X	Red from box Y	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$
Blue from box X	Red from box Y	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$
Blue from box X	White from box Y	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

- For independent events A and B, what is the connection between $P(A \cap B)$, $P(A)$ and $P(B)$?

Independent Events: If A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$.

Multiple Independent Events: If A, B and C are independent events, then $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$.

Questions:

- 10F - 1, 2, 3 (Page 261)
- 10F - 5, 6 (Page 262)
- 10F - 8, 9, 11 (Page 263)

3 Notes to Thursday 16/1

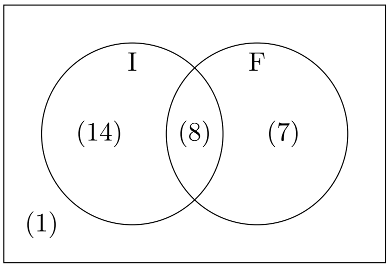
Dependent events:

- Sampling example:
  - 5 Red and 3 blue tickets in a box. One ticket is drawn, colour noted and not put back, and a second ticket is drawn.
  - Whats the chance that it is red?
  - First ticket is red, $P(\text{second is red}) = \frac{4}{7}$
  - First ticket is blue, $P(\text{second is red}) = \frac{5}{7}$
  - The probability of the second ticket being red depends on the what the first ticket was!
- Definition:
- Two or more events are **Dependent** if the occurrence of one of the events does affect the occurence of the other events.
- Events are **dependent** is they are **not independent**.
- If A and B are dependent events then $P(A \cap B) = P(A) \times P(\text{B given that A has occurred})$.
- Sampling:
  - **Without Replacement** we have dependent events.
  - **Without Replacement** we have independent events.

Questions:

- 10G - 1,2 (Page 265)
- 10G - 5,6 (Page 266)
- 10G - 7, 10 (Page 267)
- 10G - 15 (Page 268)

Conditional probability:

- **Example:** The probability that a randomly chosen student who studies french, also studies italian?
- 

- $P(\text{I given that F has occured}) = \frac{8}{15} = \frac{\text{number students that studies Italian and French.}}{\text{number of students who study French.}}$

Conditional probability definition: For events A and B, we use the notation "A|B" to represent the event "A given that B has occured".

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} \tag{9}$$

If the outcomes of the events are equally likely,

$$\frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A\cap B)}{n(U)}}{\frac{n(B)}{n(U)}} = \frac{P(A \cap B)}{P(B)} \tag{10}$$

Conditional probability formula:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{11}$$

Questions:

- 10H - 1, 2, 4 (Page 269)
- 10H - 7 (Page 270)
- 10H - 9 (Page 271)