GAL - Curs 4 14. 03. 2024

Spații vectoriale. Repere. Schimbare de coordonate. Operații cu subspații

$(V,+,\cdot)\big|_{\mathbb{K}}$ sp. vect.

$S=\{v_1, v_2, \dots, v_n\}$ s.n.

1) SLI $\Leftrightarrow \forall a_1, \dots, a_n \in \mathbb{K}$: $a_1x_1+\dots+a_nx_n=0_V \Rightarrow a_i=0\ \forall i=\overline{1,n}$
$\forall x_1, \dots, x_n \in S$

2) SLD $\Leftrightarrow \exists x_1, \dots, x_n \in S$ a.î. $a_1x_1+\dots+a_nx_n \neq 0_V$
$\exists a_1, \dots, a_n \in \mathbb{K}$, nu toți nuli

3) SG $\Leftrightarrow \forall x \in V, \exists a_1, \dots, a_n \in \mathbb{K}$ a.î. $x=a_1x_1+\dots+a_nx_n$
$V=\langle S\rangle$

V s.n. finit generat $\Leftrightarrow \exists$ S finit SG i.e. $\langle S\rangle=V$

4) bază $\Leftrightarrow \begin{cases} 1) \text{ SLI} \\ 2) \text{ SG} \end{cases}$

Teorema schimbului

$(V,+,\cdot)\big|_{\mathbb{K}}$ sp. vect, f. generat

$\{x_1, \dots, x_n\}$ SG
$\{y_1, \dots, y_m\}$ SLI
$\Rightarrow \{y_1, \dots, y_m\}$ SG

Dem

$V=\langle\{x_1, \dots, x_n\}\rangle \Rightarrow \exists a_1, \dots, a_n \in \mathbb{K}$ a.î. $y_1=a_1x_1+\dots+a_nx_n$
Ψ
y_1

Pp. abs. $a_1 = \dots = a_n = 0_K \Rightarrow y_1 = 0_V$

$\{0_V, y_2, \dots, y_n\}$ SLI (din ip)

$a_1 \cdot 0_V + 0 \cdot y_2 + \dots + 0 y_n = 0_V \Rightarrow$ SLD Contradictoriu

$a_1 \neq 0_K$

Pp. e falsă. Fie $a_1 \neq 0_K$

Tema 1: 27
Tema 2: După

$y_1 = a_1 x_1 + \dots + a_n x_n \,/\cdot a_1^{-1} \Rightarrow x_1 = a_1^{-1} y_1 - a_1^{-1} a_2 x_2 - \dots a_1^{-1} a_n x_n$

$x_1 \in <\{y_1, x_2, \dots, x_n\}> \Rightarrow <\{x_1, \dots, x_n\}> = <\{y_1, x_2, \dots, x_n\}>$
$= V$

$y_2 \in V \Rightarrow \exists\, k_1, a_2, \dots, a_n \in K$ a.î. $y_2 = k_1 y_1 + a_2 x_2 + \dots + a_n x_n$

Pp. abs. $a_2 = \dots = a_n = 0_K \Rightarrow y_2 = k_1 y_1 \Rightarrow$

$k_1 y_1 - \underset{\neq 0_K}{1_K} \cdot y_2 + 0 \cdot y_3 + \dots + 0 \cdot y_n = 0_V \Rightarrow \{y_1, \dots, y_n\}$ SLD $\not\downarrow$

Pp. e falsă. Fie $a_2 \neq 0_K$ (altfel renumerotăm indicii)

$x_2 = a_2^{-1}(y_2 - k_1 y_1 - a_3 x_3 - \dots - a_n x_n)$

$x_2 \in <\{y_1, y_2, x_3, \dots, x_n\}> \Rightarrow <\{x_1, \dots, x_n\}> = <\{y_1, y_2, x_3, \dots, x_n\}>$

Analog, după un nr. finit de pași $\Rightarrow$

$V = <\{x_1, \dots, x_n\}> = <\{y_1, \dots y_n\}> \Rightarrow \{y_1, \dots, y_n\}$ SG

Prop.
card. $\forall$ SG (finit) $\geq$ card $\forall$ SLI

Dem
Fie $\{x_1, \dots, x_n\}$ SG și
$\{y_1, \dots, y_n, y_{n+1}\}$

Dem. că $\{y_1, \dots, y_n\}$ e SLD

1) $\{y_1, \dots, y_n\}$ SLI $\xrightarrow{\text{Th. Schimbului}}$ $\{y_1, \dots, y_n\}$ SG

$V = \langle \{y_1, \dots, y_n\} \rangle \Rightarrow \exists a_1, \dots a_n \in \mathbb{K}$ a.î.

Ψ

y_{n+1}

$y_{n+1} = a_1 y_1 + \dots + a_n y_n$

$a_1 y_1 + \dots + a_n y_n - 1_{\mathbb{K}} y_{n+1} = 0_V$

$1_{\mathbb{K}} \neq 0_{\mathbb{K}}$

$\Rightarrow \{y_1, \dots, y_n, y_{n+1}\}$ SLD.

2) $\{y_1, \dots, y_n\}$ SLD $\Rightarrow \{y_1, \dots, y_n\} \cup \{y_{n+1}\}$ (o supramulțime) SLD

Teoremă

$(V, +, \cdot)|_{\mathbb{K}}$ sp. vect., finit generat

$\Rightarrow \forall B_1, B_2$ baze: card B_1 = card $B_2 = n = \dim_{\mathbb{K}} V$ (dimensiunea peste $\mathbb{K}$ a lui V)

$|B_1| = |B_2|$

① B_1 SG, B_2 SLI $\Rightarrow |B_1| \geq |B_2|$

② B_2 SG, B_1 SLI $\Rightarrow |B_2| \geq |B_1|$

$\Rightarrow |B_1| = |B_2| = n$ (este un invariant)

Def $(V, +, \cdot)|_{\mathbb{K}}$ sp. vect, $B = \{e_1, \dots, e_n\}$ bază

B o.n. reper dacă este o bază ordonată

Prop

$(V, +, \cdot)|_{\mathbb{K}}$ sp. vect., $R = \{e_1, \dots, e_n\}$ reper

$\forall x \in V, \exists! (x_1, \dots, x_n) \in \mathbb{K}^n$

coordonatele lui x în raport cu reperul R (componentele)

Dem

$V = \langle R \rangle \Rightarrow \exists\, x_1, \dots, x_n \in \mathbb{K}$ a.î. $X = \underline{x_1 e_1 + \dots + x_n e_n}$ (1)

Pp. abs. $\exists\, x_1', \dots, x_n' \in \mathbb{K}$ a.î.

$$X = \underline{x_1' e_1 + \dots + x_n' e_n} \quad (2)$$

Din (1) (2) $\Rightarrow (x_1 - x_1') e_1 + \dots + (x_n - x_n') e_n = 0_V \overset{R\ e\ SLI}{\Longrightarrow}$

$$\Rightarrow \begin{matrix} x_1 - x_1' = 0 \\ \vdots \\ x_n - x_n' = 0 \end{matrix} \quad \Rightarrow x_k = x_k' \ \forall k = \overline{1,n}$$

Modificarea coordonatelor la schimbarea de reper

$(V, +, \cdot)|_{\mathbb{K}}$ sp. vect., finit generat

$R = \{e_1, \dots, e_n\} \xrightarrow{A = (a_{ij})_{i,j=\overline{1,n}}} R' = \{e_1', \dots, e_n'\}$ repere în V

$$e_i' = \sum_{j=1}^{n} a_{ji}\, e_j \quad \forall i = \overline{1,n}$$

$$\forall x = \sum_{j=1}^{n} x_j e_j = \sum_{i=1}^{n} x_i' e_i' = \sum_{i=1}^{n} x_i' \left(\sum_{j=1}^{n} a_{ji} e_j \right) =$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} x_i' \right) \cdot e_j$$

$$\Rightarrow x_j = \sum_{i=1}^{n} a_{ji} x_i', \ \forall j = \overline{1,n}$$

$$X = AX' \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad X' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}, \quad A = (a_{ij})_{i,j=\overline{1,n}}$$

Prop. $(V,+,\cdot)$ sp. vect finit generat

$$\begin{tikzcd} R=\{e_1,\dots,e_n\} \arrow[r, "A"] \arrow[rr, bend right, "C"'] & R'=\{e_1',\dots,e_n'\} \arrow[r, "B"] & R''=\{e_1'',\dots,e_n''\} \end{tikzcd}$$

Atunci $C=AB$

Dem

$$e_i''=\sum_{k=1}^{n} c_{ki}e_k,\ \forall i=\overline{1,n}$$

$$e_i''=\sum_{j=1}^{n} b_{ji}e_j' = \sum_{j=1}^{n} b_{ji}\left(\sum_{k=1}^{n} a_{kj}e_k\right)=\sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{kj}b_{ji}\right)e_k$$

$$c_{ki}=\sum_{j=1}^{n} a_{kj}b_{ji} \Rightarrow C=AB$$

Prop $(V,+,\cdot)\big|_{\mathbb{K}}$ sp. v., f. gen.

$R=\{e_1,\dots,e_n\} \xrightarrow{A} R'=\{e_1',\dots,e_n'\}$ repere

$\Rightarrow A\in GL(n,\mathbb{K})$

(matricea de trecere de la R la R' este inversabilă)

Dem

$$\begin{tikzcd} R \arrow[r, "A"] \arrow[rr, bend right, "I_n"'] & R' \arrow[r, "B"] & R \end{tikzcd} \qquad AB=I_n$$

$$\Rightarrow B=A^{-1}\Rightarrow A\in GL(n,\mathbb{K})$$

$$\begin{tikzcd} R' \arrow[r, "B"] \arrow[rr, bend right, "I_n"'] & R \arrow[r, "A"] & R' \end{tikzcd} \qquad BA=I_n$$

Def R și R' s.n. repere $\begin{cases}\text{la fel orientate} \iff \det A>0\\ \text{opus orientate} \iff \det A<0\end{cases}$

<u>Obs</u> Relația „a fi coorientate" este o relație de echivalență

a) Reflexivă $R \xrightarrow{I_n} R$ $\quad \det(I_n) > 0$

b) Simetrică $R \xrightarrow{A} R'$ $\quad \det(A) > 0 \Rightarrow \det(A^{-1}) > 0$

$R' \xrightarrow{A} R$ $\quad \det(A^{-1}) = \dfrac{1}{\det A}$

c) Tranzitivă

$$\begin{tikzcd} R \arrow[r, "A"] \arrow[rr, bend right, "C"'] & R' \arrow[r, "B"] & R'' \end{tikzcd}$$

$\det A > 0$, $\det B > 0$ $\Rightarrow \det C = \det(A \cdot B) = \det A \cdot \det B > 0$

Pe mulțimea reperelor se consideră 2 clase de echivalență.

A alege o orientare = a preciza un reper pozitiv orientat.

<u>Criterii de LI</u>

Fie $(V, +, \cdot)\big|_{\mathbb{K}}$ sp.v., f. gen, $\dim_{\mathbb{K}} V = n$.

$S = \{v_1, \dots, v_m\} \subset V$ sistem de vectori $m \leq n$

$\Rightarrow S$ este SLI $\Leftrightarrow$ matricea componentelor vectorilor din S, în raport cu $\forall$ reper, are rangul maxim $= m$

<u>Dem</u>: Fie $R = \{e_1, \dots, e_n\}$ reper în V

$$v_i = \sum_{j=1}^{n} v_{ji} e_j \quad \forall i = \overline{1,m}$$

$$S \text{ e SLI} \Leftrightarrow \left[\forall a_1, \dots, a_m \in \mathbb{K} : \sum_{i=1}^{m} a_i v_i \Rightarrow a_1 = \dots = a_m = 0_{\mathbb{K}}\right]$$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_i v_{ji} e_j\right)$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_i v_{ji}\right) e_j = 0_V \xrightarrow[R \text{ e SLI}]{} \sum_{i=1}^{m} v_{ji}\, a_i = 0, \ \forall j = \overline{1,n}$$

(*) SLO de n ec. cu m $(a_1, \dots, a_m)$ necunoscute

(*) are numai sol. nulă $\Leftrightarrow$ rg $C = m$ = maxim

$$C = (v_{ji})_{\substack{j=\overline{1,n} \\ i=\overline{1,m}}}$$

Obs: $R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e_1', \dots, e_n'\}$

$$v_i = \sum_{k=1}^{n} v_{ki}' e_k' = \sum_{k=1}^{n} v_{ki}' \sum_{j=1}^{n} a_{jk} e_j =$$

$$\left.\begin{array}{l} = v_i = \displaystyle\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{jk} v_{ki}'\right)\cdot e_j \\ \text{dar } v_i = \displaystyle\sum_{j=1}^{n} v_{ji} e_j \end{array}\right\} \Rightarrow v_{ji} = \sum_{k=1}^{n} a_{jk} v_{ki}'$$

$$C = AC' \qquad C' = (v_{ki}')_{\substack{k=\overline{1,n} \\ i=\overline{1,m}}}$$

$$\left.\begin{array}{l} \text{rg } C = \text{rg}(AC') \\ A \in GL(n, \mathbb{K}) \end{array}\right\} \Rightarrow \text{rg } C = \text{rg } C' = m = \text{maxim}$$

Aplicație

$(\mathbb{R}^2, +, \cdot)|_{\mathbb{R}}$, $R_0 = \{e_1 = (1,0), e_2(0,1)\}$ reperul canonic (poz. orientat)
convenție.

Fie $R' = \{e_1' = (2,1), e_2' = (3,0)\}$

a) R' este reper în $\mathbb{R}^2$

b) $R_0 \xrightarrow{A} R'$, $R' \xrightarrow{B} R_0$, $A, B = ?$

R_0, R' sunt la fel orientate?

c) Fie $X = (1,2)$. Să se afle coordonatele în raport cu R'.

SOL

$e_1' = (2,1) = 2 \cdot e_1 + 1 \cdot e_2$

$e_1' = a_{11} e_1 + a_{21} \cdot e_2$

$A = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} \end{pmatrix}$

$e_2' = (3,0) = 3 \cdot e_1 + 0 \cdot e_2$

$rg\, A = 2 =$ maxim $\Rightarrow R'$ este SLI

dar $|R'| = 2 = \dim_{\mathbb{R}} \mathbb{R}^2$ $\Bigg| \Rightarrow R'$ reper

b) $R_0 \xrightarrow{A} R'$ $\quad R' \xrightarrow{A'} R_0$

$\det A = -3 < 0$ $\quad$ sunt opus orientate R', R_0

c) $X = (1,2) = 1 \cdot e_1 + 2 \cdot e_2$ $\quad (1,2)$ coord. în rap. cu R_0

$X = x_1' \cdot e_1' + x_2' \cdot e_2' = x_1' \cdot (2,1) + x_2' \cdot (3,0) = (2x_1' + 3x_2'; x_1') =$

$= \begin{cases} 2x_1' + 3x_2' = 1 \\ x_1' = 2 \end{cases} \Rightarrow \begin{cases} 4 + 3x_2' = 1 \\ x_1' = 2 \end{cases} \Rightarrow \begin{cases} x_1' = 2 \\ x_2' = -1 \end{cases} \Rightarrow (2,1)$ coord. lui X în rap cu reperul R'

SAU $X = AX'$, $X' = A^{-1} X$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Operații cu subspații vectoriale

$(V, +, \cdot)/_{K}$ sp. v. f. gen., $V' \subset V$ subm. nevidă

$V' \subset V$ subsp. vec. $\Leftrightarrow \forall x_1, \dots, x_n \in V'$, $a_1, \dots, a_n \in K$: $a_1 x_1 + \dots + a_n x_n \in V'$

Prop. $V_1, V_2 \subset V$ subsp. vect. $\Rightarrow V_1 \cap V_2$ subsp. vect.

Dem $\forall x, y \in V_1 \cap V_2$ $\quad x, y \in V_1$ şi $x, y \in V_2$

$\forall a, b \in \mathbb{K}$ $\quad \Downarrow \qquad \Downarrow$

$ax+by \in V_1$ şi $ax+by \in V_2$

$\Rightarrow ax+by \in V_1 \cap V_2$

Obs În general, $V_1 \cup V_2$ nu e sosp vect.

Considerăm $\underset{\text{not } V_1+V_2}{\overset{}{\langle V_1 \cup V_2 \rangle}} = \left\{ \sum_{i=1}^{n} a_i x_i, \; x_i \in V_1 \cup V_2, \; i = \overline{1,n}, \; a_i \in \mathbb{K} \right\}$

Prop $\underset{\langle V_1 \cup V_2 \rangle}{V_1+V_2} = \{ v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2 \}$

Dem: „$\subseteq$" Fie $v \in V_1 + V_2 = \langle V_1 \cup V_2 \rangle$

$$v = \sum_{i=1}^{n} a_i v_i, \quad v_i \in V_1 \cup V_2, \; i = \overline{1,n}$$

Renumerotăm indicii a.î. $v_1, \dots, v_k \in V_1$

$v_{k+1}, \dots, v_n \in V_2$

$$v = \underbrace{\sum_{i=1}^{k} a_i v_i}_{v_1 \in V_1} + \underbrace{\sum_{j=k+1}^{n} a_j v_j}_{v_2 \in V_2} = v_1 + v_2$$

„$\supseteq$" $\underset{v_1 \in V_1, v_2 \in V_2}{v_1 + v_2} \in \langle V_1 \cup V_2 \rangle$

(combinație liniară particulară)

Def $(V, +, \cdot) /_{\mathbb{K}}$, $V_1, V_2 \subseteq V$ subsp. vect.

Sp. că $V_1 + V_2$ este sumă directă şi notăm $V_1 \oplus V_2$ dacă

$V_1 \cap V_2 = \{0_V\}$

Prop: V_1+V_2 este sumă directă $\iff \forall\, x \in V_1+V_2 \Rightarrow \exists!\, v_1 \in V_1$, $v_2 \in V_2$ a.î.

$x = v_1+v_2$

Dem: $V_1 \oplus V_2 \iff V_1 \cap V_2 = \{0_V\}$

Pp. abs. $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ a.î. $\underset{V_1+V_2}{x \in} = v_1 + v_2 = v_1' + v_2'$

$$\underset{V_1}{v_1 - v_1'} = \underset{V_2}{v_2' - v_2} \in \underset{\{0_V\}}{V_1 \cap V_2}$$

$v_1 - v_1' = 0_V \Rightarrow v_1' = v_1$

$v_2 - v_2' = 0_V \Rightarrow v_2' = v_2$

„$\Leftarrow$" Pp. abs. $\exists\, u \in V_1 \cap V_2$, $u \neq 0_V$

$x = v_1 + v_2 = \underset{V_1}{(v_1 + u)} + \underset{V_2}{(v_2 - u)}$

scrierea nu e unică

Contradicție.

Pp. este falsă $V_1 \cap V_2 = \{0_V\}$