Tutorial # 3

1. Let $Q[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Q\}$. That $Q[\sqrt{3}]$ is a commutative ring with identity. Prove that $Q[\sqrt{3}]$ is a field.

2. Let Q be the field of rational numbers then show that
$$Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$$

3. Find a basis of $Q(\sqrt[5]{3})$ over Q.

4. Gaussian integer is a complex number such that its real and imaginary parts are both integers. $Z[i] = \{a + bi \mid a, b \in Z\}$ is a ring of Gaussian integers. Prove that the ring of Gaussian integers modulo 3 is a field. Also find its characteristic.

5. Is $\sqrt{2} + \sqrt[3]{7}$ algebraic over the field of rational numbers? Justify.

6. Let F be the field of rational numbers and $f(x) = x^4 + x^2 + 1 \in F[x]$. Show that $F(\omega)$ where ω is cube root of unity is a splitting field of $f(x)$. Also determine the degree of the splitting field of $f(x)$ over F.

7. Show that $\sqrt{2 + \sqrt{3}}$ is algebraic over Q.

8. Prove that $F_3[x]/x^2{+}1$ is a field. How many elements does the field have?

9. Prove that every non-zero element in $GF(2^n)$ possesses a unique multiplicative inverse.

10. Construct the field F_{49}.

11. Find the number of monic irreducible polynomials in $F_3[x]$ of degree 12.

12. If a is an algebraic integer and m is an ordinary integer, prove
    (a) $a + m$ is an algebraic integer.
    (b) ma is an algebraic integer.

13. (a) Let α be a root of $x^2 + 1 = 0$, and K be the field $F_3[\alpha]$. Write down a basis for K, considered as a vector space over F_3. Write out the elements of F_1 explicitly.
    (b) Deduce that if you repeat the construction in (a) with a different quadratic polynomial irreducible over F_3 (instead of $x^2 + 1$), you get the same field K.

14. Find all the primitive elements of the field $GF(3^2) = GF(3)/(x^2 + x + 2)$.