Pg. 12

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(33) a) $\chi_A = \det\begin{pmatrix} x-1 & -4 \\ 2 & x-7 \end{pmatrix} = x^2-8x+15 = (x-3)(x-5) \Rightarrow D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$$\left.\begin{aligned} E_3 &= \text{Kern}(A-3I_2) = \text{Kern}\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle . \\ E_5 &= \text{Kern}(A-5I_2) = \text{Kern}\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle . \end{aligned}\right\} \Rightarrow S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

b) $\chi_A = \det\begin{pmatrix} x-1 & -1 \\ -1 & x \end{pmatrix} = x^2-x-1 \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{1-4(-1)}}{2}$, sei $\lambda_1 = \frac{1+\sqrt{5}}{2}$ und $\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\left.\begin{aligned} E_{\lambda_1} &= \text{Kern}\begin{pmatrix} \lambda_1-1 & -1 \\ -1 & \lambda_1 \end{pmatrix} = \left\langle \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \right\rangle \\ \text{Analog, } E_{\lambda_2} &= \left\langle \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \right\rangle \end{aligned}\right\} \Rightarrow S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \text{ und } D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(34) Ansatz: $\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \dots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Aus T33 b): $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$a_n = (0 \;\; 1) \cdot \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = (0 \;\; 1) \cdot A^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0 \;\; 1) \cdot S \cdot D^n \cdot S^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{(0 \;\; 1) \cdot \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}}_{(1 \;\; 1)} \cdot D^n \cdot \frac{1}{\lambda_1-\lambda_2} \cdot \underbrace{\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} =$$

$$= \frac{1}{\lambda_1-\lambda_2} (1 \;\; 1) \cdot \underbrace{\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\begin{pmatrix} \lambda_1^n \\ -\lambda_2^n \end{pmatrix}} = \frac{1}{\lambda_1-\lambda_2} (1 \;\; 1) \cdot \begin{pmatrix} \lambda_1^n \\ -\lambda_2^n \end{pmatrix} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1-\lambda_2} = a_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

(35) $W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\alpha = \frac{1}{2}$

1) Wir bilden $H = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

2) Keine Nullzeilen, also $S = H$

3) $G = (1-\alpha) \cdot S + \frac{\alpha}{3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, mit $\alpha = \frac{1}{2} \Rightarrow G = \frac{1}{12} \begin{pmatrix} 4 & 4 & 4 \\ 8 & 2 & 2 \\ 2 & 8 & 2 \end{pmatrix}$

$G^T \cdot p = p \Leftrightarrow \frac{1}{12} \cdot \begin{pmatrix} 4 & 8 & 2 \\ 4 & 2 & 8 \\ 4 & 2 & 2 \end{pmatrix} \cdot p = p \Leftrightarrow \begin{pmatrix} 4 & 8 & 2 \\ 4 & 2 & 8 \\ 4 & 2 & 2 \end{pmatrix} \cdot p = 12p$. Wir berechnen den Eigenraum zum Eigenwert 12 von $12G^T$.

$$\text{Kern}(12G^T - 12I_3) = \text{Kern}\begin{pmatrix} -8 & 8 & 2 \\ 4 & -10 & 8 \\ 4 & 2 & -10 \end{pmatrix} = \text{Kern}\begin{pmatrix} 4 & -10 & 8 \\ -8 & 8 & 2 \\ 4 & 2 & -10 \end{pmatrix} \begin{matrix} \\ (+2) \\ (-1) \end{matrix} = \text{Kern}\begin{pmatrix} 4 & -10 & 8 \\ 0 & -12 & 18 \\ 0 & 12 & -18 \end{pmatrix} \begin{matrix} \\ \\ (+1) \end{matrix} = \text{Kern}\begin{pmatrix} 4 & -10 & 8 \\ 0 & -12 & 18 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow p = \frac{1}{17} \cdot \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix}.$$