Seminar 1 ~~S2~~ S1.151. GA. 2024

1. a) $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$, $\det A = ?$

b) $\det A = 0 \Leftrightarrow a+b+c = 0$ sau $a = b = c$

$$\det A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \xlongequal{L_1' = L_1 + L_2} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} =$$

$$= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \xlongequal[C_3' = C_3 - C_1]{C_2' = C_2 - C_1} (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} =$$

$$= (a+b+c) \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} = (a+b+c) \cdot \left((b-a)^2 - (a-b)(a-c)\right) =$$

$$= -(a+b+c) \cdot (a^2 + b^2 + c^2 - ab - ac - bc)$$

$$= -(a+b+c) \cdot \frac{1}{2} \cdot \left((a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + (b^2 - 2bc + c^2)\right)$$

$$= -\frac{1}{2}(a+b+c) \cdot \left((a-b)^2 + (a-c)^2 + (b-c)^2\right)$$

$\square$

b) $\det A = 0 \Rightarrow a+b+c = 0$ sau $(a-b)^2 + (a-c)^2 + (b-c)^2 = 0 \Leftrightarrow$

$\Leftrightarrow a+b+c = 0$ sau $a = b = c$

$\square$

2. $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$$\det \cancel{(V(a,b)} V(a,b,c)) = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & \cancel{b^2}\ b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$$

$\square$

3. — // —

Vezi poza atașată!

4. $A=\begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

a) determinați m a.î. $A^{-1} \in \cancel{\mathcal{M}_3(\mathbb{R})}\ \mathcal{M}_3(\mathbb{Z})$

Sol:

$A, A^{-1} \in \mathcal{M}_3(\mathbb{R}) \Rightarrow \det A, \det A^{-1} \in \mathbb{Z} \Rightarrow (\det A) \mid 1 \Rightarrow$
$\Rightarrow \det A \in \{-1, 1\}$

$$\det A = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} \underset{C_2' = C_2 - C_1}{=\!=\!=} \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} =$$

$$= (-1) \cdot (-1)^4 \cdot \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} = (-1) \cdot \left(-3 - (3m+4)(m-1)\right)$$

$$= 3 + 3m^2 - 3m + 4m - 4$$

$$= 3m^2 + m - 1$$

I. $\det A = 1 \Leftrightarrow 3m^2 + m - 1 = 1$

$\Leftrightarrow 3m^2 + m - 2 = 0$

$\Leftrightarrow 3m^2 + 3m - 2m - 2 = 0$

$\Leftrightarrow 3m(m+1) - 2(m+1) = 0$

$\Leftrightarrow (m+1)(3m-2) = 0$

$\Leftrightarrow m \in \{-1, \frac{2}{3}\} \cap \mathbb{Z}$

$\Leftrightarrow m = \underline{-1}$

II. $\det A = -1 \Leftrightarrow 3m^2 + m = 0$

$\Leftrightarrow m(3m+1) = 0$

$\Leftrightarrow m = \underline{0}$

$m \in \{-1, 0\}$.

$\square$

Matrice Determinant, Rang Forma eșalon

3. $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det A^* = ?$

$$\begin{array}{cccccc} 1 & 1' & 2 & 2' & 3 & 3' \end{array}$$

$$\det A = \begin{vmatrix} 1+a^2 & 0+ab & 0+ca \\ 0+ab & 1+b^2 & 0+bc \\ 0+ac & 0+bc & 1+c^2 \end{vmatrix} =$$

$= |1\ 2\ 3| + |1\ 2\ 3'| + |1\ 2'\ 3| + \cancel{|1\ 2'\ 3'|} + |1'\ 2\ 3| + \cancel{|1'\ 2\ 3'|} + \cancel{|1'\ 2'\ 3|} + \cancel{|1'\ 2'\ 3'|}$

$\cancel{|1\ 2\ 3|}$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & bc \\ 0 & 0 & c^2 \end{vmatrix} + \begin{vmatrix} 1 & ab & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} + \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} =$$

$$= 1 + c^2 + b^2 + a^2$$

$$\det A^* = (\det A)^{n-1} \underset{n=3}{=} (\det A)^2 = (a^2+b^2+c^2+1)^2$$

b) $m=0$, $A^{-1}=?$

Sol:

• $m=0 \Rightarrow A=\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}$

$\det A = -1$

$A^t=\begin{pmatrix}2 & 1 & -1\\ -1 & 0 & -1\\ 4 & 1 & 0\end{pmatrix}$

$\Gamma_{11}=(-1)^{1+1}\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}=1$

$\Gamma_{12}=-1\cdot\begin{vmatrix}-1 & -1\\ 4 & 0\end{vmatrix}=-4$

$\Gamma_{13}=1\cdot\begin{vmatrix}-1 & 0\\ 4 & 1\end{vmatrix}=-1$

$\Gamma_{21}=-1\cdot\begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix}=-1$

$\Gamma_{22}=1\cdot\begin{vmatrix}2 & -1\\ 4 & 0\end{vmatrix}=4$

$\Gamma_{23}=-1\cdot\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix}=2$

$\Gamma_{31}=1\cdot\begin{vmatrix}1 & -1\\ 0 & -1\end{vmatrix}=-1$

$\Gamma_{32}=-1\cdot\begin{vmatrix}2 & -1\\ -1 & -1\end{vmatrix}=3$

$\Gamma_{33}=1\cdot\begin{vmatrix}2 & 1\\ -1 & 0\end{vmatrix}=1$

$A^*=\begin{pmatrix}1 & -4 & -1\\ -1 & 4 & 2\\ -1 & 3 & 1\end{pmatrix}$

$A^{-1}=\begin{pmatrix}-1 & 4 & 1\\ 1 & -4 & -2\\ 1 & -3 & -1\end{pmatrix}$

• Gauss-Jordan:

$(A\,|\,I_3)\sim(I_3\,|\,A^{-1})$

$\left(\begin{array}{ccc|ccc}2 & -1 & 4 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0\\ -1 & -1 & 0 & 0 & 0 & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}\boxed{1} & 0 & 1 & 0 & 1 & 0\\ 2 & -1 & 4 & 1 & 0 & 0\\ -2 & -1 & 0 & 0 & 0 & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}\boxed{1} & 0 & 1 & 0 & 1 & 0\\ 0 & \boxed{-1} & 2 & 1 & -2 & 0\\ 0 & -1 & 1 & 0 & 1 & 1\end{array}\right)$

$L_2'=L_2-2L_1$

$L_3'=L_3+L_1$

$$\underset{L_3'=L_3-L_2}{\sim}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & -2 & 0 \\ 1 & 0 & -1 & -1 & 3 & 1 \end{array}\right) \underset{\substack{L_2'=L_2\cdot(-1) \\ L_3'=L_3\cdot(-1)}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 1 & -3 & -1 \end{array}\right)$$

$$\underset{\substack{L_1'=L_1-L_3 \\ L_2'=L_2+2L_3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 1 \\ 0 & 1 & 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc} -1 & 4 & 1 \\ 1 & -4 & -2 \\ 1 & -3 & -1 \end{array}\right) = A^{-1}$$

$\square$

5. $A \in \mathcal{M}_2(\mathbb{C})$

dacă $\exists k \in \mathbb{N}$, $k \geq 2$ a.î. $A^k = O_2$, at. $A^2 = O_2$

a) $A^2 = \alpha \cdot A$
$A^n = \alpha^{n-1} \cdot A$
b) $tr(\alpha A) = \alpha \cdot trA$

$A^k = O_2 \quad | \det$

$\det(A^k) = \det(O_2)$

$(\det A)^k = 0$

$\det A = 0$

$A^2 - trA \cdot A + \det A \cdot I_2 = O_2$

$A^2 = \cancel{trA}\ trA \cdot A$ (*)

$\underset{O_2}{A^k} = \underbrace{(trA)^{k-1}}_{\alpha} \cdot A \quad | \ Tr \Rightarrow$

$\Rightarrow 0 = (trA)^k$

$trA = 0 \Rightarrow A^2 = O_2$

$\square$

11. $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b = ?$ a.î. $rg A = 2$

Sol:

$\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \neq 0 \Rightarrow rg A \geq 2$

dar $rg A \leq \min(3, 4) = 3$

Fie $\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix}$. Vrem $\Delta_1 = 0$; $\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix}$. Vrem $\Delta_2 = 0$.

$$\Delta_1 = 6 + \cancel{4a} - a - 12 = \overset{-6-a}{\cancel{3(a-2)}}$$
$$\Delta_2 = 2 - 1 - 4b = 1 - 4b$$
$$\Rightarrow a = -6,\ b = \frac{1}{4}.$$

13. $$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$$

$$\parallel$$

$$\begin{vmatrix} a^3-1 & 3a^2-3 & 3a-3 & 0 \\ a^2-1 & a^2+2a-3 & 2a-2 & 0 \\ a-1 & 2a-2 & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

$$(a-1)^3 = \begin{vmatrix} a^2+a+1 & 3(a+1) & 3 & 0 \\ a & a+1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

$$(a-1)^3 = \begin{vmatrix} a^2-1 & 2(a+1) & 2 & 0 \\ a-1 & a-1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{vmatrix}$$

$$(a-1)^3 = \begin{vmatrix} a^2+a+2 & 3(a-1) & 2 & 0 \\ a-1 & a-1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{vmatrix}$$

$$(a-1)^5 = \underbrace{\begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix}}_{a+1}$$

Exe.

Prop: $(\mathbb{Z}_n, +, \cdot)$ este un inel

$A \in \mathcal{M}_{m,m}(\mathbb{Z}_n)$ inel $\Leftrightarrow \det A \in U(\mathbb{Z}_n)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\{\hat{a} \in \mathbb{Z}_n \mid (a,n)=1\}}$

$A = \begin{pmatrix} \hat{2} & \hat{2} \\ \hat{1} & \hat{3} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_6)$ inversabilă ?

$\det A = \widehat{2\cdot 3 - 2\cdot 1} = -\hat{2} = \hat{4} \in U(\mathbb{Z}_6) \Rightarrow U(\mathbb{Z}_6) = \{1, 5\}$. $\square$

$(\mathbb{Z}_p, +, \cdot)$ este un corp, p = prim. $U(\mathbb{Z}_p) = \mathbb{Z}_p^*$

$A \in \mathcal{M}_{m,m}(\mathbb{Z}_p)$ inversabilă $\Leftrightarrow \det A \neq \hat{0}$.

$A = \begin{pmatrix} \hat{1} & \hat{2} \\ \hat{3} & \hat{4} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_5)$

$\det A = \widehat{4\cdot 1 - 3\cdot 2} = -\hat{2} = \hat{3}$

$(\hat{3})^{-1} = \hat{2} \Rightarrow A^{-1} = \hat{2} \cdot \begin{pmatrix} \hat{4} & -\hat{2} \\ -\hat{3} & \hat{1} \end{pmatrix} = \hat{2} \cdot \begin{pmatrix} \hat{4} & \hat{3} \\ \hat{2} & \hat{1} \end{pmatrix} = \begin{pmatrix} \hat{8} & \hat{6} \\ \hat{4} & \hat{2} \end{pmatrix} =$

$= \begin{pmatrix} \hat{3} & \hat{1} \\ \hat{4} & \hat{2} \end{pmatrix} \Rightarrow U\mathbb{Z}_6 = \{1, 5\}$ $\square$

8. $A \in \mathcal{M}_n(\mathbb{R})$

$A = A^2 = O_n$

$I_n - A, I_n + A$ inv.

$I_n = I_n^2 - A^2 = (I_n - A)(I_n + A) = (I_n + A)(I_n - A) \Rightarrow (I_n - A)^{-1} = I_n + A$

$(I_n + A)^{-1} = I_n - A$

$A^3 = O_n$

$I_n = I_n^3 - A^3 = (I_n - A)(I_n + A + A^2) = (I_n + A + A^2)(I_n - A) \Rightarrow$

$\Rightarrow (I_n - A)^{-1} = I_n + A + A^2$

Analog $(I_n + A)^{-1} = I_n - A + A^2$. $\square$