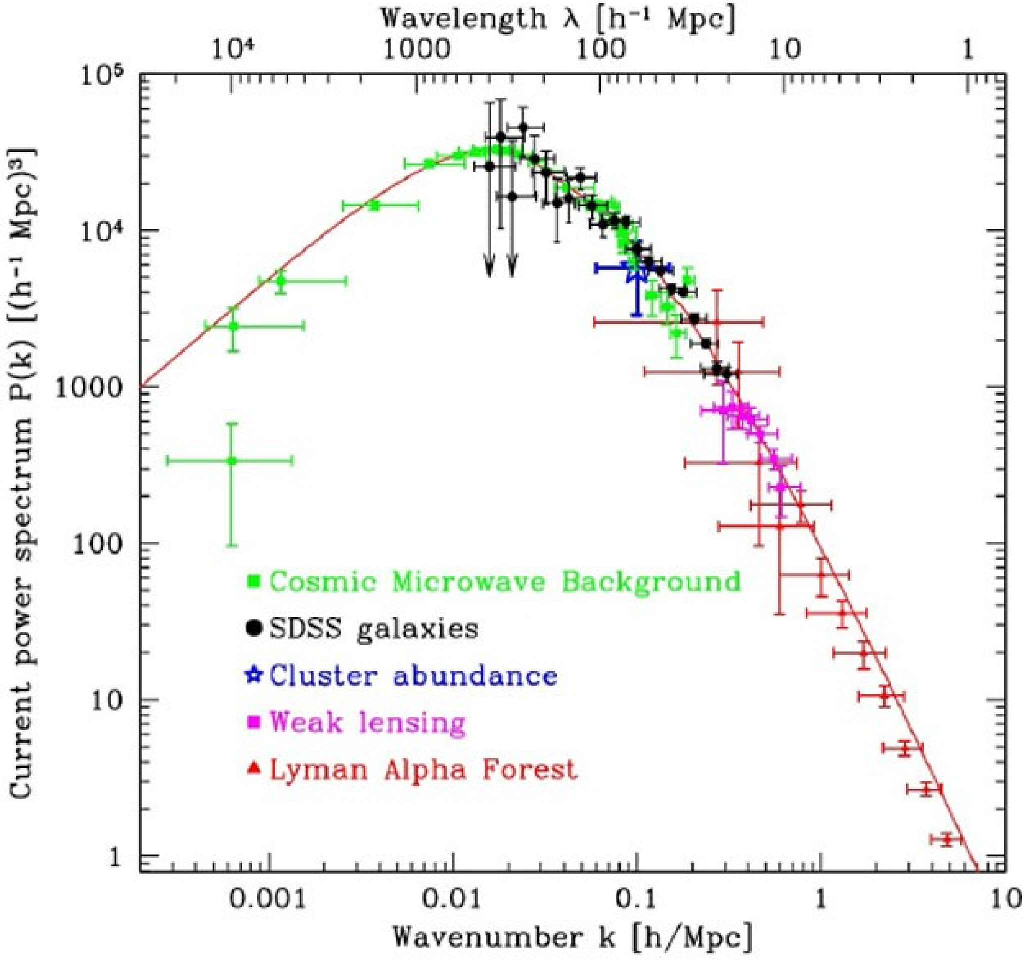
2-pt statistics



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Density perts

$$\rho = \bar{\rho}(1+\delta) \qquad\qquad \delta > -1$$

$$\langle f \rangle \equiv \int d\gamma Prob(\gamma) f()$$

From statistical homogeneity

$$\langle \rho(x) \rangle = \langle \rho \rangle \quad \Rightarrow \quad \boxed{\langle \delta(x) \rangle = 0}$$

2pt correlation function

$$\xi^{(2)}(\mathbf{r}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle$$

From statistical isotropy $\xi^{(2)}(\mathbf{r}) = \xi^{(2)}(r)$

Variance $\langle \delta^2 \rangle - \langle \delta \rangle^2 = \langle \delta^2 \rangle = \xi^{(2)}(0)$

Fourier

Cubic volume $V = L^d$ with periodic boundary conditions

$$f(\mathbf{x}) = \sum_k f_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \qquad k_i = n_i \frac{2\pi}{L}, \quad n_i = 1, 2, 3, \ldots$$

Coefficients are $$f_\mathbf{k} = \frac{1}{V} \int_V f(x) e^{-i\mathbf{k}\cdot\mathbf{x}} d^d\mathbf{x}$$

Reality implies $f_{-\mathbf{k}} = f_\mathbf{k}^*$

k=0 implies $f_\mathbf{0} = \bar{f}$

Orthogonality

$$\int dV \left(e^{i\mathbf{k}\cdot\mathbf{x}}\right)^* \left(e^{i\mathbf{k}'\cdot\mathbf{x}}\right) = V \delta_{\mathbf{k}\mathbf{k}'}$$

Completeness $$\frac{1}{V} \sum_\mathbf{k} \left(e^{i\mathbf{k}\cdot\mathbf{x}}\right)^* \left(e^{i\mathbf{k}\cdot\mathbf{x}'}\right) = \delta_D^d(\mathbf{x}' - \mathbf{x})$$

Fourier

Convolution

$$(f * g)(\mathbf{x}) = \int_V f(\mathbf{x}')g(\mathbf{x} - \mathbf{x}')d^d x' = V \sum_k f_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

Plancherel formula

$$\frac{1}{V}\int_V f(\mathbf{x})g(\mathbf{x})d^d x = V \sum_k f^*_{\mathbf{k}} g_{\mathbf{k}}$$

In the infinite V limit

$$L^d f_{\mathbf{k}} \;\to\; f(\mathbf{k})$$

$$\left(\frac{2\pi}{L}\right)^d \sum_{\mathbf{k}} \;\to\; \int d^d k$$

$$\left(\frac{2\pi}{L}\right)^d \delta_{\mathbf{k}\mathbf{k}'} \;\to\; \delta(\mathbf{k} - \mathbf{k}')$$

Power spectrum

Fourier delta

$$\delta_{\mathbf{k}} = \frac{1}{V}\int_V \delta(\mathrm{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^d\mathbf{x}$$

Statistical Homogeneity $\langle\delta(x)\rangle = 0 \quad \Rightarrow \quad \langle\delta_{\mathbf{k}}\rangle = 0$

Power spectrum

$$\begin{aligned}
\langle\delta^*_{\mathbf{k}}\delta_{\mathbf{k}'}\rangle &= \frac{1}{V^2}\int d^dx e^{i\mathbf{k}\cdot\mathbf{x}} \int d^dx' e^{-i\mathbf{k}'\cdot\mathbf{x}'}\langle\delta(\mathbf{x})\delta(\mathbf{x}')\rangle \\
&= \frac{1}{V^2}\int d^dx e^{i\mathbf{k}\cdot\mathbf{x}} \int d^dr e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle \\
&= \frac{1}{V^2}\int d^dr e^{-i\mathbf{k}'\cdot\mathbf{r}}\xi(\mathbf{r}) \int d^dx e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= \frac{1}{V}\delta_{\mathbf{k}\mathbf{k}'}\int d^dr e^{-i\mathbf{k}\cdot\mathbf{r}}\xi(\mathbf{r}) \equiv \frac{1}{V}\delta_{\mathbf{k}\mathbf{k}'}P(\mathbf{k}),
\end{aligned}$$

$$\boxed{P(\mathbf{k}) \equiv V\langle|\delta_{\mathbf{k}}|^2\rangle = \int d^dr\, e^{-i\mathbf{k}\cdot\mathbf{r}}\xi(\mathbf{r})}$$

$$\boxed{\xi(\mathbf{r}) = \frac{1}{(2\pi)^d}\int d^dk\, e^{i\mathbf{k}\cdot\mathbf{r}}P(\mathbf{k})}$$

Power spectrum

Fourier transform

$$\xi(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_V P(\mathrm{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^d\mathbf{k}$$

Dimensionless

$$\Delta^2(\mathbf{k}) \sim k^d P(\mathbf{k})$$

Dimensionfull

	Correlation function	Power spectrum
Stat. Isotropy	$\xi(\mathbf{r}) = \xi(r)$	$P(\mathbf{k}) = P(k)$
	Intuitive	**Theory**
Modes (linear)	dependent	independent

“Systematics” drive apart these Fourier cousins

Power spectrum

Variance

$$\langle \delta^2 \rangle \equiv \xi(0) = \frac{1}{(2\pi)^d} \int P(k) d^d k = \int_{-\infty}^{\infty} \Delta^2(k) d \ln k$$

$$\Delta^2(\mathbf{k}) \sim k^d P(\mathbf{k})$$

The opposite if $P(k) \to 0$ as $k \to 0$

$$P(0) = V \langle \delta_0^2 \rangle = \int \xi(r) d^d r \to 0$$

Integral constraint

2PCF should become negative at some point.

A distance from an overdense region would get an underdensity.

Power spectrum

For Istropic functions in 1D, 2D and 3D

$$P(k) = \int_0^\infty \xi(r) \cos kr \, 2dr$$

$$P(k) = \int_0^\infty \xi(r) J_0(kr) \, 2\pi r \, dr$$

$$P(k) = \int_0^\infty \xi(r) \frac{\sin kr}{kr} 4\pi r^2 dr$$

Damped Oscillations

Power law spectra

spectral index *n*

For some scales

$$\xi(r) = \left(\frac{r}{r_0}\right)^\gamma \quad \Rightarrow \quad P(k) = A^2 \left(\frac{k}{k_p}\right)^{\gamma - d}$$

Observationally in 3D for *r<10Mpc/h*

$$\gamma \simeq 1.8 \quad \Rightarrow \quad n = -1.2$$

Power-law spectra

Variance

$$\langle \delta^2 \rangle = \xi(0) = \int_0^\infty \mathcal{P}(k) \frac{dk}{k} \propto \int_0^\infty k^{n+d-1} dk = \frac{1}{n+d} \left[k^{n+d} \right]_0^\infty \qquad \text{for} \quad n \neq -d$$

Finite for $n > -d$ as $k \to 0$ large scales

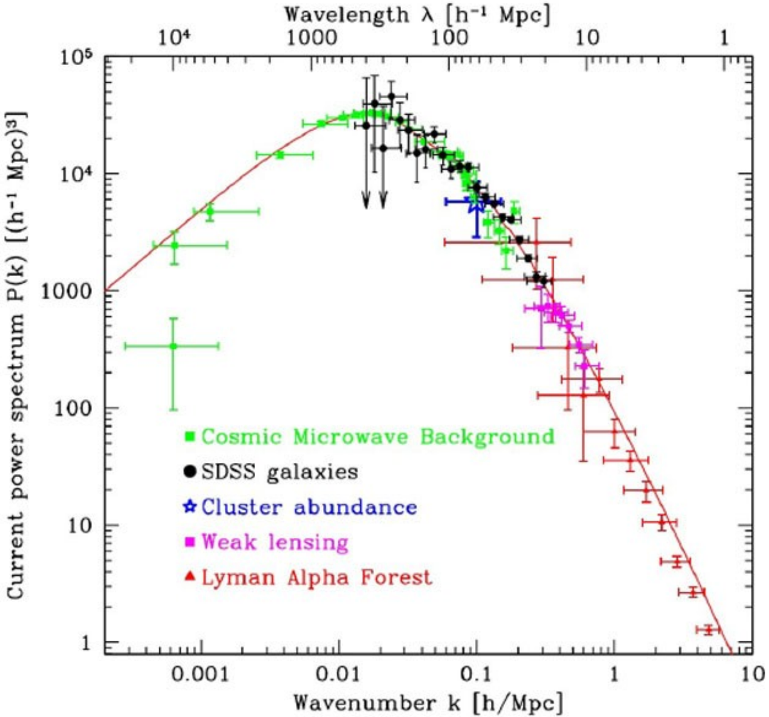
$n < -d$ as $k \to \infty$ small scales

n=-d scale invariant spectrum

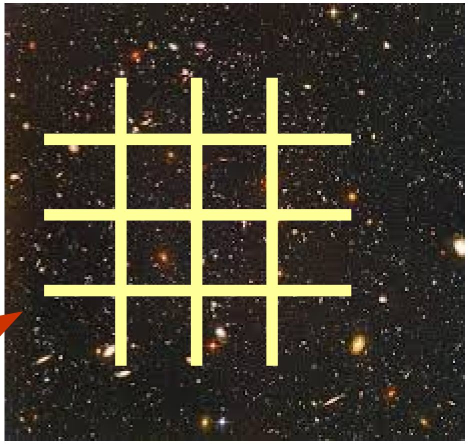
No asymptotic homogeneity

n=0 implies $\xi = 0$

Poisson distribution!



In practice



Fourier transform of discrete samples

Small grid such that it cell has 1 or 0 galaxies

$$\rho_{\mathbf{k}} = \frac{1}{V}\int_V \rho(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^d x = \frac{1}{V}\sum_j (n_j/\delta V) e^{-i\mathbf{k}\cdot\mathbf{x}_j} \delta V = \frac{1}{V}\sum_j n_j e^{-i\mathbf{k}\cdot\mathbf{x}_j},$$

$$\rho_k = \frac{1}{V}\sum_{i=gal} e^{-ik\cdot x}$$

$$\delta_k = \frac{1}{\langle N \rangle}\sum_{i=gal}^{N} e^{-ik\cdot x}$$

$$\delta_0 = \bar{\delta} = \frac{N - \langle N \rangle}{\langle N \rangle}$$

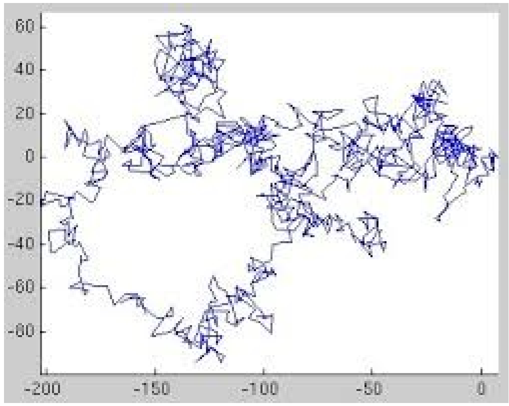
$$\langle N \rangle \equiv \langle \rho \rangle V$$

Power spectrum of Poisson distr.

Uncorrelated delta's

$$\delta_k = \frac{1}{\langle N \rangle} \sum_{i=gal}^{N} e^{-ik\cdot x}$$

Random walk in complex plane



$$|\delta_{\mathbf{k}}|^2 \;=\; \delta_{\mathbf{k}}^* \delta_{\mathbf{k}} \;=\; \frac{1}{\langle N \rangle^2} \sum_{ij} e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} \;=\; \frac{1}{\langle N \rangle^2} \left(\sum_{i \neq j} e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} + \sum_i 1 \right)$$

$$= \;\frac{1}{\langle N \rangle^2} \left(2 \sum_{\text{pairs}} \cos(\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)) + N \right) \,.$$

Power spectrum

$$\boxed{P(k) \;=\; V \langle |\delta_{\mathbf{k}}|^2 \rangle \;=\; V \frac{\langle N \rangle}{\langle N \rangle^2} \;=\; \frac{V}{\langle N \rangle} \;=\; \frac{1}{\langle \rho \rangle} \,,}$$

Vanishing spectral index

Power spectrum for matter distr.

Delta's are now correlated

$$n_i^2 = n_i$$

(Split the i=j contribution)

$$\begin{aligned}
\langle \rho^*_{\mathbf{k}} \rho_{\mathbf{k}'} \rangle &= \frac{1}{V^2} \sum_j \langle n_j \rangle e^{i(\mathbf{k}-\mathbf{k}')\cdot \mathbf{x}_j} + \frac{1}{V^2} \sum_{i \neq j} \langle n_i n_j \rangle e^{i\mathbf{k}\cdot\mathbf{x}_i} e^{-i\mathbf{k}'\cdot\mathbf{x}_j} \\
&= \frac{1}{V^2} \sum_j \langle \rho \rangle \delta V \, e^{i(\mathbf{k}-\mathbf{k}')\cdot \mathbf{x}_j} + \frac{1}{V^2} \sum_{i \neq j} \langle \rho \rangle^2 \delta V^2 \left[1 + \xi(\mathbf{x}_j - \mathbf{x}_i)\right] e^{i\mathbf{k}\cdot\mathbf{x}_i} e^{-i\mathbf{k}'\cdot\mathbf{x}_j} \\
&= \frac{\langle \rho \rangle}{V^2} \int_V e^{i(\mathbf{k}-\mathbf{k}')\cdot \mathbf{x}} d^d x + \frac{\langle \rho \rangle^2}{V^2} \int_V d^d x e^{i\mathbf{k}\cdot\mathbf{x}} \int_V d^d x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} \\
&\quad + \frac{\langle \rho \rangle^2}{V^2} \int_V d^d x d^d x' \xi(\mathbf{x}' - \mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \\
&= \frac{\langle \rho \rangle}{V} \delta_{\mathbf{k}\mathbf{k}'} + 0 + \langle \rho \rangle^2 \frac{1}{V} \delta_{\mathbf{k}\mathbf{k}'} P(\mathbf{k}) \,,
\end{aligned}$$

$$\tilde{P}(\mathbf{k}) \equiv \frac{V}{\langle \rho \rangle^2} \langle |\rho_{\mathbf{k}}|^2 \rangle = \frac{1}{\langle \rho \rangle} + P(\mathbf{k}) \,,$$

Shot noise

$$\boxed{\hat{P}(\mathbf{k}) = V \left| \frac{1}{N} \sum_{i=gal}^{N} e^{-i\mathbf{k}\cdot\mathbf{x}} \right|^2 - \frac{V}{N}}$$