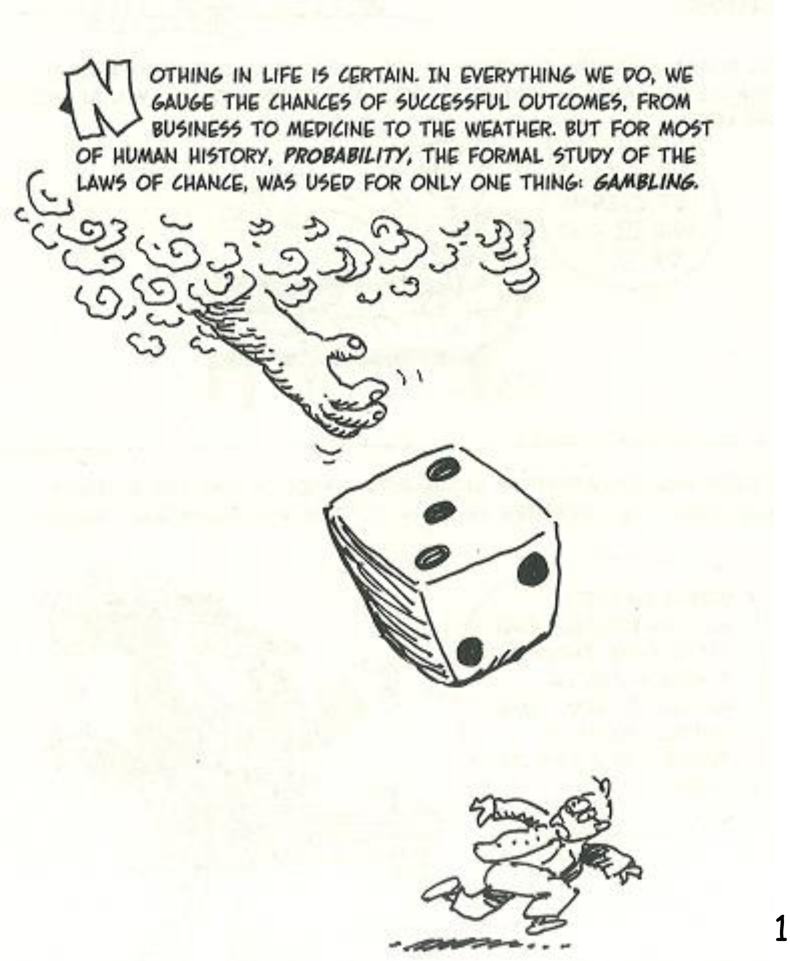
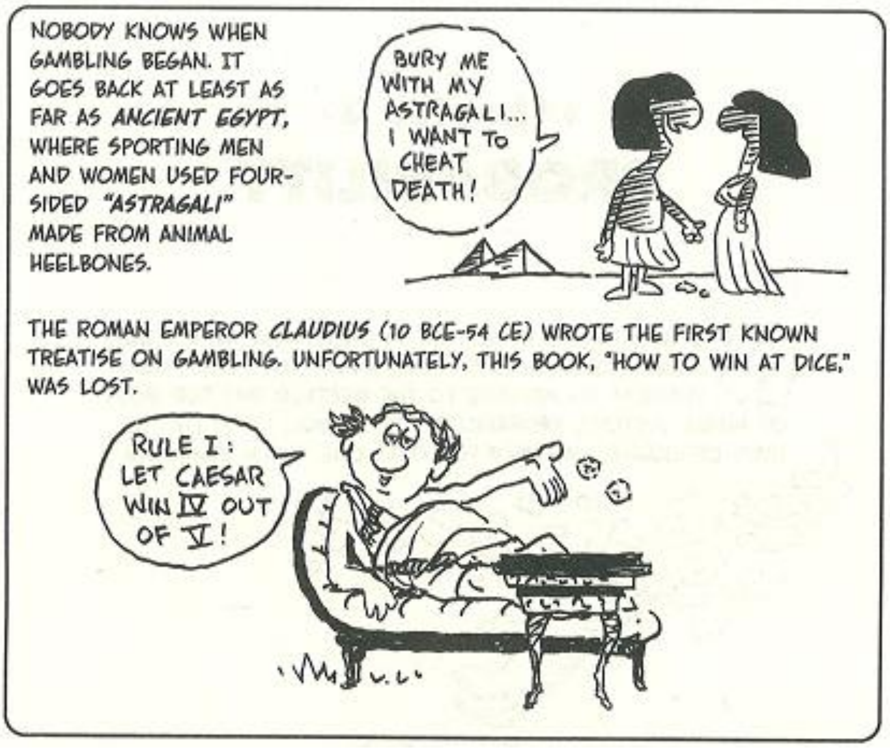
Probability



[1]

1 The Cartoon Guide to Statistics by Larry Gonick and Woollcott Smith p. .

Introduction



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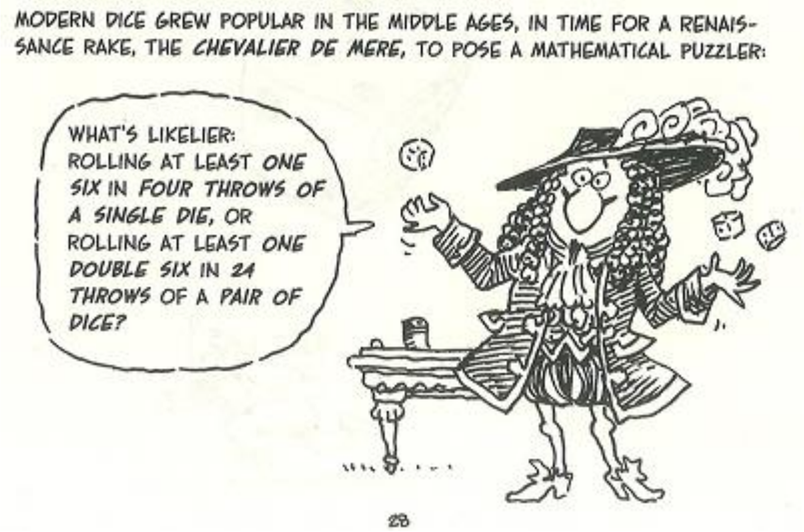

[3]

2 The Cartoon Guide to Statistics by Larry Gonick and Woollcott Smith.

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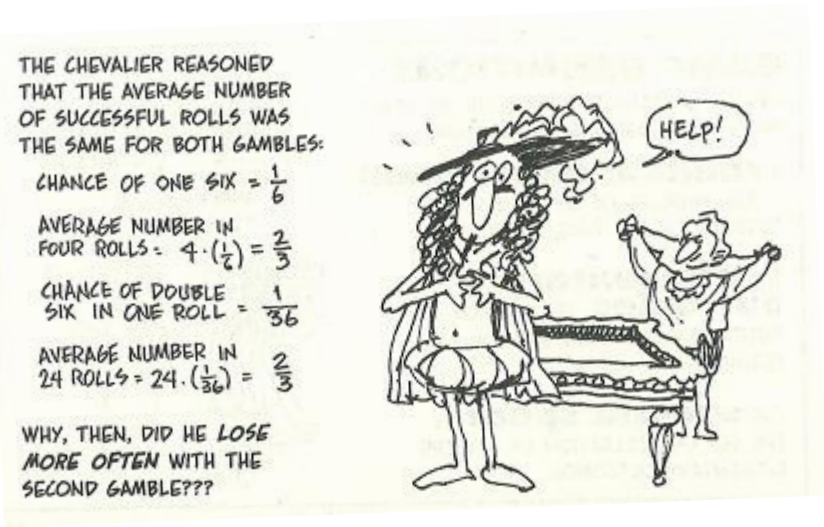

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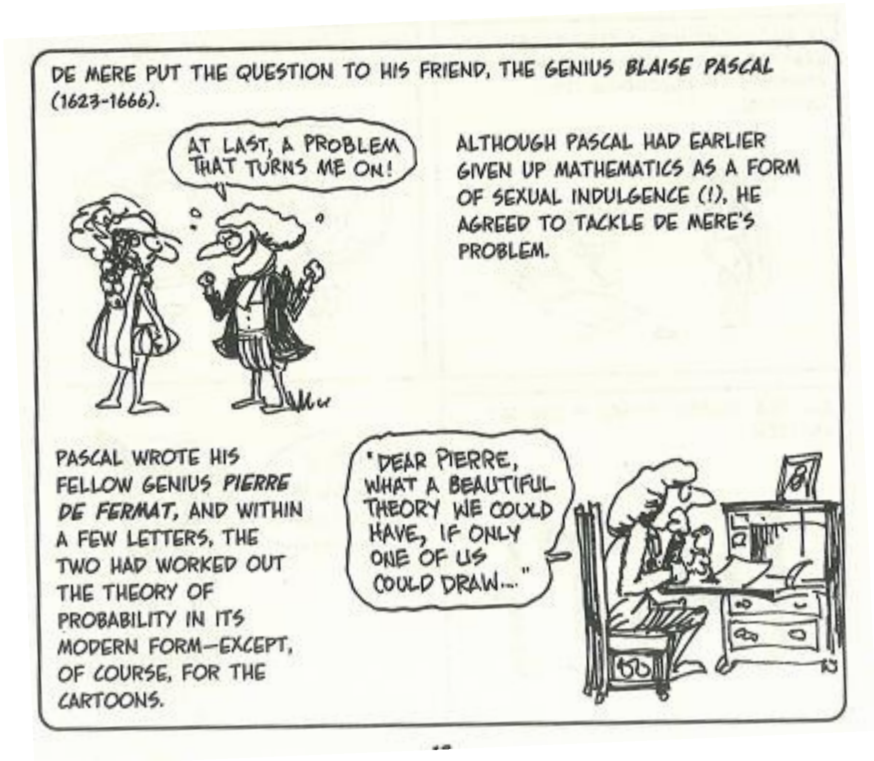

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4 The Cartoon Guide to Statistics by Larry Gonick and Woollcott Smith.

5 The Cartoon Guide to Statistics by Larry Gonick and Woollcott Smith.

Probability theory is the study of the chance (or likelihood) of events happening.

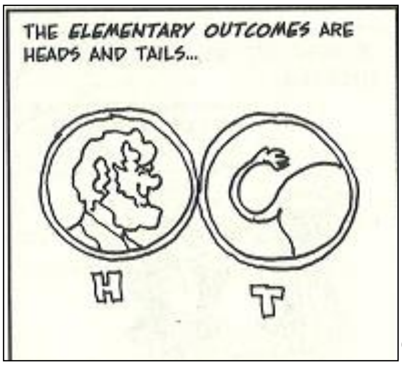
Probability is the measure of how likely an event is.

Why we need to study probability and where is it used:

- Gambling
- Predictions
  - Virology
  - Medicine
  - Weather
- Banking
  - Interests rates
  - Stocks
  - Property
- Quality control
  - Malfunctions of machines
  - Broken cookies
  - Milk cartons (what is the volume)
- Production planning
- Sports
  - Training
  - Results

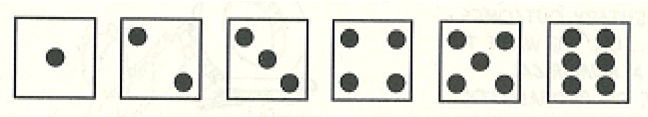
Common knowledge:

- Coin has two sides; head (H) and tail (T).

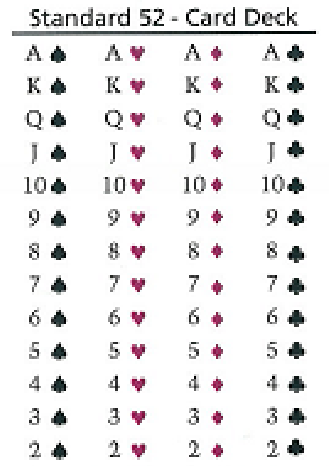


[6]

- A die has six sides. (Note that word die is singular and word dice is plural, however dice is sometimes used even if one die is thrown.)



- A deck of cards has 52 cards and has 4 suits; diamonds, hearts, clubs and spades.



Standard 52 - Card Deck

A ♠	A ♥	A ♦	A ♣
K ♠	K ♥	K ♦	K ♣
Q ♠	Q ♥	Q ♦	Q ♣
J ♠	J ♥	J ♦	J ♣
10 ♠	10 ♥	10 ♦	10 ♣
9 ♠	9 ♥	9 ♦	9 ♣
8 ♠	8 ♥	8 ♦	8 ♣
7 ♠	7 ♥	7 ♦	7 ♣
6 ♠	6 ♥	6 ♦	6 ♣
5 ♠	5 ♥	5 ♦	5 ♣
4 ♠	4 ♥	4 ♦	4 ♣
3 ♠	3 ♥	3 ♦	3 ♣
2 ♠	2 ♥	2 ♦	2 ♣

- Calendar, remember that the calendar year is not a regular cycle. Calendar questions involving months or years have to be calculated from 4 year cycles, because of the leap year.
- How to find if the calendar year is a leap year? Remember the millennium needs to be divisible by 4 and 400 to be a leap year.

6 The Cartoon Guide to Statistics by Larry Gonick and Woollcott Smith.

Terminology:

An **experiment** is a situation involving chance or probability that leads to results called outcomes.

An **outcome** is the result of a single trial of an experiment.

The number of **trials** is the total number of times the experiment is repeated.

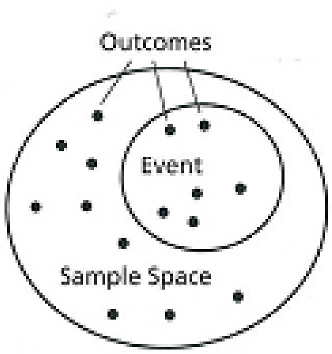
An **event** is one or more outcomes of an experiment. An event is the outcome you are hoping to experience.

The **theoretical probability** of a particular event is a measure of the chance of that event occurring in any trial of the experiment.

The **frequency** of a particular outcome is the number of times that this outcome is observed.

A **sample space (U)** is the set of all possible outcomes of an experiment.

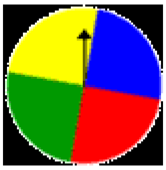
n(U) is the **number of elements** in a sample space.



1. Example:

A coin has two sides; head (H) and tail (T) so the sample space U ={H, T}, n(U) = 2.
A die has six sides so the sample space U = {1, 2, 3, 4, 5, 6} and n(U) = 6.

2. Example:



The experiment is spinning the spinner.

The possible outcomes are landing on yellow, blue, green or red.

One event of this experiment is landing on blue.

The probability of landing on blue is one fourth.

Note that in general you are not allowed to assume that all outcomes are equally likely especially if you have been given enough information to calculate their probabilities.

Different ways to represent the sample space (all the possible outcomes):

1) Listing – this is the best way if you are doing just <u>one</u> experiment or one trial

If the sample space is one dimensional then it is easy to just list the possible elements.

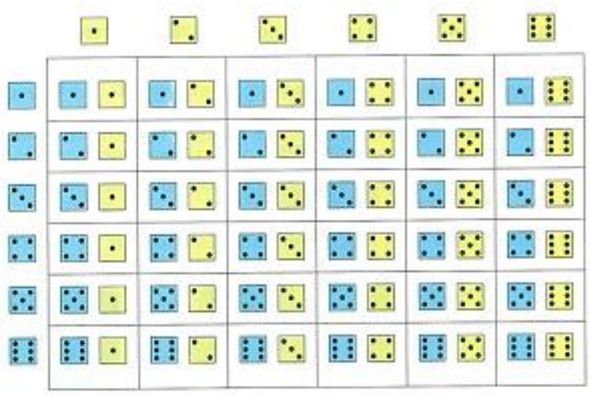
For example: U = {1, 2, 3, 4, 5, 6} or

Tossing a coin U = {H, T}

2) Grid – use this method if you are doing <u>two</u> experiments or two trials

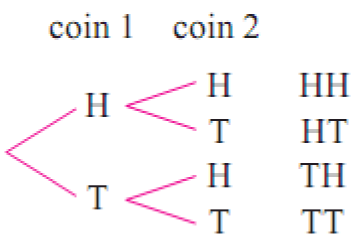
If you are doing **two** operations, the easiest way to demonstrate the sample space is to draw a grid.

	Head (H)	Tail (T)
Head (H)	HH	TH
Tail (T)	HT	TT



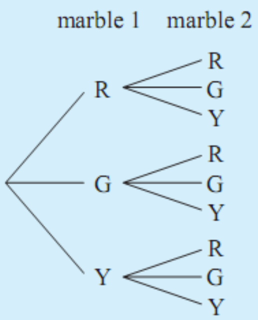
3) Tree diagram –use this method if you are doing <u>two or more</u> experiments or trials

If you have to work with 2 or more trials, a tree diagram is a god way to demonstrate the sample space.



Example 4 | **Self Tutor**

Illustrate, using a tree diagram, the possible outcomes when drawing two marbles from a bag containing a number of red, green and yellow marbles.



Haese and Harris MYP5 Plus book, 2nd edition

4) Table of outcomes are tables which compare two categorical (color, gender, type of car etc.) variables. These are often results from surveys.

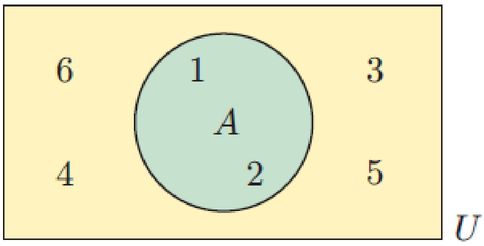
	Car	Bicycle	Bus
Male	37	10	10
Female	30	5	13

13 female teachers catch the bus to school.

It is useful to extend the table to include the totals of each row and each column.

5) Venn diagrams are a useful way of representing the events in a sample space. The diagrams consist of a rectangle which represents the complete sample space U, and circles within it represents particular events.

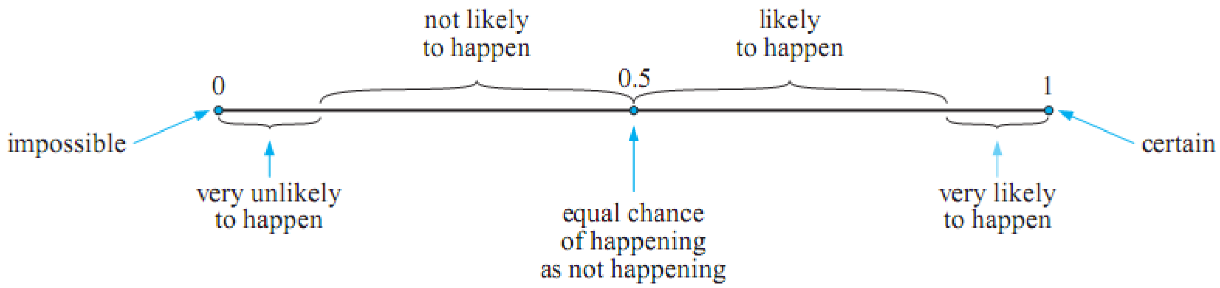
Example:
When you roll an ordinary die, the sample space or universal set is $U = \{1, 2, 3, 4, 5, 6\}$.
If the event A is "a number less than three", then there are two outcomes which satisfy event A. This can be written $A = \{1, 2\}$.



An event which has 0% chance of happening (i.e. is impossible) is assigned a probability of 0.

An event which has a 100% chance of happening (i.e. certain) is assigned a probability of 1.

All other events can then be assigned a probability between 0 and 1.



$0 \le P(A) \le 1$ This means that the probability of an event *A* is between 0 and 1 inclusive.

Finding the Probability of an Event

Probability of an event *A*:

$$P(A) = \frac{\text{The Number Of Ways Event A Can Occur}}{\text{The Total Number Of Possible Outcomes}}$$

$$P(A) = \frac{n(A)}{n(U)}$$

3. **Example:**

You are tossing a coin once, what is the probability to get

a) a head?

$P(H) = \frac{1}{2} = 0.5$

b) a head or a tail

$P(H\ or\ T) = \frac{2}{2} = 1$

Note that all probability calculations have to be started with P (just like area calculations with A). In side the parenthesis you write a description of an event you are hoping to experience.

The answers can be given as a fraction, a decimal number or a percentage. Exact answer (a fraction) is always the most desirable one.

4. **Example:**

You are dialing a friend's phone number, but can't remember the last digit. If you choose a digit at random, what is the **probability that** you dial **the correct number**?

$$P(correct\ phone\,number) = \frac{Number\ of\ correct\ digits}{Number\ of\ \ possible\ digits} = \frac{1}{10}$$

5. **Example:**

On a multiple choice test, you know the answer to question 8 is not a or d, but you are not sure about b, c and e. If you guess, what is the **probability** that **you are wrong**?

$$P(correct\ answer) = \frac{Number\ of\ wrong\ answers}{Number\ of\ \ possible\ answer} = \frac{2}{3}$$

6. **Example:**

Two dice are rolled and the sum of the numbers on the uppermost faces are added. What is the probability of getting a 7 ?

Solution:

		1st die					
	+	1	2	3	4	5	6
2nd die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

$$P(7) = \frac{\cancel{6}^{\,1}}{\cancel{36}_{\,6}} = \frac{1}{6}$$

7. **Example:**

Two dice are rolled and the score is defined as the product of the numbers showing on the uppermost faces. Write out the possibility space and use it to find the probability of scoring 12 or more.

S1: draw a grid

Solution:

		1st die					
	×	1	2	3	4	5	6
2nd die	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

S2: Highlight the values you are looking for

Solution:

		1st die					
	×	1	2	3	4	5	6
2nd die	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

S3: Write down an answer with using appropriate mathematical notation.

$$P(12\, or\, more) = P(X \geq 12) = \frac{17}{36}$$

Complementary events:

If A is an event, then A' is the complementary event of A. So if P(A) is the probability of an event A occurring. P(A') is the probability of an event A not occurring.

$$P(A') + P(A) = 1$$

Note that sometimes text books are using notation P(A) = p and P(A') = 1-p = q.
A useful rearrangement is:

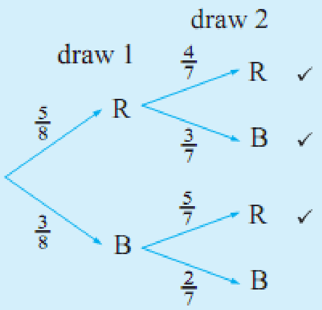
$P(A') = 1 - P(A)$ P(A not occurring) = 1- P(A occurring)

Often used in questions "not A"

"at least 1"

Example 17

A bag contains 5 red and 3 blue marbles. Two marbles are drawn simultaneously from the bag. Determine the probability that at least one is red.

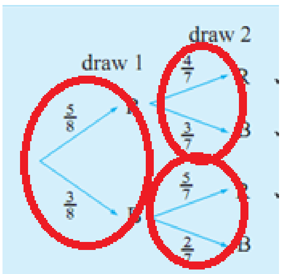


$$\begin{aligned}\text{P(at least one red)} &= \text{P(RR or RB or BR)}\\ &= \tfrac{5}{8}\times\tfrac{4}{7} + \tfrac{5}{8}\times\tfrac{3}{7} + \tfrac{3}{8}\times\tfrac{5}{7}\\ &= \frac{20+15+15}{56}\\ &= \frac{50}{56}\\ &= \frac{25}{28}\end{aligned}$$

Note: *Alternatively,* P(at least one red) $= 1 -$ P(no reds) {complementary events}

$= 1 -$ P(BB), etc

Note that when you go along the branches you need to multiply the probabilities and when you take alternative branches you need to add them.

Note also that the  branches which start from the same point have to add up to 1.

Independent events:

Independent events are events where the occurrence of one of the events does not affect the occurrence of the other event.

A and B are independent events (if and only if) P(A and B) = (P(A)) (P(B))

Example:

- throwing a die (a die does not have a memory)
- tossing a coin
- experiments with replacement (choosing a card and returning it to the deck, choosing a marble and returning it back to a box)

Often used in questions "and"

8. **Example:**

You are tossing a coin twice, what is the probability to get

b) a head **and** a tail
Note: draw a grid
$P(H\ and\ T) = \frac{2}{4} = \frac{1}{2}$

c) a head **and then** a tail
Note: the order is important!
$P(H\ and\ then\ T) = \frac{1}{4}$

Dependent events:

Two or more events are **dependent** if they are not independent. Dependent events are events where the occurrence of one of the events <u>does affect</u> the occurrence of the other event.

This means that after first event has happened the probability of the second event has changed.

If A and B are dependent events then
P(A then B) = (P(A))(P(B **given that** A has occurred))

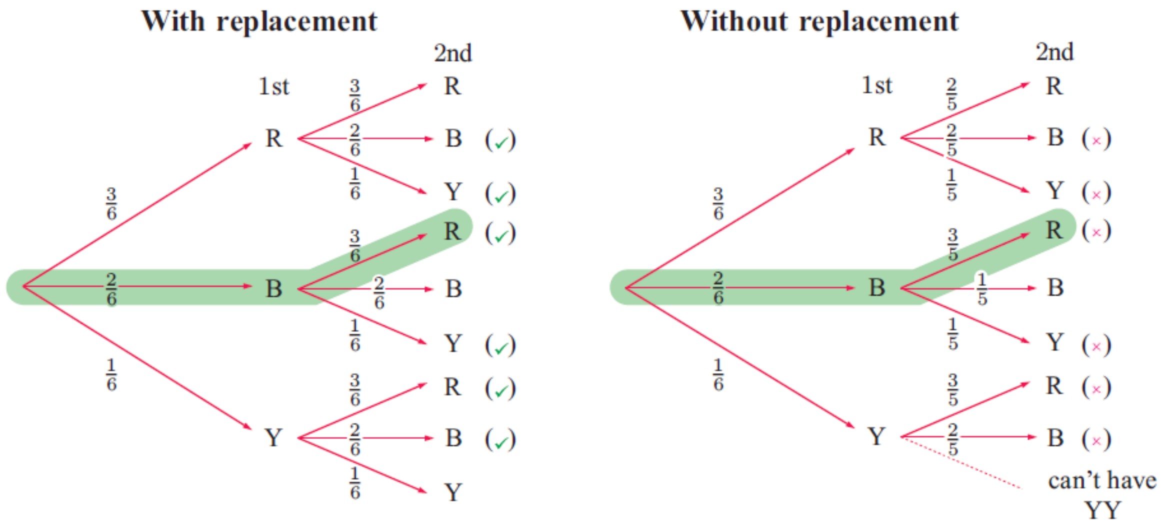
Example:

- Selecting items <u>without replacement</u>
- Deck of cards
- Eating cookies from a cookie jar
- Taking elements simultaneously

Consider a box containing 3 red, 2 blue, and 1 yellow marble. If we sample two marbles, we can do this either:

- **with replacement** of the first before the second is drawn, or
- **without replacement** of the first before the second is drawn.

Examine how the tree diagrams differ:



The highlighted branch represents a blue marble with the first draw and a red marble with the second draw. We write this as BR.

Notice that:

- with replacement (independent events), P(two reds) $= \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$
- without replacement (dependent events), P(two reds) $= \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$

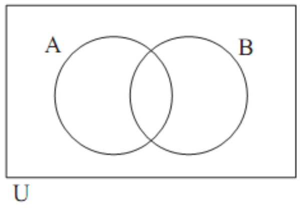
Note that with replacement the denominator does not change on the second selection, but without replacement the denominator changes because there are less objects available.

Note that when objects are drawn simultaneously it means the same as taking objects without replacement.

LAWS OF PROBABILITY

Combined events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



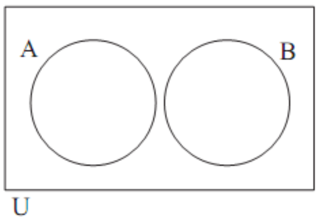
Often used in questions "or"
P(either A or B) = P(A)+P(B)-P(both A and B)
Often used in questions "or"

Mutually exclusive events

Events A and B are mutually exclusive if $A \cap B = \varnothing$.
Therefore $P(A \cap B) = 0$ and so

$$P(A \cup B) = P(A) + P(B)$$

Often used in questions "or"



Independent events (2)

Independent events are events where the occurrence of one of the events does not affect the occurrence of the other event.

A and B are independent events (if and only if) $\Leftrightarrow \quad P(A \cap B) = P(A)P(B)$

If A and B are independent events then P(A and B) = (P(A))(P(B))

Example:

- throwing a die
- <u>experiments with replacement</u> (choosing a card and returning it to the deck)

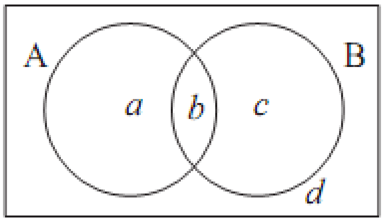
Often used in questions "and"

Conditional probability:

Conditional probability is used to represent that "A occurs knowing that B has occurred"
Often used in questions "given that P(B)"

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Proof:



$$P(A|B) = \frac{b}{b+c}$$

$$= \frac{b/(a+b+c+d)}{(b+c)/(a+b+c+d)}$$

$$= \frac{P(A \cap B)}{P(B)}$$

Example 24

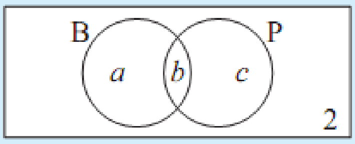
In a class of 40, 34 like bananas, 22 like pineapples and 2 dislike both fruits. If a student is randomly selected, find the probability that the student:

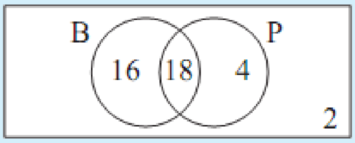
a likes both fruits

b likes at least one fruit

c likes bananas given that he/she likes pineapples

d dislikes pineapples given that he/she likes bananas.





B represents students who like bananas

P represents students who like pineapples

We are given that

$$\begin{aligned} a + b &= 34 \\ b + c &= 22 \\ a + b + c &= 38 \end{aligned}$$

$$\therefore \quad c = 38 - 34 \quad \text{and so} \quad b = 18$$
$$= 4 \qquad\qquad \text{and} \quad a = 16$$

a P(likes both) $= \frac{18}{40}$ $= \frac{9}{20}$

b P(likes at least one) $= \frac{38}{40}$ $= \frac{19}{20}$

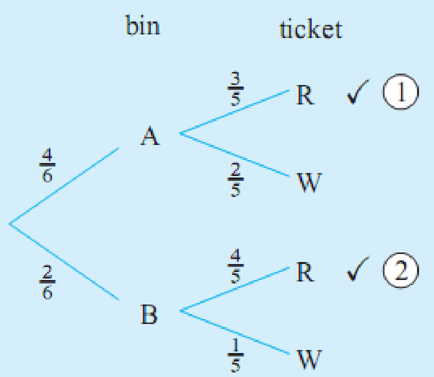
c $P(B \mid P)$ $= \frac{18}{22}$ $= \frac{9}{11}$

d $P(P' \mid B)$ $= \frac{16}{34}$ $= \frac{8}{17}$

Example 25

Bin A contains 3 red and 2 white tickets. Bin B contains 4 red and 1 white. A die with 4 faces marked A and two faces marked B is rolled and used to select bin A or B. A ticket is then selected from this bin. Determine the probability that:

a the ticket is red **b** the ticket was chosen from B given it is red.



a $P(R)$

$= \frac{4}{6} \times \frac{3}{5} + \frac{2}{6} \times \frac{4}{5}$ {the ✓ paths}

$= \frac{20}{30}$

$= \frac{2}{3}$

b $P(B \mid R) = \dfrac{P(B \cap R)}{P(R)}$

$= \dfrac{\frac{2}{6} \times \frac{4}{5}}{\frac{2}{3}}$ ← path 2

$= \frac{2}{5}$

In a family with 2 children it is known that

a) one of the children is a boy.

b) one of the children is called Bob (we know that Bob is a boys name).

What is the probability that the other child is also a boy?

a) P(a boy given that one of the children is a boy)= $\frac{1}{3}$

	B	G
B	BB	GB
G	BG	GG

b) P(other child is a boy if it is know that there is a child called Bob)= $\frac{1}{2}$

	Bob
B	
G	

	B	**G**
Bob		