Chapter 1

2015 Problem 5: Two Balloons

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In this work, we observed the different directions of air flow when two balloons are collected through preliminary experiments. We use phenomenological theory to describe the relation between pressure and elongation of balloons. A phase diagram is made according to the pressure and elongation relation from which the direction of air flow can be predicted. In experiment, we measured the pressure difference of balloons and plotted the pressure-elongation diagram to verify the theory. Besides, we take other factors that may influence the direction of air flow into account, like inflating history, Mullins effect, rubber material. Finally, we discuss the stability of equilibrium between two balloons.
Keywords: rubber elasticity; Mullins effect; Mooney-Rivlin model; rubber hysteresis

1. Introduction

Problem Statement:

> Two rubber balloons are partially inflated with air and connected together by a hose with a valve. It is found that depending on initial balloon volumes, the air can flow in different directions. Investigate this phenomenon.

In this article, preliminary experiments are done at first to reproduce the phenomena and investigate the relation between balloon volumes and directions of air flow. The air flow direction depends on the pressure difference between the two balloons. So we use phenomenological theory based on the works of Mooney, Rivlin and Hart Smith[1] to describe the

Fig. 1. Air can flow in different directions when two balloons are connected.

relation between pressure and elongation of balloons. A phase diagram relating the elongations of both balloons and the flow direction is plotted. Systematic experiments were done to verify our theory. Comparisons show that the experimental data are consistent with our theoretical predictions. We will also discuss some other factors that affect the pressure inside a balloon. Inflation history, Mullins effect[2] and material factors will be taken into account to predict how they affect the directions of air flow. As the pressure equation includes material constants, which will change with the amounts of extensions, uniaxial extension experiments of rubber, inflating and deflating balloon experiments are done to verify these properties. Hysteresis loop of balloon's pressure will be observed. Finally, we will also analyze the stability of two balloons' equilibrium using the predictions we have made.
Before theoretical analysis, preliminary experiments were done to show that air could flow in different directions*.

2. Theoretical Analysis

The air flow direction is determined by the pressure differences between two balloons. From pre-experiments, we suggest that pressure is not monotonously dependent on balloon's volume, we need to explore the exact relationship between them. In our research, we use elongation to describe the changes of volume, as volume is approximately proportional to cubic of elongation.

*See supplementary materials, Two Balloons, video 1, 2, 3.

2.1. *Phenomenological Theory*

We apply phenomenological theory of rubber which is concise and feasible for large strain. The two basic assumptions of the theory are that rubber is incompressible and is isotropic in unstrained state. The value of poissons ratio of rubber[3] (about 0.499) indicates that rubber is nearly incompressible. The condition for isotropy requires that the strain-energy function W be symmetrical with respect to the three principal extension ratios λ_1, λ_2, λ_3, we can write the three simplest possible even-powered functions which satisfy these requirements.[1] I_1, I_2, I_3 in the functions are called strain invariants.

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{1}$$

Because the deformation of a spherical balloon is an equi-biaxial tension, the first two principle elongations are the same.

$$\lambda_1 = \lambda_2 = \lambda \tag{2}$$

The condition for incompressibility or constant volume during deformation can be expressed by the relation,

$$I_3 = 1 \tag{3}$$

The combination of Eq.(1) to Eq.(3) yields:

$$\begin{aligned} I_1 &= 2\lambda^2 + \frac{1}{\lambda^4} \\ I_2 &= \lambda^4 + \frac{2}{\lambda^2} \end{aligned} \tag{4}$$

Rivlin derives the expression of the principal stresses t_i,[1] which involves the partial derivatives of the strain-energy function with respect to the independent variables I_1, I_2.

$$t_i = 2({\lambda_i}^2\frac{\partial W}{\partial I_1} + {\lambda_i}^{-2}\frac{\partial W}{\partial I_2}) + P(i = 1, 2, 3) \tag{5}$$

where P is an arbitrary hydrostatic stress.[1]

By subtraction, P may be eliminated to give the three principal stress differences.

$$\begin{aligned} t_1 - t_2 &= 2({\lambda_1}^2 - {\lambda_2}^2)(\frac{\partial W}{\partial I_1} + {\lambda_3}^2 \frac{\partial W}{\partial I_2}) \\ t_2 - t_3 &= 2({\lambda_2}^2 - {\lambda_3}^2)(\frac{\partial W}{\partial I_1} + {\lambda_1}^2 \frac{\partial W}{\partial I_2}) \\ t_3 - t_1 &= 2({\lambda_3}^2 - {\lambda_1}^2)(\frac{\partial W}{\partial I_1} + {\lambda_2}^2 \frac{\partial W}{\partial I_2}) \end{aligned} \tag{6}$$

On the surface of a balloon, $t_3 = 0$, $t_1 = t_2$. Substitute these relations into Eq.(6), it yields:

$$t = t_1 = t_2 = 2(\lambda^2 - \frac{1}{\lambda^4})(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}) \tag{7}$$

Given non-Gaussian theory, Hart Smith[1] suggested three-constant formula to explain a variety of data on the inflation of balloons.

$$\frac{\partial W}{\partial I_1} = G e^{k_1 (I_1 - 3)^2} \tag{8}$$

$$\frac{\partial W}{\partial I_2} = G \frac{k_2}{I_2} \tag{9}$$

Substitute Eq.(8) and Eq.(9) to Eq.(7) we have:

$$t = 2(\lambda^2 - \frac{1}{\lambda^4})(G e^{k_1 (I_1 - 3)^2} + \lambda^2 G \frac{k_2}{I_2}) \tag{10}$$

Then combine with Eq.(4), we find the expression of principle stress with respect to extension ratio λ

$$t = 2G(\lambda^2 - \frac{1}{\lambda^4})[e^{k_1 (2\lambda^2 + \frac{1}{\lambda^4} - 3)^2} + \frac{k_2 \lambda^4}{\lambda^6 + 2}] \tag{11}$$

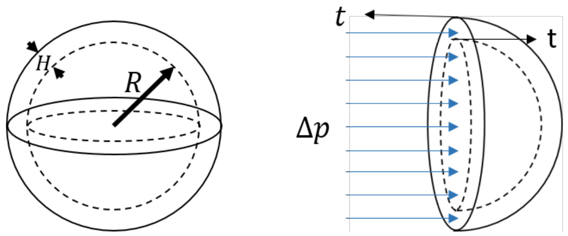
Consider force equilibrium of a half spherical balloon 2 as shown in Fig.2, we have

$$(\pi (H + R)^2 - \pi R^2) t = \pi R^2 \Delta p \tag{12}$$

Using $H \ll R$, we get

$$\Delta p = \frac{2H}{R} t \tag{13}$$

Where $\Delta p = p - p_0$, p_0 is atmospheric pressure. Note that here $H = \frac{H_0}{\lambda^2}$, $R = \lambda R_0$.



(a) Balloon Model (b) Force Analysis

Fig. 2. The pressure inside a balloon is proportional to the stress of rubber.

Finally Δp can be written as a function of λ.

$$\Delta p = \frac{4H_0G}{R_0}(\frac{1}{\lambda} - \frac{1}{\lambda^7})[e^{C_1(2\lambda^2+\frac{1}{\lambda^4}-3)^2} + \frac{C_2\lambda^4}{\lambda^6+2}] \tag{14}$$

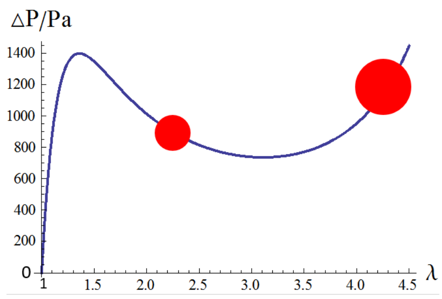
where C_1,C_2,G are material constants dependent on rubber materials, H_0 is the initial thickness of the balloon, R_0 is initial radius, λ is extension ratio used to describe elongation.

2.2. *Analysis of Air Flow Directions*

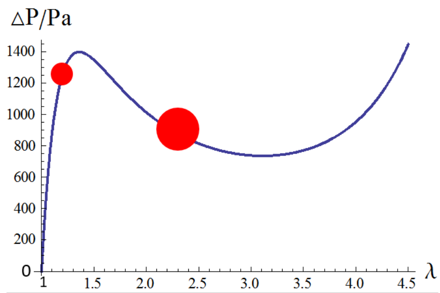
Based on the the pressure function with respect to elongation we have derived above, we can explain the phenomena in preliminary experiments. Fig. 3 is a theoretical curve of pressure with respect to elongation with parameters taken as $\frac{4H_0G}{R_0} = 1700Pa$, $C_1 = 0.001$, $C_2 = 0.8$. The dots represent two balloons of different initial conditions. In Fig. 3(a), the larger balloon have greater pressure so that air flows from it to the smaller balloon. In Fig. 3(b), the smaller balloon have greater pressure so that air flows in the opposite direction. In Fig. 3(c), the two balloons have the same pressure, even though their elongations are different, the air doesn't flow either way.

3. Experiment

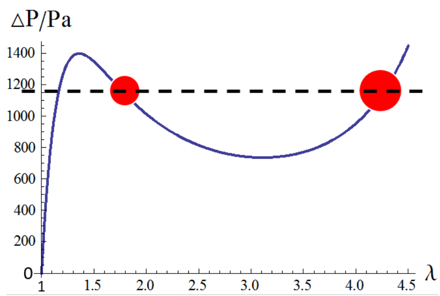
To verify the theoretical predictions, we need to measure the pressure function with respect to elongation in experiment. An U-tube barometer is connected to the balloon to measure the pressure inside. At the same time, we used ABviewer to measure the diameters of balloons along two directions, see Fig. 4. The average radius of the balloon is defined as $R = \frac{(D_1D_2^2)^{\frac{1}{3}}}{2}$, then elongation is determined by $\lambda = \frac{(D_1D_2^2)^{\frac{1}{3}}}{2R_0}$.



(a) Big to small



(b) Small to big



(c) Equilibrium

Fig. 3. Theoretical curve of pressure with respect to elongation. The direction of air flow depends on pressure difference between the balloons.

In Fig. 5, the dots are experimental data, the curve is a theoretical result by Eq.(14). The material constants in Eq.(14) were obtained by by fitting the equation to the experimental data, here $G = 36645Pa$, $C_1 = 0.0012$, $C_2 = 0.79$.
The experimental dada are in good agreement with the theoretical function

Fig. 4. Measurement of the diameters of balloons along two directions. Average radius of the balloon $R = \frac{(D_1 D_2^2)^{\frac{1}{3}}}{2}$.

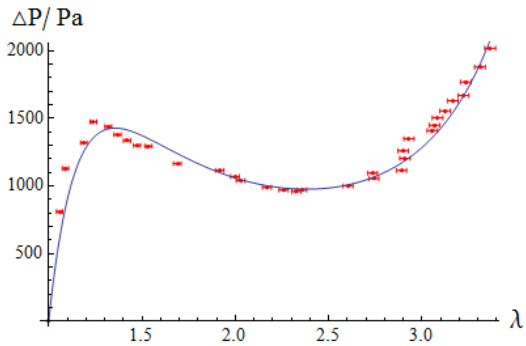


Fig. 5. Pressure-elongation(Δp-λ) curve.

and support our explanation for the airflow direction. For example, in Fig. 5 the pressure reaches a maximum when the elongation is near 1.4, which explains when airflow from small balloons to big ones may occur.
The overall condition for airflow direction and elongations of both balloons can be summarized in a "phase diagram" according to Eq.(14), see Fig. 6 and compared to experimental results. Note that the horizontal axis

represents the elongation of the bigger balloon, while the vertical axis represents that of the smaller one. So we only need to consider the region below the diagonal line. In this diagram, the grey area corresponds to airflow from bigger to smaller balloons. The white area corresponds to the opposite. On the boundary of these areas, two balloons are almost in equilibrium.

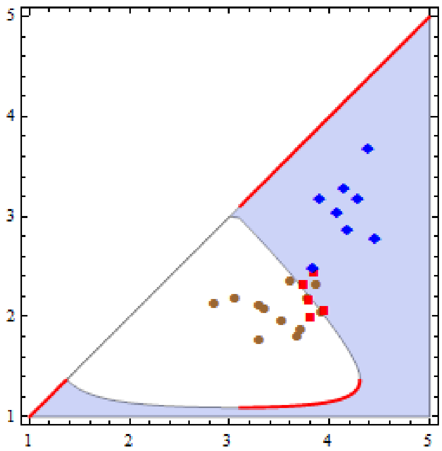


Fig. 6. Phase Diagram: Airflow direction depends on elongations of both balloons. The grey area corresponds to airflow from bigger to smaller balloons. The white area corresponds to the opposite. On the boundary of these areas, two balloons are almost in equilibrium.

Numerous experiments were done to verify the conditions for airflow direction and elongations, the results were also shown in Fig. 6. The diamonds are "bigger to smaller", which all located in the gray area. The dots are "smaller to bigger" ones, which all located in the white area. The squares near the boundary are in equilibrium.
Till now the conditions for airflow direction and balloon elongations are confirmed.

4. Other Factors

By now, we've discussed how elongation influence the pressure inside a balloon. However, there are also many other factors that will affect the pressure. In order to predict the airflow direction more precisely, we need to take these factors into account.

4.1. *Inflation History*

When we connect two similar balloons of the same size, if one has been inflated many times while the other one is new, we will find that the used balloon get inflated further even if they both have the same initial volume†. To verify this property of rubber, we conducted a uniaxial extension experiment. The experimental setup is shown in Fig. 7. It includes a force sensor connected to a balloon rubber sample, both installed on a rail with scales.

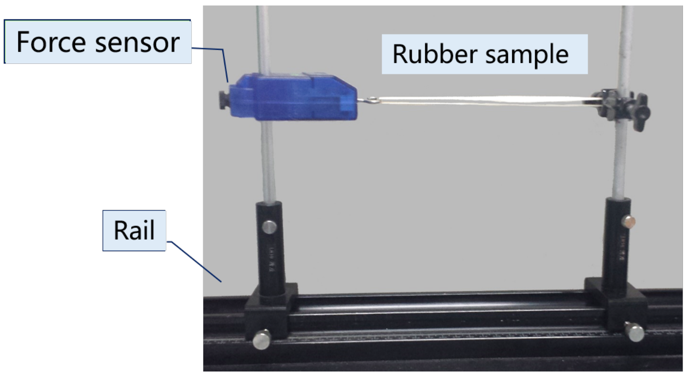


Fig. 7. Experiment setup for uniaxial extension.

Fig.8 is the force and extension curve of the first five times of extensions for one rubber sample.
From the experimental data in Fig. 8, we find that the extension of a new rubber produces the biggest stress. Then the stress of rubber changes with extension times, then tends to become stable. We can observe that the 4th and 5th curves almost overlap. According to the force analysis of the balloon in Fig. 2, the extra pressure inside a balloon is proportional to the stress of the rubber. Consequently, new balloons tend to have higher pressure than the used ones at the same elongations.

4.2. *Mullins Effect*

The Mullins effect describes the mechanical response in filled rubbers in which the stress-strain curve depends on the maximum loading previously applied. In our experiments, we observed that the deflating and the

†See supplementary materials, Two Balloons, video 4.

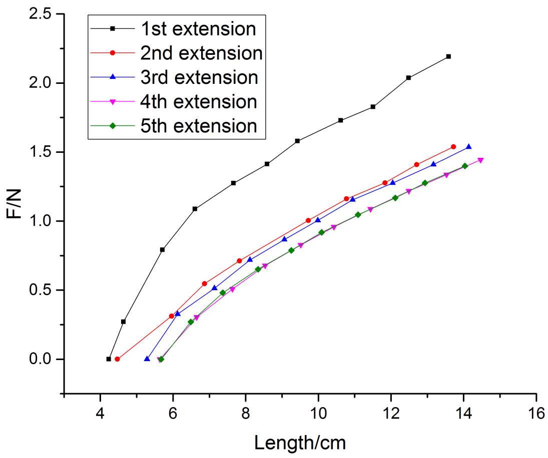


Fig. 8. Force- extension curve of the first five times of extensions.

inflating balloon have different pressures. In our video ‡, the two balloons are both stretched many times with nearly the same elongation. A deflating balloon on the left is connected to a inflating balloon on the right. Then air flows from the inflating balloon to the deflating one.

To investigate this effect, we conducted a pressure-elongation experiment during inflation and deflation. It forms a hysteresis loop, see Fig. 9. In Fig. 9, the upper curve corresponds to inflation, the lower curve corresponds to deflation.

4.3. *Balloon Material*

The third factor is rubber material of balloons. We used three kinds of balloons for comparison.

Fig. 10(b) is the pressure-elongation curves of the three balloon samples, called A, B and C. The trends are similar, but the material constants are dependent on the rubber they're made of. For small elongation, the pressure of Sample A is larger than the other two samples. So air will flow from it to the others. When balloons are inflated bigger, the pressure of Sample A becomes smaller than others, air will flow in the opposite direction.

‡See supplementary materials, Two Balloons, video 5.

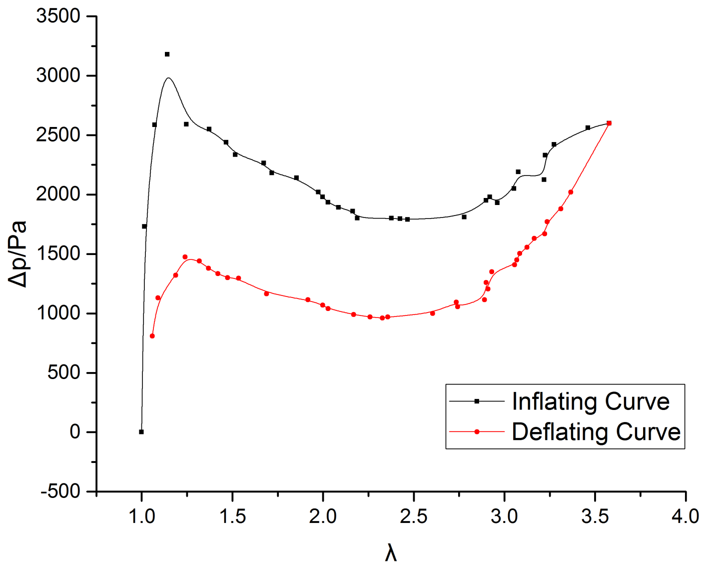


Fig. 9. Mullins effect, pressure-elongation during inflation and deflation forms a hysteresis loop.

In our experimental videos§, two balloons made of rubber A and B are connected. Although they have similar initial elongation, the air flows from A to B. When A and C are connected, air flows from A to C due to higher pressure level.

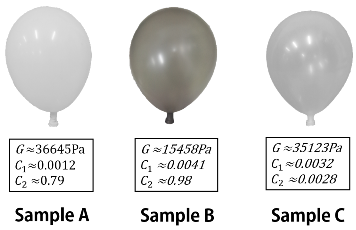
5. Stability of Equilibrium

Now we start to investigate the stability of equilibrium in the two-balloon experiments. The equilibrium can be classified as similar elongations and different elongations.

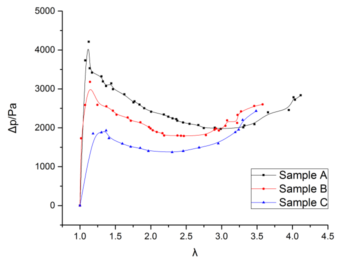
5.1. *Similar Elongations*

If the balloon elongations are the same, there are three different situations according to initial elongations. In the first situation, see Fig. 11, if one

§See supplementary materials, Two Balloons, video 6.



(a) Three Kinds of Balloons.



(b) Pressure-Elongatin(ΔP-λ) diagram of three samples.

Fig. 10. Different Balloon Materials.

balloon is given a small perturbation and becomes bigger, it has greater pressure than the other, so that and air flows from it to inflate the smaller one. In this case, the equilibrium is stable.

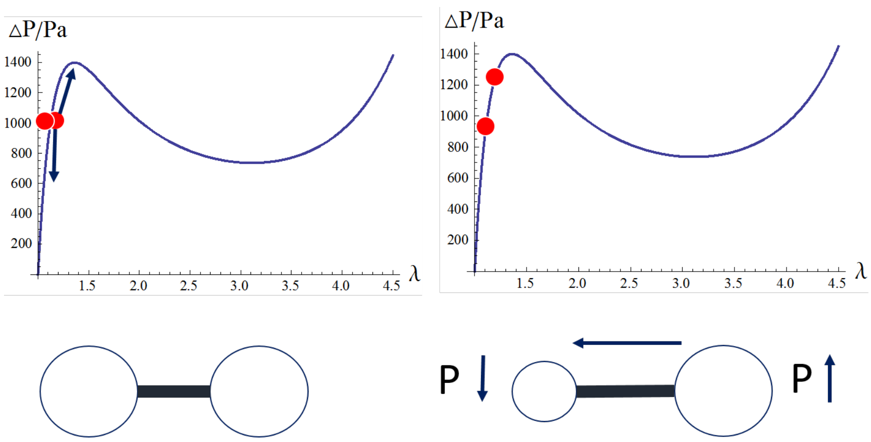


Fig. 11. Stable Equilibrium. If one balloon becomes slightly bigger it has greater pressure, so that air flows from it to the smaller one.

In the second situation, see Fig. 12, give one balloon a perturbation, the balloon that becomes bigger will have smaller pressure and air flows from the smaller balloon to the big one. In this case, the equilibrium is unstable. Similar to the first situation, we can easily find that equilibrium is stable in the third situation, see Fig. 13,

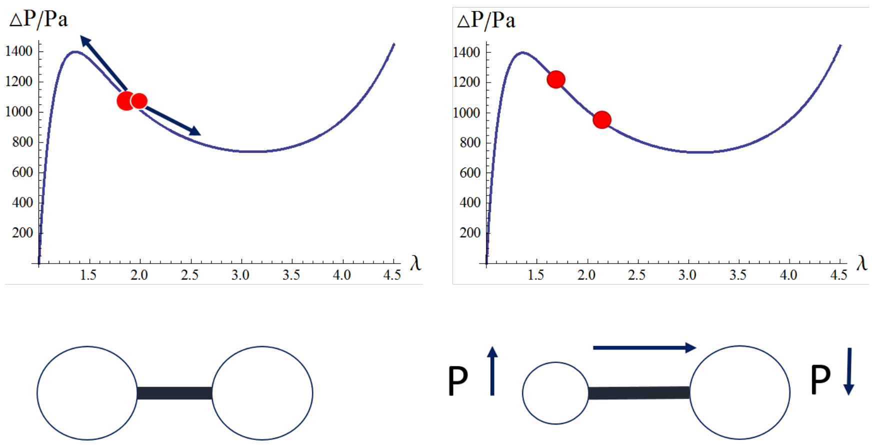


Fig. 12. Unstable Equilibrium. If one balloon becomes slightly bigger it has smaller pressure, so that air flows from the smaller one to it.

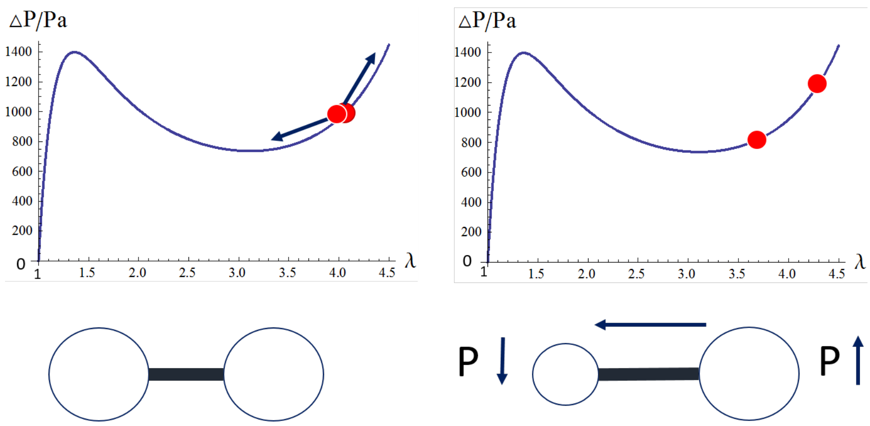
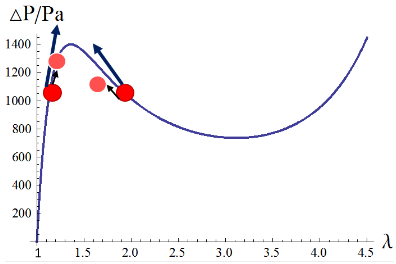


Fig. 13. Stable Equilibrium.

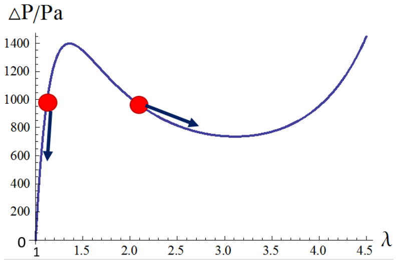
5.2. *Different Elongations*

Mullins effect needs to be taken into account in order to investigate the equilibrium stability of two balloons with different elongations. For two balloons of different elongations, there are also three situations. It is more complicated when taking two ways of perturbation into account.

When we connect two balloons at equilibrium, the total air volume is nearly constant. The volume of one balloon is approximately proportional to cubic



(a) Stable Equilibrium.



(b) Stable Equilibrium.

Fig. 14. Equilibrium is stable in Fig. 14(a) and in Fig. 14(b). Generally, the equilibrium is stable for balloons with different elongations.

of its elongation. When the small one got slightly inflated, the big balloons elongation change is less than the other. Besides, their pressure both increases, though, because of Mullins effect, the pressure of the deflating balloon will be less than the curve shows. Therefore, the big balloons pressure will be less than the smaller ones after perturbation. So the small balloon will inflate the big one and return to equilibrium. In Fig. 14, equilibrium is stable. Generally, we can find that equilibrium is stable for balloons with different elongations.

6. Conclusion

The direction of air flow is determined by pressure difference between the balloons. We use elongation as a parameter governing the pressure inside a balloon. To investigate the relation between pressure and elongation, Mooney-Rivlin model is used. Based on our theory, pressure-elongation curve and phase diagram with respect to elongation are plotted and verified experimentally.

Then, we take other factors such as Mullins effect, inflating history and material parameters into account to predict the direction of air flow. The inside pressure in balloons of similar material is determined by its elongation. Pressure is also influenced by whether the balloon is inflating or deflating, and the biggest elongation of the present extension. Inflating history also affect the pressure. For instance, the pressure in a balloon that has been extended many times is smaller than that of a new one. Balloons of different materials have different material constants, so they have different pressure in same extension ratio. Finally, several situations concerning the stability of balance are discussed.

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