Radioactive Decay Analysis

1. Mathematical Derivation

1.1. Introduction

Radioactive decay is the process by which an unstable atomic nucleus loses energy by emitting radiation. This process is random at the individual atomic level, meaning it's impossible to predict when a specific atom will decay. However, for a large number of identical atoms, the average rate of decay is predictable and follows an exponential pattern. This phenomenon is fundamental to understanding nuclear physics, nuclear energy, and various applications such as radiometric dating and medical imaging.

1.2. The Decay Law

The rate of radioactive decay is proportional to the number of undecayed radioactive nuclei present. Mathematically, this can be expressed as:

$\frac{dN}{dt} = -\lambda N$

Where:

- N is the number of radioactive nuclei at time t.
- $\frac{dN}{dt}$ is the rate of change of the number of nuclei with respect to time (i.e., the decay rate).
- λ (lambda) is the decay constant, a positive constant that is characteristic of the particular radioactive isotope. It represents the probability per unit time that a nucleus will decay.

The negative sign indicates that the number of radioactive nuclei decreases over time.

This is a first-order linear differential equation. We can solve it by separating variables:

$\frac{dN}{N} = -\lambda dt$

Integrating both sides:

$\int \frac{dN}{N} = \int -\lambda dt$

$\ln(N) = -\lambda t + C$

Where C is the integration constant. To find C, we use the initial condition that at time $t = 0$, the number of radioactive nuclei is N_0. So, when $t = 0, N = N_0$:

$\ln(N_0) = -\lambda(0) + C$

$C = \ln(N_0)$

Substituting C back into the integrated equation:

$\ln(N) = -\lambda t + \ln(N_0)$

$\ln(N) - \ln(N_0) = -\lambda t$

$\ln\left(\frac{N}{N_0}\right) = -\lambda t$

Exponentiating both sides:

$\frac{N}{N_0} = e^{-\lambda t}$

$N(t) = N_0 e^{-\lambda t}$

This equation is known as the **Radioactive Decay Law**. It describes the exponential decrease in the number of radioactive nuclei over time.

1.3. Half-Life ($T_{1/2}$)

The half-life ($T_{1/2}$) of a radioactive isotope is the time it takes for half of the initial number of radioactive nuclei to decay. At $t = T_{1/2}$, $N(t) = \frac{N_0}{2}$. Substituting this into the decay law:

$\frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}}$

$\frac{1}{2} = e^{-\lambda T_{1/2}}$

Taking the natural logarithm of both sides:

$\ln\left(\frac{1}{2}\right) = -\lambda T_{1/2}$

$$-\ln(2) = -\lambda T_{1/2}$$

$$T_{1/2} = \frac{\ln(2)}{\lambda}$$

This equation shows the relationship between the half-life and the decay constant. A larger decay constant means a shorter half-life, indicating a faster decay rate.

1.4. Mean Lifetime (τ)

The mean lifetime (τ) or average lifetime of a radioactive nucleus is the average time an atom exists before it decays. It is the reciprocal of the decay constant:

$$\tau = \frac{1}{\lambda}$$

Substituting this into the half-life equation, we can also relate half-life to mean lifetime:

$$T_{1/2} = \tau \ln(2) \approx 0.693\tau$$

1.5. Activity (A)

Activity is the rate of decay of a radioactive sample, defined as the number of decays per unit time. It is directly proportional to the number of radioactive nuclei present:

$$A = \left|\frac{dN}{dt}\right| = \lambda N$$

Substituting the decay law for N:

$$A(t) = \lambda N_0 e^{-\lambda t}$$

Since $A_0 = \lambda N_0$ (the initial activity at $t = 0$):

$$A(t) = A_0 e^{-\lambda t}$$

This shows that the activity of a radioactive sample also decays exponentially with the same decay constant as the number of nuclei.

2. Simulation Code

2.1. Introduction

This section provides a Python-based simulation of radioactive decay, demonstrating the exponential decrease in the number of radioactive nuclei over time. The simulation uses the mathematical model derived in the previous section and visualizes the decay process, including the concept of half-life.

2.2. Code Description

The simulation code is structured as follows:

1. **Parameters:** We define the `half_life` of the substance, from which the `decay_constant` (lambda) is calculated using the relationship $\lambda = \frac{\ln(2)}{T_{1/2}}$. The `initial_nuclei` represents the starting number of radioactive atoms.

2. **Simulation Time:** A time array (`time`) is created using `numpy.linspace` to cover a period of several half-lives, allowing us to observe the decay over an extended duration.

3. **Nuclei Calculation:** The number of radioactive nuclei at each time point is calculated using the Radioactive Decay Law: $N(t) = N_0 e^{-\lambda t}$.

4. **Plotting:** `matplotlib.pyplot` is used to plot the number of nuclei as a function of time. The plot includes:

5. The exponential decay curve.
6. Dashed lines indicating the half-life points, showing how the number of nuclei halves at each successive half-life period.
7. Labels for the axes, a title for the plot, and a legend.

2.3. Python Code

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
half_life = 5.0  # Half-life of the radioactive substance (e.g., years)
decay_constant = np.log(2) / half_life  # Decay constant (lambda)
initial_nuclei = 1000  # Initial number of radioactive nuclei

# Simulation time
time_start = 0
time_end = half_life * 4  # Simulate for 4 half-lives
num_steps = 1000

time = np.linspace(time_start, time_end, num_steps)  # Time points for
simulation

# Calculate the number of nuclei at each time point using the decay law
nuclei = initial_nuclei * np.exp(-decay_constant * time)

# Plotting
plt.figure(figsize=(10, 6))
plt.plot(time, nuclei, label=f'N(t) = N₀ × e^(-λt) (Half-life = {half_life}
units)',
         linewidth=2, color='blue')

# Mark half-life points
for i in range(1, int(time_end / half_life) + 1):
    t_half = i * half_life
    n_half = initial_nuclei * (0.5)**i
    plt.plot([t_half, t_half], [0, n_half], 'k--', linewidth=0.8, alpha=0.7)
    plt.plot([0, t_half], [n_half, n_half], 'k--', linewidth=0.8, alpha=0.7)
    plt.text(t_half + 0.1, n_half, f'{i}T½', verticalalignment='bottom',
             fontsize=9, fontweight='bold')

plt.title('Radioactive Decay Simulation', fontsize=14, fontweight='bold')
plt.xlabel('Time (arbitrary units)', fontsize=12)
plt.ylabel('Number of Radioactive Nuclei', fontsize=12)
plt.grid(True, alpha=0.3)
plt.legend(fontsize=11)
plt.tight_layout()
plt.show()
```

3. Conclusion

Radioactive decay is a fundamental nuclear process characterized by an exponential decrease in the number of unstable nuclei over time. The mathematical model, $N(t) = N_0 e^{-\lambda t}$, accurately describes this phenomenon, with the decay constant λ dictating the rate of decay. The concept of half-life $(T_{1/2} = \frac{\ln(2)}{\lambda})$ provides a practical measure of how quickly a radioactive substance diminishes. The simulation code presented visually reinforces these theoretical concepts, illustrating the exponential decay curve and the consistent halving of nuclei at each half-life interval. Understanding radioactive decay is crucial for various scientific and technological applications, from nuclear power generation and medical diagnostics to archaeological dating.

4. References

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