T. Hamilton - Cayley. Polinomul caracteristic.

Fie $A \in \mathcal{M}_n(\mathbb{C})$

$$P_A(x) = \det(A - x\cdot I_n) = \begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-x & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & \dots & \dots & a_{nn}-x \end{vmatrix} =$$

$$=(-1)^n \cdot \left[x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \dots + (-1)^n \cdot \sigma_n\right.$$

polinomul caracteristic asociat matricei A, unde

$\sigma_K =$ suma minorilor diagonali de ordin K, $K = \overline{1,n}$

i.e. $\sigma_1 = T_r A$

$$\sigma_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \quad (C_n^2 \text{ minori în sumă})$$

$$\sigma_3 = \sum \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \quad (C_n^3 \text{ minori în sumă})$$

$\vdots$

$\sigma_n = \det(A)$

Cazuri particulare

1) $n=2$ $\quad P_A(x) = \begin{vmatrix} a_{11}-x & a_{12} \\ a_{21} & a_{22}-x \end{vmatrix} = x^2 - \sigma_1 x + \sigma_2 = x^2 - T_r(A)\cdot x + \det(A)$

2) $n=3$ $\quad P_A(x) = \begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{vmatrix} = (-1)^3 \cdot (x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3)$

$$\sigma_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = Tr(A^*)$$

$\sigma_3 = \det(A)$; $\sigma_1 = Tr(A)$

Teorema Hamilton Cayley

$\forall A \in M_n(\mathbb{K})$

$(*)$ $P_A(A) = (-1)^n \left(A^n - \sigma_1 A^{n-1} + \sigma_2 A^{n-2} - \ldots + (-1)^n \sigma_n I_n\right) = O_n$

Cazuri particulare

1) $n=2$ $P_A(A) = A^2 - \sigma_1 A + \sigma_2 I_2 = O_2$

2) $n=3$ $P_A(A) = -\left(A^3 - \underset{\underset{Tr(A)}{\parallel}}{\sigma_1} A^2 + \underset{\underset{Tr(A^*)}{\parallel}}{\sigma_2} A - \underset{\underset{\det(A)}{\parallel}}{\sigma_3} I_3\right) = O_3$

Aplicații THC

(1) Calculul lui A^{-1} $(\det A \neq 0)$

$(*)$ $/ \cdot A^{-1} \Rightarrow A^{-1}$

Exemplu

$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow A^{-1}$

Th H-C $A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$

$\sigma_1 = Tr(A) = 3$

$\sigma_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 + 0 + 0 = 1$

$\sigma_3 = \det(A) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \overset{[C_3 - C_2]}{=\!=} \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$

$A^3 - 3A^2 + A - I_3 = O_3 \;/ \cdot A^{-1}$

$$A^2 - 3A + I_3 - A^{-1} = O_3 \Rightarrow A^{-1} = A^2 - 3A + I_3 = \dots = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

② Calcul pentru puteri

Exemplu

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A^2 - Tr(A)\,A + \det(A)\cdot I_2 = O_2$$

$$A^2 = Tr(A)\,A - \det(A)\cdot I_2$$

$$A^n = x_n \cdot A + y_n \cdot I_2, \ \forall n \geq 1$$

$$x_1 = 1 \qquad y_1 = 0$$

$$x_2 = Tr(A) \qquad y_2 = -\det(A) = -\Gamma_2$$

$$A^{n+1} = A^n \cdot A$$

$$x_{n+1}\cdot A + y_{n+1}\cdot I_2 = x_n \cdot A^2 + y_n \cdot A$$

$$= x_n(\Gamma_1 \cdot A - \Gamma_2 \cdot I_2) + y_n \cdot A$$

$$= (\Gamma_1 x_n + y_n)A - x_n \Gamma_2 I_2$$

$$\begin{cases} x_{n+1} = \Gamma_1 x_n + y_n \\ y_{n+1} = -x_n \Gamma_2 \Rightarrow y_n = -x_{n-1}\Gamma_2 \end{cases}$$

$\boxed{x_{n+1} - \Gamma_1 x_n + x_{n-1}\Gamma_2 = 0}$ recurență de ord. al 2-lea

- Exemplu

Fie $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$; a) $A^n = x_n A + y_n \cdot I_2$, $x_n, y_n = ?$

$$\Gamma_1 = Tr(A) = 3$$

$$\Gamma_2 = 2$$

$$x_{n+1} - 3x_n + 2x_{n-1} = 0 \quad \forall n \geq 2, \ x_1 = 1, \ x_2 = Tr(A) = 3$$

Asociem ecuația caracteristică : $t^2-3t+2=0$ $\to t_1=1$, $\to t_2=2$

$x_n = C_1 t_1^n + C_2 t_2^n = C_1 + C_2 \cdot 2^n, \ \forall n \in \mathbb{N}^*$

$$\begin{array}{ll} n=1 & C_1+2C_2=1 \\ n=2 & C_1+4C_2=3 \end{array} \Bigg| \Rightarrow 2C_2=2 \Rightarrow C_2=1,\quad C_1=-1$$

$x_n=-1+2^n$

$y_n = -2\cdot(-1+2^{n-1}) = -2^n+2$

$A^n = (-1+2^n)A + (2-2^n)\cdot I_2 \quad \forall n \in \mathbb{N}^*$

b) $B = A^4 + A^3 + A^2 + A + I_2$

det. $a, b \in \mathbb{R}$ a.î. $B = a\cdot A + b\cdot I_2$

Obs: $p = x^4+x^3+x^2+x+1 \qquad g = x^2-3x+2$

$P = (x-1)(x-2)\,C + \underbrace{(ax+b)}_{R}$

$$\begin{array}{l} x=1:\ 5 = a+b \\ x=2:\ 31 = 2a+b \end{array} \Bigg| \Rightarrow \boxed{a=26},\ \boxed{b=5-26=-21}$$

$R = 26X-21$

$B = P(A) = aA + bI_2 = 26A - 21 I_2$

③ Rezolvare de ecuații matriceale binome din $M_2(\mathbb{C})$

Exemplu: $X^4 = A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$

$X=?\qquad X \in M_2(\mathbb{C})$

$\det(A) = 0 = \det(X^4) = (\det X)^4 = 0 \Rightarrow \det(X)=0$

Th H–C: $X^2 - Tr(X)X + \underbrace{\det X}_{0} \cdot I_2 = O_2$

$X^2 = Tr(X)\, X$

Prop :

$$X^2 = \alpha X,\ \forall X \in \mathcal{M}_n(\mathbb{C}) \Rightarrow X^k = \alpha^{k-1} X \quad \forall k \geq 2$$

$$X^4 = (Tr(X))^3 X \ \Big/\ Tr \Rightarrow \underset{1}{\underset{\|}{Tr(A)}} = Tr(X)^3 \cdot Tr(X) = Tr$$

$z^4 - 1 = 0 \Rightarrow z \in \{\pm 1; \pm i\}$

$X = \dfrac{1}{(Tr(X))^3} \cdot A$

$X_{1,2} = \pm A$

$X_{3,4} = \dfrac{1}{\pm i} A = \mp i \cdot A$

Teorema Laplace

Fie $A \in \mathcal{M}_n(\mathbb{K})$

a) minor de ordin p ; $p \subseteq n$

$$\det(A_{J,\mathcal{J}}) = \begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_p} \\ \vdots & & \\ a_{i_p j_1} & \dots & a_{i_p j_p} \end{vmatrix}, \quad J = \{i_1, \dots, i_p\} \quad 1 \leq i_1 < \dots < i_p \leq n$$

$$\mathcal{J} = \{j_1, \dots, j_p\} \quad 1 \leq j_1 < \dots < j_p \leq n$$

b) minor complementar lui $\det(A_{J,\mathcal{J}})$

$\det(A_{\bar{J},\bar{\mathcal{J}}})$ $\quad \bar{J} = \{1, \dots, n\} \setminus \{i_1, \dots, i_p\}$

$\bar{\mathcal{J}} = \{1, \dots, n\} \setminus \{j_1, \dots, j_p\}$

(se obț. din A suprimând liniile l_{i_1}, l_{i_p}
coloanele $c_{j_1}, \dots, c_{j_p}$)

c) complementul algebric al minorului $\det(A_{J,\mathcal{J}})$

$$c = (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \cdot \det(A_{\bar{J},\bar{\mathcal{J}}})$$

Teorema Laplace

Determinantul lui A este suma produselor minorilor de ordinul p cu complemenții algebrici corespunzători pentru p linii fixate $l_{i_1},\dots,l_{i_p}$ (respectiv p coloane fixate $c_{j_1},\dots,c_{j_p}$)

$$\det(A) = \sum_{1\le j_1<\dots<j_p\le n} (-1)^{i_1+\dots+i_p+j_1+\dots+j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}}) =$$

$$= \sum_{1\le i_1<\dots<i_p\le n} (-1)^{i_1+\dots+i_p+j_1+\dots+j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}})$$

Obs: pt $p=1 \Rightarrow$ dezvoltarea det. de pe o linie, resp. o coloană.

Exemplu:

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$$

$\det A = ?$ (Th L)

$p=2$, l_1, l_2 fixate

$(1,2)+(1,3)+(1,4)+(2,3)+(2,4)+(3,4)$

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} +$$

$$(-1)^{1+2+1+4} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2+2+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} +$$

$$(-1)^{1+2+2+4} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2+3+4} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = -5$$

Sisteme de ecuații algebrice de ordinul I cu mai multe necunoscute

(sistem de m ecuații cu n necunoscute)

(*) $AX = B$

$A \in \mathcal{M}_{m,n}(\mathbb{K})$, $X \in \mathcal{M}_{n,1}(\mathbb{K})$, $B \in \mathcal{M}_{m,1}(\mathbb{K})$

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ $\quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

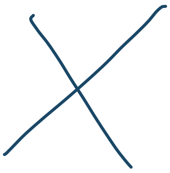
$S(A) = \{ \underset{(x_1,\dots,x_n)}{X} \in \mathbb{K}^n \mid AX = B \}$ mulțimea soluțiilor pt (*)

<u>Obs:</u> ① $S(A) \neq \emptyset$ → S.C.D. (sol. unică)
→ S.C.N (mai multe sol. / o inf.)

② $S(A) = \emptyset$ → S.I ($\nexists$ sol.)

Cazuri particulare

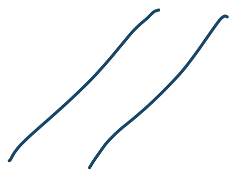
1) $n=2$ $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ $\cap$ 2 drepte în plan



SCD

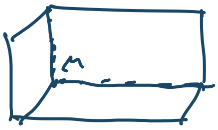
$d_1 = d_2$

SCN

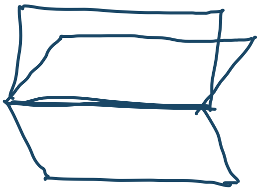


SI

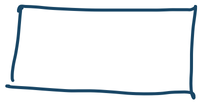
2) $n=3$ $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$



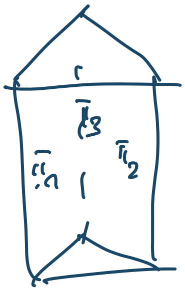
SCD



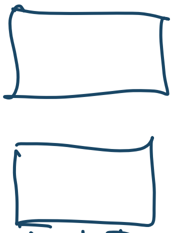
SC simplu N



SC dublu N

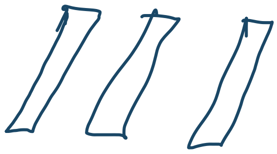


Si



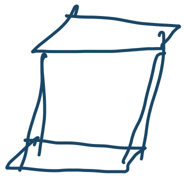
$\pi_1 \parallel \pi_3$

Si



$\pi_1 \parallel \pi_2 \parallel \pi_3$

Si



$\pi_2 \parallel \pi_3$

Si

Cazul general

(*) $AX=B$ $\quad \bar{A} = (A \mid B)$ matricea extinsă

• $n=m$ $\quad A \in \mathcal{M}_n(\mathbb{K})$; $\Delta = \det(A) \neq 0 \Rightarrow$

$$\Rightarrow \underbrace{A^{-1}\cdot A}_{\parallel}\cdot X = A^{-1}\cdot B \Rightarrow (x_1,\dots,x_n) = \left(\frac{\Delta_{x_1}}{\Delta}, \dots, \frac{\Delta_{x_n}}{\Delta}\right)$$

X

soluție unică — M. Cramer

Δ_{x_k} se obține înlocuind col. c_k cu coloana termenilor liberi

Teorema Kronecker - Capelli

Sistemul e compatibil $\Leftrightarrow$ rang A = rang $\bar{A}$

Teorema Rouché

Sistemul este compatibil $\Leftrightarrow$ toți minorii caracteristici sunt nuli.