Summary of My Summer Research in PITT PACC

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Abstract

This is a summary for my work during Jul 11 to Oct 31 in PITT PACC. I really appreciate the opportunity to work with Prof. Tao Han as well as other brilliant guys in PITT PACC and the friendship built up with them. During this period I tried to grab every chance to absorb whatever about physics. Fortunately, I had an experienced supervisor Tao and reliable collaborators such as Xing and Ahmed that I learned a lot from them by discussing as much as possible everyday.

Generally, I participated in the n-plet DM project after solving several warm-up questions Tao provided. In technical aspect, I've learned how to use Linux system and HEP packages such as MadGraph, Feynrules, FeynCalc.

As I learned a lot in this period, I also realized that I still have a long way to go to become a mature researcher. I hope I could become stronger accompanied by this sense of hunger. I value the experience in Pitt also because it enables me to ensure my passion for particle physics and the life as a researcher. After all, you must be cautious when considering whether you'd like to continue to study physics in the following several years or the rest of your life.

Anyway, I really enjoy my time in PITT PACC as a visiting student. Thank you again, Tao!

Contents

1 Cross Section under N-dimension SU(2) Group

1.1 Lagrangian and Generators

The covariant derivative of $SU(2) \bigotimes U(1)$ group is

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_w} Z_\mu (T^3 - \sin^2\theta_w Q) - ieA_\mu Q$$

Considering the n-dimension representation of SU(2) group, the generators of SU(2) group T^+, T^- have the effect of:

$$\begin{aligned} T^+|\frac{n-1}{2}, t^3\rangle &= \sqrt{(-t^3 + \frac{n-1}{2})(t^3 + \frac{n-1}{2} + 1)}\hbar|\frac{n-1}{2}, t^3 + 1\rangle \\ T^-|\frac{n-1}{2}, t^3\rangle &= \sqrt{(t^3 + \frac{n-1}{2})(-t^3 + \frac{n-1}{2} + 1)}\hbar|\frac{n-1}{2}, t^3 - 1\rangle \end{aligned} \tag{1}$$

So,

$$(T^\pm)^{t_3'}{}_{t_3} = \sqrt{(\mp t^3 + \frac{n-1}{2})(\pm t_3 + \frac{n-1}{2} + 2)}\hbar\delta^{t_3'}{}_{t_3 \pm 1}$$

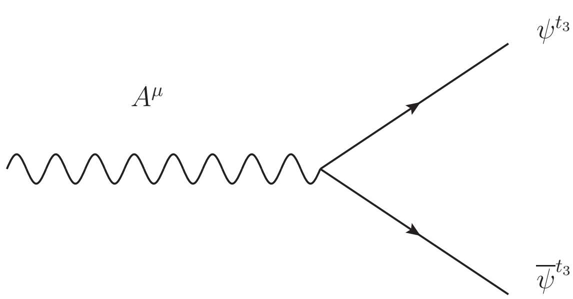
In the Lagrangian, the interactive term is

$$\begin{aligned} \mathcal{L}_{fermion} &\supset \overline{\psi}(i\not{D} - m)\overline{\psi} \\ &= \frac{g}{\cos\theta_w}(T^3 - \sin^2\theta_w Q)\overline{\psi^{t_3}}\not{Z}\psi^{t_3} + eQ\overline{\psi^{t_3}}\not{A}\psi^{t_3} \\ &\quad + \frac{g}{\sqrt{2}}\sqrt{(-t^3 + \frac{n-1}{2})(t^3 + \frac{n-1}{2} + 1)}\,\overline{\psi^{t_3+1}}\not{W}^+\psi^{t_3} + h.c \end{aligned} \tag{2}$$

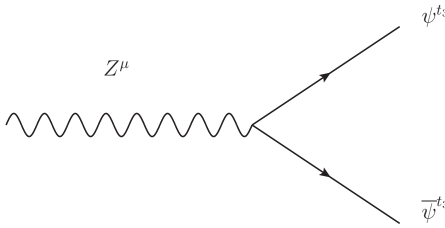
Then we could calculate the amplitude and cross section of $q\overline{q'} \to \chi\chi$ process based on this interaction term, where χ is one of the n-plet particles.

1.2 Calculation of Cross Section

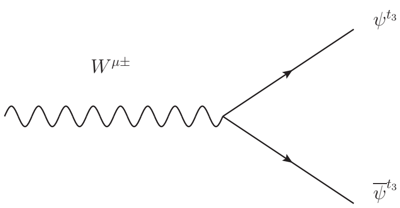
There are 3 kinds of vertices:



$$= ie(t_3 + y)\gamma^\mu$$



$$= i\frac{g}{\cos\theta_w}(T^3 - \sin^2\theta_w Q)\gamma^\mu$$



$$= i\frac{g}{\sqrt{2}}\sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 1)}\gamma^\mu$$

1.2.1 $u\overline{d} \to W^+ \to \psi^{t_3+1}\overline{\psi}^{t_3}$

$$\begin{aligned} i\mathcal{M} &= \overline{v}(p_2) i\frac{g}{\sqrt{2}}\gamma^\mu p_L u(p_1)\frac{i}{\hat{s} - m_w^2} i\frac{g}{\sqrt{2}}\sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 1)}\,\overline{u}(k_1)\gamma_\mu v(k_2) \\ \sum |M|^2 &= \frac{g^4}{4}(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 1)\frac{1}{(\hat{s} - m_w^2)^2} \times [2(k_1 \cdot p_1)(k_2 \cdot p_2) + 2(k_1 \cdot p_2)(k_2 \cdot p_1) + m^2\hat{s}] \end{aligned} \tag{3}$$

Embedding scattering amplitude into the center of mass frame:

$$\begin{aligned} p_1 =& \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, 1) \\ p_2 =& \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, -1) \\ k_1 =& \frac{\sqrt{\hat{s}}}{2}(1, \beta\sin\theta, 0, \beta\cos\theta) \\ k_2 =& \frac{\sqrt{\hat{s}}}{2}(1, -\beta\sin\theta, 0, -\beta\cos\theta) \end{aligned} \tag{4}$$

where $\beta = \sqrt{1 - \frac{4m^2}{\hat{s}}}$. Then the scalar products are:

$$\begin{aligned} k_1 \cdot p_1 = k_2 \cdot p_2 = \frac{\hat{s}}{4}(1 - \beta\cos\theta) \\ k_1 \cdot p_2 = k_2 \cdot p_1 = \frac{\hat{s}}{4}(1 + \beta\cos\theta) \end{aligned} \tag{5}$$

Therefore,

$$\sum |M|^2 = \frac{g^4}{4}(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 1)\frac{\hat{s}^2}{(\hat{s} - m_w^2)^2}(2 - \beta^2 + \beta^2\cos^2\theta)$$

Now the differential cross section as well as cross section can be written out as:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2\hat{s}}\frac{\beta}{32\pi^2}\frac{1}{4}\frac{1}{3}\sum_{spin}|M|^2 \\
\hat{\sigma} &= \frac{1}{2\hat{s}}\frac{\beta}{32\pi^2}\frac{1}{4}\frac{1}{3}\int d\phi \int d\cos\theta \sum_{spin}|M|^2 \\
&= \frac{\pi\alpha^2}{36\sin^4\theta_w}(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 1\frac{\hat{s}^2}{(\hat{s}-m_w^2)^2}\beta(3-\beta^2) \\
\sum_{t_3}\hat{\sigma} &= \frac{\pi\alpha^2}{216\sin^4\theta_w}n(n^2-1)\beta(3-\beta^2)
\end{aligned}
\tag{6}
$$

1.2.2 $q\overline{q} \to \gamma/Z \to \psi^{t_3+1}\overline{\psi}^{t_3}$

$$
\begin{aligned}
i\mathcal{M} =&\overline{v}(p_2)ieQ_q\gamma^\mu u(p_1)\frac{-i}{\hat{s}}\overline{u}(k_1)ie(t_3+y)\gamma_\mu v(k_2) \\
&+\overline{p}_2\frac{ig}{\cos\theta_w}\gamma^\mu(c_Lp_L + c_Rp_R)u(p_1)\frac{-i}{\hat{s}-m_z^2}\overline{u}(k_1)i\frac{g}{\cos\theta_w}(\cos^2\theta_w t_3 - \sin^2\theta_w y)\gamma_\mu v(k_2)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\text{When} \quad q = u : Q = \frac{2}{3}, \quad c_L = \frac{1}{2} - \frac{2}{3}\sin^2\theta_w, \quad c_R = -\frac{2}{3}\sin^2\theta_w \\
&\text{When} \quad q = d : Q = -\frac{1}{3}, \quad c_L = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_w, \quad c_R = \frac{1}{3}\sin^2\theta_w
\end{aligned}
\tag{8}
$$

Then the scattering amplitude is

$$
\begin{aligned}
\sum|M|^2 &= 32[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(p_2\cdot k_1) + m^2(p_1\cdot p_2)] \times e^4\frac{F}{\hat{s}^2} \\
&= 4e^4F(2-\beta^2+\beta^2\cos^2\theta)
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
e^4\frac{F}{\hat{s}^2} \equiv& \frac{e^4Q_q^2(t_3+y)^2}{\hat{s}^2} + \frac{(c_L^2+c_R^2)g^4}{2\cos^4\theta_w}(t_y\cos^2\theta_w - y\sin^2\theta_w)^2\frac{1}{(\hat{s}-m_z^2)^2} \\
&+ \frac{e^2g^2Q_q}{\cos^2\theta_w}(c_L+c_R)(t_3\cos^2\theta_w - y\sin^2\theta_w)\frac{1}{\hat{s}(\hat{s}-m_Z^2)}
\end{aligned}
\tag{10}
$$

So the cross section can be derived,

$$
\begin{aligned}
\hat{\sigma} =& \frac{1}{2\hat{s}}\frac{\beta}{32\pi^2}\frac{1}{4}\frac{1}{3}\int d\phi\int d\cos\theta\sum|M|^2 \\
=& \frac{2\pi\alpha^2}{9\hat{s}}F\beta(3-\beta^2)
\end{aligned}
\tag{11}
$$

2 Construction of n-plet-singlet Mixing

2.1 Lagrangian

For Dirac spinor $\Psi = (\psi, \overline{\psi}^{+})$,

$$\begin{aligned}\mathcal{L}_m ix = \frac{1}{\Lambda^{n-2}} d_I^{i_1 \cdots i_{n-1}} \times (\lambda' \phi_{i_1} \cdots \phi_{i_{\frac{n-1}{2}+y}} \phi^{+}_{i_{\frac{n+1}{2}+y}} \cdots \phi^{+}_{i_{n-1}} \chi \overline{\psi}^I \\ - \lambda \phi^{+}_{i_1} \cdots \phi^{+}_{i_{\frac{n-1}{2}+y}} \phi_{i_{\frac{n+1}{2}+y}} \cdots \phi_{i_{n-1}} \chi \psi^I + h.c) \\ = -\frac{1}{\Lambda^{n-2}} (\frac{1}{\sqrt{2}})^{n-1} \sqrt{\frac{(\frac{n-1}{2}+y)!(\frac{n-1}{2}+y)!(\frac{n-1}{2}-y)!}{(n-1)!}} \times \left[\lambda v^{n-1} \chi \psi^{-y} - \lambda' v^{n-1} \chi \overline{\psi}^{-y}\right]\end{aligned} \tag{12}$$

2.2 Dimension-5 Operator

For Dirac fermion $(\psi, \overline{\psi}^{+})$,

$\mathcal{L}_{5D} \supset \frac{\kappa}{\Lambda}(\phi^+\phi)(\overline{\psi}\psi) + \frac{\zeta_1}{\Lambda}(\phi^+\tau^a\phi)(\overline{\psi}T^a\psi) + h.c$

If $y = \frac{1}{2}$, there are two more operators,

$\mathcal{L}_{5D}^{y=\frac{1}{2}} \subset \frac{\kappa}{\Lambda}(\phi^+\phi)(\overline{\psi}\psi) + \frac{\zeta_1}{\Lambda}(\phi^+\tau^a\phi)(\overline{\psi}T^a\psi) - \frac{\zeta_2}{\Lambda}(\phi\tau^a\phi)(\overline{\psi}T^a\overline{\psi}) - \frac{\zeta_3}{\Lambda}(\phi^+\tau^a\phi^+)(\psi T^a\psi) + h.c$

For Majorana fermion ψ,

$\mathcal{L}_{5D} \subset \frac{\kappa}{\Lambda}(\phi^+\phi)(\psi\psi) + h.c$

2.3 Mass Spectrum

1. For $t_3 > \frac{n-1}{2} - 2y$**,** $Q = t_3 + y = \frac{n-1}{2} - y + 1, \cdots, \frac{n-1}{2} + y$

$$\begin{aligned}\mathcal{L} &\subset -(M + \frac{\zeta_1}{4\Lambda} y v^2)\overline{\psi_I}\psi^I + \frac{\zeta_1}{\Lambda}(\phi^+\tau^a\phi)(\overline{\psi}T^a\psi) + h.c \\ &\subset -\left[M + \frac{\zeta_1}{4\Lambda}(t_3 + y)v^2\right]\overline{\psi}_{t_3}\psi^{t_3} + h.c\end{aligned}$$

2. For $-y < t_3 \le \frac{n-1}{2} - 2y, Q = 1, \cdots, \frac{n-1}{2} - y$

$$
\begin{aligned}
\mathcal{L} \subset & -(M + \frac{\zeta_1}{4\Lambda} y v^2)\overline{\psi_I}\psi^I + \frac{\zeta_1}{\Lambda}(\phi^+\tau^a\phi)(\overline{\psi}T^a\psi) \\
& - \frac{\zeta_2}{\Lambda}(\phi\tau^a\phi)(\overline{\psi}T^a\overline{\psi}) - \frac{\zeta_3}{\Lambda}(\phi^+\tau^a\phi^+)(\psi T^a\psi) + h.c \\
& (\zeta_2, \quad \zeta_3 = 0 \text{ for } y \neq \frac{1}{2}) \\
\Rightarrow \mathcal{L} \subset & \left\{ -[M + \frac{\zeta_1}{4\Lambda}(t_3 + y)]\overline{\psi}_{t_3}\psi^{t_3} + (t_3 \to -t_3 - 2y) \right\} \\
& - \left\{ \frac{\zeta_2}{4\Lambda} v^2 \sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 2y)}\,\overline{\psi}_{t_3}\overline{\psi}^{t_3+2y} + (t_3 \to -t_3 - 2y) \right\} \\
& + \left\{ \frac{\zeta_3}{4\Lambda} v^2 \sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 2y)}\,\psi_{t_3+2y}\psi^{t_3} + (t_3 \to -t_3 - 2y) \right\} + h.c
\end{aligned}
\tag{13}
$$

Then

$$
\begin{aligned}
\mathcal{L} \subset & -\left[M + \frac{\zeta_1}{4\Lambda}(t_3 + y)v^2 \right] \overline{\Psi}_{t_3}\Psi^{t_3} \\
& - \left[M - \frac{\zeta_1}{4\Lambda}(t_3 + y)v^2 \right] (-1)^{2y}\overline{\Psi}^{t_3+2y}\Psi_{t_3+2y} \\
& - \frac{\zeta_2}{2\Lambda} v^2 \sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 2y)}\,\overline{\Psi}_{t_3}\overline{\Psi}^{t_3+2y} \\
& + \frac{\zeta_3}{2\Lambda} v^2 \sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 2y)}\,\Psi_{t_3+2y}\Psi^{t_3} + h.c
\end{aligned}
\tag{14}
$$

As $\mathcal{L} \subset -(\Psi_{t_3+2y}, \overline{\Psi}_{t_3})\tilde{M}(\Psi^{t_3}, \overline{\Psi}^{t_3+2y})^T$, we get

$$
\tilde{M} = \begin{pmatrix} -Z_3 & (-1)^{2y}(M - Z_1) \\ M + Z_1 & Z_2 \end{pmatrix}
$$

where

$$
\begin{aligned}
Z_1 &= \frac{\zeta_1}{4\Lambda}(t_3 + y)v^2 \\
Z_2 &= \frac{\zeta_2}{2\Lambda} v^2 \sqrt{(\frac{n-1}{2} - t_3)(\frac{n-1}{2} + t_3 + 2y)} \\
Z_3 &= \frac{\zeta_3}{\zeta_2} Z_2
\end{aligned}
\tag{15}
$$

When $y \neq \frac{1}{2}$, $\tilde{M} = \begin{pmatrix} 0 & (-1)^{2y}(M - Z_1) \\ M + Z_1 & 0 \end{pmatrix}$.

2.4 Mass Spectrum Check of Quadruplet-Singlet Model

In order to verify whether the mass spectrum results from our model are consistent with those in literature[2], I calculated the mass spectrum when $n = 4, y = \frac{1}{2}$.

2.4.1 Notation

The Lagrangian I use is exactly from literature

$$\begin{aligned}\mathcal{L} =& i\psi^\dagger\overline{\sigma}^\mu D_\mu\psi + i\overline{\psi}^\dagger\overline{\sigma}^\mu D_\mu\overline{\psi} + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - (M_0\psi_I\overline{\psi}^I + \frac{1}{2}m\chi\chi + h.c.)\\ &+ (\frac{1}{2}\frac{\kappa}{\Lambda}\phi^\dagger\phi\chi\chi + \frac{\kappa'}{\Lambda}\phi^\dagger\phi\psi_I\overline{\psi}^I + h.c.)\\ &+ (\frac{\zeta_1}{\Lambda}(\phi^\dagger\tau^a\phi)(\overline{\psi}t^a\psi) - \frac{\zeta_2}{\Lambda}(\phi_i\tau^{ai}{}_j\phi^j)(\overline{\psi}_I t^{aI}{}_J\overline{\psi}^J) - \frac{\zeta_3}{\Lambda}(\phi_i^\dagger\tau^{ai}{}_j\phi^{\dagger j})(\psi_I t^{aI}{}_J\psi^J) + h.c.)\\ &+ (\frac{\lambda}{\Lambda^2}\epsilon_{jl}\epsilon_{km}\phi_i^\dagger\phi^l\phi^m d_I^{ijk}\chi\overline{\psi}^I - \frac{\lambda'}{\Lambda^2}\epsilon_{kl}\phi_i^\dagger\phi_j^\dagger\phi^l d_I^{ijk}\chi\psi^I + h.c.)\end{aligned}$$

$$(\varepsilon_{IJ}) = (-\varepsilon^{IJ}) = \begin{pmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$(\psi^I) = \begin{pmatrix}\psi^{++}\\ \psi^+\\ \psi^0\\ \tilde{\psi}^-\end{pmatrix} \qquad (\psi_I) = \begin{pmatrix}\tilde{\psi}^-\\ -\psi^0\\ \psi^+\\ -\psi^{++}\end{pmatrix} \qquad (\overline{\psi}^J) = \begin{pmatrix}\tilde{\psi}^+\\ \tilde{\psi}^0\\ \psi^-\\ \psi^{--}\end{pmatrix} \qquad (\overline{\psi}_J) = \begin{pmatrix}\psi^{--}\\ -\psi^-\\ \tilde{\psi}^0\\ -\tilde{\psi}^+\end{pmatrix}$$

2.4.2 Calculation

Calculate the contribution to mass of the Lagrangian part by part as

$$
\begin{aligned}
\mathcal{L}_1 =& -(M_0 \psi_I \overline{\psi}^I + \frac{1}{2} m\chi\chi + h.c.) \\
=& -M_0 \varepsilon_{IJ} \psi^J \overline{\psi}^I - \frac{1}{2} m\chi\chi + h.c \\
=& -M_0(\tilde{\psi}^- \tilde{\psi}^+ - \psi^0 \tilde{\psi}^0 + \psi^+ \psi^- - \psi^{++}\psi^{--}) - \frac{1}{2} m\chi\chi + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_2 =& \frac{1}{2}\frac{\kappa}{\Lambda}\phi^\dagger \phi \chi\chi + \frac{\kappa'}{\Lambda}\phi^\dagger \phi \psi_I \overline{\psi}^I + h.c. \\
=& \frac{\kappa v^2}{2\Lambda}\chi\chi + \frac{\kappa' v^2}{2\Lambda}(\tilde{\psi}^- \tilde{\psi}^+ - \psi^0 \tilde{\psi}^0 + \psi^+ \psi^- - \psi^{++}\psi^{--}) + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_3 =& \frac{\zeta_1}{\Lambda}(\phi^\dagger \tau^a \phi)(\overline{\psi} t^a \psi) + h.c. \\
=& \frac{1}{4}\frac{\zeta_1}{\Lambda}(-v^2)(\frac{3}{2}\psi^{--}\psi^{++} - \frac{1}{2}\psi^- \psi^+ - \frac{1}{2}\tilde{\psi}^0 \psi^0 + \frac{3}{2}\tilde{\psi}^+ \tilde{\psi}^-) + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_4 =& -\frac{\zeta_2}{\Lambda}(\phi_i \tau^{ai}{}_j \phi^j)(\overline{\psi}_I t^{aI}{}_J \overline{\psi}^J) + h.c. \\
=& -\frac{1}{2}\frac{\zeta_2}{\Lambda}v^2(\sqrt{3}\tilde{\psi}^+ \psi^- - \tilde{\psi}^0 \tilde{\psi}^0) + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_4 =& -\frac{\zeta_3}{\Lambda}(\phi_i^\dagger \tau^{ai}{}_j \phi^{\dagger j})(\psi_I t^{aI}{}_J \psi^J) + h.c. \\
=& \frac{1}{2}\frac{\zeta_3}{\Lambda}v^2(-\sqrt{3}\tilde{\psi}^- \psi^+ + \psi^0 \psi^0) + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_5 =& \frac{\lambda}{\Lambda^2}\epsilon_{jl}\epsilon_{km}\phi_i^\dagger \phi^l \phi^m d_I^{ijk} \chi \overline{\psi}^I + h.c. \\
=& \frac{1}{\sqrt{3}}\frac{1}{2\sqrt{2}}\frac{\lambda}{\Lambda^2}v^3 \chi \tilde{\psi}^0 + h.c.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_6 =& -\frac{\lambda'}{\Lambda^2}\epsilon_{kl}\phi_i^\dagger \phi_j^\dagger \phi^l d_I^{ijk} \chi \psi^I + h.c. \\
=& -\frac{1}{\sqrt{3}}\frac{1}{2\sqrt{2}}\frac{\lambda'}{\Lambda^2}v^3 \chi \psi^0 + h.c.
\end{aligned}
$$

Redefine M and m by assuming the contribution from neutral sector is zero, that is $M = -M_0 + \frac{\kappa' v^2}{2\Lambda} - \frac{1}{8}\frac{\zeta_1 v^2}{\Lambda}$, and $\frac{1}{2}m = \frac{1}{2}m - \frac{\kappa}{2\Lambda}v^2$. Then we find the masses of different mixing terms are

$$
\begin{aligned}
M(\psi^0\tilde{\psi}^0) =& -M_0 + \frac{\kappa' v^2}{2\Lambda} - \frac{1}{8}\frac{\zeta_1 v^2}{\Lambda} = M \\
M(\psi^{++}\psi^{--}) =& -M_0 + \frac{\kappa' v^2}{2\Lambda} + \frac{3}{8}\frac{\zeta_1 v^2}{\Lambda} = M + \frac{\zeta_1 v^2}{2\Lambda} \\
M(\psi^{+}\psi^{-}) =& M_0 - \frac{\kappa' v^2}{2\Lambda} - \frac{1}{8}\frac{\zeta_1 v^2}{\Lambda} = -M - \frac{\zeta_1 v^2}{4\Lambda} \\
M(\tilde{\psi}^{+}\tilde{\psi}^{-}) =& M_0 - \frac{\kappa' v^2}{2\Lambda} + \frac{3}{8}\frac{\zeta_1 v^2}{\Lambda} = -M + \frac{\zeta_1 v^2}{4\Lambda} \\
M(\tilde{\psi}^{+}\psi^{-}) =& -\frac{\sqrt{3}\zeta_2 v^2}{2\Lambda} \\
M(\tilde{\psi}^{-}\psi^{+}) =& -\frac{\sqrt{3}\zeta_3 v^2}{2\Lambda} \\
M(\chi\chi) =& \frac{1}{2}m - \frac{\kappa}{2\Lambda}v^2 = \frac{1}{2}m \\
M(\chi\psi^0) =& \frac{\lambda' v^3}{2\sqrt{6}\Lambda^2} \\
M(\chi\tilde{\psi}^0) =& -\frac{\lambda v^3}{2\sqrt{6}\Lambda^2} \\
M(\psi^0\psi^0) =& \frac{\zeta_3 v^2}{2\Lambda} \\
M(\tilde{\psi}^0\tilde{\psi}^0) =& \frac{\zeta_2 v^2}{2\Lambda}
\end{aligned}
$$

From the calculation above the mass of the doubly charged state is

$$m_{\chi^{\pm\pm}} = M + \frac{\zeta_1 v^2}{2\Lambda}$$

while in literature it is

$$m_{\chi^{\pm\pm}} = M + \frac{\zeta_1 v^2}{\Lambda}$$

The singly charged mass matrix, which is consistent with result from Xing as $\mathcal{L} \supset -(\tilde{\psi}^+, \psi^+)\mathcal{M}_c(\psi^-, \tilde{\psi}^-)^T$, is

$$\mathcal{M}_c = \begin{pmatrix} -\frac{\sqrt{3}\zeta_2 v^2}{2\Lambda} & -M + \frac{\zeta_1 v^2}{4\Lambda} \\ -M - \frac{\zeta_1 v^2}{4\Lambda} & -\frac{\sqrt{3}\zeta_3 v^2}{2\Lambda} \end{pmatrix}$$

while in literature it is

$$\mathcal{M}_c = \begin{pmatrix} \frac{\sqrt{3}\zeta_2 v^2}{2\Lambda} & -M - \frac{\zeta_1 v^2}{2\Lambda} \\ -M + \frac{\zeta_1 v^2}{2\Lambda} & \frac{\sqrt{3}\zeta_3 v^2}{2\Lambda} \end{pmatrix}$$

The neutral mass matrix as $\mathcal{L} \supset -\frac{1}{2}(\chi, \tilde{\psi}^0, \psi^0)\mathcal{M}_n(\chi, \tilde{\psi}^0, \psi^0)^T$ from my calculation is

$$\mathcal{M}_n = \begin{pmatrix} m & -\frac{\lambda v^3}{2\sqrt{6}\Lambda^2} & \frac{\lambda' v^3}{2\sqrt{6}\Lambda^2} \\ -\frac{\lambda v^3}{2\sqrt{6}\Lambda^2} & \frac{\zeta_2 v^2}{\Lambda} & M \\ \frac{\lambda' v^3}{2\sqrt{6}\Lambda^2} & M & \frac{\zeta_3 v^2}{\Lambda} \end{pmatrix}$$

It's consistent with Xing's result when $n = 4, \quad y = \frac{1}{2}$. The difference of convention in our calculation has been taken into account. While in literature the matrix is

$$\mathcal{M}_n = \begin{pmatrix} m & -\frac{\lambda v^3}{\sqrt{3}\Lambda^2} & -\frac{\lambda' v^3}{\sqrt{3}\Lambda^2} \\ -\frac{\lambda v^3}{\sqrt{3}\Lambda^2} & -\frac{\zeta_2 v^2}{\Lambda} & M \\ -\frac{\lambda' v^3}{\sqrt{3}\Lambda^2} & M & -\frac{\zeta_3 v^2}{\Lambda} \end{pmatrix}$$

3 Relic Density

3.1 Analytical expression of σv

In the $SU(2)_L$-symmetric limit, the dominant $\chi\chi^*$ annihilation channel is into $SU(2)_L \bigotimes U(1)_Y$ vector bosons: $\chi\chi^* \to AA$. $\chi\chi^* \to WW$ has u, t and s diagrams' contribution as $\chi\chi^* \to WB$ and $\chi\chi^* \to BB$ only have u and t. Assuming χ is a fermion, these 3 channels evaluates to

$\chi\chi^* \to WW$

$$\begin{aligned} iM_{WW} = \quad & (ig)^2 \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\nu \frac{-i(\not{p}_1 - \not{k}_1 + m)}{t - m^2}\gamma^\mu u(p_1) T^a T^b \\ & + (ig)^2 \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\mu \frac{-i(\not{p}_1 - \not{k}_2 + m)}{u - m^2}\gamma^\nu u(p_1) T^b T^a \\ & + (ig) g f^{abc}[g^{\mu\nu}(k_2 - k_1) - g^{\nu\rho}(k_1 + 2k_2) + g^{\rho\mu}(2k_1 + k_2)] \frac{-i}{s^2}\overline{v}(p_2)\gamma_\rho u(p_1)\epsilon^*_\mu \epsilon^*_\nu(p_2) T^c \end{aligned}$$

$\chi\chi^* \to WB$

$$\begin{aligned} iM_{WB} = \quad & (ig)(g'Y) \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\nu \frac{-i(\not{p}_1 - \not{k}_1 + m)}{t - m^2}\gamma^\mu u(p_1) T^a \\ & + (ig)(g'Y) \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\mu \frac{-i(\not{p}_1 - \not{k}_2 + m)}{u - m^2}\gamma^\nu u(p_1) T^b \end{aligned}$$

$\chi\chi^* \to BB$

$$\begin{aligned} iM_{BB} = \quad & (g'Y)^2 \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\nu \frac{-i(\not{p}_1 - \not{k}_1 + m)}{t - m^2}\gamma^\mu u(p_1) \\ & + (g'Y)^2 \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\overline{v}(p_2)\gamma^\mu \frac{-i(\not{p}_1 - \not{k}_2 + m)}{u - m^2}\gamma^\nu u(p_1) \end{aligned}$$

Calculation about generators' trace

In dimension-n representation of SU(2), $C_2(R) = \frac{(n^2-1)}{4}$, $d(R) = n$, $d(G) = 3$, $C_2(G) = 2$. Using the relation

$$d(R)C_2(R) = T(R)d(G)$$

We have

$$T(R) = \frac{n}{12}(n^2 - 1)$$

So

$$\begin{aligned}
\mathrm{Tr}[T^aT^b] =& T(R)\delta^{ab} = \delta^{ab}\frac{n}{12}(n^2-1) \\
\mathrm{Tr}[T^aT^a] =& \frac{n}{4}(n^2-1) \\
\mathrm{Tr}[T^aT^aT^bT^b] =& \frac{n}{16}(n^2-1)^2
\end{aligned}$$

As for $\mathrm{Tr}[T^aT^bT^aT^b]$, we need to use the relation

$$T^aT^bT^a = \left(C_2(R) - \frac{1}{2}C_2(G)\right)T^b$$

Then

$$\begin{aligned}
\mathrm{Tr}[T^aT^bT^aT^b] =& \mathrm{Tr}[\left(C_2(R) - \frac{1}{2}C_2(G)\right)T^bT^b] \\
=& \left(C_2(R) - \frac{1}{2}C_2(G)\right)\mathrm{Tr}[T^bT^b] \\
=& \left(\frac{(n^2-1)}{4} - \frac{1}{2}\times 2\right)\frac{n}{4}(n^2-1) \\
=& \frac{n}{16}(n^2-5)(n^2-1)
\end{aligned}$$

We also have

$$\begin{aligned}
f^{abc}f^{abd}T^cT^d =& C_2(G)\delta^{cd}T^cT^d \\
\Longrightarrow \mathrm{Tr}[f^{abc}f^{abd}T^cT^d] =& C_2(G)\mathrm{Tr}[T^cT^c] = \frac{n(n^2-1)}{2} \\
f^{abc}T^aT^bT^c =& \frac{1}{2}iC_2(G)T^cT^c \\
\Longrightarrow \mathrm{Tr}[f^{abc}T^aT^bT^c] =& i\frac{n(n^2-1)}{4}
\end{aligned}$$

Amplitude and σv

I use FeynCalc to calculate the amplitude as follows

$$\frac{1}{4n^2}\sum_{spin,sum}|M|^2 = \frac{1}{4n^2}\sum_{spin,sum}\left(|M_{WW}|^2 + |M_{WB}|^2 + |M_{BB}|^2\right) \tag{14}$$

Since

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{CM} =& \frac{1}{2E_1 2E_2 v}\frac{|p_1|}{(2\pi)^2 4E_{cm}}|M|^2 \\
=& \frac{1}{2m2mv}\frac{m}{(2\pi)^2 4\cdot 2m}|M|^2
\end{aligned}$$

My final result for σv is as follows, the code SigmaV.nb has been put in dropbox/ewdm/Debugging.

$$\sigma v = \frac{2g_2^2 g_1^2 \left(n^2 - 1\right) nY^2 + g_1^4 \left(n^2 - 1\right)^2 n + 8g_2^4 Y^4}{128\pi m^2 n^2}$$

While in Strumia's paper[3] it is

$$\langle \sigma v \rangle \simeq \frac{g_2^4(3 - 4n^2 + n^4) + 16Y^4 g_Y^4 + 8g_2^2 g_Y^2 Y^2(n^2 - 1)}{64\pi M^2 g_\chi}$$

where $g_\chi = 4n$ for a Dirac fermion and $g_\chi = 2n$ for a Majorana fermion.

4 Decay Rate

4.1 Decay Rate in Quintuplet

I set χ_0^1 in our model as the dark matter. The masses of χ_0^1 and χ_0^2 are set to 100GeV and 120GeV respectively.

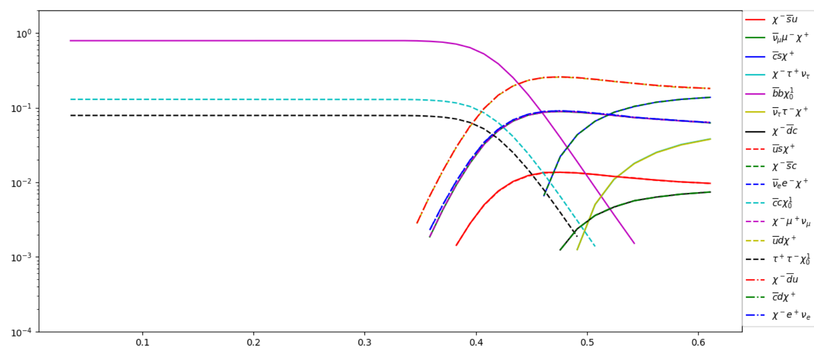


Figure 1: χ_0^2 decay.

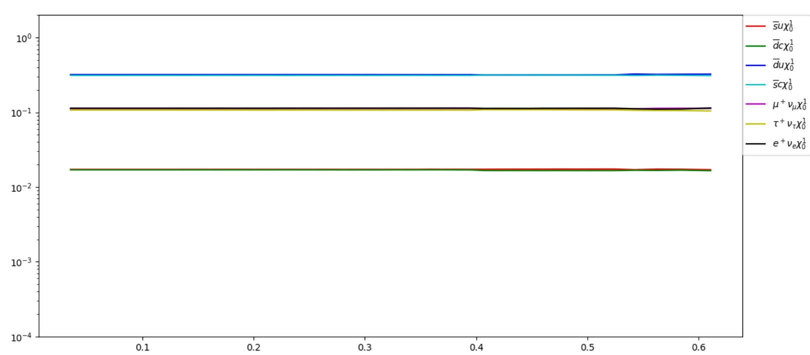


Figure 2: χ^+ decay.

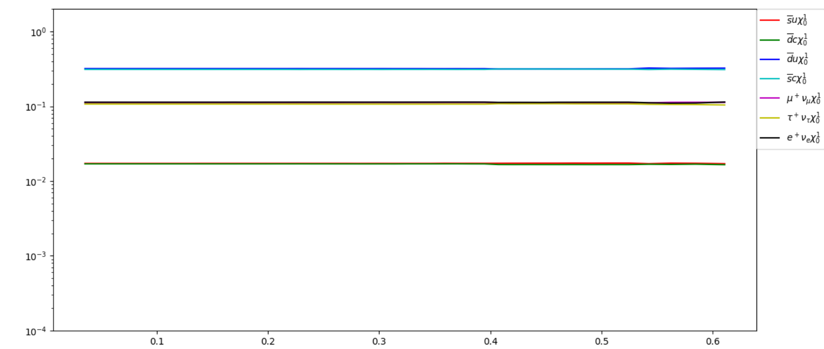


Figure 3: χ^{++} decay.

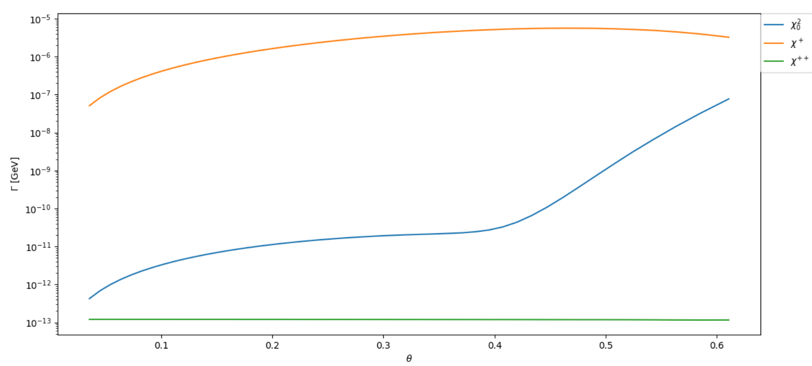


Figure 4: Total Decay Width.

4.2 Muon Decay Analytic Calculation

$$
\begin{aligned}
i\mathcal{M} &= i\frac{ig_{\mu}\theta\nu}{M_w^2}\overline{u}(k_2,s_2)\left(-i\frac{g}{\sqrt{2}}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)\right)\overline{u}(k_4,s_4)\left(-i\frac{g}{\sqrt{2}}\gamma^{\nu}\frac{1}{2}(1-\gamma^5)\right)v(k_3,s_3)\\
&= \frac{g^2}{8M_w^2}g_{\mu\nu}\overline{u}(k_2,s_2)\gamma^{\mu}(1-\gamma^5)u(k_1,s_1)\overline{u}(k_4,s_4)\gamma^{\nu}(1-\gamma^5)v(k_3,s_3)\\
&= \frac{G}{\sqrt{2}}\overline{u}(k_2)\gamma^{\mu}(1-\gamma^5)u(k_1)\overline{u}(k_4)\gamma_{\mu}(1-\gamma^5)v(k_3)
\end{aligned}
$$

$$|\mathcal{M}|^2 = 64G^2(k_1\cdot k_3)(k_2\cdot k_4)$$

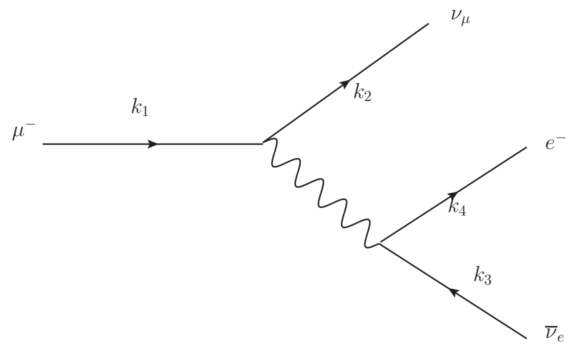


Figure 5: Feynman diagram of muon decay.

$$
\begin{aligned}
d\Gamma &= |\mathcal{M}|^2 d\text{LIPS} \\
&= \frac{1}{2m} \cdot 64G^2 (k_1 \cdot k_3)(k_2 \cdot k_4) \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \frac{d^3k_3}{(2\pi)^3 2E_{k_3}} \frac{d^3k_4}{(2\pi)^3 2E_{k_4}} (2\pi)^4 \delta^4(k_1 - k_2 - k_3 - k_4) \\
&= \frac{G^2}{8m\pi^5} (k_1 \cdot k_3)(k_2 \cdot k_4) \frac{d^3k_2}{E_{k_2}} \frac{d^3k_3}{E_{k_3}} \frac{d^3k_4}{E_{k_4}} \delta^4(k_1 - k_2 - k_3 - k_4) \\
&= \frac{G^2}{8m\pi^5} (k_1 \cdot k_3)(k_2 \cdot k_4) \frac{d^3k_2}{|\overrightarrow{k_2}|} \frac{d^3k_3}{|\overrightarrow{k_3}|} \frac{d^3k_4}{|\overrightarrow{k_4}|} \delta(m - |\overrightarrow{k_2}| - |\overrightarrow{k_3} - |\overrightarrow{k_4}|) \delta^3(k_2 + k_3 + k_4)
\end{aligned}
$$

Embedding momentum into a frame where $k_1 = (m, 0, 0, 0)$. Thus

$$k_1 \cdot k_3 = mE_3$$

$$k_2 \cdot k_4 = \frac{(k_2 + k_4)^2}{2} = \frac{(k_1 - k_3)^2}{2} = \frac{m^2 - 2mE_3}{2}$$

$$
\begin{aligned}
d\Gamma &= \frac{mG^2|\overrightarrow{k_3}|}{16\pi^5} (m - 2|\overrightarrow{k_3}|) \frac{d^3k_3 d^3k_4}{|\overrightarrow{k_3} + \overrightarrow{k_4}||\overrightarrow{k_3}||\overrightarrow{k_4}|} \delta(m - |\overrightarrow{k_3 + k_4}| - |\overrightarrow{k_3} - |\overrightarrow{k_4}|) \\
&= \frac{mG^2|\overrightarrow{k_3}|}{16\pi^5} (m - 2|\overrightarrow{k_3}|) \frac{|\overrightarrow{k_3}|^2 \sin\theta d|\overrightarrow{k_3}| d\theta d\phi d^3k_4}{|\overrightarrow{k_3} + \overrightarrow{k_4}||\overrightarrow{k_3}||\overrightarrow{k_4}|} \delta(m - |\overrightarrow{k_3 + k_4}| - |\overrightarrow{k_3} - |\overrightarrow{k_4}|) \\
&= \frac{mG^2|\overrightarrow{k_3}|^2}{16\pi^5} (m - 2|\overrightarrow{k_3}|) \frac{\sin\theta d|\overrightarrow{k_3}| d\theta d\phi d^3k_4}{(|\overrightarrow{k_3}|^2 + |\overrightarrow{k_4}|^2 + 2|\overrightarrow{k_3}||\overrightarrow{k_4}|\cos\theta)|\overrightarrow{k_4}|} \delta(m - |\overrightarrow{k_3} + \overrightarrow{k_4}| - |\overrightarrow{k_3} - |\overrightarrow{k_4}|)
\end{aligned}
$$

Use $u = |\overrightarrow{k_3} + \overrightarrow{k_4}| \Rightarrow 2udu = -2|\overrightarrow{k_3}||\overrightarrow{k_4}| \sin\theta d\theta$,

$$
\begin{aligned}
d\Gamma &= \frac{mG^2|\overrightarrow{k_3}|}{16\pi^5}(m-2|\overrightarrow{k_3}|)\frac{-udud|\overrightarrow{k_3}|d^3k_4}{u|\overrightarrow{k_4}|^2}\delta(m-|\overrightarrow{k_3+k_4}|-|\overrightarrow{k_3}|-|\overrightarrow{k_4}|)\\
&= \frac{mG^2|\overrightarrow{k_3}|}{16\pi^5}(m-2|\overrightarrow{k_3}|)\frac{d|\overrightarrow{k_3}|d^3k_4}{|\overrightarrow{k_4}|^2}\int du\delta(m-u-|\overrightarrow{k_3}|-|\overrightarrow{k_4}|)\\
&= \frac{mG^2}{8\pi^4}(m-2|\overrightarrow{k_3}|)\frac{|\overrightarrow{k_3}|}{|\overrightarrow{k_4}|^2}d|\overrightarrow{k_3}|d^3k_4\\
&= \frac{mG^2}{8\pi^4}\frac{1}{|\overrightarrow{k_4}|^2}d^3k_4\int_{\frac{m}{2}-|\overrightarrow{k_4}|}^{\frac{m}{2}}|\overrightarrow{k_3}|(m-2|\overrightarrow{k_3}|)d|\overrightarrow{k_3}|\\
&= \frac{mG^2}{8\pi^4}d^3k_4(\frac{m}{2}-\frac{2|\overrightarrow{k_3}|}{3})\\
&= \frac{mG^2}{2\pi^3}|\overrightarrow{k_4}|^2(\frac{m}{2}-\frac{2|\overrightarrow{k_3}|}{3})d|\overrightarrow{k_4}|
\end{aligned}
$$

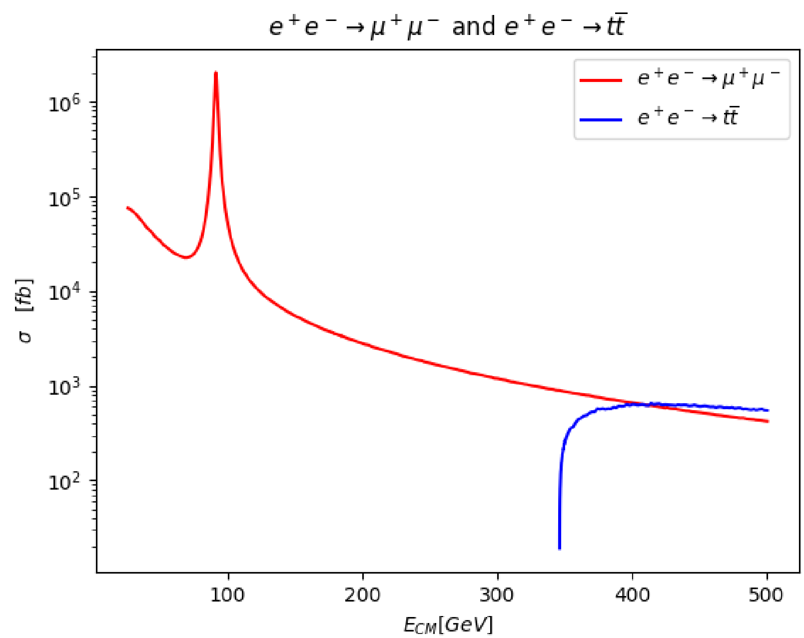
Now we get

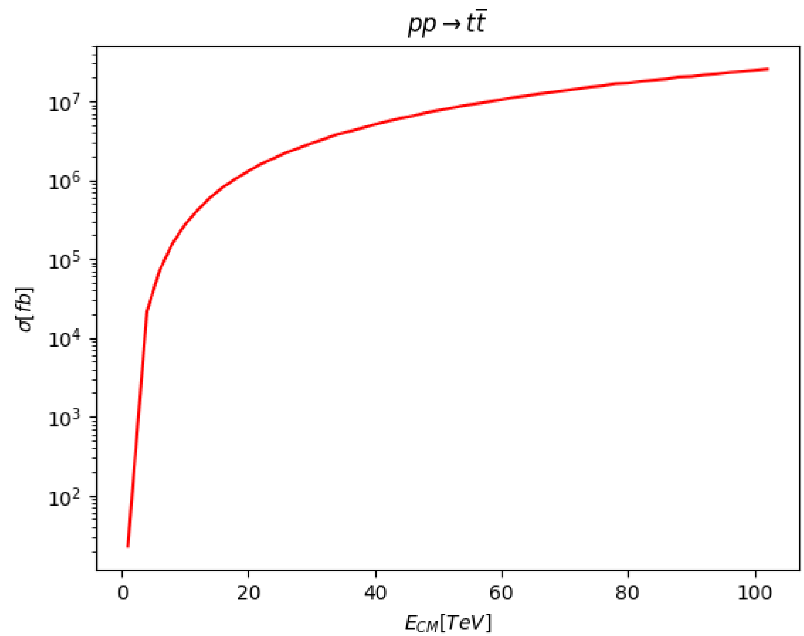
$$
\begin{aligned}
\Rightarrow \frac{d\Gamma}{dE} &= \frac{m^2G^2}{4\pi^3}E^2(1-\frac{4E}{3m})\\
\Gamma &= \frac{m^5G^2}{192\pi^3}\\
\tau &= \frac{192\pi^3}{m^5G^2}
\end{aligned}
$$

5 Warm-up

5.1 Cross Section Calculation Using Simulation Package

I generated events by Madgraph. Shell script is used to get cross sections corresponding to E_{CM} from $10 - 500 GeV$ and grab data from files. Finally, I plotted the "Cross section-Center of mass energy" diagram of the two progresses by python.





5.2 Splitting Function

By both classical and quantum computation, we know that during the scattering of an electron, the total possibility of radiating a very soft photon is infinite. Splitting function describes the probability to find a particle of longitudinal fraction z in the incident particle. In this note I give the process of derivation of splitting function.

5.2.1 Distribution function in QED

Matrix Element for Electron Splitting

The three 4-momentum of incoming and outgoing electron, photon in Fig. (6) can be written as

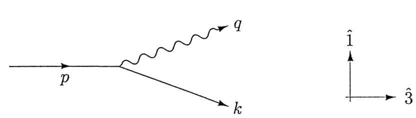


Figure 6: Kinematics of the vertex for emission of a collinear electron or photon

$$
\begin{aligned}
p &= (p, 0, 0, p), \\
q &= (zp, p_\perp, 0, zp - \frac{p_\perp^2}{2zp}), \\
k &= ((1-z)p, -p_\perp, 0, (1-z)p + \frac{p_\perp^2}{2zp})
\end{aligned}
$$

such that the photon is real and the electron is virtual, we have

$$
q^2 = 0, \qquad k^2 = -\frac{p_\perp^2}{z} \tag{14}
$$

Then the photon emission vertex is given by

$$
\begin{aligned}
iM &= \overline{u}_L(k)(-ie\gamma_\mu)u_L(p)\epsilon_T^{*\mu}(q) \\
&= ie\sqrt{2(1-z)p}\sqrt{2p}\epsilon^\dagger(k)\sigma^i\epsilon(p)\epsilon_T^{*i}(q)
\end{aligned}
$$

To order $p_\perp$, the left-handed spinors are

$$
\epsilon(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \epsilon(k) = \begin{pmatrix} p_\perp/2(1-z)p \\ 1 \end{pmatrix} \tag{14}
$$

The polarization vectors for the photon are

$$
\epsilon_L^{*i}(q) = \frac{1}{\sqrt{2}}(1, i, -\frac{p_\perp}{zp}), \qquad \epsilon_R^{*i}(q) = \frac{1}{\sqrt{2}}(1, -i, -\frac{p_\perp}{zp}) \tag{14}
$$

Then we find

$$
iM(e_L^- \to e_L^-\gamma_R) = ie\frac{\sqrt{2(1-z)}}{z}p_\perp \tag{14}
$$

$$
iM(e_L^- \to e_L^-\gamma_L) = ie\frac{\sqrt{2(1-z)}}{z(1-z)}p_\perp \tag{14}
$$

The squared matrix element, averaged over initial helicities, is therefore

$$
\frac{1}{2}\sum_{pols.}|M|^2 = \frac{2e^2p_\perp^2}{z(1-z)}[\frac{1+(1-z)^2}{z}] \tag{14}
$$

The Equivalent Photon Approximation

Call the initial state on the right-hand side of the diagram X and the final state Y, and let $M_{\gamma X}$ represent the matrix element for the scattering of the photon from X. Then the complete diagrams gives a cross section with a virtual photon

$$
\sigma = \frac{1}{(1+v_X)2p2E_X}\int\frac{d^3k}{(2\pi)^3}\frac{1}{3k^0}\int d\prod_Y[\frac{1}{2}\sum|M|^2](\frac{1}{q^2})^2|M_{\gamma X}|^2 \tag{14}
$$

After some simplification, our final result is

$$\sigma(e^- X \to e^- Y) = \int_0^1 dz \int \frac{dp_\perp^2}{p_\perp^2}\frac{\alpha}{2\pi}[\frac{1+(1-z)^2}{z}] \cdot \sigma(\gamma X \to Y)$$
$$= \int_0^1 dz \frac{\alpha}{2\pi}\log\frac{s}{m^2}[\frac{1+(1-z)^2}{z}] \cdot \sigma(\gamma X \to Y)$$

Then the probability to find a photon of longitudinal fraction z in the incident electron is

$$f_\gamma(z) = \frac{\alpha}{2\pi}\log\frac{s}{m^2}[\frac{1+(1-z)^2}{z}] \tag{14}$$

where m is the electron.

There are two singularities in Eq. (5.2.1). While zp represents the energy of emitted phton, when $z \to 0$ the energy tends to be zero, too. So this singularity is called **soft singularity**. $f_\gamma(z)$ also diverges near $s \gg m^2$. While $\log\frac{s}{m^2}$ is the integral result of $\frac{dp_\perp^2}{p_\perp^2}$, the integral over $p_\perp^2$ runs from momentum transfers of order s down to the electron mass m^2, which cuts off the singularity. So in fact, this singularity means the perpendicular momentum of photon and outgoing electron tends to be zero so that they are collinear to the incoming electron. Therefore, this singularity is known as **collinear singularity**.

The Electron Distribution

If the emitted photon is real and the electron is virtual, the process can be treated the same way. The distribution function is like substitute $x = 1 - z$ of Eq. (5.2.1)

$$f_e^{(1)}(x) = \frac{\alpha}{2\pi}\log\frac{s}{m^2}[\frac{1+x^2}{1-x}] \tag{14}$$

Considering the zeroth-order parton distribution $f_e^{(0)}(x) = \delta(1-x)$ and the IR positive contributions to the total rate from the emission of soft photons are balanced by negative contributions from diagrams with soft virtual photons. The parton distribution for electrons in the electron should have the form

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\frac{s}{m^2}[\frac{1+x^2}{(1-x)} - A\delta(1-x)] \tag{14}$$

A is determined by the condition that the electron contain exactly one electron parton

$$\int_0^1 dx \quad f_e(x) = 1 \tag{14}$$

As there is a singular denominator in Eq. (5.2.1), in order to make Eq. (5.2.1) integrable we define a distribution

$$\frac{1}{(1-x)_+} \tag{14}$$

to agree with the function 1/(1-x) for all values of x less than 1, and to have a singularity at $x = 1$. Then the integral of this distribution with any smooth function f(x) gives

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x) - f(1)}{(1-x)} \tag{14}$$

So that

$$\int_0^1 dx \frac{1+x^2}{(1-x)_+} = \int_0^1 dx \frac{x^2 - 1}{(1-x)} = -\frac{3}{2} \tag{14}$$

Thus our final form of the electron distribution is

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi} \log \frac{s}{m^2} [\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)] \tag{14}$$

5.2.2 Splitting Functions in QED

In order to account for emission of many collinear photons we can derive the evolution equations for $f_\gamma(x)$ and $f_e(x)$ and take the effects of pair creation into account. The matrix element of photon splitting into pairs is

$$\frac{1}{2}\sum_{pols.} |M|^2 = \frac{2e^2 p_\perp^2}{z(1-z)}[z^2 + (1-z)^2] \tag{14}$$

The normalization factor of a negative term is

$$\int_0^1 dz(z^2 + (1-z)^2) = \frac{2}{3} \tag{14}$$

Then the evolution equations takes the form

$$\begin{aligned}
\frac{d}{d\log Q} f_\gamma(x,Q) &= \frac{\alpha}{\pi}\int_x^1 \{P_{\gamma\leftarrow e}(z)[f_e(\frac{x}{z},Q) + f_{\bar{e}}(\frac{x}{z},Q)] + P_{\gamma\leftarrow\gamma}(z) f_\gamma(\frac{x}{z},Q)\} \\
\frac{d}{d\log Q} f_e(x,Q) &= \frac{\alpha}{\pi}\int_x^1 \{P_{e\leftarrow e}(z) f_e(\frac{x}{z},Q) + P_{e\leftarrow\gamma}(z) f_\gamma(\frac{x}{z},Q)\} \\
\frac{d}{d\log Q} f_{\bar{e}}(x,Q) &= \frac{\alpha}{\pi}\int_x^1 \{P_{e\leftarrow e}(z) f_{\bar{e}}(\frac{x}{z},Q) + P_{e\leftarrow\gamma}(z) f_\gamma(\frac{x}{z},Q)\}
\end{aligned}$$

The splitting functions $P_{i\leftarrow j}(z)$ are given by

$$\begin{aligned}
P_{e\leftarrow e}(z) &= \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \\
P_{\gamma\leftarrow e}(z) &= \frac{1+(1-z)^2}{z} \\
P_{e\leftarrow\gamma}(z) &= z^2 + (1-z)^2 \\
P_{\gamma\leftarrow\gamma}(z) &= -\frac{2}{3}\delta(1-z)
\end{aligned}$$

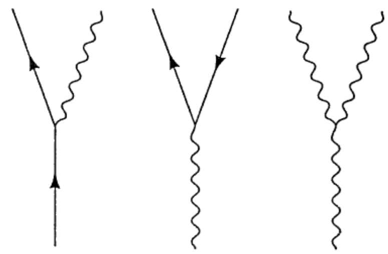


Figure 7: The three vertices that contribute to parton evolution in QCD.

5.2.3 Splitting Functions in QCD

The first vertex of Fig. 7, representing the splitting of a quark into a quark and a gluon, receives the color factor

$$\frac{1}{3}tr[t^a t^a] = C_2(r) = \frac{4}{3} \tag{14}$$

The second vertex, representing the splitting of a gluon into a quark-antiquark pair, receives the factor

$$\frac{1}{8}tr[t^a t^a] = \frac{1}{2} \tag{14}$$

Then the first three splitting functions can be taken from Eqs. (5.2.2), multiplied by the color factors computed in Eqs. (5.2.3) and (5.2.3):

$$\begin{aligned}
P_{q\leftarrow q}(z) &= \frac{4}{3}[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)] \\
P_{g\leftarrow q}(z) &= \frac{4}{3}[\frac{1+(1-z)^2}{z}] \\
P_{q\leftarrow g}(z) &= \frac{1}{2}[z^2+(1-z)^2]
\end{aligned}$$

The matrix element of the third vertex in Fig. 7 can be computed and the final result is

$$\frac{1}{2}\cdot\frac{1}{8}\sum_{spin,color}|M|^2 = \frac{2e^2p_\perp^2}{z(1-z)}\cdot P^{(0)}_{g\leftarrow g}(z) \tag{14}$$

Where

$$P_{g\leftarrow g} = 6\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right] + (\frac{11}{2} - \frac{n_f}{3})\delta(1-z) \tag{14}$$

Which is the fourth splitting function de The general procedure ofing unpolarized cross sections for QED processes is elaborated in Chapter 5 of the QFT textbook written by Peskin and Schroeder [4]. Here, we take the simple interaction process $e^+e^- \to \mu^+\mu^-$ as a specific example of the general calculation process.

5.3 $e^+e^- \to \mu^+\mu^-$

5.3.1 Draw the Feynman Diagram

First, we draw the leading order Feynman diagram of $e^+e^- \to \mu^+\mu^-$ process as Fig.(8). In this process, the contribution to amplitude is only from s-channel diagram since we only observe this kind of interaction from experiments.

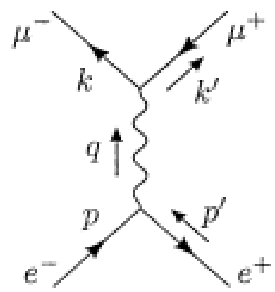


Figure 8: Feynman diagram of $e^+e^- \to \mu^+\mu^-$ process.

5.3.2 Feynman Rules

Using perturbation theory we could the scattering amplitude of one interaction process from the QED Lagrangian. Then we could derive its Feynman rules to simplify calculation by associating analytic expressions with pieces of Feynman diagrams such as external lines, vertex, propagators. For $e^+e^- \to \mu^+\mu^-$ process, we can write down the amplitude M quickly as following using Feynman rules:

$$iM = \overline{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)(\frac{-g_{\mu\nu}}{q^2})\overline{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \tag{14}$$

where μ, ν are Einstein summation indices while r, s denote the spin of particles.

5.3.3 Square the Amplitude

Finding the conjugate of M then we can obtain the squared matrix element:

$$|M|^2 = \frac{e^4}{q^4}(\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p'))(\overline{u}(k)\gamma_\mu v(k')\overline{v}(k')\gamma_\nu u(k)) \tag{14}$$

In most experiments the electron and positron beams are unpolarized, so the measured cross section is an average over the electron and positron spins s and s'. Muon detectors are normally blind to polarization, so the measured crosssection is a sum over the muon spins r and r'. Then what we want to compute is

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'}|M(s, s' \to r, r')|^2 \tag{14}$$

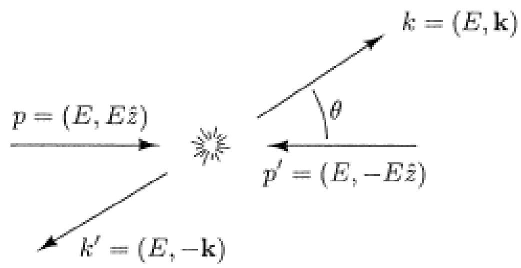


Figure 9: Center of mass frame.

Using the completeness relations

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m; \qquad \sum_s v^s(p)\overline{v}^s(p) = \not{p} - m \tag{14}$$

we can simplify the squared amplitude to

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{e^4}{4q^4} tr\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] tr\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right] \tag{14}$$

5.3.4 Trace Technology

Using trace technology and set $m_e = 0$ as $m_e/m_\mu \approx 1/200$ we can simplify the result to

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right] \tag{14}$$

5.3.5 Plug into Specific Frame of Reference

To obtain a more explicit formula we must specialize to a particular frame of reference.The simplest choice is evaluating cross sections in the center-of-mass frame in Fig.(9) In this frame of reference we can rewrite Eq.5.3.4 in terms of E and θ:

$$\begin{aligned}\frac{1}{4}\sum_{spins}|M|^2 &= \frac{8e^4}{16E^4}\left[E^2(E - |\mathbf{k}|cos\theta)^2 + E^2(E + |\mathbf{k}|cos\theta)^2 + 2m_\mu^2E^2\right] \\ &= e^4\left[(1 + \frac{m_\mu^2}{E^2}) + (1 - \frac{m_\mu^2}{E^2})cos^2\theta\right]\end{aligned}$$

5.3.6 Plug into Cross Section Formula

Then we can write the differential cross section formula in the center of mass frame:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{cm} &= \frac{1}{2E_A 2E_B|v_A v_B|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}}|M|^2 \\ &= \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[(1+\frac{m_\mu^2}{E^2})+(1-\frac{m_\mu^2}{E^2})cos^2\theta\right]\end{aligned}$$

Integrating over $d\Omega$, we find the total cross section:

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}(1+\frac{1}{2}\frac{m_\mu^2}{E^2}) \tag{14}$$

5.3.7 High-energy Limit and Nonrelativistic Limit

In the high-energy limit $E \geq m_\mu$, Eq.(5.3.6) and (5.3.6) reduce to

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\underset{E\geq m_\mu}{\longrightarrow} \frac{\alpha^2}{4E_{cm}^2}(1+cos^2\theta) \\ \sigma_{total} &\underset{E\geq m_\mu}{\longrightarrow} \frac{4\pi\alpha^2}{3E_{cm}^2}(1-\frac{3}{8}(\frac{m_\mu}{E})^4-\cdots)\end{aligned}$$

When E is barely larger than m_μ, Eq.(5.3.6) becomes

$$\frac{d\sigma}{d\Omega} \underset{|\mathbf{k}|\to 0}{\longrightarrow} \frac{\alpha^2}{2E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}} = \frac{\alpha^2}{2E_{cm}^2}\frac{|\mathbf{k}|}{E} \tag{14}$$

5.4 $e^-\mu^- \to e^-\mu^-$

Electron-Muon scattering is a different but closely related process to $e^+e^- \to \mu^+\mu^-$ process, understanding which could help us to be familiar with some tricks in calculation of QED processes such as crossing symmetry and Mandelstam variables. Draw the Feynman diagram as in Fig.(10), compute the squared amplitude, averaged and summed over spins we obtain

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{e^4}{4q^4}tr\,[(\not{p}_1'+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu]\,tr\,[(\not{p}_2'+m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu] \tag{14}$$

The results are exactly the same as our results for $e^+e^- \to \mu^+\mu^-$ process, with the replacements

$$p \to p_1, \qquad p' \to -p_1', \qquad k \to p_2', \qquad k' \to -p_2. \tag{14}$$

Put Eq.(5.4) into a particular frame of reference and cross section formula we get the differential cross section expression

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E+k)^2(1-cos\theta)^2}\left((E+k)^2+(E+kcos\theta)^2-m_\mu^2(1-cos\theta)\right) \tag{14}$$

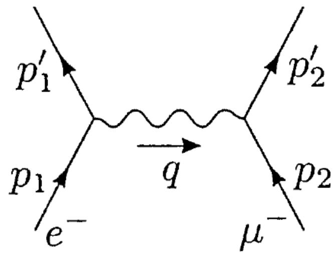


Figure 10: Feynman diagram of $e^-\mu^- \to e^-\mu^-$ process

5.5 Crossing Symmetry

The relation between $e^+e^- \to \mu^+\mu^-$ and $e^-\mu^- \to e^-\mu^-$ is the example of crossing symmetry. That is

$$M(\phi(p) + \cdots \to \cdots) = M(\cdots \to \cdots + \overline{\phi}(p)) \tag{14}$$

where $\overline{\phi}$is the antiparticle of ϕ and $k = -p$.

Label the four external momentum as

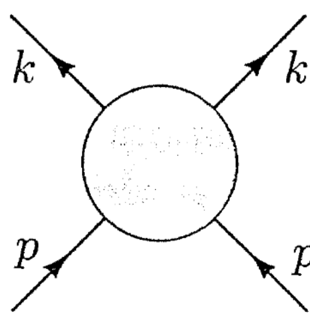


Figure 11: Four external momentum

We now define three Mandelstam variables:

$$\begin{aligned} s &= (p + p')^2 = (k + k')^2 \\ t &= (k - p)^2 = (k' - p')^2 \\ u &= (k' - p)^2 = (p' - k)^2 \end{aligned}$$

Then our previous results could be expressed in terms of s,t,u.

In $e^+e^- \to \mu^+\mu^-$, Eq.(5.3.4) can be expressed as

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{8e^4}{s^2}[(\frac{t}{2})^2 + (\frac{u}{2})^2] \tag{14}$$

In $e^-\mu^- \to e^-\mu^-$ process we can write down

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{8e^4}{t^2}[(\frac{s}{2})^2 + (\frac{u}{2})^2] \tag{14}$$

5.6 Compton Scattering

There are two diagrams which has contribution to the amplitude of Compton scattering process $\gamma e^- \to \gamma e^-$:

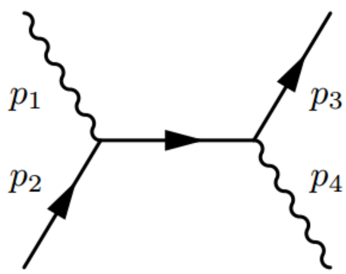


Figure 12: S-channel diagram.

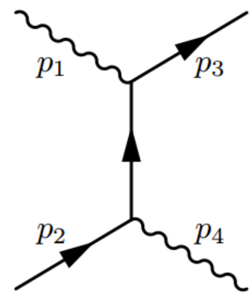


Figure 13: T-channel diagram.

$$iM_s = (-ie)^2\epsilon_1^\mu\epsilon_4^{\star\mu}\overline{u}(p_3)\gamma^\nu\frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1+p_2)^2 - m^2}\gamma^\mu u(p_2) \tag{14}$$

$$iM_s = (-ie)^2\epsilon_1^\mu\epsilon_4^{\star\mu}\overline{u}(p_3)\gamma^\mu\frac{i(\not{p}_2 - \not{p}_4 + m)}{(p_2-p_4)^2 - m^2}\gamma^\nu u(p_2) \tag{14}$$

There is a similar trick for summing over photon polarization vectors

$$\sum_{polarizations}\epsilon_\mu^\star\epsilon_\nu \to -g_{\mu\nu} \tag{14}$$

After approximately similar but more complicated mathematics we get the result

$$\frac{d\sigma}{dcos\theta} = \frac{\pi\alpha^2}{m^2}(\frac{\omega'}{\omega})^2[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta] \tag{14}$$

We could also obtain its approximation in the low and high-energy limit.

QCD Process I'm still trying to figure out the calculation of cross section in QCD processes. Generally, we need to take more diagrams into account in QCD and the calculations would be more difficult.

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