Chapter VIII

SERIES

Sec. 1. Number Series

1°. Fundamental concepts. A number series

$$a_1+a_2+\ldots+a_n+\ldots=\sum_{n=1}^{\infty} a_n \tag{1}$$

is called *convergent* if its *partial sum*

$$S_n=a_1+a_2+\ldots+a_n$$

has a finite limit as $n \longrightarrow \infty$. The quantity $S=\lim\limits_{n\to\infty} S_n$ is then called the *sum* of the series, while the number

$$R_n=S-S_n=a_{n+1}+a_{n+2}+\ldots$$

is called the *remainder* of the series. If the limit $\lim\limits_{n\to\infty} S_n$ does not exist (or is infinite), the series is then called *divergent*.

If a series converges, then $\lim\limits_{n\to\infty} a_n=0$ (*necessary condition for convergence*). The converse is not true.

For convergence of the series (1) it is necessary and sufficient that for any positive number ε it be possible to choose an N such that for $n>N$ and for any positive p the following inequality is fulfilled:

$$|a_{n+1}+a_{n+2}+\ldots+a_{n+p}|<\varepsilon$$

(*Cauchy's test*).

The convergence or divergence of a series is not violated if we add or subtract a finite number of its terms.

2°. Tests of convergence and divergence of positive series.

a) **Comparison test I.** If $0 \leqslant a_n \leqslant b_n$ after a certain $n=n_0$, and the series

$$b_1+b_2+\ldots+b_n+\ldots=\sum_{n=1}^{\infty} b_n \tag{2}$$

converges, then the series (1) also converges. If the series (1) diverges, then (2) diverges as well.

It is convenient, for purposes of comparing series, to take a *geometric progression:*

$$\sum_{n=0}^{\infty} aq^n \qquad (a \neq 0),$$

which converges for $|q| < 1$ and diverges for $|q| \geqslant 1$, and the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent series.

Example 1. The series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\ldots+\frac{1}{n\cdot 2^n}+\ldots$$

converges, since here

$$a_n=\frac{1}{n\cdot 2^n}<\frac{1}{2^n},$$

while the geometric progression

$$\sum_{n=1}^{\infty} \frac{1}{2^n},$$

whose ratio is $q=\frac{1}{2}$, converges.

Example 2. The series

$$\frac{\ln 2}{2}+\frac{\ln 3}{3}+\ldots+\frac{\ln n}{n}+\ldots$$

diverges, since its general term $\frac{\ln n}{n}$ is greater than the corresponding term $\frac{1}{n}$ of the harmonic series (which diverges).

b) **Comparison test II.** If there exists a finite and nonzero limit $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ (in particular, if $a_n \sim b_n$), then the series (1) and (2) converge or diverge at the same time.

Example 3. The series

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\ldots$$

diverges, since

$$\lim_{n\to\infty}\left(\frac{1}{2n-1}:\frac{1}{n}\right)=\frac{1}{2}\neq 0,$$

whereas a series with general term $\frac{1}{n}$ diverges.

Example 4. The series

$$\frac{1}{2-1}+\frac{1}{2^2-2}+\frac{1}{2^3-3}+\ldots+\frac{1}{2^n-n}+\ldots$$

converges, since

$$\lim_{n\to\infty}\left(\frac{1}{2^n-n}:\frac{1}{2^n}\right)=1, \quad \text{i.e.,} \quad \frac{1}{2^n-n}\sim\frac{1}{2^n},$$

while a series with general term $\frac{1}{2^n}$ converges.

c) **D'Alembert's test.** Let $a_n > 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = q.$$

Then the series (1) converges if $q < 1$, and diverges if $q > 1$. If $q = 1$, then it is not known whether the series is convergent or not.

Example 5. Test the convergence of the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n} + \ldots$$

Solution. Here,

$$a_n = \frac{2n-1}{2^n}, \quad a_{n+1} = \frac{2n+1}{2^{n+1}}$$

and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2n+1)\,2^n}{2^{n+1}(2n-1)} = \frac{1}{2} \lim_{n \to \infty} \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} = \frac{1}{2}.$$

Hence, the given series converges.

d) **Cauchy's test.** Let $a_n \geqslant 0$ (after a certain n) and let there be a limit

$$\lim_{n \to \infty} \sqrt[n]{a_n} = q.$$

Then (1) converges if $q < 1$, and diverges if $q > 1$. When $q = 1$, the question of the convergence of the series remains open.

e) **Cauchy's integral test.** If $a_n = f(n)$, where the function $f(x)$ is positive, monotonically decreasing and continuous for $x \geqslant a \geqslant 1$, the series (1) and the integral

$$\int_a^{\infty} f(x)\,dx$$

converge or diverge at the same time.

By means of the integral test it may be proved that the *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{3}$$

converges if $p > 1$, and diverges if $p \leqslant 1$. The convergence of a large number of series may be tested by comparing with the corresponding Dirichlet series (3)

Example 6. Test the following series for convergence

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots + \frac{1}{(2n-1)\,2n} + \ldots$$

Solution. We have

$$a_n = \frac{1}{(2n-1)\,2n} = \frac{1}{4n^2}\,\frac{1}{1-\frac{1}{2n}} \sim \frac{1}{4n^2}.$$

Since the Dirichlet series converges for $p=2$, it follows that on the basis of comparison test II we can say that the given series likewise converges.

3°. Tests for convergence of alternating series. If a series

$$|a_1|+|a_2|+\ldots+|a_n|+\ldots, \tag{4}$$

composed of the absolute values of the terms of the series (1), converges, then (1) also converges and is called *absolutely convergent*. But if (1) converges and (4) diverges, then the series (1) is called *conditionally (not absolutely) convergent*.

For investigating the absolute convergence of the series (1), we can make use [for the series (4)] of the familiar convergence tests of positive series. For instance, (1) converges absolutely if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1 \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|}<1.$$

In the general case, the divergence of (1) does not follow from the divergence of (4). But if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}>1$, then not only does (4) diverge but the series (1) does also.

Leibniz test If for the alternating series

$$b_1-b_2+b_3-b_4+\ldots \quad (b_n\geqslant 0) \tag{5}$$

the following conditions are fulfilled: 1) $b_1\geqslant b_2\geqslant b_3\geqslant\ldots$; 2) $\lim\limits_{n\to\infty} b_n=0$, then (5) converges.

In this case, for the remainder of the series R_n the evaluation

$$|R_n|\leqslant b_{n+1}$$

holds.

Example 7. Test for convergence the series

$$1-\left(\frac{2}{3}\right)^2-\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+(-1)^{\frac{n(n-1)}{2}}\left(\frac{n}{2n-1}\right)^n+\ldots$$

Solution. Let us form a series of the absolute values of the terms of this series:

$$1+\left(\frac{2}{3}\right)^2+\left(\frac{3}{5}\right)^3+\left(\frac{4}{7}\right)^4+\ldots+\left(\frac{n}{2n-1}\right)^n+\ldots$$

Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n-1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2},$$

the series converges absolutely.

Example 8. The series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+(-1)^{n+1}\cdot\frac{1}{n}+\ldots$$

converges, since the conditions of the Leibniz test are fulfilled. This series converges conditionally, since the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

diverges (harmonic series).

Note. For the convergence of an alternating series it is not sufficient that its general term should tend to zero. The Leibniz test only states that an alternating series converges if the absolute value of its general term tends to zero *monotonically*. Thus, for example, the series

$$1-\frac{1}{5}+\frac{1}{2}-\frac{1}{5^2}+\frac{1}{3}-\ldots+\frac{1}{k}-\frac{1}{5^k}+\ldots$$

diverges despite the fact that its general term tends to zero (here, of course, the monotonic variation of the absolute value of the general term has been violated). Indeed, here, $S_{2k}=S'_k+S''_k$, where

$$S'_k=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}, \quad S''_k=-\left(\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^k}\right),$$

and $\lim\limits_{k\to\infty} S'_k=\infty$ (S'_k is a partial sum of the harmonic series), whereas the limit $\lim\limits_{k\to\infty} S''_k$ exists and is finite (S''_k is a partial sum of the convergent geometric progression), hence, $\lim\limits_{k\to\infty} S_{2k}=\infty$.

On the other hand, the Leibniz test is not necessary for the convergence of an alternating series: an alternating series may converge if the absolute value of its general term tends to zero in nonmonotonic fashion

Thus, the series

$$1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^2}+\ldots+\frac{1}{(2n-1)^3}-\frac{1}{(2n)^2}+\ldots$$

converges (and it converges absolutely), although the Leibniz test is not fulfilled: though the absolute value of the general term of the series tends to zero, it does not do so monotonically.

4°. **Series with complex terms** A series with the general term $c_n=a_n+ib_n$ $(i^2=-1)$ converges if, and only if, the series with real terms $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge at the same time; in this case

$$\sum_{n=1}^{\infty} c_n=\sum_{n=1}^{\infty} a_n+i\sum_{n=1}^{\infty} b_n. \tag{6}$$

The series (6) definitely converges and is called *absolutely convergent*, if the series

$$\sum_{n=1}^{\infty} |c_n|=\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2},$$

whose terms are the moduli of the terms of the series (6), converges.

5°. **Operations on series.**

a) A convergent series may be multiplied termwise by any number k; that is, if

$$a_1+a_2+\ldots+a_n+\ldots=S,$$

then

$$ka_1+ka_2+\ldots+ka_n+\ldots=kS.$$

b) By the *sum (difference)* of two convergent series

$$a_1+a_2+\ldots+a_n+\ldots=S_1, \tag{7}$$

$$b_1+b_2+\ldots+b_n+\ldots=S_2 \tag{8}$$

we mean a series

$$(a_1\pm b_1)+(a_2\pm b_2)+\ldots+(a_n\pm b_n)+\ldots=S_1\pm S_2.$$

c) The *product* of the series (7) and (8) is the series

$$c_1+c_2+\ldots+c_n+\ldots, \tag{9}$$

where $c_n=a_1b_n+a_2b_{n-1}+\ldots+a_nb_1 (n=1, 2, \ldots)$.

If the series (7) and (8) converge absolutely, then the series (9) also converges absolutely and has a sum equal to S_1S_2.

d) If a series converges absolutely, its sum remains unchanged when the terms of the series are rearranged. This property is absent if the series converges conditionally.

Write the simplest formula of the nth term of the series using the indicated terms:

2401. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$

2402. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$

2403. $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\ldots$

2404. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$

2405. $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\ldots$

2406. $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\frac{8}{14}+\ldots$

2407. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\ldots$

2408. $1+\frac{1\cdot3}{1\cdot4}+\frac{1\cdot3\cdot5}{1\cdot4\cdot7}+\frac{1\cdot3\cdot5\cdot7}{1\cdot4\cdot7\cdot10}+\ldots$

2409. $1-1+1-1+1-1+\ldots$

2410. $1+\frac{1}{2}+3+\frac{1}{4}+5+\frac{1}{6}+\ldots$

In Problems 2411-2415 it is required to write the first 4 or 5 terms of the series on the basis of the known general term a_n.

2411. $a_n=\frac{3n-2}{n^2+1}$.

2412. $\frac{(-1)^n n}{2^n}$.

2413 $a_n=\frac{2+(-1)^n}{n^2}$.

2414. $a_n=\frac{1}{[3+(-1)^n]^n}$.

2415. $a_n=\frac{\left(2+\sin\frac{n\pi}{2}\right)\cos n\pi}{n!}$.

Test the following series for convergence by applying the comparison tests (or the necessary condition):

2416. $1-1+1-1+\ldots+(-1)^{n-1}+\ldots$

2417. $\frac{2}{5}+\frac{1}{2}\left(\frac{2}{5}\right)^2+\frac{1}{3}\left(\frac{2}{5}\right)^3+\ldots+\frac{1}{n}\left(\frac{2}{5}\right)^n+\ldots$

2418. $\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots+\frac{n+1}{2n+1}+\ldots$

2419. $\frac{1}{\sqrt{10}}-\frac{1}{\sqrt[3]{10}}+\frac{1}{\sqrt[4]{10}}-\ldots+\frac{(-1)^{n+1}}{\sqrt[n+1]{10}}+\ldots$

2420. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}+\ldots$

2421. $\frac{1}{11}+\frac{1}{21}+\frac{1}{31}+\ldots+\frac{1}{10n+1}+\ldots$

2422. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots+\frac{1}{\sqrt{n(n+1)}}+\ldots$

2423. $2+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^n}{n}+\ldots$

2424. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$

2425. $\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\ldots+\frac{1}{(3n-1)^2}+\ldots$

2426. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\ldots+\frac{\sqrt[3]{n}}{(n+1)\sqrt{n}}+\ldots$

Using d'Alembert's test, test the following series for convergence:

2427. $\frac{1}{\sqrt{2}}+\frac{3}{2}+\frac{5}{2\sqrt{2}}+\ldots+\frac{2n-1}{(\sqrt{2})^n}+\ldots$

2428. $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(3n-1)}{1\cdot 5\cdot 9\ldots(4n-3)}+\ldots$

Test for convergence, using Cauchy's test:

2429. $\frac{2}{1}+\left(\frac{3}{3}\right)^2+\left(\frac{4}{5}\right)^3+\ldots+\left(\frac{n+1}{2n-1}\right)^n+\ldots$

2430. $\frac{1}{2}+\left(\frac{2}{5}\right)^3+\left(\frac{3}{8}\right)^5+\ldots+\left(\frac{n}{3n-1}\right)^{2n-1}+\ldots$

Test for convergence the positive series:

2431. $1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$

2432. $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots+\frac{1}{(n+1)^2-1}+\ldots$

2433. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}+\ldots$

2434. $\frac{1}{3}+\frac{4}{9}+\frac{9}{19}+\ldots+\frac{n^2}{2n^2+1}+\ldots$

2435. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\ldots+\frac{n}{n^2+1}+\ldots$

2436. $\frac{3}{2^2\cdot 3^3}+\frac{5}{3^2\cdot 4^2}+\frac{7}{4^2\cdot 5^2}+\ldots+\frac{2n+1}{(n+1)^2(n+2)^2}+\ldots$

2437. $\frac{3}{4}+\left(\frac{6}{7}\right)^2+\left(\frac{9}{10}\right)^3+\ldots+\left(\frac{3n}{3n+1}\right)^n+\ldots$

2438. $\left(\frac{3}{4}\right)^{\frac{1}{2}}+\frac{5}{7}+\left(\frac{7}{10}\right)^{\frac{3}{2}}+\ldots+\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{2}}+\ldots$

2439. $\frac{1}{e}+\frac{8}{e^2}+\frac{27}{e^3}+\ldots+\frac{n^3}{e^n}+\ldots$

2440. $1+\frac{2}{2^2}+\frac{4}{3^3}+\ldots+\frac{2^{n-1}}{n^n}+\ldots$

2441. $\frac{1!}{2+1}+\frac{2!}{2^2+1}+\frac{3!}{2^3+1}+\ldots+\frac{n!}{2^n+1}+\ldots$

2442. $1+\frac{2}{1!}+\frac{4}{2!}+\ldots+\frac{2^{n-1}}{(n-1)!}+\ldots$

2443. $\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{4\cdot 8\cdot 12\ldots 4n}+\ldots$

2444. $\frac{(1!)^2}{2!}+\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}+\ldots+\frac{(n!)^2}{(2n)!}+\ldots$

2445. $1000+\frac{1000\cdot 1002}{1\cdot 4}+\frac{1000\cdot 1002\cdot 1004}{1\cdot 4\cdot 7}+\ldots$

$$\ldots+\frac{1000\cdot 1002\cdot 1004\ldots(998+2n)}{1\cdot 4\cdot 7\ldots(3n-2)}+\ldots$$

2446. $\frac{2}{1}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ldots+\frac{2\cdot 5\cdot 8\ldots(6n-7)(6n-4)}{1\cdot 5\cdot 9\ldots(8n-11)(8n-7)}+\ldots$

2447. $\frac{1}{2}+\frac{1\cdot 5}{2\cdot 4\cdot 6}+\ldots+\frac{1\cdot 5\ldots(4n-3)}{2\cdot 4\cdot 6\ldots(4n-4)(4n-2)}+\ldots$

2448. $\frac{1}{1!}+\frac{1\cdot 11}{3!}+\frac{1\cdot 11\cdot 21}{5!}+\ldots+\frac{1\cdot 11\cdot 21\ldots(10n-9)}{(2n-1)!}+\ldots$

2449. $1+\frac{1\cdot 4}{1\cdot 3\cdot 5}+\frac{1\cdot 4\cdot 9}{1\cdot 3\cdot 5\cdot 7\cdot 9}+\ldots+\frac{1\cdot 4\cdot 9\ldots n^2}{1\cdot 3\cdot 5\cdot 7\cdot 9\ldots(4n-3)}+\ldots$

2450. $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$.

2451. $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$.

2452. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$.

2453. $\sum_{n=1}^{\infty}\ln\frac{n^2+1}{n^2}$.

2454. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$.

2455. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

2456. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$.

2457. $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n\cdot\ln\ln n}$.

2458. $\sum_{n=2}^{\infty}\frac{1}{n^2-n}$.

2459. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$.

2460. $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)(n+2)}}$.

2461. $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n+\sqrt{\ln^3 n}}$.

2462. $\sum\limits_{n=2}^{\infty}\frac{1}{n\sqrt[3]{n}-\sqrt{n}}$.

2463. $\sum\limits_{n=1}^{\infty}\frac{\sqrt[3]{n}}{(2n-1)(5\sqrt[3]{n}-1)}$.

2464. $\sum\limits_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right)$.

2465. $\sum\limits_{n=1}^{\infty}\frac{n!}{n^n}$.

2466. $\sum\limits_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

2467. $\sum\limits_{n=1}^{\infty}\frac{3^n n!}{n^n}$.

2468*. $\sum\limits_{n=1}^{\infty}\frac{e^n n!}{n^n}$.

2469. Prove that the series $\sum\limits_{n=2}^{\infty}\frac{1}{n^p\ln^q n}$:

1) converges for arbitrary q, if $p>1$, and for $q>1$, if $p=1$;
2) diverges for arbitrary q, if $p<1$, and for $q\leqslant 1$, if $p=1$.

Test for convergence the following alternating series. For convergent series, test for absolute and conditional convergence.

2470. $1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n-1}}{2n-1}+\ldots$

2471. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots+\frac{(-1)^{n-1}}{\sqrt{n}}+\ldots$

2472. $1-\frac{1}{4}+\frac{1}{9}-\ldots+\frac{(-1)^{n-1}}{n^2}+\ldots$

2473. $1-\frac{2}{7}+\frac{3}{13}-\ldots+\frac{(-1)^{n-1}n}{6n-5}+\ldots$

2474. $\frac{3}{1\cdot 2}-\frac{5}{2\cdot 3}+\frac{7}{3\cdot 4}-\ldots+(-1)^{n-1}\frac{2n+1}{n(n+1)}+\ldots$

2475. $-\frac{1}{2}-\frac{2}{4}+\frac{3}{8}+\frac{4}{16}-\ldots+(-1)^{\frac{n^2+n}{2}}\cdot\frac{n}{2^n}+\ldots$

2476. $-\frac{2}{2\sqrt{2}-1}+\frac{3}{3\sqrt{3}-1}-\frac{4}{4\sqrt{4}-1}+\ldots+$

$$+(-1)^n\frac{n+1}{(n+1)\sqrt{n+1}-1}+\ldots$$

2477. $-\frac{3}{4}+\left(\frac{5}{7}\right)^2-\left(\frac{7}{10}\right)^3+\ldots+(-1)^n\left(\frac{2n+1}{3n+1}\right)^n+\ldots$

2478. $\frac{3}{2}-\frac{3\cdot 5}{2\cdot 5}+\frac{3\cdot 5\cdot 7}{2\cdot 5\cdot 8}-\ldots+(-1)^{n-1}\frac{3\cdot 5\cdot 7\ldots(2n+1)}{2\cdot 5\cdot 8\ldots(3n-1)}+\ldots$

2479. $\frac{1}{7}-\frac{1\cdot 4}{7\cdot 9}+\frac{1\cdot 4\cdot 7}{7\cdot 9\cdot 11}-\ldots+(-1)^{n-1}\frac{1\cdot 4\cdot 7\ldots(3n-2)}{7\cdot 9\cdot 11\ldots(2n+5)}+\ldots$

2480. $\frac{\sin\alpha}{\ln 10}+\frac{\sin 2\alpha}{(\ln 10)^2}+\ldots+\frac{\sin n\alpha}{(\ln 10)^n}+\ldots$

2481. $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

2482. $\sum_{n=1}^{\infty}(-1)^{n-1}\tan\frac{1}{n\sqrt{n}}$.

2483. Convince yourself that the d'Alembert test for convergence does not decide the question of the convergence of the series $\sum_{n=1}^{\infty}a_n$, where

$$a_{2k-1}=\frac{2^{k-1}}{3^{k-1}},\quad a_{2k}=\frac{2^{k-1}}{3^k}\quad (k=1,\ 2,\ \ldots),$$

whereas by means of the Cauchy test it is possible to establish that this series converges.

2484*. Convince yourself that the Leibniz test cannot be applied to the alternating series a) to d). Find out which of these series diverge, which converge conditionally and which converge absolutely:

a) $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}-1}-\frac{1}{\sqrt{4}+1}+\ldots$

$$\left(a_{2k-1}=\frac{1}{\sqrt{k+1}-1},\quad a_{2k}=-\frac{1}{\sqrt{k+1}+1}\right);$$

b) $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2^{k-1}},\quad a_{2k}=-\frac{1}{3^{2k-1}}\right);$$

c) $1-\frac{1}{3}+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{5}-\frac{1}{3^3}+\ldots$

$$\left(a_{2k-1}=\frac{1}{2k-1},\quad a_{2k}=-\frac{1}{3^k}\right);$$

d) $\frac{1}{3}-1+\frac{1}{7}-\frac{1}{5}+\frac{1}{11}-\frac{1}{9}+\ldots$

$$\left(a_{2k-1}=\frac{1}{4k-1},\quad a_{2k}=-\frac{1}{4k-3}\right).$$

Test the following series with complex terms for convergence:

2485. $\sum_{n=1}^{\infty}\frac{n(2+i)^n}{2^n}$.

2486. $\sum_{n=1}^{\infty}\frac{n(2i-1)^n}{3^n}$.

2487. $\sum_{n=1}^{\infty}\frac{1}{n(3+i)^n}$.

2488. $\sum_{n=1}^{\infty}\frac{i^n}{n}$.

2489. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+i}$.

2490. $\sum_{n=1}^{\infty}\frac{1}{(n+i)\sqrt{n}}$.

2491. $\sum_{n=1}^{\infty} \frac{1}{[n+(2n-1)i]^2}$.   2492. $\sum_{n=1}^{\infty} \left[\frac{n(2-i)+1}{n(3-2i)-3i}\right]^n$.

2493. Between the curves $y=\frac{1}{x^3}$ and $y=\frac{1}{x^2}$ and to the right of their point of intersection are constructed segments parallel to the y-axis at an equal distance from each other. Will the sum of the lengths of these segments be finite?

2494. Will the sum of the lengths of the segments mentioned in Problem 2493 be finite if the curve $y=\frac{1}{x^2}$ is replaced by the curve $y=\frac{1}{x}$?

2495. Form the sum of the series $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n-n}{3^n}$. Does this sum converge?

2496. Form the difference of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ and test it for convergence.

2497. Does the series formed by subtracting the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ from the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

2498. Choose two series such that their sum converges while their difference diverges.

2499. Form the product of the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Does this product converge?

2500. Form the series $\left(1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}+\ldots\right)^2$. Does this series converge?

2501. Given the series $1+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}+\ldots$ Estimate the error committed when replacing the sum of this series with the sum of the first four terms, the sum of the first five terms. What can you say about the signs of these errors?

2502*. Estimate the error due to replacing the sum of the series

$$\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)^3+\ldots+\frac{1}{n!}\left(\frac{1}{2}\right)^n+\ldots$$

by the sum of its first n terms.

2503. Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=10$.

2504.** Estimate the error due to replacing the sum of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

by the sum of its first n terms. In particular, estimate the accuracy of such an approximation for $n=1{,}000$.

2505.** Estimate the error due to replacing the sum of the series

$$1+2\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)^4+\ldots+n\left(\frac{1}{4}\right)^{2n-2}+\ldots$$

by the sum of its first n terms.

2506. How many terms of the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ does one have to take to compute its sum to two decimal places? to three decimals?

2507. How many terms of the series $\sum\limits_{n=1}^{\infty}\frac{n}{(2n+1)\,5^n}$ does one have to take to compute its sum to two decimal places? to three? to four?

2508*. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$

2509. Find the sum of the series

$$\sqrt[3]{x}+(\sqrt[5]{x}-\sqrt[3]{x})+(\sqrt[7]{x}-\sqrt[5]{x})+\ldots+(\sqrt[2k+1]{x}-\sqrt[2k-1]{x})+\ldots$$

Sec. 2. Functional Series

1°. Region of convergence. The set of values of the argument x for which the functional series

$$f_1(x)+f_2(x)+\ldots+f_n(x)+\ldots \tag{1}$$

converges is called the *region of convergence* of this series. The function

$$S(x)=\lim_{n\to\infty} S_n(x),$$

where $S_n(x)=f_1(x)+f_2(x)+\ldots+f_n(x)$, and x belongs to the region of convergence, is called the *sum* of the series; $R_n(x)=S(x)-S_n(x)$ is the *remainder* of the series.

In the simplest cases, it is sufficient, when determining the region of convergence of a series (1), to apply to this series certain convergence tests, holding x constant.

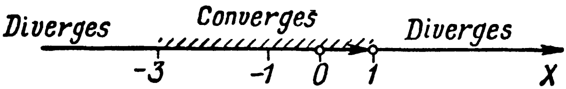

Fig. 104

Example 1. Determine the region of convergence of the series

$$\frac{x+1}{1\cdot 2}+\frac{(x+1)^2}{2\cdot 2^2}+\frac{(x+1)^3}{3\cdot 2^3}+\ldots+\frac{(x+1)^n}{n\cdot 2^n}+\ldots \tag{2}$$

Solution. Denoting by u_n the general term of the series, we will have

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{|x+1|^{n+1}2^n n}{2^{n+1}(n+1)|x|^n}=\frac{|x+1|}{2}.$$

Using d'Alembert's test, we can assert that the series converges (and converges absolutely), if $\frac{|x+1|}{2}<1$, that is, if $-3<x<1$; the series diverges, if $\frac{|x+1|}{2}>1$, that is, if $-\infty<x<-3$ or $1<x<\infty$ (Fig. 104). When $x=1$ we get the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$, which diverges, and when $x=-3$ we have the series $-1+\frac{1}{2}-\frac{1}{3}+\ldots$, which (in accord with the Leibniz test) converges (conditionally).

Thus, the series converges when $-3\leqslant x<1$.

2°. **Power series.** For any *power series*

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots \tag{3}$$

(c_n and a are real numbers) there exists an interval (the *interval of convergence*) $|x-a|<R$ with centre at the point $x=a$, with in which the series (3) converges absolutely; for $|x-a|>R$ the series diverges. In special cases, the *radius of convergence* R may also be equal to 0 and ∞. At the end-points of the interval of convergence $x=a\pm R$, the power series may either converge or diverge. The interval of convergence is ordinarily determined with the help of the d'Alembert or Cauchy tests, by applying them to a series, the terms of which are the absolute values of the terms of the given series (3).

Applying to the series of absolute values

$$|c_0|+|c_1||x-a|+\ldots+|c_n||x-a|^n+\ldots$$

the convergence tests of d'Alembert and Cauchy, we get, respectively, for the radius of convergence of the power series (3), the formulas

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|c_n|}}\quad\text{and}\quad R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.$$

However, one must be very careful in using them because the limits on the right frequently do not exist. For example, if an infinitude of coefficients c_n

vanishes [as a particular instance, this occurs if the series contains terms with only even or only odd powers of $(x-a)$], one cannot use these formulas. It is then advisable, when determining the interval of convergence, to apply the d'Alembert or Cauchy tests directly, as was done when we investigated the series (2), without resorting to general formulas for the radius of convergence.

If $z=x+iy$ is a complex variable, then for the power series

$$c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots+c_n(z-z_0)^n+\ldots \tag{4}$$

$(c_n=a_n+ib_n,\ z_0=x_0+iy_0)$ there exists a certain circle *(circle of convergence)* $|z-z_0|<R$ with centre at the point $z=z_0$, inside which the series converges absolutely; for $|z-z_0|>R$ the series diverges. At points lying on the circumference of the circle of convergence, the series (4) may both converge and diverge. It is customary to determine the circle of convergence by means of the d'Alembert or Cauchy tests applied to the series

$$|c_0|+|c_1|\cdot|z-z_0|+|c_2|\cdot|z-z_0|^2+\ldots+|c_n|\cdot|z-z_0|^n+\ldots,$$

whose terms are absolute values of the terms of the given series. Thus, for example, by means of the d'Alembert test it is easy to see that the circle of convergence of the series

$$\frac{z+1}{1\cdot 2}+\frac{(z+1)^2}{2\cdot 2^2}+\frac{(z+1)^3}{3\cdot 2^3}+\ldots+\frac{(z+1)^n}{n\cdot 2^n}+\ldots$$

is determined by the inequality $|z+1|<2$ [it is sufficient to repeat the calculations carried out on page 305 which served to determine the interval of convergence of the series (2), only here x is replaced by z]. The centre of the circle of convergence lies at the point $z=-1$, while the radius R of this circle (the radius of convergence) is equal to 2.

3°. **Uniform convergence.** The functional series (1) converges uniformly on some interval if, no matter what $\varepsilon>0$, it is possible to find an N such that does not depend on x and that when $n>N$ for all x of the given interval we have the inequality $|R_n(x)|<\varepsilon$, where $R_n(x)$ is the remainder of the given series.

If $|f_n(x)|\leqslant c_n$ $(n=1, 2, \ldots)$ when $a\leqslant x\leqslant b$ and the number series $\sum\limits_{n=1}^{\infty} c_n$ converges, then the functional series (1) converges on the interval $[a, b]$ absolutely and uniformly *(Weierstrass' test)*.

The power series (3) converges absolutely and uniformly on any interval lying within its interval of convergence. The power series (3) may be termwise differentiated and integrated within its interval of convergence (for $|x-a|<R$); that is, if

$$c_0+c_1(x-a)+c_2(x-a)^2+\ldots+c_n(x-a)^n+\ldots=f(x), \tag{5}$$

then for any x of the interval of convergence of the series (3), we have

$$c_1+2c_2(x-a)+\ldots+nc_n(x-a)^{n-1}+\ldots=f'(x), \tag{6}$$

$$\int_{x_0}^{x} c_0dx+\int_{x_0}^{x} c_1(x-a)\,dx+\int_{x_0}^{x} c_2(x-a)^2\,dx+\ldots+\int_{x_0}^{x} c_n(x-a)^n\,dx+\ldots=$$

$$=\sum_{n=0}^{\infty} c_n\,\frac{(x-a)^{n+1}\,(x_0-a)^{n+1}}{n+1}=\int_{x_0}^{x} f(x)\,dx \tag{7}$$

[the number x_0 also belongs to the interval of convergence of the series (3)]. Here, the series (6) and (7) have the same interval of convergence as the series (3).

Find the region of convergence of the series:

2510. $\sum_{n=1}^{\infty} \frac{1}{n^x}$.

2511. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$.

2512. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\ln x}}$.

2513. $\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^2}$.

2514. $\sum_{n=0}^{\infty} 2^n \sin \frac{x}{3^n}$.

2515**. $\sum_{n=0}^{\infty} \frac{\cos nx}{e^{nx}}$.

2516. $\sum_{n=0}^{\infty} (-1)^{n+1} e^{-n \sin x}$.

2517. $\sum_{n=1}^{\infty} \frac{n!}{x^n}$.

2518. $\sum_{n=1}^{\infty} \frac{1}{n!\, x^n}$.

2519. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)\, x^n}$.

2520. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(x-2)^n}$.

2521. $\sum_{n=0}^{\infty} \frac{2n+1}{(n+1)^5 x^{2n}}$.

2522. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n (x-5)^n}$.

2523. $\sum_{n=1}^{\infty} \frac{n^n}{x^{n^n}}$.

2524*. $\sum_{n=1}^{\infty} \left(x^n + \frac{1}{2^n x^n}\right)$.

2525. $\sum_{n=-1}^{\infty} x^n$.

Find the interval of convergence of the power series and test the convergence at the end-points of the interval of convergence:

2526. $\sum_{n=0}^{\infty} x^n$.

2527. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$.

2528. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$.

2529. $\sum_{n=1}^{\infty} \frac{2^{n-1} x^{2n-1}}{(4n-3)^2}$.

2530. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$.

2531. $\sum_{n=0}^{\infty} \frac{(n+1)^5 x^{2n}}{2n+1}$.

2532. $\sum_{n=0}^{\infty} (-1)^n (2n+1)^2 x^n$.

2533. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

2534. $\sum_{n=1}^{\infty} n!\, x^n$.

2535. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

2536. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{2n-1}x^n.$

2537. $\sum_{n=0}^{\infty}3^{n^2}x^{n^2}.$

2538. $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n.$

2539. $\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n}.$

2540. $\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\cdot 3^n\cdot\ln n}.$

2541. $\sum_{n=1}^{\infty}x^{n!}.$

2542**. $\sum_{n=1}^{\infty}n!\,x^{n!}.$

2543*. $\sum_{n=1}^{\infty}\frac{x^{n^2}}{2^{n-1}n^n}.$

2544*. $\sum_{n=1}^{\infty}\frac{x^{n^n}}{n^n}.$

2545. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-5)^n}{n\cdot 3^n}.$

2546. $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 5^n}.$

2547. $\sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{n\cdot 9^n}.$

2548. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{2n}.$

2549. $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n^2}.$

2550. $\sum_{n=1}^{\infty}n^n(x+3)^n.$

2551. $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-1}}{2n\cdot 4^n}.$

2552. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\,2^n}.$

2553. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$

$\times\frac{(2n-1)^{2n}(x-1)^n}{(3n-2)^{2n}}.$

2554. $\sum_{n=1}^{\infty}\frac{n!\,(x+3)^n}{n^n}.$

2555. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{(n+1)\ln^2(n+1)}.$

2556. $\sum_{n=1}^{\infty}\frac{(x-3)^{2n}}{(n+1)\ln(n+1)}.$

2557. $\sum_{n=1}^{\infty}(-1)^{n+1}\times$

$\times\frac{(x-2)^n}{(n+1)\ln(n+1)}.$

2558. $\sum_{n=1}^{\infty}\frac{(x+2)^{n^2}}{n^n}.$

2559*. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-1)^n.$

2560. $\sum_{n=1}^{\infty}\frac{(2n-1)^n(x+1)^n}{2^{n-1}\cdot n^n}.$

2561. $\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt[3]{n+2}}{n+1}\times$

$\times(x-2)^n.$

2562. $\sum_{n=0}^{\infty}\frac{(3n-2)(x-3)^n}{(n+1)^2\,2^{n+1}}.$

2563. $\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{(2n+1)\sqrt{n+1}}.$

Determine the circle of convergence:

2564. $\sum\limits_{n=0}^{\infty} i^n z^n$.

2565. $\sum\limits_{n=0}^{\infty} (1+ni)\,z^n$.

2566. $\sum\limits_{n=1}^{\infty} \frac{(z-2i)^n}{n\cdot 3^n}$.

2567. $\sum\limits_{n=0}^{\infty} \frac{z^{2n}}{2^n}$.

2568. $(1+2i)+(1+2i)(3+2i)\,z+\ldots+$
$$+(1+2i)(3+2i)\ldots(2n+1+2i)\,z^n+\ldots$$

2569. $1+\frac{z}{1-i}+\frac{z^2}{(1-i)(1-2i)}+\ldots$
$$\ldots+\frac{z^n}{(1-i)(1-2i)\ldots(1-ni)}+\ldots$$

2570. $\sum\limits_{n=0}^{\infty} \left(\frac{1+2ni}{n+2i}\right)^n z^n$.

2571. Proceeding from the definition of uniform convergence, prove that the series

$$1+x+x^2+\ldots+x^n+\ldots$$

does not converge uniformly in the interval $(-1, 1)$, but converges uniformly on any subinterval within this interval.

Solution. Using the formula for the sum of a geometric progression, we get, for $|x|<1$,

$$R_n(x)=x^{n+1}+x^{n+2}+\ldots=\frac{x^{n+1}}{1-x}.$$

Within the interval $(-1, 1)$ let us take a subinterval $[-1+\alpha, 1-\alpha]$, where α is an arbitrarily small positive number. In this subinterval $|x|\leqslant 1-\alpha$, $|1-x|\geqslant\alpha$ and, consequently,

$$|R_n(x)|\leqslant\frac{(1-\alpha)^{n+1}}{\alpha}.$$

To prove the uniform convergence of the given series over the subinterval $[-1+\alpha, 1-\alpha]$, it must be shown that for any $\varepsilon>0$ it is possible to choose an N dependent only on ε such that for any $n>N$ we will have the inequality $|R_n(x)|<\varepsilon$ for *all* x of the subinterval under consideration.

Taking any $\varepsilon>0$, let us require that $\frac{(1-\alpha)^{n+1}}{\alpha}<\varepsilon$; whence $(1-\alpha)^{n+1}<\varepsilon\alpha$, $(n+1)\ln(1-\alpha)<\ln(\varepsilon\alpha)$, that is, $n+1>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}$ [since $\ln(1-\alpha)<0$] and $n>\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$. Thus, putting $N=\frac{\ln(\varepsilon\alpha)}{\ln(1-\alpha)}-1$, we are convinced that when $n>N$, $|R_n(x)|$ is indeed less than ε for all x of the subinterval $[-1+\alpha, 1-\alpha]$ and the uniform convergence of the given series on any subinterval within the interval $(-1, 1)$ is thus proved.

As for the entire interval $(-1, 1)$, it contains points that are arbitrarily close to $x=1$, and since $\lim\limits_{x\to 1} R_n(x)=\lim\limits_{x\to 1}\frac{x^{n+1}}{1-x}=\infty$, no matter how large n is, points x will be found for which $R_n(x)$ is greater than any arbitrarily large number Hence, it is impossible to choose an N such that for $n > N$ we would have the inequality $|R_n(x)| < \varepsilon$ at *all* points of the interval $(-1, 1)$, and this means that the convergence of the series in the interval $(-1, 1)$ is not uniform.

2572. Using the definition of uniform convergence, prove that:

a) the series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$$

converges uniformly in any finite interval;

b) the series

$$\frac{x^2}{1}-\frac{x^4}{2}+\frac{x^6}{3}-\ldots+\frac{(-1)^{n-1}x^{2n}}{n}+\ldots$$

converges uniformly throughout the interval of convergence $(-1, 1)$;

c) the series

$$1+\frac{1}{2^x}+\frac{1}{3^x}+\ldots+\frac{1}{n^x}+\ldots$$

converges uniformly in the interval $(1+\delta, \infty)$ where δ is any positive number;

d) the series

$$(x^2-x^4)+(x^4-x^6)+(x^6-x^8)+\ldots+(x^{2n}-x^{2n+2})+\ldots$$

converges not only within the interval $(-1, 1)$, but at the extremities of this interval, however the convergence of the series in $(-1, 1)$ is nonuniform.

Prove the uniform convergence of the functional series in the indicated intervals:

2573. $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^2}$ on the interval $[-1, 1]$.

2574. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{2^n}$ over the entire number scale.

2575. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{\sqrt{n}}$ on the interval $[0, 1]$.

Applying termwise differentiation and integration, find the sums of the series:

2576. $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots$

2577. $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$

2578. $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n-1}}{2n-1}+\ldots$

2579. $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots$

2580. $1+2x+3x^2+\ldots+(n+1)x^n+\ldots$

2581. $1-3x^2+5x^4-\ldots+(-1)^{n-1}(2n-1)x^{2n-2}+\ldots$

2582. $1\cdot 2+2\cdot 3x+3\cdot 4x^2+\ldots+n(n+1)x^{n-1}+\ldots$

Find the sums of the series:

2583. $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\ldots+\frac{n}{x^n}+\ldots$

2584. $x+\frac{x^5}{5}+\frac{x^9}{9}+\ldots+\frac{x^{4n-3}}{4n-3}+\ldots$

2585*. $1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\ldots+\frac{(-1)^{n-1}}{(2n-1)\,3^{n-1}}+\ldots$

2586. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}+\ldots$

Sec. 3. Taylor's Series

1°. Expanding a function in a power series. If a function $f(x)$ can be expanded, in some neighbourhood $|x-a|<R$ of the point a, in a series of powers of $x-a$, then this series (called *Taylor's series*) is of the form

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots \qquad (1)$$

When $a=0$ the Taylor series is also called a *Maclaurin's series*. Equation (1) holds if when $|x-a|<R$ the *remainder term* (or simply remainder) of the Taylor series

$$R_n(x)=f(x)-\left[f(a)\sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]\longrightarrow 0$$

as $n\longrightarrow\infty$.

To evaluate the remainder, one can make use of the formula

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],\ \text{where } 0<\theta<1 \qquad (2)$$

(Lagrange's form).

Example 1. Expand the function $f(x)=\cosh x$ in a series of powers of x.

Solution. We find the derivatives of the given function $f(x)=\cosh x$, $f'(x)=\sinh x$, $f''(x)=\cosh x$, $f'''(x)=\sinh x$, ...; generally, $f^{(n)}(x)=\cosh x$, if n is even, and $f^{(n)}(x)=\sinh x$, if n is odd. Putting $a=0$, we get $f(0)=1$, $f'(0)=0$, $f''(0)=1$, $f'''(0)=0$, ...; generally, $f^{(n)}(0)=1$, if n is even, and $f^{(n)}(0)=0$ if n is odd. Whence, from (1), we have:

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots \qquad (3)$$

To determine the interval of convergence of the series (3) we apply the d'Alembert test. We have

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}:\frac{x^{2n}}{(2n)!}\right|=\lim_{n\to\infty}\frac{x^2}{(2n+1)(2n+2)}=0$$

for any x. Hence, the series converges in the interval $-\infty < x < \infty$. The remainder term, in accord with formula (2), has the form:

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\cosh\theta x, \text{ if } n \text{ is odd, and}$$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\sinh\theta x, \text{ if } n \text{ is even.}$$

Since $0 > \theta > 1$, it follows that

$$|\cosh\theta x| = \frac{e^{\theta x}+e^{-\theta x}}{2} \leqslant e^{|x|}, \qquad |\sinh\theta x| = \left|\frac{e^{\theta x}-e^{-\theta x}}{2}\right| \leqslant e^{|x|},$$

and therefore $|R_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}e^{|x|}$. A series with the general term $\frac{|x|^n}{n!}$ converges for any x (this is made immediately evident with the help of d'Alembert's test); therefore, in accord with the necessary condition for convergence,

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0,$$

and consequently $\lim\limits_{n\to\infty} R_n(x) = 0$ for any x. This signifies that the sum of the series (3) for any x is indeed equal to $\cosh x$.

2°. **Techniques employed for expanding in power series.** Making use of the principal expansions

$$\text{I. } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \quad (-\infty < x < \infty),$$

$$\text{II. } \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots \quad (-\infty < x < \infty),$$

$$\text{III. } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots \quad (-\infty < x < \infty),$$

$$\text{IV. } (1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \ldots$$
$$\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots \quad (-1 < x < 1)^{*}),$$

$$\text{V } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \quad (-1 < x \leqslant 1),$$

and also the formula for the sum of a geometric progression, it is possible, in many cases, simply to obtain the expansion of a given function in a power series, without having to investigate the remainder term. It is sometimes advisable to make use of termwise differentiation or integration when expanding a function in a series. When expanding rational functions in power series it is advisable to decompose these functions into partial fractions.

---

*) On the boundaries of the interval of convergence (i. e., when $x=-1$ and $x=1$) the expansion IV behaves as follows: for $m \geqslant 0$ it converges absolutely on both boundaries; for $0 > m > -1$ it diverges when $x=-1$ and conditionally converges when $x=1$; for $m \leqslant -1$ it diverges on both boundaries.

Example 2. Expand in powers of x *) the function

$$f(x)=\frac{3}{(1-x)(1+2x)}.$$

Solution. Decomposing the function into partial fractions, we will have

$$f(x)=\frac{1}{1-x}+\frac{2}{1+2x}.$$

Since

$$\frac{1}{1-x}=1+x+x^2+\ldots=\sum_{n=0}^{\infty} x^n \tag{4}$$

and

$$\frac{1}{1+2x}=1-2x+(2x)^2-\ldots=\sum_{n=0}^{\infty}(-1)^n 2^n x^n, \tag{5}$$

it follows that we finally get

$$f(x)=\sum_{n=0}^{\infty} x^n+2\sum_{n=0}^{\infty}(-1)^n 2^n x^n=\sum_{n=0}^{\infty}[1+(-1)^n 2^{n+1}]\, x^n. \tag{6}$$

The geometric progressions (4) and (5) converge, respectively, when $|x|<1$ and $|x|<\frac{1}{2}$; hence, formula (6) holds for $|x|<\frac{1}{2}$, i.e., when $-\frac{1}{2}<x<\frac{1}{2}$.

3°. **Taylor's series for a function of two variables.** Expanding a function of two variables $f(x, y)$ into a *Taylor's series* in the neighbourhood of a point (a, b) has the form

$$f(x, y)=f(a, b)+\frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+\right.$$
$$\left.+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)+\ldots+\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a, b)+\ldots \tag{7}$$

If $a=b=0$, the Taylor series is then called a *Maclaurin's series*. Here the notation is as follows:

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a, b)=\left.\frac{\partial f(x, y)}{\partial x}\right|_{\substack{x=a\\ y=b}}(x-a)+\left.\frac{\partial f(x, y)}{\partial y}\right|_{\substack{x=a\\ y=b}}(y-b);$$

$$\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a, b)=\left.\frac{\partial^2 f(x, y)}{\partial x^2}\right|_{\substack{x=a\\ y=b}}(x-a)^2+$$
$$+2\left.\frac{\partial^2 f(x, x)}{\partial x\,\partial y}\right|_{\substack{x=a\\ y=b}}(x-a)(y-b)+\left.\frac{\partial^2 f(x, y)}{\partial y^2}\right|_{\substack{x=a\\ y=b}}(y-b)^2 \text{ and so forth.}$$

*) Here and henceforward we mean "in positive integral powers".

The expansion (7) occurs if the remainder term of the series

$$R_n(x,\ y)=f(x,\ y)-\left\{f(a,b)+\sum_{k=1}^{n}\frac{1}{k!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^k f(a,\ b)\right\}\longrightarrow 0$$

as $n\longrightarrow\infty$. The remainder term may be represented in the form

$$R_n(x,\ y)+\frac{1}{(n+1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n+1} f(x,y)\Bigg|_{\substack{x=a+\theta(x-a)\\ y=b+\theta(y-b)}},$$

where $0<\theta<1$.

Expand the indicated functions in positive integral powers of x, find the intervals of convergence of the resulting series and investigate the behaviour of their remainders:

2587. $a^x\ (a>0)$.

2588. $\sin\left(x+\frac{\pi}{4}\right)$.

2589. $\cos(x+a)$.

2590. $\sin^2 x$.

2591*. $\ln(2+x)$.

Making use of the principal expansions I-V and a geometric progression, write the expansion, in powers of x, of the following functions, and indicate the intervals of convergence of the series:

2592. $\frac{2x-3}{(x-1)^2}$.

2593. $\frac{3x-5}{x^2-4x+3}$.

2594. xe^{-2x}.

2595. e^{x^2}.

2596. $\sinh x$.

2597. $\cos 2x$.

2598. $\cos^2 x$.

2599. $\sin 3x+x\cos 3x$.

2600. $\frac{x}{9+x^2}$.

2601. $\frac{1}{\sqrt{4-x^2}}$.

2602. $\ln\frac{1+x}{1-x}$.

2603. $\ln(1+x-2x^2)$.

Applying differentiation, expand the following functions in powers of x, and indicate the intervals in which these expansions occur:

2604. $(1+x)\ln(1+x)$.

2605. $\arctan x$

2606. $\arcsin x$.

2607. $\ln(x+\sqrt{1+x^2})$.

Applying various techniques, expand the given functions in powers of x and indicate the intervals in which these expansions occur:

2608. $\sin^2 x\cos^2 x$.

2609. $(1+x)e^{-x}$.

2610. $(1+e^x)^3$.

2611. $\sqrt[3]{8+x}$.

2612. $\frac{x^2-3x+1}{x^2-5x+6}$.

2613. $\cosh^3 x$.

2614. $\frac{1}{4-x^4}$.

2615. $\ln(x^2+3x+2)$.

2616. $\int_0^x \frac{\sin x}{x}\,dx$.

2617. $\int_0^x e^{-x^2}\,dx$.

2618. $\int_0^x \frac{\ln(1+x)\,dx}{x}$.

2619. $\int_0^x \frac{dx}{\sqrt{1-x^4}}$.

Write the first three nonzero terms of the expansion of the following functions in powers of x:

2620. $\tan x$.
2621. $\tanh x$.
2622. $e^{\cos x}$.
2623. $\sec x$.
2624. $\ln\cos x$.
2625. $e^x \sin x$.

2626*. Show that for computing the length of an ellipse it is possible to make use of the approximate formula

$$s \approx 2\pi a\left(1-\frac{\varepsilon^2}{4}\right),$$

where ε is the eccentricity and $2a$ is the major axis of the ellipse.

2627. A heavy string hangs, under its own weight, in a catenary line $y=a\cosh\frac{x}{a}$, where $a=\frac{H}{q}$ and H is the horizontal tension of the string, while q is the weight of unit length. Show that for small x, to the order of x^4, it may be taken that the string hangs in a parabola $y=a+\frac{x^2}{2a}$.

2628. Expand the function x^3-2x^2-5x-2 in a series of powers of $x+4$.

2629. $f(x)=5x^3-4x^2-3x+2$. Expand $f(x+h)$ in a series of powers of h

2630. Expand $\ln x$ in a series of powers of $x-1$.

2631. Expand $\frac{1}{x}$ in a series of powers of $x-1$.

2632. Expand $\frac{1}{x^2}$ in a series of powers of $x+1$.

2633. Expand $\frac{1}{x^2+3x+2}$ in a series of powers of $x+4$.

2634. Expand $\frac{1}{x^2+4x+7}$ in a series of powers of $x+2$.

2635. Expand e^x in a series of powers of $x+2$.

2636. Expand $\sqrt{x}$ in a series of powers of $x-4$.

2637. Expand $\cos x$ in a series of powers of $x-\frac{\pi}{2}$.

2638. Expand $\cos^2 x$ in a series of powers of $x-\frac{\pi}{4}$.

2639*. Expand $\ln x$ in a series of powers of $\frac{1-x}{1+x}$.

2640. Expand $\frac{x}{\sqrt{1+x}}$ in a series of powers of $\frac{x}{1+x}$.

2641. What is the magnitude of the error if we put approximately

$$e \approx 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\,?$$

2642. To what degree of accuracy will we calculate the number $\frac{\pi}{4}$, if we make use of the series

$$\operatorname{arc\,tan} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots,$$

by taking the sum of its first five terms when $x=1$?

2643*. Calculate the number $\frac{\pi}{6}$ to three decimals by expanding the function $\arcsin x$ in a series of powers of x (see Example 2606).

2644. How many terms do we have to take of the series

$$\cos x = 1-\frac{x^2}{2!}+\ldots,$$

in order to calculate $\cos 18°$ to three decimal places?

2645. How many terms do we have to take of the series

$$\sin x = x-\frac{x^3}{3!}+\ldots,$$

to calculate $\sin 15°$ to four decimal places?

2646. How many terms of the series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$$

have to be taken to find the number e to four decimal places?

2647. How many terms of the series

$$\ln(1+x) = x-\frac{x^2}{2}+\ldots,$$

do we have to take to calculate $\ln 2$ to two decimals? to 3 decimals?

2648. Calculate $\sqrt[3]{7}$ to two decimals by expanding the function $\sqrt[3]{8+x}$ in a series of powers of x.

2649. Find out the origin of the approximate formula $\sqrt{a^2+x} \approx a+\frac{x}{2a}$ $(a>0)$, evaluate it by means of $\sqrt{23}$, putting $a=5$, and estimate the error.

2650. Calculate $\sqrt[4]{19}$ to three decimals.

2651. For what values of x does the approximate formula

$$\cos x \approx 1 - \frac{x^2}{2}$$

yield an error not exceeding 0.01? 0.001? 0.0001?

2652. For what values of x does the approximate formula

$$\sin x \approx x$$

yield an error that does not exceed 0.01? 0.001?

2653. Evaluate $\int_0^{1/2} \frac{\sin x}{x} dx$ to four decimals.

2654. Evaluate $\int_0^1 e^{-x^2} dx$ to four decimals.

2655. Evaluate $\int_0^1 \sqrt[3]{x} \cos x \, dx$ to three decimals.

2656. Evaluate $\int_0^1 \frac{\sin x}{\sqrt{x}} dx$ to three decimals.

2657. Evaluate $\int_0^{1/4} \sqrt{1+x^3} \, dx$ to four decimals.

2658. Evaluate $\int_0^{1/9} \sqrt{x} e^x \, dx$ to three decimals.

2659. Expand the function $\cos(x-y)$ in a series of powers of x and y, find the region of convergence of the resulting series and investigate the remainder.

Write the expansions, in powers of x and y, of the following functions and indicate the regions of convergence of the series:

2660. $\sin x \cdot \sin y$.

2661. $\sin(x^2+y^2)$.

2662*. $\frac{1-x+y}{1+x-y}$.

2663*. $\ln(1-x-y+xy)$.

2664*. $\arctan \frac{x+y}{1-xy}$.

2665. $f(x, y) = ax^2 + 2bxy + cy^2$. Expand $f(x+h, y+k)$ in powers of h and k.

2666. $f(x, y) = x^3 - 2y^3 + 3xy$. Find the increment of this function when passing from the values $x=1$, $y=2$ to the values $x=1+h$, $y=2+k$.

2667. Expand the function e^{x+y} in powers of $x-2$ and $y+2$.

2668. Expand the function $\sin(x+y)$ in powers of x and $y-\frac{\pi}{2}$.

Write the first three or four terms of a power-series expansion in x and y of the functions:

2669. $e^x \cos y$.

2670. $(1+x)^{1+y}$.

Sec. 4. Fourier Series

1°. Dirichlet's theorem. We say that a function $f(x)$ satisfies the *Dirichlet conditions* in an interval (a, b) if, in this interval, the function

1) is uniformly bounded; that is $|f(x)| \leqslant M$ when $a < x < b$, where M is constant;

2) has no more than a finite number of points of discontinuity and all of them are of the first kind [i.e., at each discontinuity ξ the function $f(x)$ has a finite limit on the left $f(\xi - 0) = \lim\limits_{\varepsilon \to 0} f(\xi - \varepsilon)$ and a finite limit on the right $f(\xi + 0) = \lim\limits_{\varepsilon \to 0} f(\xi + \varepsilon)$ $(\varepsilon > 0)$];

3) has no more than a finite number of points of strict extremum.

Dirichlet's theorem asserts that a function $f(x)$, which in the interval $(-\pi, \pi)$ satisfies the Dirichlet conditions at any point x of this interval at which $f(x)$ is continuous, may be expanded in a trigonometric *Fourier series*:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots, \quad (1)$$

where the *Fourier coefficients* a_n and b_n are calculated from the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \; (n = 0, 1, 2, \ldots); \; b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \; (n = 1, 2, \ldots).$$

If x is a point of discontinuity, belonging to the interval $(-\pi, \pi)$, of a function $f(x)$, then the sum of the Fourier series $S(x)$ is equal to the arithmetical mean of the left and right limits of the function:

$$S(x) = \frac{1}{2} [f(x-0) + f(x+0)].$$

At the end-points of the interval $x = -\pi$ and $x = \pi$,

$$S(-\pi) = S(\pi) = \frac{1}{2} [f(-\pi + 0) + f(\pi - 0)].$$

2°. Incomplete Fourier series. If a function $f(x)$ is even [i. e., $f(-x) = f(x)$], then in formula (1)

$$b_n = 0 \; (n = 1, 2, \ldots)$$

and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots).$$

If a function $f(x)$ is odd [i.e., $f(-x)=-f(x)$], then $a_n=0$ $(n=0, 1, 2 \ldots)$ and

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad (n=1, 2, \ldots).$$

A function specified in an interval $(0, \pi)$ may, at our discretion, be continued in the interval $(-\pi, 0)$ either as an even or an odd function; hence, it may be expanded in the interval $(0, \pi)$ in an incomplete Fourier series of sines or of cosines of multiple arcs.

3°. **Fourier series of a period $2l$.** If a function $f(x)$ satisfies the Dirichlet conditions in some interval $(-l, l)$ of length $2l$, then at the discontinuities of the function belonging to this interval the following expansion holds:

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+b_2\sin\frac{2\pi x}{l}+\ldots$$
$$\ldots+a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}+\ldots,$$

where

$$\left.\begin{aligned} a_n&=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad (n=0, 1, 2, \ldots),\\ b_n&=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \quad (n=1, 2, \ldots).\end{aligned}\right\} \quad (2)$$

At the points of discontinuity of the function $f(x)$ and at the end-points $x=\pm l$ of the interval, the sum of the Fourier series is defined in a manner similar to that which we have in the expansion in the interval $(-\pi, \pi)$.

In the case of an expansion of the function $f(x)$ in a Fourier series in an arbitrary interval $(a, a+2l)$ of length $2l$, the limits of integration in formulas (2) should be replaced respectively by a and $a+2l$

Expand the following functions in a Fourier series in the interval $(-\pi, \pi)$, determine the sum of the series at the points of discontinuity and at the end-points of the interval ($x=-\pi$, $x=\pi$), construct the graph of the function itself and of the sum of the corresponding series [outside the interval $(-\pi, \pi)$ as well]:

2671. $f(x)=\begin{cases} c_1 \text{ when } -\pi<x\leqslant 0,\\ c_2 \text{ when } 0<x<\pi.\end{cases}$

Consider the special case when $c_1=-1$, $c_2=1$.

2672. $f(x)=\begin{cases} ax \text{ when } -\pi<x\leqslant 0,\\ bx \text{ when } 0\leqslant x<\pi.\end{cases}$

Consider the special cases: a) $a=b=1$; b) $a=-1$, $b=1$; c) $a=0$, $b=1$; d) $a=1$, $b=0$.

2673. $f(x)=x^2$.
2674. $f(x)=e^{ax}$.
2675. $f(x)=\sin ax$.
2676. $f(x)=\cos ax$.
2677. $f(x)=\sinh ax$.
2678. $f(x)=\cosh ax$.

2679. Expand the function $f(x)=\frac{\pi-x}{2}$ in a Fourier series in the interval $(0, 2\pi)$.

2680. Expand the function $f(x)=\frac{\pi}{4}$ in sines of multiple arcs in the interval $(0, \pi)$. Use the expansion obtained to sum the number series:

a) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$; b) $1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\ldots$;

c) $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\ldots$

Take the functions indicated below and expand them, in the interval $(0, \pi)$, into incomplete Fourier series: a) of sines of multiple arcs, b) of cosines of multiple arcs. Sketch graphs of the functions and graphs of the sums of the corresponding series in their domains of definition.

2681. $f(x)=x$. Find the sum of the following series by means of the expansion obtained:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

2682. $f(x)=x^2$. Find the sums of the following number series by means of the expansion obtained:

$$1)\ 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots;\quad 2)\ 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

2683. $f(x)=e^{ax}$.

2684. $f(x)=\begin{cases} 1 \text{ when } 0<x<\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}\leqslant x<\pi. \end{cases}$

2685. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ \pi-x \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

Expand the following functions, in the interval $(0, \pi)$, in sines of multiple arcs:

2686. $f(x)=\begin{cases} x \text{ when } 0<x\leqslant\frac{\pi}{2}, \\ 0 \text{ when } \frac{\pi}{2}<x<\pi. \end{cases}$

2687. $f(x)=x(\pi-x)$.

2688. $f(x)=\sin\frac{x}{2}$.

Expand the following functions, in the interval $(0, \pi)$, in cosines of multiple arcs:

2689. $f(x)=\begin{cases} 1 \text{ when } 0<x\leqslant h, \\ 0 \text{ when } h<x<\pi. \end{cases}$

2690. $f(x)=\begin{cases} 1-\frac{x}{2h} & \text{when } 0<x\leqslant 2h, \\ 0 & \text{when } 2h<x<\pi. \end{cases}$

2691. $f(x)=x\sin x$.

2692. $f(x)=\begin{cases} \cos x & \text{when } 0<x\leqslant\frac{\pi}{2}, \\ -\cos x & \text{when } \frac{\pi}{2}<x<\pi. \end{cases}$

2693. Using the expansions of the functions x and x^2 in the interval $(0, \pi)$ in cosines of multiple arcs (see Problems 2681 and 2682), prove the equality

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{3x^2-6\pi x+2\pi^2}{12} \quad (0\leqslant x\leqslant\pi).$$

2694.** Prove that if the function $f(x)$ is even and we have $f\left(\frac{\pi}{2}+x\right)=-f\left(\frac{\pi}{2}-x\right)$, then its Fourier series in the interval $(-\pi, \pi)$ represents an expansion in cosines of odd multiple arcs, and if the function $f(x)$ is odd and $f\left(\frac{\pi}{2}+x\right)=f\left(\frac{\pi}{2}-x\right)$, then in the interval $(-\pi, \pi)$ it is expanded in sines of odd multiple arcs.

Expand the following functions in Fourier series in the indicated intervals:

2695. $f(x)=|x| \quad (-1<x<1)$.

2696. $f(x)=2x \quad (0<x<1)$.

2697. $f(x)=e^x \quad (-l<x<l)$.

2698. $f(x)=10-x \quad (5<x<15)$.

Expand the following functions, in the indicated intervals, in incomplete Fourier series: a) in sines of multiple arcs, and b) in cosines of multiple arcs:

2699. $f(x)=1 \quad (0<x<1)$.

2700. $f(x)=x \quad (0<x<l)$.

2701. $f(x)=x^2 \quad (0<x<2\pi)$.

2702. $f(x)=\begin{cases} x & \text{when } 0<x\leqslant 1, \\ 2-x & \text{when } 1<x<2. \end{cases}$

2703. Expand the following function in cosines of multiple arcs in the interval $\left(\frac{3}{2}, 3\right)$:

$$f(x)=\begin{cases} 1 & \text{when } \frac{3}{2}<x\leqslant 2, \\ 3-x & \text{when } 2<x<3. \end{cases}$$