# Filling in Details of Alard 2007 MNRAS Paper

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## 1 Intro

Quick reminder for below: $r_s, y, R_0, r_0$ are in the source plane, while $r, x, \theta$ are in the image plane.

## 2 Basic Ideas

To get to eq.1 of Alard 2007, use $\vec{r_s} = \vec{r} - \vec{\alpha}$, where $\vec{r_s}$ is the angular position of the point source in the source plane (see fig. in Narayan and Bartlemann for this), $\vec{r} = r\hat{r}$ is the angular position of the point source in the image plane, and $\vec{\alpha}$ is the deflection angle, which is equal to the radial gradient of the 2D projected potential $\phi$, which in cylindrical coordinates is

$$\nabla_r \phi = \frac{\partial \phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta}. \tag{1}$$

(also, we use $\hat{r}, \hat{\theta}$ in place of Alard's $\hat{u_r}, \hat{u_\theta}$).
Thus

$$\begin{aligned} \vec{r_s} &= r\hat{r} - \left(\frac{\partial \phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta}\right) & (2) \\ &= \left(r - \frac{\partial \phi}{\partial r}\right)\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta} & (3) \end{aligned}$$

Now let's go to the simplest case of an on-axis point source with perfectly axially symmetric potential $\phi = \phi_0$. In this case $r_s = 0$, $\frac{\partial \phi_0}{\partial \theta} = 0$ since there is no angular variation of the potential.
Thus, eq.(3) reduces to

$$0 = r - \frac{\partial \phi}{\partial r} \tag{4}$$

or

$$r = \left.\frac{\partial \phi}{\partial r}\right|_{r_E} = r_E \tag{5}$$

Where the derivative of the potential is to be evaluated at the Einstein radius, $r_E$.
This indicates the point source is spread into a circle of radius $r_E$ in the image plane.
We now take the first step of perturbing the situation:

$$\begin{aligned} r_s &= \epsilon y & (6) \\ \phi &= \phi_0 + \epsilon\psi & (7) \end{aligned}$$

which imply we have moved the source off-axis, and added a non-axially symmetric potential, both to the exact same order $\epsilon$ by assumption. Note that the perturbed distance $y$ is in the source plane.
Note that this changes the notation of the first eq in Alard eq.3 (which is fairly nonsensical, of course).
Eq.(6) leads to a modification of eq.(5) into

$$r = r_E + \epsilon x \tag{8}$$

which changes the notation in Alard eq.4 (which is somewhat weird notation, since his $dr$ is not necessarily a small distance), and we do not follow his convention of setting $r_E \equiv 1$.
Note that now the perturbed distance $x$ is in the image plane.
To recap: $r_s, y$ are in the source plane, while $r, x$ are in the image plane.

$$\phi = \phi_0 + \epsilon\psi = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n\phi_0}{dr^n}\right|_{r_E} (r - r_E)^n + \epsilon \frac{1}{n!} \left.\frac{d^n\psi(\theta)}{dr^n}\right|_{r_E} (r - r_E)^n \tag{9}$$

or

$$\phi = \sum_{n=0}^{\infty} \left[ \frac{1}{n!} \left.\frac{d^n\phi_0}{dr^n}\right|_{r_E} + \epsilon \frac{1}{n!} \left.\frac{d^n\psi(\theta)}{dr^n}\right|_{r_E} \right] (r - r_E)^n \tag{10}$$

And now define

$$C_n \equiv \frac{1}{n!} \left.\frac{d^n\phi_0}{dr^n}\right|_{r_E} \tag{11}$$

$$f_n(\theta) \equiv \frac{1}{n!} \left.\frac{d^n\psi(\theta)}{dr^n}\right|_{r_E} \tag{12}$$

So that we can write eq.(10) as

$$\phi = \sum_{n=0}^{\infty} [C_n + \epsilon f_n] (r - r_E)^n \tag{13}$$

where $f_n = f_n(\theta)$.
Now let's go back and substitute eq.(6) and eq.(8) into eq.(3) to find

$$\epsilon y = \left( r_E + \epsilon x - \frac{\partial\phi}{\partial r} \right) \hat{r} - \frac{1}{r_E + \epsilon x} \frac{\partial\phi}{\partial\theta} \hat{\theta} \tag{14}$$

Now substitute eq.(13) into this to find

$$\epsilon y = \left( r_E + \epsilon x - \frac{\partial}{\partial r} \left( \sum_{n=0}^{\infty} [C_n + \epsilon f_n] (r - r_E)^n \right) \right) \hat{r} - \\ \frac{1}{r_E + \epsilon x} \frac{\partial}{\partial\theta} \left( \sum_{n=0}^{\infty} [C_n + \epsilon f_n] (r - r_E)^n \right) \hat{\theta} \tag{15}$$

Remembering that $C_n, f_n$ are independent of $r$, and using $r - r_E = \epsilon x$ :

$$\epsilon y = \left( r_E + \epsilon x - \sum_{n=0}^{\infty} [C_n + \epsilon f_n] n (\epsilon x)^{n-1} \right) \hat{r} - \\ \frac{1}{r_E + \epsilon x} \frac{\partial}{\partial\theta} \left( \sum_{n=0}^{\infty} [C_n + \epsilon f_n] (\epsilon x)^n \right) \hat{\theta} \tag{16}$$

Let us expand eq.(16) in orders, first the zero'th order piece in $\epsilon$:

$$0 = (r_E - C_1) \hat{r} \rightarrow r_E = C_1 = \left.\frac{\partial\phi}{\partial\theta}\right|_{r_E} \tag{17}$$

Next the first order piece in $\epsilon$:

$$\epsilon y = (\epsilon x - (2C_2\epsilon x + \epsilon f_1)) \hat{r} - \frac{1}{r_E} \epsilon \frac{\partial f_0}{\partial\theta} \hat{\theta} \tag{18}$$

where the coefficient of $C_2$ was expanded to order with $n = 2$, that of the $f_1$ term with $n = 1$, and that of the $\frac{\partial f_1}{\partial\theta}$ term with $n = 0$. Bit involved, but this is how to get each term proportional to $\epsilon$.

Now divide through by $\epsilon$ and collect terms:

$$y = [(1 - 2C_2)x - f_1] \hat{r} - \frac{1}{r_E} \frac{\partial f_0}{\partial\theta} \hat{\theta} \tag{19}$$

or, defining $\kappa_2 = 1 - 2C_2$

$$y = [\kappa_2 x - f_1] \hat{r} - \frac{1}{r_E} \frac{\partial f_0}{\partial\theta} \hat{\theta} \tag{20}$$

Figure 1: Circular source in the source plane.

# 3 Reconstruction of Images

## 3.1 Circular Source Contours

In this section we study the mapping of a circular contour using this formalism.

Notation: $\vec{r}_0$ points from the origin (i.e center of the source plane, looking directly on-axis from us through the lens to the source) to the center of the actual source object (like a small circular galaxy), $\vec{R}_0$ is the vectorial radius of this source from its center, and $\vec{r}_s$ is the vector from the origin to some point on the edge of the circular source. As a first stab, see Fig.1 for this configuration.

So:

$$\vec{r}_s = \vec{r}_0 + \vec{R}_0, \quad \rightarrow \vec{R}_0 = \vec{r}_s - \vec{r}_0, \tag{21}$$

Replacing $r_s$ by eq.(20), we obtain:

$$\vec{R_0} = [\kappa_2 x - f_1(\theta)]\,\hat{r} - \frac{1}{r_E}\frac{\partial f_0(\theta)}{\partial\theta}\hat{\theta} - \vec{r_0}. \tag{22}$$

Note that $\vec{r}_0 = (x_0, y_0)$ lies in the source plane, but we can write it in terms of the cylindrical unit vectors

$$\hat{r} = (\cos\theta, \sin\theta) \tag{23}$$

$$\hat{\theta} = (-\sin\theta, \cos\theta) \tag{24}$$

as following

$$\vec{r}_0 = (\vec{r}_0 \cdot \hat{\theta})\hat{\theta} + (\vec{r}_0 \cdot \hat{r})\hat{r} \tag{25}$$

or

$$\vec{r}_0 = [x_0\,\cos\theta + y_0\,\sin\theta]\,\hat{r} + [-x_0\,\sin\theta + y_0\,\cos\theta]\,\hat{\theta}. \tag{26}$$

So that eq.(22) becomes

$$\vec{R_0} = [\kappa_2 x - f_1(\theta)]\,\hat{r} - \frac{1}{r_E}\frac{\partial f_0(\theta)}{\partial\theta}\hat{\theta} - [x_0\,\cos\theta + y_0\,\sin\theta]\,\hat{r} + [-x_0\,\sin\theta + y_0\,\cos\theta]\,\hat{\theta} \tag{27}$$

Defining

$$\overline{f}_i(\theta) = f_i(\theta) + (x_0\cos\theta + y_0\sin\theta)r_E^{1-i}, \quad i = 0, 1 \tag{28}$$

this becomes

$$\vec{R}_0 = \left[\kappa_2 x - \overline{f}_1(\theta)\right]\hat{r} - \frac{1}{r_E}\frac{\partial\overline{f}_0(\theta)}{\partial\theta}\hat{\theta}. \tag{29}$$

(Alard eq.11).

Now, taking the square of this we obtain

Solving this for $x(\theta)$, the radius of the arc in the image plane as a function of theta, the following two solutions are obtained:

$$x = \frac{1}{\kappa_2}\left[\overline{f}_1(\theta) \pm \sqrt{R_0^2 - \left(\frac{1}{r_E}\frac{\partial \overline{f}_0(\theta)}{\partial\theta}\right)^2}\,\right]. \tag{31}$$

corresponding to the inner and outer edges of the arc.

Thus, given the source radius $R_0$ and position $(x_0, y_0)$ we can draw the arcs using eq.(31) by varying $\theta$ between $[0, 2\pi[$ and taking the corresponding $x$.

### 3.2 Elliptical Source Contours

We now extend the discussion of previous section to elliptical contours.

First we can consider the equation of an elliptical contour aligned to the main axis (note that this is a very limited set of ellipses),

$$(1-\eta)x_s^2 + (1+\eta)y_s^2 = R_0^2. \tag{32}$$

Now, using eq.(32)and eq.(23) (considering the sources at origin), defining $S = 1 - \eta\cos(2\theta)$ and after some algebra we have: (boring according to Gabriel :-) )

$$x = \frac{1}{\kappa_2}\left\{f_1(\theta) + \frac{\eta\sin(2\theta)}{S\,r_E}\frac{\partial f_0(\theta)}{\partial\theta} \pm \frac{1}{S}\sqrt{SR_0^2 - (1-\eta^2)\left[\frac{1}{r_E}\frac{\partial f_0(\theta)}{\partial\theta}\right]^2}\,\right\}. \quad (8) \tag{33}$$

(Alard eq.15).

### 3.3 Conditions for the Validity of the Approximation

At first order in $\epsilon$ we may define

$$q \equiv \frac{r}{r_{\rm E}}[1 - \epsilon\, g(\theta)]. \tag{34}$$

As the functional $\phi_0(q)$ represents the general expression for the perturbed potential, with $q$ defined above, we may expand $\phi_0(q)$

$$\phi_0(q) = \phi_0\left(\frac{r}{r_{\rm E}} - \epsilon\frac{r}{r_{\rm E}}\,g(\theta)\right)$$

in a Taylor Series around $\epsilon = 0$, in the following way

$$\begin{aligned}
\phi_0(q) &\approx \phi_0\left(\frac{r}{r_{\rm E}}\right) + \epsilon\left(\frac{d\phi_0}{d\epsilon}\right)\bigg|_{\epsilon=0} && (35)\\
&= \phi\left(\frac{r}{r_{\rm E}}\right) + \epsilon\left(\frac{d\phi_0}{dq}\right)\bigg|_{\epsilon=0}\left(\frac{\partial q}{\partial\epsilon}\right) && (36)\\
&\approx \phi\left(\frac{r}{r_{\rm E}}\right) - \frac{r}{r_{\rm E}}\phi_0'\left(\frac{r}{r_{\rm E}}\right)g(\theta)\epsilon && (37)
\end{aligned}$$

Where we have defined $\phi_0'\left(\frac{r}{r_{\rm E}}\right) \equiv \left(\frac{d\phi_0}{dq}\right)\Big|_{\epsilon=0}$. Note, that we have $\phi_0(q) = \phi_0\left(\frac{r}{r_{\rm E}}\right)$ when $\epsilon = 0$ and similarly with its derivatives. Therefore, taking the partial derivatives of Eq. (37), it is straightforward to verify

$$\begin{aligned}
r_{\rm E}\frac{\partial\phi}{\partial r} &= \phi_0'\left(\frac{r}{r_{\rm E}}\right) - \epsilon\left[g(\theta)\phi_0'\left(\frac{r}{r_{\rm E}}\right) + \frac{r}{r_{\rm E}}g(\theta)\phi_0''\left(\frac{r}{r_{\rm E}}\right)\right] && (38)\\
\frac{r_{\rm E}}{r}\frac{\partial\phi}{\partial\theta} &= -\phi_0'\left(\frac{r}{r_{\rm E}}\right)\frac{dg(\theta)}{d\theta}\epsilon && (39)
\end{aligned}$$

(Alard eq.19).

### 3.4 Numerical Testing

### 3.5 Comparison with Ray Tracing

### 3.6 Inverse Modelling

We'll skip to here, for now.

To get Alard eq.29, take Alard eq. 12:

$$x = \frac{1}{\kappa_2}\left[\overline{f}_1(\theta) \pm \sqrt{R_0^2 - \left(\frac{1}{r_E}\frac{\partial \overline{f}_0(\theta)}{\partial \theta}\right)^2}\right]. \tag{40}$$

And separate the two solutions for the inner and outer edge of the arc:

$$x_i = \frac{1}{\kappa_2}\left[\overline{f}_1(\theta) - \sqrt{R_0^2 - \left(\frac{1}{r_E}\frac{\partial \overline{f}_0(\theta)}{\partial \theta}\right)^2}\right]. \tag{41}$$

$$x_o = \frac{1}{\kappa_2}\left[\overline{f}_1(\theta) + \sqrt{R_0^2 - \left(\frac{1}{r_E}\frac{\partial \overline{f}_0(\theta)}{\partial \theta}\right)^2}\right]. \tag{42}$$

Take

$$x_o + x_i = \frac{2}{\kappa_2}\overline{f}_1 \tag{43}$$

And solve for $\overline{f}_1$:

$$\overline{f}_1 = \frac{\kappa_2}{2}(x_o + x_i) \tag{44}$$

which is exactly Alard eq. 29a, without his additional constant (whose provenance I don't get at the moment).

Now take

$$x_o - x_i = \frac{1}{\kappa_2}\left(2\sqrt{R_0^2 - \left(\frac{1}{r_E}\frac{d\overline{f}_0(\theta)}{d\theta}\right)^2}\right) \tag{45}$$

And solve for $\frac{d\overline{f}_0(\theta)}{d\theta}$ (few straight steps of algebra here):

$$\frac{d\overline{f}_0(\theta)}{d\theta} = r_E\sqrt{R_0^2 - \frac{\kappa_2^2}{4}(x_o - x_i)^2} \tag{46}$$

which is our form of Alard eq. 29b.

## 4 Caustics in the Perturbative Approach

Going from Alard eq.29' (Jacobian) to A. eq.30:

$$x = \frac{1}{\kappa_2}\left(\overline{f}_1 + \frac{d^2\overline{f}_0(\theta)}{d\theta^2}\right) \tag{47}$$

seemed fairly nontrivial to me, when i started the calculation.

But then getting eq.31 was not so bad, starting from A. eq. 13:

$$x_s = (\kappa_2 x - \overline{f}_1)\cos\theta + \frac{d\overline{f}_0(\theta)}{d\theta}\sin\theta \tag{48}$$

$$y_s = (\kappa_2 x - \overline{f}_1)\sin\theta - \frac{d\overline{f}_0(\theta)}{d\theta}\cos\theta \tag{49}$$

And simply sticking in A.eq.30 in for x, pretty straight.

Getting A.eq 33 from 31 is straight, and then I got most of the way to A.eq.34 from 33 and 32, with some small hiccups..