Seminar 1

Sem 1) Ex. $\boxed{109}$ ~~carte~~

1) R_1, R_2 comutative

i) $R = R_1 \times R_2$, $I \leq R \Leftrightarrow (\exists) I_1 \leq R_1$, $(\exists) I_2 \leq R_2$ a.î. $I = I_1 \times I_2$

Ex 1: $R = \mathbb{Z}_4 \times \mathbb{Q}$

i) Idealele lui R?

ii) Inelele factor ale lui R?

ii) $R/_I = R_1 \times R_2/_{I_1 \times I_2} \simeq R_1/_{I_1} \times R_2/_{I_2}$

i) „$\Leftarrow$" $I = I_1 \times I_2$, $I_1 \leq R_1, I_2 \leq R_2$: Ar. că $I \leq R_1 \times R_2$.

fie $x, y \in I \Rightarrow x - y \in I$

$x \in I, a \in R \Rightarrow ax \in I$

$x = (x_1, x_2)$, $\boxed{x_1 \in I_1}$ $\boxed{x_2 \in I_2}$

$y = (y_1, y_2)$, $\boxed{y_1 \in I_1}$ $\boxed{y_2 \in I_2}$

$x - y = \cancel{(x_1, y_1)} (\underbrace{x_1 - y_1}_{\in I_1}, \underbrace{x_2 - y_2}_{\in I_2}) \in I_1 \times I_2 = I$

$a = (a_1, a_2)$ $a_i \in R_i$, $a_2 \in R_2$

$ax = (a_1, a_2)(x_1, x_2) = (\underbrace{a_1 x_1}_{\in I_1}, \underbrace{a_2 x_2}_{\in I_2}) \in I_1 \times I_2 = I$

„$\Rightarrow$" Fie $I \leq R_1 \times R_2$

$I_1 = \{a_1 \in R_1 \mid (a_1, *) \in I\}$

$I_2 = \{a_2 \in R_2 \mid (*, a_2) \in I\}$

$\boxed{I \overset{?}{=} I_1 \times I_2}$?

$p_1 : R_1 \times R_2 \to R_1$ $p_1(x_1, x_2) = x_1$

$p_2 : R_1 \times R_2 \to R_2$ $p_2(x_1, x_2) = x_2$

p_1, p_2 morfisme de inele

$I_1 = p_1(I)$

$I_2 = p_2(I)$

„$\subseteq$" Fie $(a_1, a_2) \in I \overset{?}{\Rightarrow} a_1 \in I_1, a_2 \in I_2$

„$\supseteq$" Fie $(a_1, a_2) \in I_1 \times I_2 \Rightarrow$

$a_1 \in I_1 \Leftrightarrow (a_1, *) \in I$

$a_2 \in I_2 \Leftrightarrow (*, a_2) \in I$

$\underbrace{(1,0)}_{\in R} \cdot \underbrace{(a_1, *)}_{\in I} + \underbrace{(0,1)}_{\in R} \cdot \underbrace{(*, a_2)}_{\in I} = (a_1, a_2) \in I$

ii)

Fii: $f: R \to R'$ morf. de inele

$R/_{\ker f} \approx \operatorname{Im} f$

f - surj. $\Rightarrow R/_{\ker f} \approx R'$

$\hat{a} \in R/_I$

$\hat{a} = \hat{0} \Leftrightarrow a \in I$

$R_1 \times R_2 \xrightarrow{f} R_1/_{I_1} \times R_2/_{I_2}$

f morfism surj.

$\ker f = I_1 \times I_2$

$f(a_1, a_2) = (\overline{a_1}, \overline{\overline{a_2}})$

$\ker f = \{(a_1, a_2) \mid f(a_1, a_2) = (\overline{0}, \overline{\overline{0}})\} =$

$= \{(a_1, a_2) \mid (\overline{a_1}, \overline{\overline{a_2}}) = (\overline{0}, \overline{\overline{0}})\} =$

$= \{(a_1, a_2) \mid$ ~~$a_1 \equiv 0 \pmod{I_1}$ și $a_2 \equiv \overline{0}$ (mod~~ $\overline{a_1} = \overline{0}$ și $\overline{\overline{a_2}} = \overline{\overline{0}}\} =$

$= \{(a_1, a_2) \mid a_1 \in I_1, a_2 \in I_2\} =$

$= I_1 \times I_2$

$R/_{\{0\}} \simeq R$

$p: R \to R/_{\{0\}}$ surj.

$\ker p = \{0\} \Rightarrow p$ inj. $\Rightarrow$ izomorfism.

$R \times \{0\} \simeq R$

$(a, 0) \to a$

Ex1: $R = \mathbb{Z}_4 \times \mathbb{Q}$

Idealele lui $\mathbb{Z}_4$: $\{\hat{0}\}, \mathbb{Z}_4, \hat{2}\mathbb{Z}_4$.

Idealele lui $\mathbb{Q}$: $\{0\}, \mathbb{Q}$.

Idealele lui R: $\{\hat{0}\} \times \{0\}, \{\hat{0}\} \times \mathbb{Q}, \mathbb{Z}_4 \times \{0\}, \mathbb{Z}_4 \times \mathbb{Q}, \hat{2}\mathbb{Z}_4 \times \{0\}, \hat{2}\mathbb{Z}_4 \times \mathbb{Q}$.

Inelele factor ale lui R:

$R/_{\{\hat{0}\} \times \{0\}} = \mathbb{Z}_4 \times \mathbb{Q}/_{\{\hat{0}\} \times \{0\}} \approx \mathbb{Z}_4/_{\{\hat{0}\}} \times \mathbb{Q}/_{\{0\}} \approx \mathbb{Z}_4 \times \mathbb{Q}$

$\mathbb{Z}_4 \times \mathbb{Q}/_{\{\hat{0}\} \times \mathbb{Q}} \approx \mathbb{Z}_4/_{\{\hat{0}\}} \times \mathbb{Q}/_{\mathbb{Q}} \approx \mathbb{Z}_4 \times \{0\} \approx \mathbb{Z}_4$

$\mathbb{Z}_4 \times \mathbb{Q}/_{\mathbb{Z}_4 \times \{0\}} \approx \mathbb{Z}_4/_{\mathbb{Z}_4} \times \mathbb{Q}/_{\{0\}} \approx \{\hat{0}\} \times \mathbb{Q} \approx \mathbb{Q}$

$\mathbb{Z}_4 \times \mathbb{Q}/_{\mathbb{Z}_4 \times \mathbb{Q}} \approx \mathbb{Z}_4/_{\mathbb{Z}_4} \times \mathbb{Q}/_{\mathbb{Q}} \approx \{\hat{0}\} \times \{0\} \approx \{0\}$

$\mathbb{Z}_4 \times \mathbb{Q}/_{\hat{2}\mathbb{Z}_4 \times \{0\}} \approx \mathbb{Z}_4/_{\hat{2}\mathbb{Z}_4} \times \mathbb{Q}/_{\{0\}} \approx \mathbb{Z}_2 \times \mathbb{Q}$

$\mathbb{Z}_4 \times \mathbb{Q}/_{\hat{2}\mathbb{Z}_4 \times \mathbb{Q}} \approx \mathbb{Z}_4/_{\hat{2}\mathbb{Z}_4} \times \mathbb{Q}/_{\mathbb{Q}} \approx \mathbb{Z}_2 \times \{0\} = \mathbb{Z}_2$

$\mathbb{Z}_n/_{\hat{d}\mathbb{Z}_n} \approx \mathbb{Z}_d$

$d \mid n$

$\mathbb{Z}_n \xrightarrow{f} \mathbb{Z}_d$

$f(\hat{a}) = \overline{\overline{a}}$ corect def. surj. nucleu

$\ker f = \{\hat{a} \in \mathbb{Z}_n \mid \overline{\overline{a}} = \overline{\overline{0}}\} =$

$= \{\hat{a} \in \mathbb{Z}_n \mid d \mid a\}$

2) $R[X]/_{(X-a)} \approx R$ $a \in R$ fixat $\boxed{Ex.\ 132}$

$$R \rightarrow R[x]$$
$1_R \downarrow \swarrow f$
$a \in R$

$X \mapsto a$

$f: R[X] \rightarrow R$

$\operatorname{Ker} f = (X-a) \Rightarrow X-a \in \operatorname{Ker} f \Rightarrow f(X-a) = 0 \Rightarrow$

$f(X) - f(a) = 0 \Rightarrow f(a) = a \Rightarrow f(X) = a$

$f(a_0 + a_1X + \dots + a_nX^n) = a_0 + a_1a + \dots + a_na^n$

$\operatorname{Ker} f = (X-a)$

$f(\dots) = 0 \iff \dots \in (X-a)$

$f(\underbrace{a_0 + a_1X + \dots + a_nX^n}_{p(X)}) = 0 \iff a_0 + a_1 \cdot a + \dots + a_na^n = 0 \iff p(a) = 0 \iff$ Bézout

$X-a \mid p$

Caz particular:

$\mathbb{Z}[X]/_{(X-3)} \approx \mathbb{Z}$

3) $\boxed{Ex.\ 122}$ $\mathbb{Z}[x]/_{(x^2-x)} \approx \mathbb{Z} \times \mathbb{Z}$

$\bar{f} = \overline{(X^2-X)q} + \bar{r}$

grad $r < 2$

$\boxed{r = aX + b}$

$\bar{f} = \bar{r}$

$\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}[x]/_{(x^2-x)}$

$(a,b) \mapsto \overline{aX+b}$

$\mathbb{Z}[x] \xrightarrow{f} \mathbb{Z} \times \mathbb{Z}$

$\operatorname{Ker} f = (X^2 - x)$

$(\alpha, \beta)^2 - (\alpha, \beta) = 0$

$\alpha^2 - \alpha = 0$ și $\beta^2 - \beta = 0 \iff$

$\alpha = 0, 1$

$\beta = 0, 1$

$f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}[x]/(x^2-x)$

$f(a,b) = \overline{ax+b}$

~~$f((a,b)f(c,d)) = f(\overline{ax+b} \cdot \overline{cx+d}) =$~~

$f(a,b)\cdot f(c,d) = \overline{ax+b} \cdot \overline{cx+d} = \overline{(ax+b)(cx+d)} = \overline{acx^2 + bcx + dax + bd} =$

$f(ac; bd) = \overline{acx+bd}$

$(x^2-x) = \{f \in \mathbb{Z}[x] \mid f(0) = f(1) = 0\}$

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}\times\mathbb{Z}$

$f(p) = (p(0), p(1))$

$f(p+q) = ((p+q)(0), (p+q)(1)) = (p(0)+q(0), p(1)+q(1)) = (p(0), p(1)) + (q(0), q(1)) =$

$= f(p) + f(q)$

$f(p\cdot q) = ((p\cdot q)(0)), (p\cdot q)(1)) = (p(0)\cdot q(0), p(1)\cdot q(1)) = (p(0), p(1))\cdot(q(0), q(1)) =$

$= f(p)\cdot f(q)$

$\operatorname{Ker} f = \{p \mid f(p) = (0,0)\} =$

$= \{p \mid (p(0), p(1)) = (0,0)\} =$

$= \{p \mid p(0) = 0$ și $p(1) = 0\} =$

$= \{p \mid x \mid p$ și $(x-1) \mid p\} =$

$= (x(x-1)) = (x^2-x)$

f surj. $(a,b) \in \mathbb{Z}\times\mathbb{Z}$

$p(0) = a, p(1) = b$

$p(x) = mx + n$

$p(0) = n = a$

$p(1) = m + n = b \Rightarrow m = b - a$

$\Rightarrow p = (b-a)x + a$

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Seminar 2

p. 94: 116 (carte) temă

) prob. 123

$$\mathbb{Q}[X]/(X^2-1) \simeq \mathbb{Q}\times\mathbb{Q} \quad \text{şi } \mathbb{Z}[X]/(X^2-1) \not\simeq \mathbb{Z}\times\mathbb{Z}$$

$$\varphi : \mathbb{Q}[X] \to \mathbb{Q}\times\mathbb{Q} \qquad \varphi(f) = (f(-1), f(1))$$

1. φ morfism

(i) $\varphi(f\cdot g) = ((f\cdot g)(-1), (f\cdot g)(1)) = \cancel{(f(-1)\cdot g(-1), f(1)\cdot g(1)) = (f(-1), f(1))\cdot(g(-1), g(1)) =}$

$(fg)(a) = f(a)\cdot g(a)$

$\cancel{= \varphi(f)\cdot\varphi(g)}$

$\varphi(f)\cdot\varphi(g) = (f(-1), f(1))\cdot(g(-1), g(1)) = (f(-1)\cdot g(-1), f(1)\cdot g(1))$

$f, g \in \mathbb{R}[X]$
$(fg)(a) = f(a)\cdot g(a)$, $a\in\mathbb{R}$.

$f = \sum_{i=0}^{m} a_i x^i$ $\qquad fg = \sum_{k=0}^{m+n} c_k x^k$

$g = \sum_{j=0}^{n} b_j x^j$ $\qquad c_k = \sum_{i+j=k} a_i b_j$

$(fg)(a) = \sum_{k=0}^{n} c_k a^k$

$f(a) = \sum_{i=0}^{m} a_i a^i$

$g(a) = \sum_{j=0}^{n} b_j a^j$ $\Rightarrow f(a)\cdot g(a) = (fg)(a)$

(ii) $\varphi(f+g) = \varphi(f)+\varphi(g)$

$((f+g)(-1), (f+g)(1)) = (f(-1)+g(-1), f(1)+g(1)) = (f(-1), f(1)) + (g(-1), g(1)) = \varphi(f)+\varphi(g)$

φ surj.

fie $(a,b)\in\mathbb{Q}\times\mathbb{Q}$ $\qquad \varphi(f) = (a,b) \Leftrightarrow f(-1) = a,\ f(1) = b.$

$\exists f\in\mathbb{Q}[X]$ a.î.

$f = \alpha X + \beta,\ \alpha, \beta \in \mathbb{Q}.$

$f(-1) = a \Leftrightarrow -\alpha+\beta = a$

$f(1) = b \Leftrightarrow \alpha+\beta = b$

$\beta = \frac{a+b}{2} \in\mathbb{Q} \Rightarrow \alpha = \frac{b-a}{2}\in\mathbb{Q}$

$\operatorname{Ker}\varphi = \{f \in \mathbb{Q}[x] \mid \varphi(f) = (0,0)\} =$
$= \{f \in \mathbb{Q}[x] \mid f(-1) = 0$ și $f(1) = 0\} =$
$= \{f \in \mathbb{Q}[x] \mid (x+1) \mid f$ și $(x-1) \mid f\} =$
$= \{f \in \mathbb{Q}[x] \mid (x^2-1) \mid f\} =$
$= (x^2-1)$

$\boxed{\begin{array}{l}(a,b)=1\\ a \mid c, b \mid c \Rightarrow ab \mid c\end{array}}$ exerciții

$\left.\begin{array}{l}x+1 \mid f\\ x-1 \mid f\end{array}\right.$ $((x+1),(x-1)) = 1$?
$\Rightarrow (x-1)(x+1) \mid f.$

$d = (x+1, x-1)$
$\left.\begin{array}{l}d \mid x+1\\ d \mid x-1\end{array}\right\} \Rightarrow d \mid (x+1)-(x-1) = 2$
$\Rightarrow d \mid 2 \Rightarrow 2 = d \cdot e$
$\Rightarrow 0 = \operatorname{grad} d + \operatorname{grad} e$
$\Rightarrow \operatorname{grad} = 0 \Rightarrow d \in \mathbb{Q}^* \Rightarrow d \sim 1$
$d \mid 1,$
$1 = d \cdot \frac{1}{d}.$

$\operatorname{Ker}\varphi = (x^2-1)$
$\operatorname{Im}\varphi = \mathbb{Q} \times \mathbb{Q}$
$\mathbb{Q}[x]/(x^2-1) \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q}$

ii) $\mathbb{Z}[x]/(x^2-1) \not\simeq \mathbb{Z} \times \mathbb{Z}$

Cons. inelul $\mathbb{Z} \times \mathbb{Z} = R.$

$e \in R$ s.n. idempotent dc. $e^2 = e.$

fie $e = (a,b) \in R.$
$e^2 = (a,b)^2 = (a,b)\cdot(a,b) = (a^2,b^2) = (a,b)$
$a^2 = a, b^2 = b, a,b \in \mathbb{Z}. \Leftrightarrow \begin{cases} a(a-1) = 0\\ b(b-1) = 0\end{cases} \Leftrightarrow \begin{cases} a = 0 \text{ sau } a = 1\\ b = 0 \text{ sau } b = 1\end{cases}$

$\{(0,0),(0,1),(1,0),(1,1)\}$
$R = \mathbb{Z}[x]/(x^2-1) = \{\bar{f} \mid f \in \mathbb{Z}[x]\}$ $\quad \bar{f} = \overline{(x^2-1)}\,\bar{q} + \bar{r}$, $\operatorname{grad} r < 2$
$\bar{f} = \bar{r}$

R/I, $a \in I$
$\bar{a} = \bar{0}$, $a \equiv b \pmod{I} \Leftrightarrow a - b \in I.$

$R/\equiv \pmod{I}$

$\bar{a} = \bar{b} \Leftrightarrow a \sim b.$
$\Leftrightarrow a \equiv b \pmod{I}$

$$\mathbb{Z}[X]/(X^2-1) = \{\widehat{aX+b} \mid a,b\in\mathbb{Z}\}$$

$$(\widehat{aX+b})^2 = \widehat{aX+b} \Leftrightarrow$$

$$\widehat{a^2X^2+2abX+b^2} = \widehat{aX+b} = \widehat{2abX+a^2+b^2} \Leftrightarrow \begin{cases} \cancel{a = 2ab} \\ \cancel{b = a^2+b^2} \end{cases} \cancel{\Leftrightarrow \begin{cases} 2b=1 \\ b(b-1)=-a^2 \end{cases}}$$

$$\begin{array}{l|l} a^2X^2+2abX+b^2 & X^2-1 \\ -a^2X^2+a^2 & a^2 \\ \hline 2abX+a^2+b^2 & \end{array}$$

$$2abX+a^2+b^2-(aX+b)\in(X^2+1) \Leftrightarrow$$

$$\Leftrightarrow 2abX+a^2+b^2-(aX+b)\in(X^2-1) \Rightarrow$$

$$(2ab-a)X+a^2+b^2-b\in(X^2-1)$$

$$(2ab-a)X+a^2+b^2-b=0$$

$$\begin{cases} 2ab-a=0 \\ a^2+b^2-b=0 \end{cases} \Leftrightarrow \begin{cases} a(2b-1)=0 \\ a^2+b^2-b=0 \end{cases} \Leftrightarrow \begin{cases} a=0 \\ b^2=b \end{cases} \Leftrightarrow \begin{cases} a=0 \\ b=0 \text{ sau } b=1 \end{cases}$$

$\Rightarrow \mathbb{Z}[X]/(X^2-1)$ are 2 el. idempotente: $\hat{0}$ și $\hat{1}$.] $\Rightarrow$ nu sunt izomorfe.

$\mathbb{Z}\times\mathbb{Z}$ are 4

temă: teorema 121 din carte (formulele lui Newton) de citit.

polinoame simetrice:

$$f=(X_1^2+X_2^2)(X_1^2+X_3^2)(X_2^2+X_3^2)\in\mathbb{C}[X_1,X_2,X_3]$$

f simetric.

$s_1 = X_1+X_2+X_3$

$s_2 = X_1X_2+X_1X_3+X_2X_3$

$s_3 = X_1X_2X_3$

f este omogen de grad $p \Leftrightarrow f(tX_1,\dots,tX_n) = t^p f(X_1,\dots,X_n)$

$$\text{"}\Rightarrow\text{"}\quad f=\sum_{i_1+\dots+i_n=p} a_{i_1\dots i_n} X_1^{i_1}\dots X_n^{i_n}$$

$$f(tX_1,\dots,tX_n)=\sum_{i_1+\dots+i_n=p} a_{i_1\dots i_n}(tX_1)^{i_1}\dots(tX_n)^{i_n} = t^p f(X_1,\dots,X_n)$$

dacă $f(tX_1,\dots,tX_n) = t^p f(X_1,\dots,X_n) \Rightarrow f$ omogen de grad p.

$X_1^{i_1}\dots X_n^{i_n}$ apare în scrierea lui f.

$i_1+\dots+i_n \neq p$.

$t^{i_1+\dots+i_n} X_1^{i_1}\dots X_n^{i_n}$

$t^p \boxed{X_1^{i_1}\dots X_n^{i_n}}$

$$f(tX_1,tX_2,tX_3) = (t^2X_1^2+t^2X_2^2)(t^2X_1^2+t^2X_3^2)(t^2X_2^2+t^2X_3^2) =$$
$$= t^6 f(X_1,X_2,X_3) \Rightarrow f \text{ este omogen de grad } 6.$$

$X_1^{i_1}X_2^{i_2}X_3^{i_3}$, $i_1+i_2+i_3=6$

$i_1 \geq i_2 \geq i_3$ ptr. termen principal

$(4,2,0) \leftrightarrow \cancel{X_1^4X_2^2}$

$(4,1,1) \leftrightarrow X_1^4X_2X_3$

$(3,3,0) \leftrightarrow X_1^3X_2^3$

$(3,2,1) \leftrightarrow X_1^3X_2^2X_3$

$(2,2,2) \leftrightarrow X_1^2X_2^2X_3^2$

$\mathrm{lt}(f) = X_1^4X_2^2$ $\quad X_1^2(X_1X_2)^2$

$f_1 = f - s_1^{j_1}s_2^{j_2}s_3^{j_3}$

$f_1 = f - s_1^2s_2^2$

f_1 omogen de grad 6? da

$(tX_1+tX_2+tX_3)^2\ \cancel{(tX_1tX_2+\dots)}^2$

$$f - s_1^2s_2^2 - as_1^3s_3 - bs_2^3 - c\cdot s_1s_2s_3 - d\cdot s_3^2 = 0$$
$$f = s_1^2s_2^2 + as_1^3s_3 + bs_2^3 + cs_1s_2s_3 + ds_3^2$$

$X_1=1,\ X_2=0,\ X_3=-1$

$s_1=0,\ s_2=-1,\ s_3=0$

$f(1,0,-1) = 1\cdot 2\cdot 1 = 2$

$2 = bs_2^3 = -b \Rightarrow \boxed{b=-2}$

$X_1=\frac{1}{2};\ X_2=\frac{1}{2};\ X_3=-1$

$s_1=0,\ s_2=-\frac{3}{4},\ s_3=\frac{1}{4}$

$f(\frac{1}{2},\frac{1}{2},-1) = \frac{1}{2}\cdot\frac{5}{4}\cdot\frac{5}{4} = \frac{25}{32}$

$(-2)(-\frac{3}{4})^3 + d\cdot\frac{1}{16} = \frac{25}{32} \Leftrightarrow \frac{27}{32} - \frac{25}{32} = -d\cdot\frac{1}{16} \Leftrightarrow$

$d\cdot\frac{1}{16} = -\frac{2}{32} \Leftrightarrow \boxed{d=-1}$

$x_1 = 2,\ x_2 = 2,\ x_3 = -1$

$s_1 = 3, s_2 = 0,\ s_3 = -4$

$f(2,2,-1) = 8\cdot 5\cdot 5 = 200$

$a\cdot 27(-4) + (-1)(-4)^2 = 200$

$-108\cdot a - 16 = 200$

$\boxed{a = -2}$

$x_1 = x_2 = x_3 = 1$

$s_1 = 3,\ s_2 = 3,\ s_3 = 1$

$81 + (-2)\cdot 27 + (-2)\cdot 27 + c\cdot 9 + (-1)1 = 8$

$81 + (-2)\cdot 27 \ \ 108 - 81 + 9(2)$

$9c = 8 + 1 + 54 + 54 - 81 = 108 - 81 + 9(2)$

$c = 12 - 9 + 1 = 4$

$\boxed{c = 4}$

$f = s_1^2 s_2^2 - 2 s_1^3 s_3 - 2 s_2^3 + 4 s_1 s_2 s_3 - s_3^2$

$f_2 = f + \underbrace{x_1^4 + x_2^4 + x_3^4}_{(s_1^2 - 2s_2)^2 - 2(s_2^2 - 2 s_1 s_3)}$ separat.

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S3 Seminar 3

săpt 4

170 → EXAMEN !!!

p. 108/169

$f = \underline{x^3} + y^3 + z^3$

$lt(f) = x^3$

$s_1 = \underline{x} + y + z$

$s_2 = \underline{xy} + yz + xz$

$s_3 = \underline{xyz}$

f pol. omogen de gr. 3.

$f(tx, ty, tz) = t^3 f(x,y,z) \Rightarrow f$ omogen de grad 3.

$lt = x^i y^j z^k \qquad i+j+k = 3$

$i \geq j \geq k$

$(3,0,0) \to x^3$

$(2,1,0) \to x^2y$

$(1,1,1) \to xyz$

$f = s_1^3 + \underline{a} \cdot s_1 s_2 + \underline{b} \cdot s_3$

$s_1 = 0 \quad (0,1,-1)$

$f(1, \frac{1}{2}, +\frac{1}{2}) = \cancel{\frac{1}{8}} \cancel{\frac{1}{8}} = 0 \; -\frac{3}{4}$

$s_1 = 0;\ s_2 = -\frac{3}{4},\ s_3 = \frac{1}{4}$

$f = b \cdot \left(\frac{1}{4}\right) = \frac{3}{4} \Rightarrow \boxed{b = 3}$

$0, 1, 1$

$s_1 = 2;\ s_2 = 1;\ s_3 = 0$

$f = 2$

$2 = 8 + a \cdot 2 = 2 \Leftrightarrow 2a = -6 \Rightarrow \boxed{a = -3}$

$f = s_1^3 - 3s_1s_2 + 3s_3$

$(x+(y+z))^3 = x^3 + 3x^2(y+z) + 3x(y+z)^2 + (y+z)^3 = \underline{x^3} + \underline{3x^2y} + \underline{3x^2z} + \underline{3xy^2} + \underline{3xz^2}$

$+ 6xyz + \underline{y^3 + z^3 + 3y^2z + 3yz^2} = x^3 + 3xz^2 + 3xy^2 + 3xz^2 + 3y^2z + 3yz^2 + 6xyz$

$= \underbrace{x^3 + y^3 + z^3}_{f} + 3x^2y + 3x^2z + 3xy^2 + 3xz^2 + 3y^2z + 3yz^2 + 6xyz$

$= f + 3yz(z+y+x) + 3xy(x+y+z) + 3xz(x+y+z) - 3xyz =$

$= f + 3(x+y+z)(xy+yz+xz) - 3xyz =$

$= f + 3s_1s_2 - 3s_3$

Teorema 121 : p. 106

$p_k = x_1^k + x_2^k + \dots + x_n^k, \ k \geq 1$

$s_1, \dots s_n \in A[x_1 \dots x_n]$ pol. sim. fund.

$p_k - s_1p_{k-1} + s_2p_{k-2} + \dots (-1)^n s_n p_{k-n} = 0, \ k \geq n$

$p_k - s_1p_{k-1} + \dots + (-1)^{k-1} s_{k-1}p_1 + (-1)^k k s_k = 0, \ 1 \leq k \leq n-1$

(165) $x^4 + x^3 + 2x^2 + x + 1 = 0$

x_1, x_2, x_3, x_4 - răd. ec.

$p_5 = x_1^5 + x_2^5 + x_3^5 + x_4^5$

$x_1^4 + x_1^3 + 2x_1^2 + x_1 + 1 = 0 \ / \cdot x_1$

$x_2^4 + x_2^3 + 2x_2^2 + x_2 + 1 = 0 \ / \cdot x_2$

$x_3^4 + x_3^3 + 2x_3^2 + x_3 + 1 = 0 \ / \cdot x_3$

$x_4^4 + x_4^3 + 2x_4^2 + x_4 + 1 = 0 \ / \cdot x_4$

(+)

$p_5 + p_4 + 2p_3 + p_2 + p_1 = 0$

$p_1 = s_1 = -1$

$p_2 = -3$

$p_3 = 2$

$p_4 = 1$

$p_4 + p_3 + 2p_2 + p_1 + 4 = 0$

$p_2 = s_1^2 - 2s_2 = 1 - 4 = -3$

$S_1 = -1$

$S_2 = 2$

$S_3 = -1$

$S_4 = 1$

$x_1^4 + \dots + x_n^4 = (x_1^2 + \dots + x_n^2) - 2\sum_{i<j} x_i^2x_j^2 =$

$= (s_1^2 - 2s_2)^2 - 2\left[\left(\sum_{i<j} x_ix_j\right)^2 - 2s_1s_3\right] =$

$= (s_1^2 - 2s_2)^2 - 2(s_2^2 - 2s_1s_3)$

$k=3 \quad p_3 - s_1p_2 + s_2p_1 - 3s_3 = 0$

$p_3 = s_1p_2 - s_2p_1 + 3s_3 = -1\cdot(-3) - 2\cdot(-1) + 3\cdot(-1) = 3 + 2 - 3 = 2$

$p_4 = -p_3 - 2p_2 - p_1 - 4 = -2 - 2(-3) + 1 - 4 = -2 + 6 - 3 = 1$

$p_5 = -p_4 - 2p_3 - p_2 - p_1 =$
$= -1 - 4 + 3 + 1 = \boxed{-1 = p_5}$

p. 99/152

$X^3 - 1 = (X-1)(X^2+X+1)$
$(X-1) \to$ ired.

$X^2+X+1 \in \mathbb{Q}[X] \to$ ired. pe $\mathbb{Q}[X]$, $\mathbb{R}[X]$, deoarece $\Delta < 0$.
/peste $\mathbb{Q}$ și $\mathbb{R}$. $\Delta = -3 < 0$.

$x_{1,2} = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1+\sqrt{3}i}{2}$

$X^2+X+1 = \left(X + \frac{1+\sqrt{3}i}{2}\right)\left(X - \frac{-1+\sqrt{3}i}{2}\right) \in \mathbb{C}[X]$.

$X^4 - 1 = (X^2+1)(X^2-1) = (X^2+1)(X-1)(X+1)$

X^2+1 - ired. peste $\mathbb{R}$ și $\mathbb{Q}$. $\Delta = -4 < 0$

$X^2+1 = (X-i)(X+i) \in \mathbb{C}[X]$

$X^5 - 1 = (X-1)(X^4+X^3+X^2+X+1)$

$f = a_0 + a_1X + \dots + a_nX^n$

$\frac{p}{q} \in \mathbb{Q}$, $(p,q) = 1$

$f\left(\frac{p}{q}\right) = 0 \Rightarrow p \mid a_0$
$q \mid a_n$

$a_0 + a_1\frac{p}{q} + a_2\left(\frac{p}{q}\right)^2 + \dots + a_n\left(\frac{p}{q}\right)^n = 0 \;\Big|\; q^n$

$a_0q^n + a_1pq^{n-1} + \dots + a_np^n = 0$

$\left.\begin{array}{l} p \mid a_0q^n \\ (p,q) = 1 \end{array}\right| \Rightarrow p \mid a_0$
$q \mid a_n$

$f = 1+X+X^2+X^3+X^4$

f are răd. raționale?

Nu știm! Dar căutăm.

Fie $\frac{p}{q}$ răd. raț. a lui f. $\Rightarrow p \mid 1$, $q \mid 1$ $\Rightarrow p = \pm 1$, $q = \pm 1$

$f(1) = 5 \neq 0$
$f(-1) = 1 \neq 0$

Știm! Nu are răd. raționale.

= ec. reciprocă $| : X^2$

$X^4+X^3+X^2+X+1 = (X^2+aX+b)(X^2+cX+d)$, $a,b,c,d \in \mathbb{Q}$.

$= X^4 + \underline{aX^3} + bX^2 + \underline{cX^3} + acX^2 + bcX + dX^2 + adX + bd$

$\frac{a}{n}, \frac{b}{n}, \frac{c}{n}, \frac{d}{n}$, $a,b,c,d \in \mathbb{Z}$, $n \in \mathbb{Z}$

$$\begin{cases} a+c = 1 \\ bd = 1 \\ ac+d+b = 1 \\ bc+ad = 1 \end{cases} \qquad c = 1-a, \quad d = \frac{1}{b}$$

$$\begin{cases} a(1-a) + \frac{1}{b} + b = 1 \\ \frac{a}{b} + b(1-a) = 1 \end{cases} \Leftrightarrow \begin{cases} ab(1-a) + 1 + b^2 = b \\ a + b^2(1-a) = b \end{cases} \Leftrightarrow$$

$$\begin{cases} ab - a^2b + b^2 + 1 = b \\ a + b^2 - b^2a = b \end{cases} \quad - (-)$$

$ab - a + 1 - a^2b + b^2a = 0$ — sol (reală) ..

$X^6 - 1 = \dots$

Temă
sem 3

$X^4 + x^3 + X^2 + X + 1 = 0 \;/ : x^2 \iff$

$\frac{1}{X^2} + X^2 + \frac{1}{x} + X + 1 = 0 \iff$

$t^2 - 2 + t + 1 = 0 \iff$

$t^2 + t - 1 = 0$

$t_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$

$\frac{1}{x} + X = t$

$t^2 = \frac{1}{x^2} + x^2 + 2$

$\frac{1}{x} + x = t \iff$

$x^2 + 1 - tx = 0$

1) $X^2 + \frac{1+\sqrt{5}}{2}x + 1 = 0$ $\quad \Delta = \frac{6+2\sqrt{5}}{4} - 4 = \frac{3+\sqrt{5}-8}{2} = \frac{-5+\sqrt{5}}{2} < 0$

2) $X^2 + \frac{1-\sqrt{5}}{2}x + 1 = 0$ $\quad \Delta = \frac{6-2\sqrt{5}}{4} - 4 = \frac{3-\sqrt{5}-8}{2} = -\frac{5+\sqrt{5}}{2} < 0$

$X_{1,2} = \cancel{\frac{\frac{-1-\sqrt{5}}{2} \pm \sqrt{\frac{-5+\sqrt{5}}{2}}}{2}} = \cancel{\frac{-1-\sqrt{5}}{4} \pm}$

$X_{1,2} = \frac{-1-\sqrt{5}}{4} \pm \frac{\sqrt{5-\sqrt{5}}\, i}{\sqrt{2}}$

$X_{3,4} = \frac{\sqrt{5}-1}{2} \pm \frac{\sqrt{5+\sqrt{5}}\, i}{\sqrt{2}}$

$X^6 - 1 = (X^3)^2 - 1 = (X^3-1)(X^3+1) = (X-1)(X^2+X+1)(X+1)(X^2-X+1)$

$X^2 - X + 1 = 0$ $\quad \Delta = 1 - 4 = -3 < 0$ – ireductibile în $\mathbb{R}$ și $\mathbb{Q}$

$X_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$

$X^2 - X + 1 = \left(X - \frac{1+\sqrt{3}i}{2}\right)\left(X - \frac{1-\sqrt{3}i}{2}\right)$ în $\mathbb{C}$

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Seminar 4

Spații vectoriale

1) p... / 193 care din submulț. sunt subsp. vect ...

b) $W = \{ f, f$ unitar, i.e are coeficientul dominant $= 1\}$ (=monic) $f \in \mathbb{Q}[X]$. subsp. vect. al lui $\mathbb{Q}[X]$

$f(x) = x^2 \quad \in W$

$g(x) = x^2 + 2x \in W$

$g(x) - f(x) = 2x \notin W$

c) $W = \{f; f$ neunitar$\}$

$f(x) = 3x^2 \quad \in W$

$g(x) = 3x^2 + x \quad \in W$

$g(x) - f(x) = x \notin W$

d) $\{f, f$ ~~de~~ grad impar$\} = W$

$f(x) = x^3 \quad \in W$

$g(x) = x^3 - x^2 \quad \in W$

$f(x) - g(x) = x^2 \notin W$

e) $\{f; f$ de grad $\leq 2\} = W$

$f, g \in W \Rightarrow f - g \in W$

$f \in W$ și $\alpha \in \mathbb{Q} \overset{?}{\Rightarrow} \alpha f \in W$

grad (αf) = grad α + grad f = $0 +$ grad f = grad $f \leq 2$

f) $\{f, f(0) = f(1)\} = W$

$\left.\begin{array}{l}(f-g)(0) = f(0) - g(0) \\ (f-g)(1) = f(1) - g(1)\end{array}\right\} \Rightarrow (f-g)(0) = (f-g)(1) \overset{f,g \in W}{\Longrightarrow} f - g \in W$

$\left.\begin{array}{l}(\alpha f)(0) = \alpha \cdot f(0) \\ (\alpha f)(1) = \alpha f(1)\end{array}\right\} \overset{f \in W}{\Rightarrow} (\alpha f)(0) = (\alpha f)(1) \Rightarrow \alpha f \in W$

a) $\{f, f(a) = f(-a), \forall a \in Q\} = W$

$(f-g)(a) = f(a) - g(a)$
$(f-g)(-a) = f(-a) - g(-a) \xrightarrow{f,g \in W} f-g \in W$

$(\alpha f)(a) = \alpha f(a)$
$(\alpha f)(-a) = \alpha f(-a) \xrightarrow{f \in W} \alpha f \in W.$

2) ex. 194 U, W subsp. R^3. Calc. suma și intersecția subsp.

$U = \langle (2,3,1), (1,2,0) \rangle \subseteq R^3$
$V = \langle (1,1,1), (0,-1,-1) \rangle \subseteq R^3$

$U + V = ?$
$U \cap V = ?$

$U + V = \langle (2,3,1), (1,2,0), (1,1,1), (0,-1,-1) \rangle \subseteq R^3$

$U = \langle x_1, \dots x_m \rangle \subseteq W$
$V = \langle y_1, \dots y_m \rangle \subseteq W$
$U + V = \langle x_1, \dots x_m, y_1, \dots y_m \rangle$
$z \in U + V \boxed{=?} = \{x + y \mid x \in U, y \in V\}$
$z = x + y = \sum_{i=1}^{m} \alpha_i x_i + \sum_{j=1}^{m} \beta_j y_j, \ \alpha_i, \beta_j \in K$

$z = a(2,3,1) + b(1,2,0) + c(1,1,1) + d(0,-1,-1) =$
$= (2a + b + c, 3a + 2b + c - d, a + c - d) =$
$= (x_1, x_2, x_3)$

$$\begin{cases} 2a + b + c = x_1 \\ 3a + 2b + c - d = x_2 \\ a + c - d = x_3 \end{cases}$$

$d = 0 \Rightarrow \begin{cases} 2a + b + c = x_1 \\ 3a + 2b + c = x_2 \\ a + c = x_3 \end{cases}$

$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $\det A = 4 + 1 - 2 - 3 = 0$

$a=0 \Rightarrow \begin{cases} b+c=x_1 \\ 2b+c-d=x_2 \\ c-d=x_3 \end{cases}$ $A=\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ $\det A=-1+2+1=2 \Rightarrow$ sist. compatibil unic det.

$\Rightarrow \mathbb{R}^3$

Sol 2. $U \cap V$.

$g \in U \cap V \Rightarrow g = a(2,3,1)+b(1,2,0)$
$g = c(1,1,1)+d(0,-1,-1)$

$(2a+b, 3a+2b, a) = (c, c-d, c-d) \Leftrightarrow$

$\begin{cases} c=2a+b \\ c-d=3a+2b \\ c-d=a \end{cases} \Rightarrow a+b=0 \Leftrightarrow a=-b$

$\begin{cases} c=2(c-d)+b \\ c-d=3(c-d)+2b \end{cases} \Leftrightarrow \begin{cases} 2c-2d+b=c \\ 2c-2d+2b=0 \end{cases} \Leftrightarrow b=2d-c$

~~2c-2d+4d~~ -

$\Downarrow$

$d=0$
$b=\cancel{2}c$
$a=c$

$g=(2a+b, 3a+2b, a)=(c,c,c)=c\cdot(1,1,1)$

$U \cap V = \langle (1,1,1) \rangle$

$V = \langle x, \cancel{\alpha x} \rangle \quad \alpha \neq 0$
$V = \langle x \rangle$
$V = \langle y_1, \dots y_m \rangle = \langle y_1, \dots y_{m-1} \rangle$
$y_m \in \langle y_1, \dots y_{m-1} \rangle$
$\alpha_1 y_1 + \dots + \alpha_m y_m = \alpha_1 y_1 + \dots + \alpha_{m-1} y_{m-1} + (\beta_1 y_1 + \dots + \beta_m$
$= (\alpha_1+\beta_1)y_1 + \dots + (\alpha_{m-1}+\beta_{m-1})y$

3) $U = \langle (1,2) \rangle$
$V = \langle (1,-1),(2,-2) \rangle = \langle (1,-1) \rangle$

$U+V = \langle (1,2),(1,-1),(2,-2) \rangle = \cancel{\langle}$
$= \{ a(1,2)+b(1,-1) \mid a,b \in \mathbb{R} \} = \mathbb{R}^2$
$= \{ (a+b, 2a-b) \mid a,b \in \mathbb{R} \}$

$\begin{cases} a+b=x_1 \\ 2a-b=x_2 \end{cases}$ $A=\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ $\det A=-1-2=-3 \neq 0 \Rightarrow$ sist. comp. unic. det.

$g \in U \cap V \Rightarrow g = a(1,2)$
$g = b(1,-1)$

$(a, 2a) = (b, -b) \Leftrightarrow \begin{cases} a=b \\ 2a=-b \end{cases} \Leftrightarrow 3a=0 \Rightarrow a=0=b$

$g=0 \Rightarrow U \cap V = (0,0)$

4) ~~ex. 200~~

$\{(2,3,1), (1,2,0), (0,-1,-1)\} = B'$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

B' bază în $\mathbb{R}^3$

$\Leftrightarrow$ liniar independentă / sist. de generatori

$\Downarrow$

$a(2,3,1) + b(1,2,0) + c(0,-1,-1) = (0,0,0) \Rightarrow a = b = c = 0$

$(2a+b,\ 3a+2b-c,\ a-c) = (0,0,0)$

$$\begin{cases} 2a+b=0 \\ 3a+2b-c=0 \\ a-c=0 \end{cases} \Leftrightarrow \begin{cases} 2a+b=0 \\ 2a+2b=0 \\ a=c \end{cases} \Leftrightarrow \begin{array}{l} 2a+b=0 \\ a+b=0 \\ \hline (-) \end{array}$$

$$\boxed{\begin{array}{l} a=0=c \\ b=0 \end{array}}$$

$(2,3,1) = 2\underset{e_1}{(1,0,0)} + 3\underset{e_2}{(0,1,0)} + \underset{e_3}{(0,0,1)} = 2e_1 + 3e_2 + e_3$

$(1,2,0) = (1,0,0) + 2(0,1,0) = e_1 + 2e_2$

$(0,-1,-1) = -e_2 - e_3$

$$U_{BB'} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

linie, linie

$p = (2,3,7) = 2e_1 + 3e_2 + 7e_3$

$(2,3,7) = p = x(2,3,1) + y(1,2,0) + z(0,-1,-1)$

$p_{B'} = U^{-1} p_B = U^{-1} (2\ 3\ 7)$

NU LA EXAMEN! (lucruri mai serioase)

$U^{-1} = \frac{1}{\det U} \cdot U^*$

~~$U^* = \begin{pmatrix} -2 & 1 & -1 \\ -2 & 2 & \end{pmatrix}$~~

$$U^* = \begin{pmatrix} -2 & +2 & -2 \\ 1 & -2 & 1 \\ -1 & +2 & 1 \end{pmatrix}$$

$$U^{**} = \begin{pmatrix} -2 & +1 & -1 \\ +2 & -2 & +2 \\ -2 & +1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & +2 & -2 \\ 1 & -2 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

$\det U = - \ldots = -2$

$= -4 - 1 + 3 =$

$$U^{-1}(2\ 3\ 7) = \begin{pmatrix} -2 & 2 & -2 \\ 1 & -2 & 1 \\ -1 & 2 & 1 \end{pmatrix} (2\ 3\ 7) = \begin{pmatrix} +2 & -2 & +2 \\ -1 & +2 & -1 \\ +1 & -2 & -1 \end{pmatrix} = (8\ -12\ -6) = (4\ -6\ -3)$$

ex. 200, 208

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Seminar 5

208/p.140 $\overbrace{(2,2,3)}^{x_1}, \overbrace{(1,-1,0)}^{x_2}$

$\underbrace{(-1,2,1)}_{x_3}$

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\alpha_1(2,2,3) + \alpha_2(1,-1,0) + \alpha_3(-1,2,1) = 0$

$\Rightarrow (2\alpha_1 + \alpha_2 - \alpha_3, 2\alpha_1 - \alpha_2 + 2\alpha_3, 3\alpha_1 + \alpha_3) = 0$

$\Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \\ 3\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_3 = -3\alpha_1 \end{cases}$

$\det A = \begin{vmatrix} 2 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = -1 \neq 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow$ formează bază

$F = \{x_1, x_2, x_3\}$

$B = \{e_1, e_2, e_3\}$

$F \to B$

$x_1 = 2e_1 + 2e_2 + 3e_3$

$U = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$

$V^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

$u^t = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$

$U^* = \begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -2 & -4 \end{pmatrix}$

$U^{-1} = \frac{1}{\det U} \cdot U^*$

$U^{-1} = \frac{1}{-1} \begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$

?

$\Rightarrow e_1 = x_1 - 4x_2 - 3x_3$

$e_2 = x_1 - 5x_2 - 3x_3$

$e_3 = -x_1 + 6x_2 + 4x_3$

$x = (1,1,1) = e_1 + e_2 + e_3$

$\Rightarrow x = x_1 - 3x_2 - 2x_3$

~~$x_{B'} = x_B$~~

$x_{B'} = U^{-1} x_B \Rightarrow$ pe coloane

$x_{B'} = x_B \cdot U^{-1} \Rightarrow$ pe linii

Ex: Două spații izomorfe au aceeași dimensiune și reciproc

Fie $f: V \to V'$ izomorfism $\Rightarrow \dim V = \dim V'$

Fie $B = \{v_1, \dots, v_n\}$ bază în V

$f(B) = f(\{v_1, \dots, v_n\}) = \{f(v_1), f(v_m), \dots, f(v_n)\}$ bază în V'

$$\left.\begin{array}{l}\sum_{i=1}^{n} \alpha_i f(v_i) = 0 \Rightarrow f\left(\sum_{i=1}^{n} \alpha_i v_i\right) = 0 \\ \qquad\qquad f \text{ izomorfism}\end{array}\right| \Rightarrow$$

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0, \forall i$$

Fie $x' \in V'$

f izomorfism $\Rightarrow x' = f(x) \Rightarrow \exists\, x \in V$ a.î. $f(x) = x'$

$$x = \sum_{i=1}^{n} \alpha_i v_i \Rightarrow x' = f\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i f(v_i)$$

Fie V, V' a.î. $\dim V = \dim V'$

$B = \{v_1, \dots, v_n\} \subset V$ bază

$B' = \{v_1', \dots, v_n'\} \subset V'$ bază

$f: V \to V'$ liniară izomorfism

$f(v_i) = v_i'$, $i = \overline{1,n}$

$$f\left(\sum_{i=1}^{n}\alpha_i v_i\right)=\sum_{i=1}^{n}\alpha_i v_i'$$

$f(x+y)=f(x)+f(y), \forall x,y\in V$
$f(\alpha x)=\alpha f(x), \forall x\in V, \forall \alpha\in K$

$$f\left(\sum_{i=1}^{n}\alpha_i v_i+\sum_{i=1}^{n}\beta_i v_i\right)=f\left(\sum_{i=1}^{n}(\alpha_i+\beta_i)v_i\right)=\sum_{i=1}^{n}(\alpha_i+\beta_i)v_i'=$$

$$=\sum_{i=1}^{n}\alpha_i v_i'+\sum\beta_i v_i'=f\left(\sum\alpha_i v_i\right)+f\left(\sum\beta_i v_i\right)$$

<u>f inj</u>

$f(x)=0 \Rightarrow x=0$

$f\left(\sum\alpha_i v_i\right)=0 \Rightarrow \sum\alpha_i v_i'=0$

$\Rightarrow \alpha_i=0, \forall i \Rightarrow x=0$

$x'\in V'$

$x'=\sum\beta_i v_i'=\sum\beta_i f(v_i)=f(\underbrace{\sum\beta_i v_i}_{x})=f(x)$

(3) $f:\mathbb{R}^2\to\mathbb{R}^2$

$f((x_1,x_2))=(x_1,0)$

$B=\{e_1,e_2\}$

$f(e_1)=\alpha_{11}e_1+\alpha_{12}e_2$

$f(e_2)=\alpha_{21}e_1+\alpha_{22}e_2$

$M(f)_B=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$

$M(f)_B=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$f(e_1)=(1,0)=e_1=1\cdot e_1+0\cdot e_2$

$f(e_2)=(0,0)=0\cdot e_1+0\cdot e_2$

$B'=\{e_1', e_2'\}$

$e_1'=(1,1)$
$e_2'=(2,1)$ $\Big\}$ $\Rightarrow$ sunt luate aleator bazele în care vrea să le treacă

$M(f)_{B'}=?$ $\qquad M(f)_{B'}=U\, M(f)_B\, U^{-1}$

$U=\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$ - luăm intrările vectorilor și $\qquad U$ cu U^{-1}.

$\det(U)=-1$

~~$U^*=($~~

$f(e_1') = \alpha_{11}' e_1' + \alpha_{12}' e_2'$

$e(e_1') = \alpha_{21}' e_1' + \alpha_{22}' e_2'$

$$M(f)_{B'} = \begin{pmatrix} \alpha_{11}' & \alpha_{12}' \\ \alpha_{21}' & \alpha_{22}' \end{pmatrix}$$

$$U^* = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$M(f)_{B'} = U M(f)_B U^{-1}$$

$$M(f)_{B'} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

(4) $V = \mathbb{R}[x] \leq 3$ (pol de grad ≤ 3)

$B = \{p_1, p_2, p_3, p_4\}$, $p_1 = 1$, $p_2 = x$, $p_3 = x^2$, $p_4 = x^3$

$B' = \{p_1', p_2', p_3', p_4'\} = p_3 + 2ap_2 + a^2 p_1$

$p_1' = 1$

$p_2' = x + a$ $\qquad a \in \mathbb{R}, a \neq 0$

$p_3' = (x+a)^2$

$p_4' = (x+a)^3$

$D : V \to V$ $\qquad$ D derivatea

$$M(D)_B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$M(D)_{B'} = ?$

$\alpha_1 p_1' + \alpha_2 p_2' + \alpha_3 p_3' + \alpha_4 p_4' = 0$

coef. dominant este α_4,

$\Rightarrow \alpha_4 = 0$

$\Rightarrow \alpha_3 = 0$

$\Rightarrow \alpha_2 = 0$

$\Rightarrow \alpha_1 = 0$

$p_1' = 1 = p_1$

$p_2' = x + a = a p_1 + p_2$

$p_3' = (x+a)^2 = p_3 + 2a p_2 + a^2 p_1$

$p_4' = (x+a)^3 = p_4 + 3a p_3 + 3a^2 p_2 + a^3 p_1$

matrice de trecere

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 2a & 1 \end{pmatrix}$$

$\det U = 1$

$$U^t = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 1 & 3a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$U^{-1} = ?$

fie V inversa

$$V = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

$U \cdot V = I_4 \Rightarrow$

$$U \cdot V = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a \cdot a_1 + b_1 & a \cdot a_2 + b_2 & a a_3 + b_3 & a a_4 + b_4 \\ a^2 a_1 + 2a b_1 + c_1 & a^2 \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

sistem $\Rightarrow$

$$\begin{cases} a_1 = 1 \\ a_2 = a_3 = a_4 = 0 \\ b_2 = 1 \end{cases}$$

$b_1 = -a$

$b_3 = 0$

$b_4 = 0$

$b_1 = \ldots$

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ a^2 & -2a & 1 & 0 \\ -a^3 & 3a^2 & -3a & 1 \end{pmatrix}$$

$3a^2 - 6a^2 + d_2 = 0$

$d_2 = 3a^2$

$$M(D)_{B'} = U\,M(D)_B \cdot U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} U^{-1} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 6a & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ a^2 & -2a & 1 & 0 \\ -a^3 & 3a^2 & -3a & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

se putea face și așa:

$D(p_1') = 0$

$D(p_2') = p_1'$

$D(p_3') = 2p_2'$

$D(p_4') = 3p_3'$

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Seminar 6

$f = X^4 + X^3 + X^2 + X + 1 \in \mathbb{Q}[X]$ - ireductibil.

$f = (X^2 + aX + b)(X^2 + cX + d) = X^4 + (a+c)X^3 + (b + ac + d)X^2 + (bc + ad)X + bd$

$$\begin{cases} a + c = 1 \\ bd = 1 \\ ac + d + b = 1 \\ bc + ad = 1 \end{cases} \quad c = 1 - a \quad \Rightarrow d = \frac{1}{b}$$

$$\begin{cases} a(1-a) + \frac{1}{b} + b = 1 \\ b(1-a) + a \cdot \frac{1}{b} = 1 \end{cases} \Leftrightarrow \begin{cases} a - a^2 + \frac{b^2+1}{b} = 1 \\ b - ab + \frac{a}{b} = 1 \end{cases} \Leftrightarrow \begin{cases} ba - ba^2 + b^2 - b + 1 = 0 \\ b^2 - ab^2 + a - b = 0 \end{cases}$$

$(1-a)b^2 - b + a = 0$

$\Delta = 1 - 4a + 4a^2 = (1-2a)^2$

$b_{1,2} = \frac{1 \pm (1-2a)}{2(1-a)} \to 1, \quad \frac{a}{1-a}$

$b^2(1+a) - a^2 b + 1 - a = 0$

$\Delta = a^4 - 4(1+a)(1-a)$

$\Delta = a^4 + 4a^2 - 4$

$\boxed{b = 1}$ $-a - a^2 + 2 = 1$

$a^2 - a - 1 = 0 \qquad \Delta = 1 + 4 = 5 \Rightarrow a \notin \mathbb{Q}$

$\longrightarrow$

$a = \frac{m_1}{n}, \ (m_1, n) = 1$

$b = \frac{m_2}{n}$

$c = \frac{m_3}{n}$

$d = \frac{m_4}{n}$

$\left(X^2 + \frac{m_1}{n}X + \frac{m_2}{n}\right)\left(X^2 + \frac{m_3}{n}X + \frac{m_4}{n}\right) = f \qquad m_1, m_2, m_3, m_4, n \in \mathbb{Z}$

$(nX^2 + m_1X + m_2)(nX^2 + m_3X + m_4) = n^2(X^4 + X^3 + X^2 + X + 1)$

$$\begin{cases} X^3: \ \cancel{n}m_3 + \cancel{n}m_1 = n^{\cancel{2}} \\ \quad m_1 + m_3 = n \quad \Rightarrow m_3 = n - m_1 \\ X^2: \ nm_4 + nm_2 + m_1m_3 = n^2 \\ X: \ m_1m_4 + m_2m_3 = n^2 \\ m_2m_4 = n^2 \quad \Rightarrow m_4 = \frac{n^2}{m_2} \end{cases}$$

~~to~~ be continued...

Ex:

Cons. $f:\mathbb{R}^3 \to \mathbb{R}^2$ urm. fc. $f(e_1) = e_1' + e_2'$

$f(e_2) = e_1' - e_2'$

$f(e_3) = e_2'$

Det. baze pentru imag. lui f ($= \operatorname{Im} f$), Ker f, spațiu generat de liniile/col. lui $M(f)$ = matricea asoc. lui f.

a) ~~bază~~ pt. Im f.

$f((1,0,0)) = (1,1)$
$f((0,1,0)) = (1,-1)$
$f((0,0,1)) = (0,1)$

$$\operatorname{Im} f = \cancel{\{(1,1),(1,-1),(0,1)\}} = \{f(x) \mid x \in \mathbb{R}^3\} = \{(f(x_1, x_2, x_3)) \mid (x_1,x_2,x_3) \in \mathbb{R}^3\} =$$

$$= \{\cancel{f(x_1 e_1 + x_2 e_2 + x_3 e_3)} \mid (x_1,x_2,x_3) \in \mathbb{R}^3\} =$$

$$= \{x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3) \mid (x_1,x_2,x_3) \in \mathbb{R}^3\} =$$

$$= \{x_1(1,1) + x_2(1,-1) + x_3(0,1) \mid x_1,x_2,x_3 \in \mathbb{R}\} = \mathbb{R}^2$$

$$= \{(x_1 + x_2,\ x_1 - x_2 + x_3) \mid x_1,x_2,x_3 \in \mathbb{R}\}$$

baza are maxim 2 elem.

$(a,b) \in \mathbb{R}^2$.

x_1, x_2, x_3 $\quad a = x_1 + x_2$

$\quad b = x_1 + x_3 - x_2$

$$\begin{cases} x_2 = 0 \\ x_1 = a \\ x_3 = b - a \end{cases}$$

$\{(1,0),(0,1)\}$ - bază.

b) bază în Ker f.

$$= \{x \in \mathbb{R}^3 \mid f(x) = 0\}$$

$$\operatorname{Ker} f = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid f((x_1,x_2,x_3)) = (0,0)\} \cancel{f((x_1,x_2,x_3)) = (0,1)}\}$$

$$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 + x_2,\ x_1 - x_2 + x_3) = (0,0)\} =$$

$$= \{x_1, x_2, x_3 \in \mathbb{R} \mid x_1 = -x_2,\ x_1 - x_2 + x_3 = 0\}$$

$$\begin{cases} x_1 = -x_2 \Leftrightarrow x_2 = -x_1 \\ x_1 - x_2 + x_3 = 0 \end{cases} \Leftrightarrow \{x_1 + x_1 + x_3 = 0 \Leftrightarrow x_3 = -2x_1$$

$$\operatorname{Ker} f = \{(x_1, -x_1, -2x_1) \mid x_1 \in \mathbb{R}\}$$

$$= \{(x, -x, -2x) \mid x \in \mathbb{R}\} = \{(-x, x, 2x) \mid x \in \mathbb{R}\}$$

$$= \{x \cdot (-1, 1, 2) \mid x \in \mathbb{R}\} = \langle (-1, 1, 2) \rangle$$

c) $M(f)$.

matricea asociată acestei ~~transformări~~ aplicații liniare: transformări

$$M(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$V = \langle (1,1), (1,-1), (0,1) \rangle = \mathbb{R}^2$$

$$\alpha(1,1) + \beta(1,-1) = (\alpha+\beta, \alpha-\beta) = (0,0) \Leftrightarrow$$

$$\begin{cases} \alpha + \beta = 0 \\ \alpha - \beta = 0 \end{cases} \Leftrightarrow 2\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow \beta = 0$$

$$\cancel{V' = \langle (1,1,0), (1,-1,1) \rangle}$$

$$V' = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \mathbb{R}^2$$

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \beta = 0 \Rightarrow \alpha = 0$$

$\rightarrow b = \frac{a}{1-a}$

$$a - a^2 + \frac{a}{1-a} + \frac{1-a}{a} = 1 \quad / \cdot a(1-a)$$

$$a^2(1-a)^2 + a^2 + (1-a)^2 = a(1-a)$$

$$a^2 - 2a^3 + a^4 + a^2 + 1 - 2a + a^2 - a + a^2 = 0$$

$$a^4 - 2a^3 + 4a^2 - 3a + 1 = 0 \quad , a \in \mathbb{R}$$

$\Downarrow$

$\boxed{a_1 = \pm 1}$ fals!

IV Ex2 : $f : \mathbb{R}^4 \to \mathbb{R}^4$

$f(e_1) = e_1 + e_2 + e_3$; $f(e_2) = e_1 + e_2 + e_4$; $f(e_3) = e_3 - e_4$; $f(e_4) = e_4 - e_3$

) Baze ptr. Im f, Ker f, sp. generat de liniile și col. lui f.

) $v \in \mathbb{R}^4$ $f(v) = e_1 - e_2 + e_3$.

Ex3 : $f : \mathbb{R}^4 \to \mathbb{R}^4$, $f((x_1, x_2, x_3, x_4)) = (2x_1 + x_2, x_3, 4x_1, x_1 + x_2)$.

rang $M(f)$ = ? dimensiunea Ker f.

$\shortparallel$

dimens. Im f

o bază ptr. Im f, Ker f.

Ex 1) $f((1,0,0,0)) = (1,1,1,0)$
$f((0,1,0,0)) = (1,1,0,1)$
$f((0,0,1,0)) = (0,0,1,-1)$
$f((0,0,0,1)) = (0,0,-1,1)$

$= (x_1+x_2)e_1 + (x_1+x_2)e_2 + (x_1+x_3-x_4)e_3 + (x_2-x_3+x_4)e_4$

$\operatorname{Im} f = \{f(x) \mid x \in \mathbb{R}^4\} = \{f((x_1,x_2,x_3,x_4)) \mid (x_1,x_2,x_3,x_4) \in \mathbb{R}^4\} =$

$= \{x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3) + x_4 f(e_4) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\} =$

$= \{x_1(1,1,1,0) + x_2(1,1,0,1) + x_3(0,0,1,-1) + x_4(0,0,-1,1) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\} =$

$= \{(x_1+x_2, x_1+x_2, x_1+x_3-x_4, x_2-x_3+x_4) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\} = \mathbb{R}^4$

$x_1(1,1,1,0) + x_2(1,1,0,1) + x_3(0,0,1,-1) + x_4(0,0,-1,1)$ ~~(e_1,e_2,e_3,e_4) bază~~

~~$(a,a,b,c) \in \mathbb{R}^4$~~

~~x_1,x_2,x_3,x_4 $a = x_1+x_2$ $b = x_1+x_2$ $b = x_1+x_3-x_4$ $c = x_2-x_3+x_4$~~

~~$x_1 =$ $x_2 =$ $x_3 =$ $x_4 =$~~

$\langle(1,1,1,0),(1,1,0,1)\rangle$ bază

$\alpha(1,1,1,0) + \beta(1,1,0,1) = (0,0,0,0) \Leftrightarrow$

$\alpha = 0, \beta = 0$

$\dim(\operatorname{Im} f) = 2$

$\operatorname{Ker} f = \{x \in \mathbb{R}^4 \mid f(x) = 0\} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid f((x_1,x_2,x_3,x_4)) = (0,0,0,0)\} =$

$= \{x \in \mathbb{R}^4 \mid (x_1+x_2, x_1+x_2, x_1+x_3-x_4, x_2-x_3+x_4) \neq 0\}$

$$\begin{cases} x_1+x_2=0 \\ x_1+x_3-x_4=0 \\ x_2-x_3+x_4=0 \end{cases} \Leftrightarrow \begin{cases} x_1+x_2=0 \\ x_1+x_2=0 \end{cases} \Leftrightarrow \boxed{\begin{cases} x_1 = -x_2 \\ x_3-x_4 = x_2 \end{cases}} \Leftrightarrow \begin{matrix} x_2 = -x_1 \\ x_3-x_4 = -x_1 \end{matrix}$$

$= \{(a,-a,-a+b,b) \mid a,b \in \mathbb{R}\}$

$= \{(-x_2, x_2, x_2+x_4, x_4) \mid x_2,x_4 \in \mathbb{R}\} = \{a(1,1,1,0)+b(0,0,1,1) \mid a,b \in \mathbb{R}\}$

$= \{x_2(-1,1,1,0) + x_4(0,0,1,1) \mid x_2,x_4 \in \mathbb{R}\}$ $= a(1,-1,-1,0)+b(0,0,1,1)$

$= \langle(-1,1,1,0),(0,0,1,1)\rangle$

$\boxed{\langle(1,-1,-1,0),(0,0,1,1)\rangle \text{ bază}}$

$\dim(\operatorname{Ker} f) = 2$

$$A = M(f) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$C(A) = V' = \left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\1\end{pmatrix} \right\rangle$

$\alpha\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}\alpha\\\alpha+\beta\\-\beta\\\beta\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow \alpha = \beta = 0$

$L(A) = V = \langle(1,1,1,0),(1,1,0,1),(0,0,1,-1),(0,0,-1,1)\rangle$

$\alpha(1,1,1,0) + \beta(1,1,0,1) = (0,0,0,0)$ ~~$+\varepsilon(0,0,-1,1) = (0,0,0,0) \Leftrightarrow$~~

$\begin{cases}\alpha+\beta=0 \\ \alpha = 0\end{cases}$ $\beta = 0$

$V = \langle(1,1,1,0),(1,1,0,1)\rangle$

$v \in \mathbb{R}^4 \quad f(v) = e_1 - e_2 + e_3 = (1, -1, 1, 0)$

~~$\text{Im } f = \{(1,-1,1,0)\}$~~

$f(v) = (x_1 + x_2, x_1 + x_2, x_1 + x_3 - x_4, x_2 - x_3 + x_4) = (1, -1, 1, 0) \Leftrightarrow$

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = -1 \\ x_1 + x_3 - x_4 = 1 \\ x_2 - x_3 + x_4 = 0 \end{cases} \quad ? \quad \nexists v$$

<u>Ex 2</u> : $f : \mathbb{R}^4 \to \mathbb{R}^4, \ f((x_1, x_2, x_3, x_4)) = (2x_1 + x_2, x_3, 4x_1, x_1 + x_2)$

$\text{Im } f = \{(2x_1 + x_2, x_3, 4x_1, x_1 + x_2) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ $= \{x_1(2,0,4,1) + x_2(1,0,0,1) + x_3(0,1,0,0) \mid x_1, x_2, x_3 \in \mathbb{R}\}$

$= \langle x_1(2,0,4,1) + \ldots$

$\langle (2,0,4,1), (1,0,0,1), (0,1,0,0) \rangle$ bază

$\alpha(2,0,4,1) + \beta(1,0,0,1) + \gamma(0,1,0,0) = (0,0,0,0)$

$$\begin{cases} 2\alpha + \beta = 0 \\ \gamma = 0 \\ 4\alpha = 0 \Rightarrow \alpha = 0 \\ \alpha + \beta = 0 \Rightarrow \beta = 0 \end{cases} \qquad \dim(\text{Im } f) = 3$$

$\text{Ker } f = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid (2x_1 + x_2, x_3, 4x_1, x_1 + x_2) = (0,0,0,0)\} \Leftrightarrow$

$= \{x_4(0,0,0,1) \mid x_4 \in \mathbb{R}\} = \{(0,0,0,x_4) \mid x_4 \in \mathbb{R}\}$

$$\begin{cases} 2x_1 + x_2 = 0 \\ x_3 = 0 \\ 4x_1 = 0 \\ x_1 + x_2 = 0 \end{cases} \Leftrightarrow \left.\begin{matrix} x_1 = 0 \\ x_3 = 0 \end{matrix}\right\} \Rightarrow x_2 = 0$$

$\text{Ker } f = \{(0,0,0,1)\} = \langle (0,0,0,1) \rangle \quad \dim(\text{Ker } f) = 1$

1) $x_1 = x_2 = 0$ $\qquad\qquad x_1 = 1$ $\qquad\qquad \dim(\text{Im } f) = 2$

$x_3 = 1$ $\qquad\qquad x_2 = 0$

$x_4 = 0$ $\qquad\qquad x_3 = x_4 = 0$

$(0,0,1,-1) \in \text{Im } f \qquad (1,1,1,0) \in \text{Im } f.$

$\alpha(0,0,1,-1) + \beta(1,1,1,0) = (0,0,0,0) \Leftrightarrow \alpha = \beta = 0$

vect. $(0,0,1,-1)$ și $(1,1,1,0)$ rep. bază ptr. $\text{Im } f$.

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Seminar 7

$v_1, v_2 \in V$ liniar indep.

$W = \langle v_1, v_2 \rangle$ bază

$W = \underbrace{\langle v_1+v_2, v_1-v_2 \rangle}_{\text{bază}}$

$a_1(v_1+v_2) + a_2(v_1-v_2) = 0$

$(a_1+a_2)v_1 + (a_1-a_2)v_2 = 0$

$\begin{cases} a_1+a_2=0 \\ a_1-a_2=0 \end{cases}$

$\overset{(+)}{\Rightarrow} 2a_1 = 0 \Rightarrow a_1 = 0 \Rightarrow a_2 = 0$

Ex 2 (din temă)

$M(f) = \left(\begin{array}{ccc|c} 2 & 0 & 4 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$f(e_1) = ?e_1 + ?e_2 + ?e_3 + ?e_4$

$f(e_2) = \text{---}$

$f(e_1) = f((1,0,0,0)) = (2,0,4,1)$

$f(e_2) = (1,0,0,1)$

$f(e_3) = (0,1,0,0)$

$f(e_4) = (0,0,0,0)$

$\mathrm{Im}\, f = \langle (2,0,4,1), (1,0,0,1), (0,1,0,0) \rangle$

~~p 110~~

Exerc. 1: $A = \begin{pmatrix} 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & 2 & -1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$ rang $A = ?$

$\begin{vmatrix} 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \cdot (-1)^5 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -$

$$\begin{vmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 4-3 = 1 \neq 0 \Rightarrow \text{rang } A = 3$$

$f_A : \mathbb{R}^5 \to \mathbb{R}^4$, $f_A(x) = A \cdot x = (-x_2 + x_4 + x_5,\ x_1 + x_2 + x_3 + x_4 - x_5,$
$2x_1 + x_2 + 2x_3 + 2x_4 - x_5,\ x_1 + x_3 + x_4)$

$\text{Ker } f_A = \{x \in \mathbb{R}^5 \mid f_A(x) = 0\} = \{x \in \mathbb{R}^5 \mid Ax = 0\} \Leftrightarrow$

$$\begin{cases} -x_2 \\ x_1 \\ 2x_1 \\ x_1 \end{cases}$$

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & 2 & -1 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 - x_5 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_5 = 0 \end{cases}$$

$\Rightarrow x_3 = -x_1$ $\Rightarrow \text{Ker } f = \{(x_1, x_2, -x_1, 0, x_2) \mid x_1, x_2 \in \mathbb{R}\}$
$x_5 = x_2$
$x_4 = 0$ $\Rightarrow \dim(\text{Ker } f) = 2 \Rightarrow \dim(\text{Im } f) = 3.$

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Seminar 8

Corpuri finite.

Dem. K = corp. finit, nr. de elem. ale lui K e o putere a unui nr. prim.

$\varphi: \mathbb{Z} \longmapsto K$ morfism de inele unitare

$n \longmapsto n \cdot 1_K$

$\underline{TFI} \Rightarrow \mathbb{Z}/_{\ker \varphi} \simeq \operatorname{Im} f \subseteq K$

$\mathbb{Z}/_{p\mathbb{Z}}$ inel integru $\Rightarrow p =$ prim

Deci, $\mathbb{Z}_p \longrightarrow K$ incluziune; K $\mathbb{Z}_p$-spațiu vectorial

Dc. $|K| < \infty \Rightarrow$ ~~dim~~ $\dim_{\mathbb{Z}_p} K < \infty \Rightarrow K \underset{\text{izo de spații vectoriale}}{\simeq} \underbrace{\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{n \text{ ori}} \Rightarrow |K| = |\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{n \text{ ori}}| = p^n$

$\Rightarrow \forall p$ prim, $\forall n \geq 1$, $\exists$ corp cu p^n elemente.

$\Rightarrow \nexists$ corpuri cu 6 (10, 14, 15) elem

$K[X]/_{(f)}$, $f \in K[X]$ polinom ireductibil cu q^n elemente.

$\simeq$ corp

f

grad $f = n$, K corp finit cu q elemente

Construcția unui corp cu 4 elemente:

$K = \mathbb{Z}_2$, grad $f = 2$

$f = aX^2 + bX + c = X^2 + X + \hat{1} \in \mathbb{Z}_2[X]$

$f(\hat{0}) = \hat{1}$, $f(\hat{1}) = \hat{1} \Rightarrow f$ ireductibil.

$\mathbb{Z}_2[X]/_{(X^2+X+1)}$ corp cu 4 elemente.

$= \{\hat{0}, \hat{1}, \bar{x}, \bar{x} + \hat{1}\}$

Corp cu 8 elemente:

$K = \mathbb{Z}_2$

$f = X^3 + X + \hat{1}$, $f \in \mathbb{Z}_2[X]$

f ireductibile.

$\mathbb{Z}_2[X] / (X^3+X+\hat{1})$ - corp cu 8 elemente $= \{\hat{0}, \hat{1}, \overline{X}, \overline{X}+\hat{1}, \overline{X}^2, \overline{X}^2+\overline{X}, \overline{X}^2+\overline{X}+\hat{1},$
$\overline{X}^2+\hat{1}\}$

Corp cu 9 elemente:

$K = \mathbb{Z}_3$

$f = X^2 + 1 \in \mathbb{Z}_3[X]$, f ireductibile (nu are rădăcini peste $\mathbb{Z}_3$)

$\mathbb{Z}_3[X] / (X^2+1) = \{\hat{0}, \hat{1}, \hat{2}, \overline{X}, \overline{X}+\hat{1}, \overline{X}+\hat{2}, \hat{2}\overline{X}, \hat{2}\overline{X}+\hat{1}, \hat{2}\overline{X}+\hat{2}\}$

corp cu 9 el.

Matrice (simpatice):

probl. 217:

$(3\times 4)(4\times 4)$

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 2 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{C_1+C_2} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{L_2-L_1} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{C_2+C_1}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{C_4+C_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{L_2 \leftrightarrow L_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cancel{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{-1 \cdot L_2}{\sim} \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix}_{I_2},$$

rang $A = 2$

$$A \xrightarrow{C_1+C_2} A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ 2 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$L_1 \; \cancel{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

op. pe linii

op. pe col.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{= V \text{ - inversabile}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

" U - inversabile

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

De. $A \in \mathcal{M}_{m \times n}(K) \Rightarrow (\exists)\, U \in \mathcal{M}_m(K)$ inversabilă, $(\exists)\, V \in \mathcal{M}_n(K)$ inversabilă a.î. $UAV = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$, unde $r = \operatorname{rang} A$.

$$U \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 & -1 \\ 2 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \overset{(-1)L_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$U \cdot A \cdot V = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Temă: Fie $A \in \mathcal{M}_{m \times n}(K)$ și $U \in \mathcal{M}_m(K)$, $V \in \mathcal{M}_n(K)$ inversabile (U, V $\not\exists$)

2) $U \cdot A$, $A \cdot V$, $U \cdot A \cdot V$ au același rang $=$ rang A.

\+ ex. 218 – ineg. lui Sylvester

$\operatorname{rang}(AB) \leq \operatorname{rang} A = r$

$$UAV = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right), \quad r = \operatorname{rang} A.$$

$\underline{AB = U^{-1}(UAV) \cdot (V^{-1}B)}$

$$\operatorname{rang}(AB) = \operatorname{rang}(UAV)(V^{-1}B) =$$

$$= \operatorname{rang}\left(\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \cdot \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right)\right) =$$

$$= \operatorname{rang}\begin{pmatrix} \overset{r}{\{} B_1 & * \\ 0 & 0 \end{pmatrix} \leq r$$

Analog ptr. B

ALGEBRĂ

Seminar 9

211/p.140
matrice eșalon ~~matr.~~ 2×3 din $\mathbb{Z}_2$.

$$\begin{pmatrix} \boxed{\hat{1}} & \hat{0} & * \\ \hat{0} & \boxed{\hat{1}} & * \end{pmatrix}, \begin{pmatrix} \boxed{\hat{1}} & * & \hat{0} \\ \hat{0} & \hat{0} & \boxed{\hat{1}} \end{pmatrix}, \begin{pmatrix} \hat{0} & \boxed{\hat{1}} & * \\ \hat{0} & \hat{0} & \boxed{\hat{1}} \end{pmatrix}$$ - rang 2

$$\begin{pmatrix} \boxed{\hat{1}} & * & * \\ \hat{0} & \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \boxed{\hat{1}} & * \\ \hat{0} & \hat{0} & \hat{0} \end{pmatrix}, \begin{pmatrix} \hat{0} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & \hat{0} \end{pmatrix}$$ - rang 1, unde $* = \hat{0}, \hat{1}$

212. $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{pmatrix} \overset{L_2-L_1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \overset{L_3-2L_1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -5 & -4 \end{pmatrix} \sim$$

$$\overset{L_2\leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -5 & -4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \overset{(-1)\cdot L_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \overset{L_1-2L_2}{\sim} \begin{pmatrix} 1 & 0 & -7 & -4 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim$$

$$\overset{+7L_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \overset{L_2-5L_3}{\sim} \begin{pmatrix} \boxed{1} & 0 & 0 & -11 \\ 0 & \boxed{1} & 0 & 9 \\ 0 & 0 & \boxed{1} & -1 \end{pmatrix} \Rightarrow \text{rang } 3$$

col: $\langle e_1, e_2, e_3 \rangle = \mathbb{R}^3$.
linii: $\langle (1,0,0,-11), (0,1,0,9), (0,0,1,-1) \rangle$

213/p. 140

$$\begin{cases} x-2y+z+t=1 \\ x-2y+z-t=-1 \\ x-2y+z+5t=5 \end{cases}$$

$$A^e=\begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{pmatrix}$$

$$A^e \overset{L_2-L_1}{\sim} \begin{pmatrix} \boxed{1} & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 1 & -2 & 1 & 5 & 5 \end{pmatrix} \overset{L_3-L_1}{\sim} \begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix} \sim$$

$$\overset{(-\frac{1}{2})L_2}{\sim} \begin{pmatrix} 1 & -2 & 1 & \boxed{1} & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 5 & 5 \end{pmatrix} \overset{L_1-L_2}{\sim} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 5 & 5 \end{pmatrix} \overset{L_3-5L_2}{\sim}$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \end{pmatrix} \overset{L_3-L_1}{\sim} \begin{pmatrix} \boxed{1} & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x-2y+z=0 \\ t=1 \end{cases} \Rightarrow x=2y-z$$

prime = pivoți
sec = restul
x, t = prime = pe col. cu pivoți.

214/p. 140

$$\begin{cases} x+y-3z=-1 \\ 2x+y-2z=1 \\ x+y+z=3 \\ x+2y-3z=1 \end{cases}$$

$$A^e=\begin{pmatrix} 1 & 1 & -3 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{pmatrix} \overset{L_2-2L_1}{\sim}$$

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{pmatrix} \underset{L_3-L_1}{\sim} \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 1 & 2 & -3 & 1 \end{pmatrix} \underset{L_4-L_1}{\sim}$$

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \overset{(-1)L_2}{\sim} \begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \overset{L_1-L_2}{\sim} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \overset{L_4-L_2}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix} \overset{\left(\frac{1}{4}\right)L_3}{\sim} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 4 & 5 \end{pmatrix} \overset{L_1-L_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 5 \end{pmatrix} \sim$$

$$\overset{L_2+4L_4}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 5 \end{pmatrix} \overset{L_4-4L_3}{\sim} \begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\substack{L_1-L_4 \\ L_2-2L_4 \\ L_3-L_4}}{\sim} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ 0 = 1 \end{cases} \Rightarrow \text{sist. incompatibil.}$$

Exerc: Curs. $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 3 & -1 & 3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

1) $A \sim B$?

2) Dc. $A \sim B$, at. $U, V = ?$ inversabile $B = UAV$

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$U'BV' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$UAV = U'BV' \Rightarrow B = \underbrace{\boxed{(U')^{-1}U}}_{U_1} A \underbrace{\boxed{V(V')^{-1}}}_{V_1} \Rightarrow A \sim B$$

$$A \overset{L_3+L_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 3 \end{pmatrix} \overset{L_3-3L_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \overset{L_3+L_2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_3-C_1}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B \overset{L_2-L_1}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix} \overset{L_3-2L_1}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \overset{L_3-L_2}{\sim} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_2+C_1}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_3-C_1}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U_1 A V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U_2 B V_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U_1 A V_1 = U_2 B V_2 \Leftrightarrow B = \overbrace{(U_2^{-1} U_1)}^{U} A \overbrace{(V_1 V_2^{-1})}^{V}$$

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \qquad U_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad V_2^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U_2 = (L_3 - L_2)(L_3 - 2L_1)(L_2 - L_1)$$

$$U_2^{-1} = (L_2 - L_1)^{-1}(L_3 - 2L_1)^{-1}(L_3 - L_2)^{-1} = (L_2 + L_1)(L_3 + 2L_1)(L_3 + L_2)$$

$(L_3 + L_1)(L_3 + 2L_1)(L_3 + L_2) I_3$ $\Rightarrow$ de la dr. la stg.

$$V_2^{-1} = [(C_2 + C_1)(C_3 - C_1)]^{-1} = (C_3 - C_1)^{-1}(C_2 + C_1)^{-1} = (C_3 + C_1)(C_2 - C_1)$$

$$\cancel{C_3 + C_1} \cdot V_2^{-1} = (C_3 + C_1)(C_2 - C_1)$$

Temă:

Ex 1) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{pmatrix}$ → rang 2 $\quad B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ → rang 1

1) Aflați $U \in \mathcal{M}_3(\mathbb{R})$ a.î. $U \cdot A = B$ $\quad U = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$

2) Ar. că U nu e inversabilă. → rang.

→ sist. de ec. → se află U.

Ex 2) Rez. sist. peste $\mathbb{Z}_5$ → Gauss.

$$\begin{cases} \hat{2}x + y + \hat{3}z = \hat{4} \\ \hat{4}x + y + \hat{4}z = \hat{1} \\ x + y = \hat{3} \end{cases}$$

Ex 3)

$$\begin{cases} x - 2y + 3z = 4 \\ 2x - 3y + az = 5 \\ 3x - 4y + 5z = b \end{cases}, \quad a, b \in \mathbb{Q} \quad \to \text{Gauss}$$

$a, b = ?$ a.î. 1) sist. să nu aibă sol

2) să aibă o sol

3) să aibă o infinitate de sol

209, 210, 215 facem data viit.

Ex 2: $\begin{cases} \hat{2}x + y + \hat{3}z = \hat{4} \\ \hat{4}x + y + \hat{4}z = \hat{1} \\ x + y = \hat{3} \end{cases}$

$$\bar{A} = \begin{pmatrix} \hat{2} & \hat{1} & \hat{3} & \hat{4} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{1} & \hat{1} & \hat{0} & \hat{3} \end{pmatrix} \underset{L_1 - L_3}{\sim} \begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{1} & \hat{1} & \hat{0} & \hat{3} \end{pmatrix} \underset{L_3 - L_1}{\sim}$$

$$\begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{0} & \hat{1} & \hat{2} & \hat{2} \end{pmatrix} \underset{L_2 + L_1}{\sim} \begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & \hat{2} & \hat{2} \\ \hat{0} & \hat{1} & \hat{2} & \hat{2} \end{pmatrix} \underset{L_3 - L_2}{\sim} \begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & \hat{2} & \hat{2} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \end{pmatrix} \Rightarrow$$

$$\begin{cases} x + \hat{3}z = \hat{1} \\ y + \hat{2}z = \hat{2} \\ \hat{0} = \hat{0} \end{cases} \Rightarrow \quad x = \hat{1} - \hat{3}z, \quad y = \hat{2} - \hat{2}z$$

Ex 1)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{pmatrix} \overset{L_2-2L_1}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 3 & 3 & 2 \end{pmatrix} \overset{L_3-3L_1}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{pmatrix} \overset{\cancel{L_2+2L_1}}{\sim}$$

$$\overset{C_2-2C_3}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \overset{L_2\cdot(-1)}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \overset{L_3+L_2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_3-C_1}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_3-C_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_2-2C_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \overset{C_3-C_1}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{L_2+L_1}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U\cdot A = B \quad , \quad U = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

$$\begin{cases} x_1+2y_1+3z_1=1 \\ 2x_1+y_1+3z_1=2 \\ x_1+y_1+2z_1=1 \end{cases} \qquad \begin{cases} x_2+2y_2+3z_2=0 \\ 2x_2+y_2+3z_2=-3 \\ x_2+y_2+2z_2=-1 \end{cases} \qquad \begin{cases} x_3+2y_3+3z_3=3 \\ 2x_3+y_3+3z_3=3 \\ x_3+y_3+2z_3=2 \end{cases}$$

$$\Lambda_1 = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \underset{\substack{L_3-L_1 \\ (-1)L_2}}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & +3 & +3 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \overset{L_2+2L_3}{\sim} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \underset{L_3+L_2}{\overset{L_1-2L_2}{\sim}}$$

$$\begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1+z_1=0 \\ y_1+z_1=0 \\ 0=0 \end{array}$$

$$U_2 = \begin{pmatrix} \boxed{1} & 2 & 3 & 0 \\ 2 & 1 & 3 & -3 \\ 1 & 1 & 2 & -1 \end{pmatrix} \underset{L_3-L_1}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & -3 \\ 0 & -1 & -1 & -1 \end{pmatrix} \underset{(-1)L_2}{\overset{L_2+2L_3}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \underset{L_3+L_2}{\overset{L_1+2L_3}{\sim}}$$

$$\begin{pmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_2+z_2=-2 \\ y_2+z_2=1 \\ 0=0 \end{array}$$

$$\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+x_2+3x_3=2 \\ x_1+x_2+2x_3=1 \end{cases} \Leftrightarrow \begin{cases} x_1+x_3=1 \\ x_2+x_3=0 \end{cases} \Leftrightarrow \begin{matrix} x_1=1-x_3 \\ x_2=-x_3 \end{matrix}$$

$$\begin{cases} y_1+2y_2+3y_3=-1 \\ 2y_1+y_2+3y_3=-2 \\ y_1+y_2+2y_3=-1 \end{cases} \Leftrightarrow \begin{cases} y_2+y_3=0 \\ y_1+y_3=-1 \end{cases} \Leftrightarrow \begin{matrix} y_1=-1-y_3 \\ y_2=-y_3 \end{matrix}$$

$$\begin{cases} z_1+2z_2+3z_3=0 \\ 2z_1+z_2+3z_3=0 \\ z_1+z_2+2z_3=0 \end{cases} \Leftrightarrow \begin{cases} z_1+z_3=0 \\ z_2+z_3=0 \end{cases} \Leftrightarrow \begin{matrix} z_1=-z_3 \\ z_2=-z_3=z_1 \end{matrix}$$

$$U=\begin{pmatrix} 1-x_3 & -x_3 & x_3 \\ -1-y_3 & -y_3 & y_3 \\ -z_3 & -z_3 & z_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{pmatrix} \Leftrightarrow$$

$$U=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$1-x_3-2x_3+3x_3=1$

$2-2x_3-x_3+3x_3=2$

$$U=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_2+L_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rang } U=1 \Rightarrow U \text{ inversabilă}$$

<u>Ex 3)</u>

$$A=\begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{pmatrix} \underset{L_3-3L_1}{\xrightarrow{L_2-2L_1}} \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{pmatrix} \xrightarrow{L_3-2L_2}$$

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{pmatrix} \xrightarrow{L_1+2L_2} \begin{pmatrix} 1 & 0 & 2a-9 & -2 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & 8-2a & b-6 \end{pmatrix}$$

a) nicio sol $= \begin{cases} 8-2a=0 \\ b-6=1 \end{cases} \Leftrightarrow \boxed{\begin{matrix} a=4 \\ b=7 \end{matrix}}$

b) o sol: $8-2a\neq 0 \Leftrightarrow a\neq 4$

$8-2a=1 \Leftrightarrow a=\frac{7}{2}$

$a-6=0 \Leftrightarrow a=6$ Fals.

c) o infinitate de sol.

$$\begin{cases} 8-2a=0 \\ b-6=0 \end{cases} \Leftrightarrow \begin{matrix} a=4 \\ b=6 \end{matrix}$$

Atunci $A = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{matrix} x-z=-2 \\ y-2z=-3 \end{matrix} \Leftrightarrow \begin{matrix} x=-2+z \\ y=-3+2z \end{matrix}$

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$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ -1 & 2 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{L_3 - L_1 \\ L_4 + L_1}]{L_2 - 2L_1} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & 4 \\ 0 & 4 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{L_2 \cdot (-1)} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 4 \\ 0 & 4 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{L_1 - 2L_2 \\ L_4 - 4L_2}]{L_3 + L_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{pmatrix} \xrightarrow{L_3 - L_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{L_4 - 2L_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cancel{\xrightarrow{L_3 - L_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \xrightarrow{L_3 - L_4} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$$(A \mid I_4) \sim \ldots \simeq (I_4 \mid A^{-1})$$

4×8

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

A - inversabilă $\Leftrightarrow$ are nr. maxim de pivoţi

$$\begin{pmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ 0 & 0 & 1 & & * & * \\ 0 & 0 & 0 & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & * \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$$

$$(A \mid I_4) \xrightarrow{\text{eşalonez}} (I_4 \mid B)$$

$$\underset{\substack{\downarrow \\ 4\times 4 \text{ inversabilă}}}{U}(A \mid I_4) = (I_4 \mid B)$$

$$\underset{\substack{\downarrow \\ 4\times 4}}{U}\underset{\substack{\downarrow \\ 4\times 8}}{(A \mid I_4)} = \underset{\substack{\downarrow \\ 4\times 8}}{(UA \mid U)} = (I_4 \mid B)$$

$UA = I_4$; $U = B \Rightarrow \boxed{B = A^{-1}}$

$$A = \left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_1 - L_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_3 - L_1 \\ L_4 - 2L_1}}{\overset{L_2 - 3L_1}{\sim}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -3 & 1 & 3 & 0 \\ 0 & 1 & 3 & 4 & -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 3 & -2 & 0 & 2 & 1 \end{array}\right) \underset{L_2 - L_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -4 & -2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 3 & -2 & 0 & 2 & 1 \end{array}\right) \underset{L_4 + L_2}{\overset{L_3 - L_2}{\sim}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -4 & -2 & 1 & 1 & 0 \\ 0 & 0 & 6 & 8 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -4 & 1 & 3 & 1 \end{array}\right) \underset{L_3 + 5L_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -4 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -19 & 4 & 16 & 5 \\ 0 & 0 & -1 & -1 & -4 & 1 & 3 & 1 \end{array}\right) \underset{L_4 + L_3}{\overset{L_2 + 3L_3}{\sim}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 5 & -59 & 13 & 48 & 15 \\ 0 & 0 & 1 & 3 & -19 & 4 & 16 & 5 \\ 0 & 0 & 0 & 2 & -23 & 5 & 19 & 6 \end{array}\right)$$

$$A \sim^{L_1 - L_2} \left(\begin{array}{cccc|cccc} -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \underset{(-1)L_1}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\underset{\substack{L_3 - L_1 \\ L_4 - 2L_1}}{\overset{L_2 - 3L_1}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 1 & -1 & 1 & 0 \\ 0 & -3 & 2 & 3 & 2 & -2 & 0 & 1 \end{array}\right) \underset{(-1)L_2}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 1 & -1 & 1 & 0 \\ 0 & -3 & 2 & 3 & 2 & -2 & 0 & 1 \end{array}\right)$$

$$\underset{L_4 + 3L_2}{\overset{L_1 - L_2}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & -7 & 4 & 0 & 1 \end{array}\right) \underset{L_3 - L_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 8 & -5 & 1 & -1 \\ 0 & 0 & 2 & 3 & -7 & 4 & 0 & 1 \end{array}\right)$$

$$\underset{L_4 - 2L_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 8 & -5 & 1 & -1 \\ 0 & 0 & 0 & 1 & -23 & 14 & -2 & 3 \end{array}\right) \underset{L_3 - L_4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 31 & -19 & 3 & -4 \\ 0 & 0 & 0 & 1 & -23 & 14 & 2 & 3 \end{array}\right)$$

Ex 2) din temă : $\mathbb{Z}_5$

$$A^e = \begin{pmatrix} \hat{2} & \hat{1} & \hat{3} & \hat{4} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{1} & \hat{1} & \hat{0} & \hat{3} \end{pmatrix} \underset{L_1-L_3}{\sim} \begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{1} & \hat{1} & \hat{0} & \hat{3} \end{pmatrix} \underset{L_3-L_1}{\overset{L_2+L_1}{\sim}} \begin{pmatrix} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & & \\ & & & \end{pmatrix}$$

$$\underset{L_3 \leftrightarrow L_1}{\sim} \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{3} \\ \hat{4} & \hat{1} & \hat{4} & \hat{1} \\ \hat{2} & \hat{1} & \hat{3} & \hat{4} \end{pmatrix} \underset{L_3-2L_1}{\overset{L_2+L_1}{\sim}} \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{3} \\ \hat{0} & \hat{2} & \hat{4} & \hat{4} \\ \hat{0} & \hat{4} & \hat{3} & \hat{3} \end{pmatrix} \underset{\hat{3}(L_2)}{\sim}$$

$$\begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{3} \\ \hat{0} & \hat{1} & \hat{2} & \hat{2} \\ \hat{0} & \hat{4} & \hat{3} & \hat{3} \end{pmatrix} \underset{L_3+L_2}{\overset{L_1-L_2}{\sim}} \begin{pmatrix} \boxed{\hat{1}} & \hat{0} & \hat{3} & \hat{1} \\ \hat{0} & \boxed{\hat{1}} & \hat{2} & \hat{2} \\ \hat{0} & \hat{0} & \hat{0} & \hat{0} \end{pmatrix}$$

$$x + \hat{3}z = \hat{1} \qquad x = \hat{1} + \hat{2}z$$
$$y + \hat{2}z = \hat{2} \qquad y = \hat{2} + \hat{3}z$$

Exemplu curs:

$$A = \begin{pmatrix} 2x^2-12x+16 & 2-x & 2x^2-12x+17 \\ 0 & 3-x & 0 \\ x^2-6x+7 & 2-x & x^2-6x+8 \end{pmatrix} \underset{L_2-L_1}{\sim}$$

Forma diagonal canonică

$$\begin{pmatrix} 2x^2-12x+16 & 2-x & 2x^2-12x+17 \\ -2x^2+12x-16 & 1 & -2x^2+12x-17 \\ x^2-6x+7 & 2-x & x^2-6x+8 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2}$$

$$\begin{pmatrix} 2x^2+12x-16 & 1 & -2x^2+12x-17 \\ 2x^2-12x+16 & 2-x & 2x^2-12x+17 \\ x^2-6x+7 & 2-x & x^2-6x+8 \end{pmatrix} \underset{C_1 \leftrightarrow C_2}{\sim}$$

$$\begin{pmatrix} 1 & -2x^2+12x-16 & -2x^2+12x-17 \\ 2-x & 2x^2-12x+18 & 2x^2-12x+17 \\ 2-x & x^2-6x+7 & x^2-6x+8 \end{pmatrix} \underbrace{L_2+(x-2)L_1}_{L_3+(x-2)L_1}$$

$$\begin{pmatrix} 1 & -2x^2+12x-16 & -2x^2+12x-17 \\ 0 & -2x^3+18x^2-52x+48 & -2x^3+18x^2-53x+51 \\ 0 & -2x^3+17x^2-46x+39 & -2x^3+17x^2-47x+42 \end{pmatrix} \underbrace{C_2(-)C_1}_{C_3-(-)C_1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2x^3+18x^2-52x+48 & -- \\ 0 & -2x^3 & -- \end{pmatrix}$$

$$\begin{pmatrix} -2x^3+18x^2-52x+48 & -2x^3+18x^2-53x+51 \\ -2x^3+17x^2-46x+39 & -2x^3+17x^2-47x+42 \end{pmatrix} \underbrace{C_1-C_2}$$

$$\begin{pmatrix} x-3 & -2x^3+18x^2-53x+51 \\ x-3 & -2x^3+17x^2-47x+42 \end{pmatrix} \underbrace{C_2+(2x^2-12x+17)C_1}$$

$$\begin{pmatrix} x-3 & 0 \\ x-3 & (x-3)(2x^2-12x+17)-2x^3+17x^2-47x+42 \end{pmatrix}^2$$

$$\approx \begin{pmatrix} x-3 & 0 \\ 0 & -(x-3)^2 \end{pmatrix} \underbrace{(-1)L_2} \begin{pmatrix} x-3 & 0 \\ 0 & (x-3)^2 \end{pmatrix}$$

$x-3, (x-3)^2$ – factorii invarianți

$$A \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-3 & 0 \\ 0 & 0 & (x-3)^2 \end{pmatrix}$$

$2x^2+12x+16)(x-2)+2x^2-12x+16 = -2x^3+12x^2-16x+4x^2-24x+32+$
$+2x^2-12x+16 =$

$= -2x^3+18x^2-52x+48 \quad -x+2+1 = -2x^3+18x^2-53x+51$

$\cancel{2x^2+12x}\ -2x^3+12x^2-16x+4x^2-24x+32+x^2-6x+7 =$

$= -2x^3+17x^2-46x+39$

$-x+3$

~~[illegible]~~

$$\begin{array}{l|l} -2x^3+18x^2-53x+51 & x-3 \\ \hline 2x^3-6x^2 & -2x^2+12x-17 \\ \quad 12x^2-53x & \\ \quad -12x^2+36x & \\ \qquad -17x+51 & \\ \qquad 17x-51 & \\ \qquad\quad 0 & \end{array}$$

$$\begin{array}{l|l} -2x^3+17x^2-47x+42 & x-3 \\ \hline 2x^3-6x^2 & -2x^2+11x-14 \\ \quad 11x^2-47x & \\ \quad -11x^2+33x & \\ \qquad -14x+42 & \\ \qquad 14x+42 & \\ \qquad / \quad / & \end{array}$$

$(x-3)(2x^2-12x+17 - 2x^2+11x-14) \qquad -x^2+6x-9$

$\shortparallel$

$-x+3$

$-X-1-X^3-3X^2+4X$

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$$A = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$I_3X - A = \begin{pmatrix} X-4 & -5 & 2 \\ 2 & X+2 & -1 \\ 1 & 1 & X-1 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_3} \begin{pmatrix} 1 & 1 & X-1 \\ 2 & X+2 & -1 \\ X-4 & -5 & 2 \end{pmatrix} \xrightarrow{L_2 - 2L_1} \begin{pmatrix} 1 & 1 & X-1 \\ 0 & X & 1-2X \\ X-4 & -5 & 2 \end{pmatrix}$$

$$\xrightarrow{L_3-(X-4)L_1} \begin{pmatrix} 1 & 1 & X-1 \\ 0 & X & 1-2X \\ 0 & -X-1 & -X^2+5X-2 \end{pmatrix} \xrightarrow[C_3-(X-1)C_1]{C_2-C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & X & 1-2X \\ 0 & -X-1 & -X^2+5X-2 \end{pmatrix}$$

$$A' = \begin{pmatrix} X & 1-2X \\ -X-1 & -X^2+5X-2 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1-2X & X \\ -X^2+5X-2 & -X-1 \end{pmatrix} \xrightarrow{C_1+2C_2} \begin{pmatrix} 1 & X \\ -X^2+3X-4 & -X-1 \end{pmatrix}$$

$$\xrightarrow{C_2 - XC_1} \begin{pmatrix} 1 & 0 \\ X^2+3X-4 & +X^3-3X^2+3X-1 \end{pmatrix} \xrightarrow{L_2-(X^2+3X-4)L_1} \begin{pmatrix} 1 & 0 \\ 0 & +(X-1)^3 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (X-1)^3 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 4 & 2 \\ -1 & -4 & -1 \\ 0 & 0 & -2 \end{pmatrix} \qquad IX - C = \begin{pmatrix} X & -4 & -2 \\ 1 & X+4 & 1 \\ 0 & 0 & X+2 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2}$$

$$\begin{pmatrix} 1 & X+4 & 1 \\ X & -4 & -2 \\ 0 & 0 & X+2 \end{pmatrix} \xrightarrow{L_2 - XL_1} \begin{pmatrix} 1 & X+4 & 1 \\ 0 & -X^2-4X-4 & -2-X \\ 0 & 0 & X+2 \end{pmatrix} \xrightarrow[C_3-C_1]{C_2-(X+4)C_1} \begin{pmatrix} 1 & 0 & (\\ 0 & -(X+2)^2 & -(\\ 0 & 0 & X+ \end{pmatrix}$$

$$\begin{pmatrix} -(X+2)^2 & -(X+2) \\ 0 & X+2 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} -(X+2) & -(X+2)^2 \\ X+2 & 0 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} X+2 & 0 \\ -(X+2) & -(X+2)^2 \end{pmatrix} \xrightarrow{L_2+L_1}$$

$$\begin{pmatrix} x+2 & 0 \\ 0 & -(x+2)^2 \end{pmatrix}$$

$$C \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & x+2 & 0 \\ 0 & 0 & -(x+2)^2 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & -1 & 3 \\ 22 & 9 & -27 \\ 5 & 2 & -6 \end{pmatrix} \qquad Ix - B = \begin{pmatrix} x+3 & 1 & -3 \\ -22 & x-9 & 27 \\ -5 & -2 & x+6 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & x+3 & -3 \\ x-9 & -22 & 27 \\ -2 & -5 & x+6 \end{pmatrix}$$

$$\underset{L_2-(x-9)L_1}{\overset{L_3+2L_1}{\sim}} \begin{pmatrix} 1 & x+3 & -3 \\ 0 & -x^2+6x+5 & 3x \\ 0 & 2x+1 & x \end{pmatrix} \underset{C_3+3C_1}{\overset{C_2-(x+3)C_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overset{-x^2+6x+5}{\cancel{(x+1)(x+5)}} & 3x \\ 0 & 2x+1 & x \end{pmatrix}$$

$$\begin{pmatrix} -(x^2+6x+5) & 3x \\ 2x+1 & x \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 2x+1 & x \\ -x^2+6x+5 & 3x \end{pmatrix} \overset{C_1-2C_2}{\sim} \begin{pmatrix} 1 & x \\ -x^2+5 & 3x \end{pmatrix} \overset{C_1 - xL_1}{\sim}$$

$$\begin{pmatrix} 1 & 0 \\ -x^2+5 & -x^3-2x \end{pmatrix} \xrightarrow{L_2-(x^2+5)L_1} \begin{pmatrix} 1 & 0 \\ 0 & +x^3-2x \end{pmatrix}$$

$$IX - F = \begin{pmatrix} x-1 & -2 & -3 & -4 \\ 0 & x-1 & -2 & -3 \\ 0 & 0 & x-1 & -2 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \overset{C_1 \leftrightarrow C_2}{\sim} \begin{pmatrix} -2 & x-1 & -3 & -4 \\ x-1 & 0 & -2 & -3 \\ 0 & 0 & x-1 & -2 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \overset{C_1 - C_3}{\sim}$$

$$\begin{pmatrix} 1 & x-1 & -3 & -4 \\ x+1 & 0 & -2 & -3 \\ 1-x & 0 & x-1 & -2 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \underset{L_3-(1-x)L_1}{\overset{L_2-(x+1)L_1}{\sim}} \begin{pmatrix} 1 & x-1 & -3 & -4 \\ 0 & -x^2+1 & 3x+1 & 4x+1 \\ 0 & (x+1)^2 & -2x+2 & -4x+2 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \underset{\substack{C_3+3C_1 \\ C_4+4C_1}}{\overset{C_2-(x-1)C_1}{\sim}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-x^2 & 3x+1 & 4x+1 \\ 0 & (x+1)^2 & -2x+2 & -4x+2 \\ 0 & 0 & 0 & x-1 \end{pmatrix}$$

$$\begin{pmatrix} -x^2+1 & 3x+1 & 4x+1 \\ (x+1)^2 & -2x+2 & -4x+2 \\ 0 & 0 & x-1 \end{pmatrix} \underset{C_1 \leftrightarrow C_3}{\sim} \begin{pmatrix} 4x+1 & 3x+1 & -x^2+1 \\ -4x+2 & -2x+2 & (x+1)^2 \\ x-1 & 0 & 0 \end{pmatrix} \underset{C_1-C_2}{\sim}$$

$$\begin{pmatrix} x & 3x+1 & -x^2+1 \\ -2x & -2x+2 & (x+1)^2 \\ x-1 & 0 & 0 \end{pmatrix} \underset{L_1-L_3}{\sim} \begin{pmatrix} 1 & 3x+1 & -x^2+1 \\ -2x & -2x+2 & (x+1)^2 \\ x-1 & 0 & 0 \end{pmatrix} \underset{\substack{L_2+(2x)L_1 \\ L_3-(x-1)L_1}}{\sim}$$

$$\begin{pmatrix} 1 & 3x+1 & -x^2+1 \\ 0 & 6x^2+2 & -2x^3+x^2+4x+1 \\ 0 & -3x^2+2x+1 & x^3-x^2-x+1 \end{pmatrix} \underset{\substack{C_2-(3x+1)C_1 \\ C_3+(1-x^2)C_1}}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6x^2+2 & -2x^3+x^2+4x+1 \\ 0 & -3x^2+2x+1 & x^3-x^2-x+1 \end{pmatrix}$$

$$\begin{pmatrix} 6x^2+2 & -2x^3+x^2+4x+1 \\ -3x^2+2x+1 & x^3-x^2-x+1 \end{pmatrix}$$

$$\begin{pmatrix} x-1 & -2 & -3 & -4 \\ 0 & x-1 & -2 & -3 \\ 0 & 0 & x-1 & -2 \\ 0 & 0 & 0 & x-1 \end{pmatrix}$$

$$\begin{vmatrix} -2 & -3 & -4 \\ x-1 & -2 & -3 \\ 0 & x-1 & -2 \end{vmatrix} = -8-4(x-1)^2-6(x-1)-6(x-1) =$$

$$= -8-4x^2+8x-4-6x+6-6x+6 =$$

$$= -4x^2-4x = -4x(x+1)$$

Fol. Δ : $\Delta_3 =$ cmmdc$\left((x-1)^3, -4x(x+1), \dots\right) = 1 \Rightarrow \Delta_1 = \Delta_2 = 1$

$\Delta_4 = \det = (x-1)^4$

$$F \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (x-1)^4 \end{pmatrix}$$

225.

$$\begin{pmatrix} x-\hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & x & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & x-\hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & x \end{pmatrix} \overset{C_1 \leftrightarrow C_2}{\sim} \begin{pmatrix} \hat{1} & x-\hat{1} & \hat{0} & \hat{1} \\ x & \hat{0} & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & x-\hat{1} & \hat{0} \\ \hat{1} & \hat{0} & \hat{0} & x \end{pmatrix} \underset{\substack{L_3-L_1 \\ L_4-L_1}}{\overset{L_2-xL_1}{\sim}} \begin{pmatrix} \hat{1} & x-\hat{1} & \hat{0} & \hat{1} \\ \hat{0} & -x^2+x & \hat{0} & \hat{1}-x \\ \hat{0} & \cancel{x} & x-\hat{1} & \hat{1} \\ \hat{0} & \hat{1}-x & \hat{0} & x-\hat{1} \end{pmatrix} \sim \dots$$

$$\begin{pmatrix} -x^2+x & \hat{0} & \hat{1}-x \\ x & x-\hat{1} & \hat{1} \\ \hat{1}-x & \hat{0} & x-\hat{1} \end{pmatrix} \sim \begin{pmatrix} \hat{1}-x & \hat{0} & x-\hat{1} \\ x & x-\hat{1} & \hat{1} \\ -x^2+x & \hat{0} & \hat{1}-x \end{pmatrix} \overset{L_1+L_2}{\sim} \begin{pmatrix} \hat{1} & x-\hat{1} & x \\ x & x-\hat{1} & \hat{1} \\ -x^2+x & \hat{0} & \hat{1}-x \end{pmatrix} \sim$$

$$\underset{\substack{L_2-xL_1 \\ L_3+(x-x^2)L_1}}{\sim} \begin{pmatrix} \hat{1} & x-\hat{1} & x \\ \hat{0} & -\hat{1}-x^2 & \hat{1}-x^2 \\ \hat{0} & +x(x-\hat{1})^2 & \hat{1}-x+x^2-x^3 \end{pmatrix} \sim$$

$$\begin{pmatrix} (\hat{1}-x)(\hat{1}+x) & -(\hat{1}-x)(\hat{1}+x) \\ x(x-\hat{1})^2 & (\hat{1}-x)^2(\hat{1}+x) \end{pmatrix} \sim \begin{pmatrix} x^2+\hat{1} & -(x^2+\hat{1}) \\ x(x^2+\hat{1}) & (x^2+\hat{1})(x+\hat{1}) \end{pmatrix} \overset{2}{\sim}$$

$$\overset{2}{\sim} \begin{pmatrix} x(x^2+\hat{1}) & (x^2+\hat{1})(x+\hat{1}) \\ x^2+\hat{1} & -(x^2+\hat{1}) \end{pmatrix} \sim \begin{pmatrix} \cancel{x}(x^2+\hat{1}) & (x^2+\hat{1})(x+\cancel{\hat{1}}) \\ -(x^2+\hat{1}) & x^2+\hat{1} \end{pmatrix} \sim$$

$$\begin{pmatrix} \hat{1} & (x^2+\hat{1})x \\ -2x^2 & x^2+\hat{1} \end{pmatrix} \sim \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & x^2+\hat{1} \end{pmatrix}$$

ALGEBRĂ

$$XI-A=\begin{pmatrix} x-2 & 1 & -1 & 0\\ 0 & x-1 & -1 & 0\\ 0 & -1 & x-2 & 1\\ 1 & -1 & 1 & x-1\end{pmatrix}\overset{C_1\leftrightarrow C_4}{\sim}\begin{pmatrix} 1 & x-2 & -1 & 0\\ x-1 & 0 & -1 & 0\\ -1 & 0 & x-2 & 1\\ -1 & 1 & 1 & x-1\end{pmatrix}\underset{C_2-(x-2)C_1}{\overset{C_3+C_1}{\sim}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ x-1 & -(x-1)(x-2) & x-2 & 0\\ -1 & x-2 & x-3 & 1\\ -1 & x-2 & 0 & x-1\end{pmatrix}\sim \dots$$

$$\begin{pmatrix} -(x-1)(x-2) & x-2 & 0\\ x-2 & x-3 & 1\\ x-2 & 0 & x-1\end{pmatrix}\underset{C_1\leftrightarrow C_3}{\overset{L_1\leftrightarrow L_2}{\sim}}\begin{pmatrix} 1 & x-3 & x-2\\ 0 & x-2 & -(x-1)(x-2)\\ x-1 & 0 & x-2\end{pmatrix}\overset{L_3-(x-1)L_1}{\sim}$$

$$\begin{pmatrix} 1 & x-3 & x-2\\ 0 & x-2 & -(x-1)(x-2)\\ 0 & -(x-1)(x-3) & (x-2)(x+2)\end{pmatrix}\sim \dots$$

$$\begin{pmatrix} x-2 & -(x-1)(x-2)\\ -(x-1)(x-3) & (x-2)(x+2)\end{pmatrix}\overset{C_1\leftrightarrow C_2}{\sim}\begin{pmatrix} (1-x)(x-2) & x-2\\ (x-2)(x+2) & -(x-1)(x-3)\end{pmatrix}\overset{C_1+xC_2}{\sim}$$

$$\begin{pmatrix} x-2 & -(x-1)(x-2)\\ (x-1)^2 & -(x-1)^2\end{pmatrix}\sim\begin{pmatrix} 1 & 0\\ 0 & \frac{\Delta_2}{\Delta_1}\end{pmatrix}=\Delta_2$$

$$\Delta_1=1$$

$$\Delta_2=-(x-2)(x-1)^2+(x-1)^3(x-2)=(x-1)^2(x-2)(-1+x-1)=$$
$$=(x-1)^2(x-2)(x-2)=(x-1)^2(x-2)^2$$

$$XI-A\sim\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & (x-1)^2(x-2)^2\end{pmatrix}$$

LGEBRĂ

Seminar 11

2211'

$$E = \begin{pmatrix} 3 & 0 & 0 \\ a & 3 & 0 \\ b & 0 & -2 \end{pmatrix}$$

$$XI_3 - E = \begin{pmatrix} X-3 & 0 & 0 \\ -a & X-3 & 0 \\ -b & 0 & X+2 \end{pmatrix} =$$

$\Delta_1 = \text{cmmdc}(X-3, \cancel{X-3}, X+2, -a, -b)$.

CI: $a=b=0 \Rightarrow \Delta_1 = 1$

$\Delta_2 = \text{cmmdc}((X-3)^2, (X+2)(X-3), (X-3)(X+2)) =$

$= X-3$

$\Delta_3 = (X-3)^2(X+2)$

$d_1 = \Delta_1$

$d_2 = \Delta_2/\Delta_1$

$d_3 = \Delta_3/\Delta_2$

fact. inv: $\overset{d_1}{X-3}$, $(X+2)(X-3) = d_2$

$$XI_3 - E \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & X-3 & 0 \\ 0 & 0 & (X-3)^2(X+2) \end{pmatrix}$$

diviz. el: $X-3, X-3, X+2$, $\cancel{(X-3)^2}$

$$J_A = \begin{pmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{-2} \end{pmatrix}$$

CII: $a \neq 0$ sau $b \neq 0$.

$\Delta_1 = 1$

$\Delta_2 = \text{cmmdc}((X-3)^2, b(X-3), (X-3)(X+2), a(X+2))$

II. 1. $a=0, b \neq 0 \Rightarrow \Delta_2 = (X-3)$

$\Delta_3 = (X-3)^2(X+2)$ $\Rightarrow d_1 = 1$, $d_2 = X-3$, $d_3 = (X-3)(X+2)$ ...

II. 2. $a \neq 0$, $\Rightarrow \Delta_2 = 1$

$\Delta_3 = (X-3)^2(X+2)$ $\Rightarrow d_1 = 1, d_2 = 1, d_3 = (X-3)^2(X+2)$

$$J_A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$(x-3)^2,\ x+2 \Rightarrow$ diviz. el.

$J_2(3)$ $\quad J_1(-2) = J_{x+2}$

221./p.170

$$XI_3 - A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x-1)^3 \end{pmatrix}$$

$$J_3(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = J_A \ ; \ J_{(x-\lambda)^3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$x^3 - 2x = x(x^2-2)$$

$$XI_3 - B \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3-2x \end{pmatrix}$$

$\mathbb{Q}$: $X,\ X^2-2$

$J_X = (0)$

$J_{X^2-2} = \cancel{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}\ C_{X^2-2} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$\phantom{J_{X^2-2} = C_{X^2-2} = {}} -a_0 \ \ -a_1$

$$J_{A_{\mathbb{Q}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

π ireduct

$$J_{\pi^k} = \begin{pmatrix} C_\pi & N & & 0 \\ & \ddots & \ddots & \\ 0 & & C_\pi & N \end{pmatrix}$$

~~$XI_3 - C \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \end{pmatrix}$~~

$\mathbb{R}$ și $\mathbb{C}$: divizori elementari: $X,\ X-\sqrt{2},\ X+\sqrt{2}$

$$J_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$XI_3 - C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & X+2 & 0 \\ 0 & 0 & (X+2)^2 \end{pmatrix} \qquad J_C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

Temă:

224, de ter, 225,
227, 228, 229, 232
233, 235, 237, 238
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din astea, unul la examen

$$J_{(X+2)^2} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

diviz. el: $X+2, (X+2)^2$

$$XI_4 - D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (X-1)^2(X-2)^2 \end{pmatrix}$$

div. el: $(X-1)^2, (X-2)^2$

$$J_D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$XI_4 - F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (X-1)^4 \end{pmatrix}$$

div. el: $(X-1)^4$

$$D = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

$$J_F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$D^4 = ?$

$J_A = UAU^{-1}$, U - inversabilă

$J_A^n = (UA\cancel{U^{-1}})(\cancel{U}A\cancel{U^{-1}}) \cdot \ldots \cdot (\cancel{U}AU^{-1}) = UA^nU^{-1}$

$\Rightarrow A^n = U^{-1} J_A^n U$

$$J_A = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \Rightarrow J_A^n = \begin{pmatrix} J_1^n & 0 \\ 0 & J_2^n \end{pmatrix} = \begin{pmatrix} 1 & n & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2^n & n \cdot 2^{n-1} \\ 0 & 0 & 0 & 2^n \end{pmatrix}$$

$$J_1^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & \overset{C_n^1}{n} \\ 0 & 1 \end{pmatrix} \qquad J_2^n = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}$$

$$J_A U = UA$$

$$U = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix} \qquad J_A \cdot U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1+y_1 & x_2+y_2 & x_3+y_3 & x_4+y_4 \\ y_1 & y_2 & y_3 & y_4 \\ 2z_1+t_1 & 2z_2+t_2 & 2z_3+t_3 & 2z_4+t_4 \\ 2t_1 & 2t_2 & 2t_3 & 2t_4 \end{pmatrix}$$

$$U \cdot A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2x_1+x_4 & x_2-x_1-x_3-x_4 & x_1+x_2+2x_3+x_4 & x_3+x_4 \\ 2y_1+y_4 & y_2-y_1-y_3-y_4 & y_1+y_2+2y_3+y_4 & y_3+y_4 \\ 2z_1+z_4 & z_2-z_1-z_3-z_4 & z_1+z_2+2z_3+z_4 & z_3+z_4 \\ 2t_1+t_4 & t_2-t_1-t_3-t_4 & t_1+t_2+2t_3+t_4 & t_3+t_4 \end{pmatrix}$$

$$(1)\ \begin{cases} \cancel{x_1}+y_1 = \cancel{2}x_1+x_4 \\ \cancel{x_2}+y_2 = \cancel{x_2}-x_1-x_3-x_4 \\ \cancel{x_3}+y_3 = x_1+x_2+\cancel{2}x_3+x_4 \\ \cancel{x_4}+y_4 = \cancel{x_4}+x_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = x_1+x_4 \\ y_2 = -x_1-x_3-x_4 \\ y_3 = x_1+x_2+x_3+x_4 \\ y_4 = x_3 \end{cases}$$

$$U = \begin{pmatrix} \alpha_1 & \alpha_2 = 0 & \alpha_3 & \alpha_4 \\ \alpha_1+\alpha_4 & -\alpha_1-\alpha_3-\alpha_4 & \alpha_1+\cancel{\alpha_2}+\alpha_3+\alpha_4 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_4 & -\beta_1-\beta_2-\beta_3-\beta_4 & \beta_1+\beta_2+\beta_4 & \beta_3-\beta_4 \end{pmatrix}$$

$$(2)\ \begin{cases} \cancel{2z_1}+t_1 = \cancel{2z_1}+z_4 \\ \cancel{2}z_2+t_2 = \cancel{z_2}-z_1-z_3-z_4 \\ \cancel{2z_3}+t_3 = z_1+z_2+\cancel{2z_3}+z_4 \\ \cancel{2}z_4+t_4 = z_3+\cancel{z_4} \end{cases} \Leftrightarrow \begin{cases} t_1 = z_4 \\ t_2 = -z_1-z_3-z_4-z_2 \\ t_3 = z_1+z_2+z_4 \\ t_4 = z_3-z_4 \end{cases}$$

$$\begin{cases} y_1 = 2y_1 + y_4 \\ y_2 = y_2 - y_1 - y_3 - y_4 \\ y_3 = y_1 + y_2 + 2y_3 + y_4 \\ y_4 = y_3 + y_4 \end{cases} \iff y_1 = -y_4 \iff \alpha_3 = -\alpha_1 - \alpha_4$$

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \Rightarrow \alpha_2 = 0$

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$

$$\begin{cases} 2t_1 = 2t_1 + t_4 \\ 2t_2 = t_2 - t_1 - t_3 - t_4 \\ 2t_3 = t_1 + t_2 + 2t_3 + t_4 \\ 2t_4 = t_3 + t_4 \end{cases} \iff t_4 = 0 \Rightarrow \beta_3 = \beta_4$$

$t_1 + t_3 + t_4 + t_2 = 0$

$t_1 + t_2 + t_4 = 0 \Rightarrow t_1 + t_2 = 0 \iff -\beta_1 - \beta_2 - \beta_4 = 0$

$\beta_1 + \beta_2 + \beta_4 = \beta_3 - \beta_4 = 0$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \qquad \det U = 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$\boxed{U = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Temă: U, A^n, 224, ...

$$U \cdot A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 2 & -3 & 2 & 1 \\ 1 & -2 & 3 & 2 \\ 2 & -2 & 0 & 0 \end{pmatrix}$$

$$J_A \cdot U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 0 & 0 \end{pmatrix}$$

$$U \cdot A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$$

$$J_A \cdot U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

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224/170.

considerăm A^t de dimens 4.

$$A^t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad x^4 - 3x + 1 \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 \end{pmatrix}$$

$f(x) = x^4 - 1$

$f(x) = (x^2+1)(x-1)(x+1) = (x-i)(x+i)(x-1)(x+i)$

$$J_A = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

225/170.a) $A = \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & \hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & \hat{0} \end{pmatrix} \in \mathcal{M}_4(\mathbb{Z}_2)$

Formă Jordan.

$$XI - A = \begin{pmatrix} X & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & X & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & X & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & X \end{pmatrix} - \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & \hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & \hat{0} \end{pmatrix}$$

$$= \begin{pmatrix} X+\hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & X & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & X+\hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & X \end{pmatrix} \sim_{L_1 \leftrightarrow L_3} \begin{pmatrix} \hat{1} & \hat{1} & X+\hat{1} & \hat{0} \\ \hat{0} & X & \hat{0} & \hat{1} \\ X+\hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{1} & \hat{0} & X \end{pmatrix}$$

$$\underset{\sim}{L_3 + (X+1)L_1} \begin{pmatrix} \hat{1} & \hat{1} & X+\hat{1} & \hat{0} \\ \hat{0} & X & \hat{0} & \hat{1} \\ 0 & X & \hat{0} & \hat{1} \\ \hat{0} & \hat{1} & \hat{0} & X \end{pmatrix} \sim \begin{pmatrix} \hat{1} & \hat{0} & \cancel{X+\hat{1}}\ \hat{0} & \hat{0} \\ \hat{0} & X & \hat{0} & \hat{1} \\ \hat{0} & X & X^2+1 & \hat{1} \\ 0 & \hat{1} & \hat{0} & X \end{pmatrix}$$

$$\sim \begin{pmatrix} X & \hat{0} & \hat{1} \\ X & X^2+1 & \hat{1} \\ \hat{1} & \hat{0} & X \end{pmatrix} \underset{\sim}{C_1 \leftrightarrow C_3} \begin{pmatrix} \hat{1} & \hat{0} & X \\ \hat{1} & X^2+1 & X \\ X & \hat{0} & \hat{1} \end{pmatrix} \begin{matrix} L_2 + L_1 \\ \sim \\ L_3 + X \end{matrix}$$

$$\sim \begin{pmatrix} \hat{1} & \hat{0} & X \\ \hat{0} & X^2+1 & 0 \end{pmatrix} \sim \begin{pmatrix} \hat{1} & \hat{0} & \hat{0} \\ \hat{0} & X^2+1 & 0 \end{pmatrix}$$

factori unu $\underline{x^2+\hat{1},\ x^2+\hat{1}} = (x+\hat{1})^2$.

$$f_A(x) = (x^4+\hat{1})$$

$$m_A(x) = x^2+\hat{1}$$

$$J_A = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

valori proprii: $\lambda_1 = \hat{1}$

$W_1 = \{ X \in \mathbb{Z}_2^4 \mid AX = \lambda X \}$

$$\begin{pmatrix} \hat{1} & \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \\ \hat{1} & \hat{1} & \hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & \hat{0} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1+x_2+x_4 = x_1 & \Rightarrow x_2+x_4 = 0. \\ x_4 = x_2 \\ x_1+x_2+x_3 = x_3. & \Rightarrow x_1+x_2 = 0. \\ x_2 = x_4 \end{cases} \Rightarrow x_1 = x_4 = x_2$$

$W_1 = \{(a, a, b, a) \mid a, b \in \mathbb{Z}_2\}$.

$<(\hat{0}, \hat{0}, \hat{1}, \hat{0}), (\hat{1}, \hat{1}, \hat{0}, \hat{1})>$

-dimensiune 2

-multiplicitati 4 (la a puterea apare ~~a~~ in desc...)

e). $\begin{pmatrix} \hat{1} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{2} & \hat{2} & \hat{2} \\ \hat{0} & \hat{1} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{1} & \hat{0} \end{pmatrix} \in M_4(\mathbb{Z}_3)$

$$xI - A = \begin{pmatrix} x+\hat{2} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & x+\hat{1} & \hat{1} & \hat{1} \\ \hat{0} & \hat{2} & x & \hat{0} \\ \hat{0} & \hat{0} & \hat{2} & x \end{pmatrix} \underset{\sim}{\xrightarrow{L_1 \leftrightarrow L_2}} \begin{pmatrix} \hat{0} & x+\hat{1} & \hat{1} & \hat{1} \\ x+\hat{2} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{2} & x & \hat{0} \\ \hat{0} & \hat{0} & \hat{2} & x \end{pmatrix}$$

$$\underset{\sim}{\xrightarrow{C_1 \leftrightarrow C_3}} \begin{pmatrix} \hat{1} & x+\hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & x+\hat{2} & \hat{0} \\ \hat{0} & \hat{2} & x & \hat{0} \\ \hat{0} & \hat{0} & \hat{2} & x \end{pmatrix} \sim \begin{pmatrix} \hat{1} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & x+\hat{2} & \hat{0} \\ \hat{0} & \hat{2} & x & \hat{0} \\ \hat{0} & \hat{0} & \hat{2} & x \end{pmatrix} \sim \begin{pmatrix} \hat{0} & x+\hat{2} & \hat{0} \\ \hat{2} & x & \hat{0} \\ \hat{0} & \hat{2} & x \end{pmatrix},$$

$\underset{\sim}{L_1+L_3}$ ... în final trebuie să dea $\begin{pmatrix} \hat{1} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{1} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & x^4-\hat{1} \end{pmatrix}$

$\hat{2} - x(x-\hat{2}) =$
$= \hat{2} - x^2 + \hat{2}x$
$= 2 - x^2 - x$

$$\begin{vmatrix} x & \hat{2} & \hat{1} & \hat{1} \\ \hat{2} & x & \hat{0} \\ \hat{0} & \hat{2} & x \end{vmatrix} = (x+\hat{1})x^2 + \hat{1} + x$$
$$= x^3 + x^2 + x + 1 = x^2(x+1) + (x+1)$$
$$= (x+1)(x^2+1)$$

$\Delta_3 =$ cmmdc$\left((x-\hat{1})(x^2+x+\hat{1}), (x+\hat{1})(x^2+\hat{1}), \dots\right) = 1.$

$\Delta_4 = x^4 - \hat{1}$

$d_1 = x^4 - \hat{1} = (x^2-\hat{1})(x^2+\hat{1}) = (x-\hat{1})(x+\hat{1})(x^2+\hat{1})$
$= (x+\hat{2})(x+\hat{1})(x^2+\hat{1})$

div elem: $(x+\hat{1})$, $(x+\hat{2})$, $(x^2+\hat{1})$

$$J_A = \begin{pmatrix} \hat{1} & \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{2} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \\ \hat{0} & \hat{0} & \hat{2} & \hat{0} \end{pmatrix}$$

234) $A=\begin{pmatrix}5 & 4 & 2\\ 4 & 5 & 2\\ 2 & 2 & 2\end{pmatrix}$

$$IX-A=\begin{pmatrix}X-5 & -4 & -2\\ -4 & X-5 & -2\\ -2 & -2 & X-2\end{pmatrix}\overset{C_1\leftrightarrow C_3}{\sim}\begin{pmatrix}-2 & -4 & X-5\\ -2 & X-5 & -4\\ X-2 & -2 & -2\end{pmatrix}.$$

$$\overset{-\frac{1}{2}C_1}{\sim}\begin{pmatrix}1 & -4 & X-5\\ 1 & X-5 & -4\\ \frac{2-X}{2} & -2 & -2\end{pmatrix}\overset{L_2-L_1}{\sim}\begin{pmatrix}1 & -4 & X-5\\ 0 & X-1 & -X+1\\ \frac{2-X}{2} & -2 & -2\end{pmatrix}$$

$$\overset{L_3-\frac{2-X}{2}L_1}{\sim}\begin{pmatrix}1 & -4 & X-5\\ 0 & X-1 & -X+1\\ 0 & 2-2X & \cancel{-X^2+3X-14}\end{pmatrix}$$

$-2-\frac{2-X}{2}\cdot(-4)=-2+2(2-X)=2-2X$

$-2-\frac{2-X}{2}\cdot(X-5)=-2-\frac{(2-X)(X-5)}{2}=\frac{-4-(2X-10-X^2+5X)}{2}$

$=\frac{-4-(2-X)(X-5)}{2}=\frac{-4-(2X-10-X^2+5X)}{2}=\frac{-4-2X+10+X^2 \ldots}{2}$

$=\frac{X^2-7X+6}{2}$

$$\sim\begin{pmatrix}1 & 0 & 0\\ 0 & X-1 & 1-X\\ 0 & 2-2X & \frac{X^2-7X+6}{2}\end{pmatrix}\sim\begin{pmatrix}X-1 & 1-X\\ -2(X-1) & \frac{X^2}{2}-\frac{7X}{2}+3\end{pmatrix}$$

$$\overset{L_2+2L_1}{\sim}\begin{pmatrix}X-1 & 0\\ 0 & \frac{X^2}{2}-\frac{11X}{2}+5\end{pmatrix}\overset{2L_2}{\sim}\begin{pmatrix}X-1 & 0\\ 0 & (X-1)(X-10)\end{pmatrix}$$

2 fact invar

$d_1=X-1$

$d_2=(X-1)(X-10)$

3 div elem: $X-1,\ X-1,\ X-10$

$$J_A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 10\end{pmatrix}$$

$P_A(X)=(X-1)^2(X-10)$, $m_A=(X-1)(X-10)$

$P_A(X)=(X-1)(X^2-11X+10)=X^3-11X^2+10X-X^2+11X-10$

$=X^3-12X^2+21X-10$.

$A^3-12A^2+21A-10I_3=0$

$P_A(X)=\det(XI-A)$; $P_A(0)=\det(-A)=(-1)^n\cdot\det(A)$.

$P_A(0)=-10 \Rightarrow$

$$A^3-12A^2+21A-10I_3=0 \quad |\cdot A^{-1}$$

$$\Rightarrow 10A^{-1}=A^2-12A+21I_3$$

$$\Rightarrow A^{-1}=\frac{1}{10}\cdot A^2-\frac{12}{10}A+\frac{21}{10}I_3$$

val proprii: $\lambda_1=1$, $\lambda_2=10$.

$W_1=\{X\in\mathbb{C}^3 \mid AX=X\}$

$$\begin{pmatrix}5&4&2\\4&5&2\\2&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\Rightarrow\begin{cases}5x_1+4x_2+2x_3=x_1\\4x_1+5x_2+2x_3=x_2\\2x_1+2x_2+2x_3=x_3\end{cases}$$

$$\Rightarrow\begin{cases}4x_1+4x_2+2x_3=0\\4x_1+4x_2+2x_3=0\\2x_1+2x_2+x_3=0\end{cases}\Rightarrow 2x_1+2x_2+x_3=0$$

$$\Rightarrow W_1=\{(a,b,-2a-2b)\mid a,b\in\mathbb{C}\}=\langle(1,0,-2),(0,1,-2)\rangle$$

$W_{10}=\{X\in\mathbb{C}^3 \mid AX=10X\}$

$$\Rightarrow\begin{pmatrix}5&4&2\\4&5&2\\2&2&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}10x_1\\10x_2\\10x_3\end{pmatrix}$$

$$\Rightarrow\begin{cases}5x_1+4x_2+2x_3=10x_1\\4x_1+5x_2+2x_3=10x_2\\2x_1+2x_2+2x_3=10x_3\end{cases}\Rightarrow\begin{cases}4x_2+2x_3-5x_1=0\\4x_1+2x_3-5x_2=0\\2x_1+2x_2-8x_3=0\end{cases}$$

$$\Rightarrow x_1+x_2-4x_3=0\Rightarrow x_1=4x_3-x_2$$

$$4x_2+2x_3-5(4x_3-x_2)=0\Rightarrow 4x_2+2x_3=20x_3+1$$

$$\Rightarrow 18x_3+6x_2=0$$

$$\Rightarrow 3x_3+x_2=0$$

$$16x_3-4x_2+2x_3-5x_2=0.$$

$$\Rightarrow 18x_3-9x_2=0\Rightarrow 2x_3-x_2=0.$$

$$\Rightarrow x_2=2x_3$$

$$x_1=2x_3$$

$$\Rightarrow W_{10}=\{(2a,2a,a)\mid a\in\mathbb{C}\}\Rightarrow W_{10}=\langle(2,2,1)$$