GAL - Curs 5 28. 03. 2024

Operații cu subspații vectoriale
Morfisme de subspații vectoriale

Teorema Grassman

$(V, +, \cdot)\big|_{\mathbb{K}}$ sp. vect. finit generat și $V_1, V_2 \subset V$ subspații vectoriale.

$\Rightarrow \dim(V_1+V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$,

unde $V_1 + V_2 = \langle V_1 \cup V_2 \rangle$

Dem

$\dim(V) = n$, $\dim V_j = n_j$, $j = \overline{1,2}$, $\dim(V_1 \cap V_2) = p$

$n_j < n$, $p < n$, $j = \overline{1,2}$

Fie $R_0 = \{e_1, \dots, e_p\}$ reper în $V_1 \cap V_2$

Extindem la $R_1 = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}\}$ reper în V_1

—— " —— $R_2 = \{e_1, \dots, e_p, g_{p+1}, \dots, g_{n_2}\}$ reper în V_2

Fie $R = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$.

Dem. că R este reper în $V_1 + V_2$.

(1) R este SLI

$\forall a_1, \dots, a_p, b_{p+1}, \dots, b_{n_1}, c_{p+1}, \dots, c_{n_2} \in \mathbb{K}$ a.î.

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0_V$$

$$\underbrace{\underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{- \sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2}}_{\in V_1 \cap V_2} = \sum_{i=1}^{p} a'_i e_i$$

$$\sum_{i=1}^{p} (a_i - a'_i) e_i + \sum_{j=p+1}^{n_1} b_j f_j = 0 \xRightarrow{R_1 \text{ e SLI}} a_i - a'_i = 0 \ \forall i = \overline{1,p}$$
$$b_j = 0, \forall j = \overline{p+1, n_1}$$

$$\sum_{i=1}^{p} a'_i e_i + \sum_{k=p+1}^{n_2} c_k g_k = 0 \xRightarrow{R_2 \text{ e SLI}} a'_i = 0 \ \forall i = \overline{1,p}$$
$$c_k = 0, \ \forall k = \overline{p+1, n_2}$$

$$\Rightarrow \left.\begin{array}{l} a_i = 0 \ \forall i = \overline{1,p} \\ b_j = 0, \ \forall j = \overline{p+1, n_1} \\ c_k = 0, \ \forall k = \overline{p+1, n_2} \end{array}\right\} \Rightarrow R \text{ e SLI}$$

2) R e SG i.e. $\langle R \rangle = V_1 + V_2$

$\forall x \in V_1 + V_2 \Rightarrow \exists \, x_1 \in V_1, x_2 \in V_2$ a.î.

$$x = x_1 + x_2 = \left(\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j\right) + \left(\sum_{i=1}^{p} a'_i e_i + \sum_{k=p+1}^{n_2} c_k g_k\right) =$$

$$= \sum_{i=1}^{p} (a_i + a'_i) e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k \in \langle R \rangle$$

„$\subseteq$" din construcție.

Deci $<R> = V_1 + V_2$

În concluzie $R = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$

reper în $V_1 + V_2$

$\dim(V_1 + V_2) = |R| = n_1 + n_2 - p = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

<u>OBS</u>: În particular

$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$.

$V_1 + V_2$ este sumă directă $\Longleftrightarrow$ $V_1 \cap V_2 = \{0_V\}$

<u>OBS</u> a) $V = V_1 \oplus V_2$, R_k reper în V_k, $k = \overline{1,2}$ $\Rightarrow$

$R = R_1 \cup R_2$ reper în V și $R = R_1 \cup R_2$ (partiție)

$V_k = <R_k>$, $k = \overline{1,2}$ $\Rightarrow V = V_1 \oplus V_2$

<u>Prop</u>:

$A \in \mathcal{M}_{m,n}(\mathbb{K})$

$S(A) = \{X \in \mathbb{K}^n \mid \underset{(m,n)}{A}\underset{(n,1)}{X} = \underset{(m,1)}{0}\} \subset \mathbb{K}^n$

(mult. sol. unui SLO) $\qquad x = \sum_{i=1}^{n} x_i e_i$

$R = \{e_1, \dots, e_n\}$ reper în $\mathbb{K}^n$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

a) $S(A) \subseteq \mathbb{K}^n$ subsp. vect.

b) $\dim S(A) = n - \text{rg}\, A$

Exercițiu: $(\mathbb{R}^3, +, \cdot)\big|_{\mathbb{R}}$

$$V' = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x - y + z = 0 \\ 2x + y - z = 0 \end{cases} \right\} = S(A)$$

a) $\dim V'$; b) Precizați un reper în V'

Sol

a) $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} & \begin{matrix} 1 \\ -1 \end{matrix} \end{pmatrix}$ $\qquad \dim V' = 3 - rg(A) = 3 - 2 = 1$

$x = 0$

$y = z$

$z = \alpha$

b) $\begin{cases} x - y = -z \\ 2x + y = z \end{cases}$ $+$

$3x = 0$

$V' = \{(0, \alpha, \alpha),\ \alpha \in \mathbb{R}\} = \langle \overbrace{\{(0, 1, 1)\}}^{R'} \rangle$

R' e SG, $\dim V' = 1 = |R'| \Rightarrow R'$ este reper în V'

Prop $(V, +, \cdot)\big|_{K}$ sp. vect., $V' \subset V$ subsp. vect.

Coord. vectorilor din V', în rap. cu $\forall$ reper, reprezintă soluțiile unui SLO i.e $\exists A \in \mathcal{M}_{m,n}(K)$ a.î. $V' = S(A)$

Ex: $(\mathbb{R}^4, +, \cdot)\big|_{\mathbb{R}}$, $V' = \langle \{ \overbrace{(1, 1, 0, 0)}^{u}, \overbrace{(1, 0, 1, -1)}^{v} \} \rangle$

a) Să se descrie V' printr-un sistem de ec. liniare (SLO)

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$ subsp. complementar lui V'

c) Să se descompună $x = (1, 1, 2, 1)$ în raport cu $\mathbb{R}^4 = V' \oplus V''$

SOL:

a) $\{u, v\}$ SLI

$$\mathrm{rg}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = 2 \xRightarrow[\text{LI}]{\text{crit}} \overset{R'}{\overset{\shortparallel}{\{u, v\}}} \text{ este SLI în } V'$$

R' este reper în V'

Fie $X \in V' \Rightarrow \exists\, a, b \in \mathbb{R}$ a.î. $X = au + bv$.

$$(x_1, x_2, x_3, x_4) = a(1,1,0,0) + b(1,0,1,-1) = (a+b, a, b, -b)$$

$$\begin{cases} a+b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4 \end{cases} \qquad \left(\begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{matrix}\right|\left.\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right)$$

$$\text{S.C.} \quad \begin{cases} \Delta_{c_1} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \\ \\ \Delta_{c_2} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & -1 & x_3 \end{vmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} -x_3 + x_1 - x_2 = 0 \\ -x_4 + x_2 - x_4 = 0 \end{cases}$$

$$\Rightarrow V' = X \in \mathbb{R}^4 \,\middle|\, \begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_4 = 0 \end{cases} \Big\}$$

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix}; \quad V' = S(A)$$

b) $\mathbb{R}^4 = V' \oplus V''$

$$\begin{array}{cccc} & & w & t \\ & & \shortparallel & \shortparallel \end{array}$$

$$\begin{pmatrix} 1 & 1 & \boxed{0} & \boxed{1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \neq 0$$

$R'' = \{w, t\}$ (este SLI) reper în V''

$V'' = \langle R'' \rangle$ (nu este unic)

c) $x = (1,1,2,1) = x' + x'' \in V' \oplus V''$
(scrierea este unică)

$R = R' \cup R'' = \{u, v, w, t\}$ reper în $\mathbb{R}^4$

$(1,1,2,1) = a(1,1,0,0) + b(1,0,1,-1) + c(0,0,0,1) + d(1,0,0,0)$

$= (a+b+d, a, b, -b+c)$

$a = 1$

$b = 2$

$-b + c = 1 \Rightarrow c = 3$

$a + b + d = 1 \Rightarrow d = \cancel{1} - \cancel{1} - 2 = -2$

$$(1,1,2,1) = \underbrace{1 \cdot (1,1,0,0) + 2(1,0,1,-1)}_{x' = (3,1,2,-2)} + \underbrace{3(0,0,0,1) - 2(1,0,0,0)}_{x'' = (-2,0,0,3)}$$

<u>Exercițiu</u>:

$(\mathbb{R}^4, +, \cdot)|_{\mathbb{R}}$, $V' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x + y - z - 3t = 0\}$

$V'' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x + y + z + 2t = 0\}$

Dem. $\mathbb{R}^4 = V' + V''$, dar suma nu e directă.

<u>Sol</u>:

$\dim V' = 4 - rg\, A' = 4 - 1 = 3$, $V' = S(A')$, $A' = (1\ \ 1\ \ -1\ \ -3)$

$\dim V' = 4 - rg\, A'' = 4 - 1 = 3$, $V'' = S(A'')$, $A'' = (1\ \ 1\ \ 1\ \ 2)$

$V' \cap V'' = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + y - z - 3t = 0 \\ x + y + z + 2t = 0 \end{cases} \right\} = S(A)$

$\dim(V' \cap V'') = 4 - rg(A) = 4 - 2 = 2$; $A = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

$\dim(V' \cup V'') = 3 + 3 - 2 = 4$

$\left.\begin{array}{l} V' + V'' \subset \mathbb{R}^4 \\ \dim(V' + V'') = \dim \mathbb{R}^4 = 4 \end{array}\right\} \Rightarrow V' + V'' = \mathbb{R}^4$

nu e $\oplus$ deoarece $V' \cap V'' \neq \{0_V\}$

<u>OBS</u>: $\left.\begin{array}{l} V' \subset V \text{ subsp. vect.} \\ \text{dacă } \dim V' = \dim V = n \end{array}\right\} \Rightarrow V' = V.$

<u>Morfisme de spații vectoriale</u> (Aplicații liniare)

<u>Def</u>: $(V_j, +, \cdot)\,\big|_{\mathbb{K}_j}$, $j = \overline{1,2}$ sp. vectoriale.

$f : V_1 \to V_2$ s.n. aplicație semi-liniară $\Leftrightarrow$

1) $f(x+y) = f(x) + f(y)$

2) $\exists\ \theta : \mathbb{K}_1 \to \mathbb{K}_2$ izomorfism de corpuri a.î.

$f(\alpha x) = \theta(\alpha) f(x)$, $\forall\, x \in V_1$, $y \in V_1$, $\alpha \in \mathbb{K}_1$.

Dacă $\underset{\mathbb{K}}{\mathbb{K}_1 = \mathbb{K}_2}$, $\theta : \mathbb{K} \to \mathbb{K}$, $\theta = id_{\mathbb{K}}$, at. f. s.n.

morfism de sp. vectoriale (aplicații liniare)

Obs: a) $(V_j, +, \cdot)|_{\mathbb{R}}$, $j=\overline{1,2}$ sp. vectoriale reale

$\theta : \mathbb{R} \to \mathbb{R}$ automorfism de corpuri $\Rightarrow \theta = id_{\mathbb{R}}$.

$\forall f : V_1 \to V_2$ apl. semi-liniară $\Rightarrow f$. apl. liniară.

b) $(\mathbb{C}^n, +, \cdot)_{\mathbb{C}}$ sp. vect.

$\theta : \mathbb{C} \to \mathbb{C}$ automorfism de corpuri, $\theta(z) = \bar{z}$

$f : \mathbb{C}^n \to \mathbb{C}^n$, $f(z_1, \dots, z_n) = (\overline{z_1}, \dots, \overline{z_n})$

f este apl. semi-liniară (nu este liniară).

$(V_j, +, \cdot)|_{K}$, $f : V_1 \to V_2$ apl. liniară $\Leftrightarrow$ 1) $f(x+y) = f(x) + f(y)$

2) $f(\alpha x) = \alpha f(x)$

$\forall x, y \in V_1$, $\alpha \in K$

Dacă în plus, f bijectivă, atunci (at.) f s.n. izomorfism de sp. vect.

Dacă $V_1 = V_2$, at. $f =$ automorfism de sp. vect.

$End(V) = \{ f : V \to V \mid f \text{ liniară} \}$

$Aut(V) = \{ f \in End(V) \mid f \text{ bij} \}$

OBS: a)
$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$$

f, g. apl. lini. $\Rightarrow h$ apl. lin.
$h = g \circ f$

b) $f : V_1 \to V_2$ apl. lin. $\Rightarrow f : (V_1, +) \to (V_2, +)$ morfism de grupuri și $f(0_{V_1}) = 0_{V_2}$

Exemple de apl. liniare:

1) $f: V \to V$, $f(x) = 0_V$, $f(x) = x$ apl. lin.

2) $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = Y$, $Y = AX$ apl. lin.

3) $f: \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $f(A) = Tr(A)$ apl. liniară

$$Tr(A+B) = Tr(A) + Tr(B)$$
$$Tr(\alpha A) = \alpha Tr(A)$$

$f(A) = \det(A)$ <u>NU</u> este apl. liniară.

Prop. de caracterizare a apl. liniare

$f: V_1 \to V_2$ apl. liniară $\Leftrightarrow$

$f(ax + by) = a f(x) + b f(y)$, $\forall x, y \in V_1$, $\forall a, b \in \mathbb{K}$

$\Rightarrow f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i)$, $\forall x_1, \dots, x_n \in V_1$
$\forall a_1, \dots, a_n \in \mathbb{K}$

Dem

„$\Rightarrow$" Ip: f liniară.

$x \in V_1$, $a \in \mathbb{K}$ $\Rightarrow a \cdot x \in V_1$

$y \in V_1$, $b \in \mathbb{K}$ $\Rightarrow b \cdot y \in V_1$

$f(ax + by) \overset{1)}{=} f(ax) + f(by) \overset{2)}{\Longrightarrow}$

$\Rightarrow a f(x) + b \cdot f(y)$

„$\Leftarrow$" Ip: $f(ax + by) = a f(x) + b f(y)$, $\forall x, y \in V_1$
$\forall a, b \in \mathbb{K}$

Fie $a, b = 1_{\mathbb{K}}$ $\quad f(1_{\mathbb{K}} \cdot x + 1_{\mathbb{K}} \cdot y) = f(x+y) = f(x) + f(y)$

Fie $b = 0_{\mathbb{K}}$ $\quad f(a\cdot x + 0_{\mathbb{K}}\cdot y) = f(a\cdot x) = a f(x)$

! <u>OBS</u> : $f : V_1 \to V_2$ liniară

Dacă $V' \subset V_1$ subsp. vect. $\Rightarrow f(V') \subset V_2$ subsp. vect.

<u>Dem</u> : Fie $y_1, y_2 \in f(V')$ $\Rightarrow a y_1 + b y_2 \in f(V')$

$a, b \in \mathbb{K}$

$\exists\, x_1, x_2 \in V'$ a.î. $y_1 = f(x_1)$, $y_2 = f(x_2)$

$$a y_1 + b y_2 = a f(x_1) + b f(x_2) = f(a x_1) + f(b x_2) = f(\underbrace{a x_1 + b x_2}_{x \in V'})$$

<u>Def</u> : $f : V_1 \to V_2$ apl. liniară

$\mathrm{Ker}(f) = \{x \in V_1 \mid f(x) = 0_{V_2}\}$ nucleul lui f

$\mathrm{Im}(f) = y \in V_2 \mid \exists x \in V_1$ a.î. $f(x) = y\}$ imaginea lui f

<u>Prop</u> :

$f : V_1 \to V_2$ apl. lin.

a) $\mathrm{Ker}(f) \subset V_1$, $\mathrm{Im}(f) \subset V_2$ subsp. vect.

b) f inj. $\Leftrightarrow$ $\mathrm{Ker}\, f = \{0_{V_1}\}$

c) f surj $\Leftrightarrow$ $\mathrm{Im}\, f = \dim V_2$

<u>Dem</u> :

a) $\mathrm{Im}\, f = f(V_1) \subset V_2$ subsp. (din OBS)

$\mathrm{Ker}\, f \subset V_1$ subsp. vect.

Fie $x_1, x_2 \in \mathrm{Ker}\, f \Rightarrow f(x_1) = f(x_2) = 0_{V_2}$

Fie $a, b \in \mathbb{K}$

$f(ax_1 + bx_2) = a \underbrace{f(x_1)}_{0_{V_2}} + b \underbrace{f(x_2)}_{0_{V_2}} = 0_{V_2} \Rightarrow ax_1 + bx_2 \in \operatorname{Ker} f$

b) f inj. $\overset{?}{\Rightarrow} \operatorname{Ker} f = \{0_{V_1}\}$

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{V_2}$

dar $f(0_{V_1}) = 0_{V_2}$

$\Rightarrow f(x) = f(0_{V_1}) \overset{inj.}{\Rightarrow} x = 0_{V_1}$

$\operatorname{Ker} f = \{0_{V_1}\} \overset{?}{\Rightarrow} f$ inj.

Fie $x_1, x_2 \in V_1$, $f(x_1) = f(x_2) \Rightarrow f(x_1 - x_2) = 0_{V_2}$

$x_1 - x_2 \in \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow x_1 = x_2$

c) f surj. $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$

„$\Rightarrow$" f surj. $\Rightarrow \operatorname{Im} f = V_2 \Rightarrow \dim \operatorname{Im} f = \dim V_2$

„$\Leftarrow$" $\dim \operatorname{Im} f = \dim V_2$

dar $\operatorname{Im} f \subseteq V_2$ subsp. vect.

$\xrightarrow{OBS} \operatorname{Im} f = V_2 \Rightarrow f$ surj.