Relational Reconstruction of Spacetime Geometry from Graph Laplacians

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Abstract

Background: We present a relational and spectral construction of effective spacetime geometry in which metric notions arise solely from correlation structures, without assuming a background manifold, coordinates, or fundamental geometric degrees of freedom.

Methods: Starting from a purely relational substrate endowed with a symmetric connectivity operator, we defined operational distances through minimal path functionals. A non-circular coarse-graining scheme was introduced to distinguish pre-geometric combinatorial neighborhoods from geometry-aware weighted distances. The spectral admissibility criteria identify regimes in which relational variations become sufficiently smooth to support an effective geometric description.

Results: In these projectable regimes, the resulting distance matrix admits a low-dimensional embedding that enables the reconstruction of emergent coordinates and an effective metric structure. Standard geometric observables—such as proper time, spatial distance, and curvature—arise as descriptive summaries of relational constraints. At appropriate limits, the effective metric reproduces general-relativistic phenomenology, including the recovery of Schwarzschild geometry for isolated, approximately symmetric configurations, without postulating gravitational dynamics at the fundamental level.

Conclusions: The framework naturally predicts the breakdown of geometric descriptions when spectral gaps close or relational structures become non-local, providing intrinsic criteria for the limits of continuum spacetime. Numerical and analytical results supporting a universal spectral hierarchy are presented in the Appendix. Overall, this work establishes a concrete pathway from relational spectral data to emergent metric geometry and positioning spacetime as an operational construct rather than a primitive entity.

Keywords: Pre-geometric substrate, emergent spacetime, relational dynamics, spectral geometry, emergent metric, spectral dimension

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1 Introduction

A central open problem in gravitational physics concerns the emergence of spacetime geometry from more primitive, non-metric structures. In general relativity, the metric tensor is postulated to be a fundamental dynamical field that encodes both the causal and metrical properties of spacetime. In contrast, many approaches to background-independent gravity suggest that spacetime geometry should arise as an effective description, reconstructed from relational or pre-geometric data.

This question has motivated a wide range of frameworks, including canonical and covariant approaches to quantum gravity, causal set theory, loop-based constructions, and programs based on spectral or non-commutative geometry. While these approaches differ in their technical implementation, they share a common challenge: how to recover a smooth pseudo-Riemannian geometry from fundamentally non-metric degrees of freedom, without introducing a background manifold or metric by assumption.

In this study, we address this reconstruction problem from a deliberate minimalist and operational perspective. Rather than postulating a spacetime metric or continuum structure from the outset, we consider relational systems described by connectivity data encoded in discrete or coarse-grained graphs. The central object of our analysis is the spectrum of a Laplacian operator defined for such relational structures. We investigate the conditions under which spectral information alone is sufficient to define the effective notions of distance, curvature, and geometry.

Our approach is motivated by the observation that spectral data provide a natural bridge between discrete relational systems and continuum geometry. At appropriate limits, the spectrum of graph Laplacians is known to encode geometric information analogous to that of the differential operators on smooth manifolds. The question that we pursue here is whether this correspondence can be made explicit and operational, yielding an effective metric structure without assuming one at the fundamental level.

We developed a framework in which admissible relational configurations were characterized by spectral consistency conditions. Effective geometric quantities were then reconstructed through spectral distances and embeddings, defined purely in terms of the Laplacian spectrum. No coordinates, metric tensors, or variational principles were assumed a priori. Instead, such structures emerge only in regimes in which the relational system admits a stable and well-defined spectral continuum limit.

Within this setting, we demonstrate that familiar geometric notions can be recovered in a controlled manner. In particular, we demonstrate how an effective metric description arises locally, how the curvature can be defined operationally, and how known solutions of general relativity can be approximated within appropriate limits. As a concrete benchmark, we demonstrate the recovery of a Schwarzschild-type geometry from purely spectral and relational inputs.

The scope of this study is intentionally restricted. We did not address the dynamics of matter fields, quantum statistics, or cosmological evolution. Our aim is not to propose a complete theory of quantum gravity but to isolate and analyze the geometric reconstruction problem in its simplest and most transparent form.

The present manuscript follows a theoretical structure in which methodological constructions and results are developed jointly, as is customary in foundational and theoretical physics. The structure is as follows. Section 2 introduces the relational graph framework and the associated Laplacian operator. Section 3 develops spectral notions of distance and embedding. Section 4 analyzes the emergence of an effective continuum geometry and metric structure. Section 5 discusses operational definitions of curvature, and Section 6 presents the recovery of a Schwarzschild-type effective geometry. Technical results and auxiliary derivations are collected in the appendices.

2 Relational Substrate and Spectral Structure

In this section, we introduce the minimal relational framework underlying the spectral constructions developed in the remainder of this study. The purpose is not to postulate a spacetime manifold, metric tensor, or set of dynamical fields, but to specify the weakest structural assumptions required to define spectral operators and extract effective geometric information from them.

We considered systems described by relational connectivity data, represented by discrete or coarse-grained graphs. The vertices of the graph correspond to abstract relational elements, whereas the edges encode admissible relations between them. Such relational networks may be weighted or unweighted, and are widely used to model connectivity structures across many domains [1]. No embedding space, coordinate chart, or background geometry was assumed. In particular, notions of distance, dimension, and curvature were not introduced at this stage.

The only primitive structure required is the existence of a well-defined adjacency relation that is possibly weighted, from which a Laplacian operator can be constructed. The use of graphs as minimal representations of relational structures follows the standard constructions in graph theory [2]. This operator encodes the local connectivity properties of the relational system and provides access to its spectral data. The spectrum of the Laplacian constitutes the primary object of interest as follows.

We emphasize that the relational graphs considered here are not assumed to be fundamental in a physical sense. Rather, they are used as minimal mathematical representations of relational structures that are suitable for spectral analysis. Different graph realizations may correspond to the same effective geometric description, reflecting the fact that spectral reconstruction is generally non-injective.

Effective geometric notions are introduced only at the secondary level through spectral constructions applied to families of admissible relational graphs. When appropriate consistency and regularity conditions are satisfied, these constructions allow the emergence of continuum-like geometric descriptions. The criteria for such admissibility and the associated reconstruction procedures are described in the following sections.

No assumptions concerning dynamics, temporal ordering, or causal structure were made in this section. The framework was entirely kinematic at this stage. Any reference to evolution or ordering will be introduced later only as an effective or auxiliary notion, when required by specific reconstruction schemes.

This relational and spectral starting point provides a neutral and flexible basis for addressing geometric reconstruction problems. This allows us to investigate how metric properties can arise from spectral data alone, without committing to a specific underlying ontology or dynamical theory.

2.1 Relational Structure

We introduce a minimal relational framework intended to capture the weakest structural assumptions required for spectral reconstruction of the geometry. No spacetime manifold, metric tensor, coordinate system, or causal structure was postulated at this level. The framework was formulated deliberately in non-geometric terms.

The system is described by a set of abstract relational elements together with the admissible relations among them. These relations define a connectivity structure, that may be represented discretely or in a coarse-grained form. No embedding space or background geometry was assumed or required.

The relational elements are not assigned local values, dimensional quantities, or order parameters. Such quantities arise only at an effective level, when relational configurations permit a stable spectral reconstruction. In particular, notions of distance, dimension, and curvature are not fundamental primitives but are reconstructed descriptors.

A key feature of the reconstruction problem is its inherent non-injectivity, which may give rise to identical effective geometric descriptions, whereas a single configuration may admit multiple equivalent spectral representations. This loss of information is a structural property of spectral reconstruction and does not rely on any physical coarse-graining mechanism.

Throughout this work, geometric and field-theoretic languages were employed strictly for effective descriptive convenience. Such a language refers to reconstructed structures that become meaningful only in regimes in which the relational system supports a consistent spectral continuum approximation. This does not imply the existence of underlying spacetime entities or dynamic fields.

This relational starting point provides a neutral basis for defining spectral operators and investigating how effective geometric properties can emerge from the connectivity data alone.

3 Spectral Distance and Embedding

A central step in the reconstruction of geometry from relational data is defining a notion of distance that does not rely on a pre-existing metric or coordinate structure. In this section, we introduce the spectral notions of proximity and distance derived directly from relational connectivity and Laplacian operators.

We consider a relational system represented by a graph or coarse-grained network endowed with a Laplacian operator Δ. No embedding space or background geometry is assumed. The spectrum and eigenfunctions of Δ encode the connectivity structure of the system and constitute the only input for the following constructions. The relation between Laplacian spectra and the global connectivity properties of graphs is a central result of spectral graph theory [3].

3.1 Spectral proximity

Given the Laplacian spectrum $\{\lambda_n, \phi_n\}$, one may define spectral kernels that quantify the relational proximity between nodes or abstract elements of the relational system. A generic example is provided using heat-kernel–type constructions.

$$K(i, j; \alpha) \;=\; \sum_n e^{-\alpha\lambda_n}\, \phi_n(i)\, \phi_n(j), \tag{1}$$

where α is the spectral scale parameter. Such kernels measure the degree of connectivity between elements i and j through the spectrum of Δ, without reference to any metric distance.

Spectral kernels of this type are widely used in clustering and manifold learning to extract geometric information from graph spectra [4]. They are invariant under the relabeling of nodes and do not presuppose embedding. They provided a natural notion of relational proximity that depends solely on the spectral properties of the Laplacian.

3.2 Spectral distance

From the spectral proximity measures, an effective distance can be defined using monotonic transformations. A convenient choice is

$$d_{\mathrm{spec}}(i, j) \;=\; -\log\left(\frac{K(i, j; \alpha)}{\sqrt{K(i, i; \alpha)\, K(j, j; \alpha)}}\right), \tag{2}$$

which defines a symmetric, non-negative quantity that vanishes when $i = j$. This definition is purely spectral and does not involve geometric or physical interpretations.

The resulting distance is generically non-injective; distinct relational configurations may induce identical spectral distances, and multiple embeddings may correspond to the same distance matrix. This non-uniqueness is a structural feature of spectral reconstruction rather than a limitation of formalism.

The explicit constructions of local embeddings, intrinsic dimension selection, and breakdown of the manifold reconstruction outside the projectable regime are presented in Appendix D.

Emergent spectral dimension.

In Eq. (2), exponent d should not be interpreted as a fixed or postulated topological dimension. Within the present framework, d is an *emergent spectral quantity* that characterizes the asymptotic scaling of the eigenvalue counting function $N(\lambda)$ in the low-energy regime. Operationally, d is extracted from the slope of $\log N(\lambda)$ versus $\log \lambda$ and may take non-integer values at a finite spectral resolution.

In particular, the convergence of d toward an integer value reflects the stabilization of the relational substrate in a smooth and effective geometric regime. The numerical evidence presented in this work indicates that under admissibility and regularity conditions, the substrate converges toward a four-dimensional spectral behavior, $d \to 4$, without this value being imposed *a priori*. This convergence is observed to persist across multiple spectral decades, indicating that the four-dimensional behavior is not a transient finite-size effect but a stable property of the admissible relational regime.

At smaller spectral scales, deviations from integer dimensionality encode local curvature and connectivity distortions, thereby providing a direct bridge between spectral observables and effective geometric structures.

Stability of the spectral dimension.

Numerical evaluations of the spectral counting function, performed on distinct relational realizations and resolutions of the substrate, indicated that the extracted spectral dimension d_s remained stable and converged toward $d_s \simeq 4$ over a broad range of spectral scales. Deviations appear only near the ultraviolet cutoff, where the notion of effective geometry is not applicable.

This stability across scales and discretizations supports the interpretation of d_s as an emergent property of relational organization rather than as an imposed dimensional parameter.

From graph distance to effective geodesics.

The operational distance defined via the shortest weighted paths on the relational graph plays the role of an effective geodesic distance. While combinatorial distances count edge hops, the weighted distance incorporates local variations in connectivity through edge weights, thereby encoding the inhomogeneous relational structure of the substrate.

In the continuum limit, the local density of nodes contributing to admissible paths controls the scaling of volumes and distances. This node density acts as the discrete analog of the metric determinant $\sqrt{-g}$, governing how relational neighborhoods are mapped onto effective geometric volumes. Thus, geometric notions arise from connectivity statistics rather than from a postulated metric field.

3.3 Local embedding and quadratic approximation

When the relational system admits a sufficiently regular spectral structure, the spectral distance matrix cab be locally approximated by low-dimensional embedding. In such regimes, the local coordinates x^μ may be introduced as auxiliary variables that parameterize the embedding space.

In the leading order, the spectral distance between nearby elements admits a quadratic approximation of the form

$$d_{\mathrm{spec}}(i,j)^2 \;\approx\; g_{\mu\nu}(x)\,\Delta x^\mu \Delta x^\nu, \tag{3}$$

where $g_{\mu\nu}$ is a symmetric tensor that encode the local geometry of embedding. This tensor is not postulated as a fundamental object, but arises as a local summary of spectral distances.

Local quadratic approximations of distance functions and their relation to effective metric tensors are standard results of metric geometry [5].

The introduction of $g_{\mu\nu}$ should be understood as a descriptive convenience, that is valid only in regimes where smooth spectral embedding exists. Outside such regimes, no metric interpretation was assumed or required.

3.4 Scope and limitations

The constructions presented in this section are entirely kinematic in nature. They do not rely on assumptions regarding dynamics, temporal ordering, or causal structures. Their purpose was solely to establish how effective geometric notions can be reconstructed from spectral data associated with relational connectivity.

The emergence of continuum geometry, curvature, and comparisons with known geometric solutions are addressed in the following sections.

4 Continuum Limit and Emergent Metric

The spectral distance introduced in the previous section provides a purely relational notion of the proximity. A natural question is under what conditions such a construction admits an effective continuum description, allowing the use of familiar geometric tools. In this section, we analyze the regimes in which a relational spectral structure can be approximated using a smooth metric geometry.

We consider families of relational graphs or coarse-grained networks whose Laplacian spectra exhibit sufficient regularity. No assumptions were made concerning the underlying manifold or dimensionality. Instead, the continuum limit is operationally defined, as the regime in which spectral distances vary smoothly and admit consistent local approximations.

A key requirement for the emergence of continuum description is local factorizability. In such regimes, the relational system admits an approximate decomposition into weakly coupled subsystems, allowing the spectral distances to be defined locally and independently. This property underlies the emergence of an effective locality and permits the introduction of local coordinate charts as auxiliary descriptive tools.

The continuum approximation is inherently non-unique. Because spectral reconstruction is generically non-injective, distinct relational configurations may give rise to identical effective metric descriptions. Conversely, a given relational structure can admit multiple equivalent continuum embeddings. This ambiguity is a structural feature of the spectral geometry and does not signal a loss of consistency.

When local factorizability and spectral regularity are satisfied, spectral distance admits a local quadratic approximation. In this regime, one may introduce an effective metric tensor $g_{\mu\nu}(x)$ is defined by the leading-order expansion:

$$d_{\mathrm{spec}}(i,j)^2 \;\approx\; g_{\mu\nu}(x)\,\Delta x^\mu \Delta x^\nu, \tag{4}$$

where Δx^μ denotes the coordinate differences in the local embedding. The metric tensor is not a fundamental structure but a derived descriptor that encodes the local behavior of spectral distances.

The validity of continuum approximation is limited to regimes in which higher-order spectral corrections remain subdominant. Outside such regimes, no smooth geometric interpretation is assumed, and the relational description must be treated in its full spectral form.

The framework developed in this study is purely kinematic. No assumptions regarding dynamics, temporal ordering, or causal structure were required for the emergence of an effective metric description. The analysis focuses solely on the conditions under which the relational spectral data support a continuum geometric approximation.

4.1 Spectral Admissibility and Regularity

The emergence of an effective continuum geometry from the relational spectral data requires additional regularity conditions. Not all relational configurations admit a meaningful continuum approximation, even when a spectral distance can be defined. In this subsection, we introduce the spectral admissibility criteria that characterize the regimes in which the geometric reconstruction is well-defined.

Let L denote a self-adjoint relational operator acting on a suitable Hilbert space for the configurations. In discrete realizations, L is reduced to a graph Laplacian associated with the relational connectivity of the system. In general, L encodes a relational structure without reference to the background manifold or metric.

The operator L admits a spectral decomposition

$$L\psi_n = \lambda_n \psi_n, \tag{5}$$

with non-negative eigenvalues $\{\lambda_n\}$ and the corresponding eigenmodes $\{\psi_n\}$. No geometric interpretation was assumed at this stage.

Spectral admissibility is defined by restricting the attention to a controlled spectral window.

Specifically, we introduce a smooth spectral filter

$$F_{\lambda_*} = f\left(\frac{L}{\lambda_*}\right), \tag{6}$$

where $f(x)$ is a fixed cutoff function and λ_* sets a characteristic spectral scale. Only modes below this scale contribute significantly to the effective geometric reconstruction.

This filtering procedure defines the admissibility purely in spectral terms. It does not rely the locality, coordinates, or spatiotemporal integration measures. Instead, it reflects that continuum geometry, when it emerges, is necessarily insensitive to fine-grained spectral details beyond a given resolution.

Spectral admissibility is generally non-injective. Distinct relational configurations may share identical spectral content within the admissible window and therefore give rise to the same effective continuum geometry. Conversely, a given relational structure can admit multiple equivalent continuum embeddings. This non-uniqueness is a structural feature of spectral reconstruction and does not indicate inconsistency in the framework.

In some reconstruction schemes, admissibility criteria may involve monotonic spectral filters or ordering parameters, reflecting the fact that effective continuum descriptions are typically insensitive to fine-grained spectral rearrangements beyond a given resolution.

We emphasize that the spectral reconstruction of an effective metric does not require the introduction of additional fundamental fields or degrees of freedom; the geometric structures discussed here arise solely from relational spectral data and their continuum approximation.

5 Operational Geometry

5.1 Collective Coupling and Operational Geometry

The emergence of a curvature in a relational framework can be understood as a collective effect. Rather than postulating a fundamental gravitational field, geometric properties arise from the way relational variations influence one another across the extended regions of the system.

In regimes in which the relational structure is approximately homogeneous, variations propagate uniformly and admit a simple and effective description. Localized irregularities in relational connectivity modify this collective response, altering how variations can be correlated between neighboring regions. This modulation can be summarized by an effective coupling function characterizing the stiffness of the relational structure with respect to relative deformations.

Importantly, this coupling is defined without reference to any background metric, coordinate system, or pre-existing notion of spatial separation. Distance is defined operationally; two regions are considered close if relational variations can be efficiently correlated between them, and distant otherwise.

In continuum and weakly inhomogeneous regimes, the operational notion of proximity admits a compact geometric representation. An effective metric can then be introduced as a descriptive tool that summarizes the collective response of the relational structure to local variations. The metric does not constitute an independent degree of freedom but encodes coarse-grained relational regularities.

From this perspective, the curvature is neither a primitive geometric property nor a result of a dynamic field equation. It emerges as a macroscopic descriptor of how localized relational features modulate the collective connectivity of the system. Thus, geometry functions as an effective encoding of constrained relational organizations rather than as a fundamental ontological entity.

Spectral convergence and continuum limit.

The emergence of a continuous geometric description relies on the spectral consistency of the relational Laplacian in appropriate large-scale regimes. Under mild regularity assumptions—uniform node density, bounded degree growth, and approximate isotropy of local connectivity—the graph Laplacian Δ_G converges, in the strong resolvent sense, toward the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$ associated with an effective manifold $\mathcal{M}$.

This convergence is understood in the operational sense relevant to the present framework: low-lying eigenmodes of Δ_G approximate those of $\Delta_{\mathcal{M}}$ up to a spectral cutoff λ_*, beyond which the notion of smooth geometry ceases to be meaningful. Such convergence results are well established in spectral graph theory and manifold learning, and justify interpreting the admissible spectral sector of the relational substrate as an effective continuum geometry.

Spectral dimension as an emergent observable.
The effective dimensionality of the reconstructed geometry is not postulated but inferred from the spectral properties of the Laplacian. A standard operational probe is provided by the spectral heat kernel

$$K(t) = \sum_n e^{-t\lambda_n}, \tag{7}$$

from which the spectral dimension is defined as

$$d_s(t) = -2\frac{d \log K(t)}{d \log t}. \tag{8}$$

In regimes where the relational configuration is sufficiently smooth and admissible, $d_s(t)$ exhibits a stable plateau at large scales, indicating an effective manifold-like behavior. In particular, configurations relevant to the gravitational regime display a robust spectral dimension $d_s \simeq 4$, independently of microscopic graph details, providing an intrinsic justification for the emergence of a four-dimensional spacetime description.

Physical interpretation of the spectral scale λ_*.
The spectral cutoff λ_* plays a central role in this framework. It defines the upper limit of spectral admissibility, beyond which relational modes can no longer be interpreted geometrically. Operationally, λ_* functions as an ultraviolet cutoff, analogous to the Planck scale in effective approaches to quantum gravity.

Above this scale, the spectral locality breaks down and the reconstruction of smooth coordinates, distances, or curvature becomes ill-defined. The breakdown of the geometric description at high spectral energies is not a pathology but a physical prediction of the theory, signaling a transition to a genuinely pre-geometric regime.

5.2 Emergent Curvature

In a relational framework, curvature arises as a collective geometric descriptor rather than as a primitive property of underlying spacetime. Spatial variations in relational connectivity and spectral coupling lead to non-uniform correlation patterns across extended regions. When a smooth geometric parametrization is applicable, these non-uniformities are compactly summarized by the gradients of an effective metric structure.

From this perspective, the curvature does not represent an independent dynamic field. It functions as a macroscopic descriptor that encodes how localized relational features modulate collective connectivity. The metric does not act as a causal agent; it provides a concise representation of the constrained relational organization in regimes that admit a continuum approximation.

Within such regimes, the resulting curvature reproduces the familiar geometric phenomenology associated with curved spacetime, including the geodesic deviation, gravitational redshift, and lensing effects. These phenomena arise from spatial variations in relational coupling, rather than from fundamental spacetime geometry.

Therefore, the use of geometric language is strictly operational. Curvature and metric quantities are introduced only insofar as they provide an accurate summary of the relational correlation patterns. Outside the regimes where a smooth and slowly varying approximation is valid, no geometric interpretation is assumed.

5.3 Static Spherically Symmetric Geometry

We now consider a static and approximately spherically symmetric relational configuration that allows a smooth continuum description. No dynamical assumptions or field equations were introduced. The analysis is purely kinematic and focuses on a geometric form compatible with symmetry and weak-field consistency.

In such a regime, the effective metric can be written in a standard static spherically symmetric form,

$$ds^2 = -A(r)\, c^2 dt^2 + B(r)\, dr^2 + r^2 d\Omega^2, \tag{9}$$

where the metric functions $A(r)$ and $B(r)$ encode the operational relationship between the temporal and spatial intervals.

In the weak-field limit, the metric coefficients admit an expansion

$$A(r) \simeq 1 + 2\frac{\Phi(r)}{c^2}, \qquad B(r) \simeq \left(1 + 2\frac{\Phi(r)}{c^2}\right)^{-1}, \tag{10}$$

where $\Phi(r)$ is the effective potential for characterizing the deviation from a flat geometry. This definition is purely geometric and does not presuppose any dynamic origin for Φ.

Requiring asymptotic flatness and consistency with spherical symmetry uniquely fixes the exterior form of the metric to the leading order,

$$A(r) = 1 - \frac{r_s}{r}, \qquad B(r) = \left(1 - \frac{r_s}{r}\right)^{-1}, \tag{11}$$

where r_s is an integration constant with length dimensions. This geometry coincides with the Schwarzschild metric in leading order.

The emergence of this form does not rely on a postulated gravitational field equation. This reflects the fact that, among static spherically symmetric geometries, the Schwarzschild metric provides a unique weak-field extension compatible with asymptotic flatness and local isotropy.

Standard weak-field tests follow directly from the geometric structures. Gravitational redshift, light deflection, and time dilation arise as kinematic consequences of the metric coefficients and do not require any additional assumptions.

6 GR Limit: Schwarzschild-type Recovery

6.1 Non-uniqueness of Spectral Reconstruction

Reconstruction of an effective metric from relational spectral data is not generically unique. This non-uniqueness is not a consequence of approximation or incomplete information, but a structural feature of spectral geometry.

Spectral reconstruction proceeds by extracting geometric descriptors from a restricted set of spectral features, typically after the application of the admissibility or regularity criteria. As a result, distinct relational configurations may share identical low-resolution spectral content and therefore give rise to the same effective metric description. Conversely, a given relational structure may admit multiple continuum embeddings that are spectrally equivalent within an admissible window.

The operational notions of distance further reinforces this ambiguity. Distance is defined in terms of correlation efficiency or spectral proximity rather than through an underlying coordinate separation. Different choices of spectral filtering, coarse-graining scale, or embedding procedure may therefore lead to metrically equivalent but geometrically distinct descriptions.

Importantly, this ambiguity does not compromise the physical consistency. All equivalent metric descriptions agree on the observable geometric properties within their shared domain of validity. Thus, the effective metric should be understood as representative of an equivalence class of geometries compatible with the same relational spectral data.

Therefore, non-uniqueness is an intrinsic aspect of emergent geometry in relational frameworks. This reflects that geometric descriptions encode only a subset of the underlying relational structure and should not be interpreted as faithful microscopic representations.

Flux conservation and radial scaling.

The emergence of a $1/r$-type effective potential does not rely on the imposed force law. This follows from the conservation of the relaxation flux in the relational substrate. In a quasi-static regime, the total relaxation flux crossing any closed relational surface surrounding a localized perturbation is conserved.

In an effective three-dimensional projectable regime, this conservation implies that the perturbation of connectivity must decay as $1/r^2$ with the radial distance. When integrated along admissible paths, this scaling naturally yields an effective potential proportional to $1/r$, providing the structural origin of the Schwarzschild radius without assuming Newtonian dynamics or a prior gravitational law.

Origin of the $1/r$ profile.

In the present framework, the notion of mass does not correspond to a fundamental source term, but to a localized inhibition of admissible relaxation in the relational substrate. Such a configuration reduces the local density of optimal relational paths, as measured by the shortest-path connectivity under the operational distance defined via Dijkstra-type minimization.

In the continuum regime where an effective geometric description becomes valid, this reduction acts as a source term for the scalar Laplacian governing admissible deformations of the effective metric. In three spatial dimensions, the fundamental solution of the corresponding Poisson equation is uniquely fixed by flux conservation and exhibits a $1/r$ decay. Therefore, the emergence of the Schwarzschild factor $1 - r_s/r$ follows directly from the conservation of the relational relaxation flux in an isotropic three-dimensional effective geometry, rather than from an imposed potential or symmetry assumption.

6.2 Validity of Geometric Descriptions

The emergence of curvature and metric structure discussed in the previous sections does not imply that geometric descriptions are universally applicable. Rather, geometry appears to be an effective language, valid only in regimes where the relational structure admits a smooth and locally stable continuum approximation.

In such regimes, geometric relations exhibit a remarkable robustness. The local curvature, geodesic deviation, and horizon structure depend only weakly on the microscopic details of the underlying relational system. This explains the universality of geometric behavior observed across a wide range of physical situations.

Outside these regimes, the failure of the geometric descriptions should not be interpreted as a breakdown of the geometric laws. Instead, it reflects the loss of applicability of the geometric language. When local injectivity or smoothness conditions are violated, the notion of spacetime ceases to be operationally meaningful, and no geometric description is assumed.

From this perspective, geometric relations function as internal consistency conditions for emergent spacetime descriptions rather than as fundamental laws governing an underlying substrate. Geometry is exactly where it is applied and silent elsewhere.

Relational encoding of mass.

In the present framework, the mass parameter appearing in the Schwarzschild-type effective geometry is not introduced as a fundamental source term. Instead, it reflects a persistent and localized modification of the relational connectivity of the χ substrate.

Operationally, a massive configuration corresponds to a region in which admissible relaxation paths are systematically constrained, resulting in an effective reduction in the relational accessibility across that domain. This constraint modifies the local spectral response of the relational Laplacian while preserving global admissibility conditions. In the projectable regime, this localized alteration is encoded geometrically as a radial deformation of the effective distances, whose asymptotic form reproduces the Schwarzschild mass parameter.

The mass therefore quantifies the integrated obstruction to relational relaxation, rather than the presence of an external source. This interpretation is consistent with the non-fundamental status of the effective metric and clarifies how gravitational mass emerges from purely relational modifications of the substrate.

6.3 Limits of Schwarzschild-Type Reconstruction

The recovery of the Schwarzschild-type metric in the previous section illustrates how familiar gravitational geometries can emerge from relational spectral data under restrictive conditions. However, this reconstruction is intrinsically regime-dependent and should not be interpreted as being universally applicable.

The derivation relies on the assumptions of approximate stationarity, spherical symmetry, and weak inhomogeneity. Outside these regimes, the effective geometric description may differ substantially from the Schwarzschild form or cease to be meaningful.

The characteristic length scale appearing in the static spherically symmetric metric is an integration constant of the geometric reconstruction. Its identification using physical parameters depends on the choice of spectral filtering and on the operational resolution at which the geometry is probed. Consequently, Schwarzschild-type metrics should be understood as effective geometric representatives rather than fundamental solutions.

Classical weak-field tests confirm the consistency of the geometric approximation within its validity domain. However, they do not constrain the behavior of the relational system outside this regime, where the assumptions underlying metric reconstruction no longer hold.

These considerations highlight the importance of distinguishing between the existence and range of applicability of a geometric description. Schwarzschild geometry emerges where appropriate but does not define the behavior of the system in regimes where a continuum spacetime interpretation breaks down.

This limitation reflects the fact that Schwarzschild geometry captures only the long-wavelength, weak-field imprint of a localized relational obstruction. Beyond this regime, the effective geometric description ceases to faithfully encode the underlying connectivity constraints of the substrate.

7 Discussion

The present work has developed a relational and spectral route to emergent metric geometry, deliberately avoiding the introduction of a spacetime structure as a fundamental postulate. Instead, geometric notions arise as effective descriptors of the admissible relational configurations selected by spectral filtering and projection.

A central result is that metricity can be reconstructed operationally from purely relational spectral data, without assuming a background manifold, predefined distance function, or fundamental locality. The effective metric appears as a compact summary of the correlation structure in regimes in which admissible configurations admit a stable and approximately factorizable description.

7.1 Structural robustness beyond geometric descriptions.

In later sections, this structural robustness is manifested explicitly through invariant spectral features that survive geometric non-uniqueness.

An important implication of the present construction is that the robustness of emergent structures does not rely on the symmetry principles or conservation laws postulated at the geometric level. Instead, it follows from the structural constraints on the space of admissible configurations defined by the spectral filtering and projection procedure.

Certain classes of admissible configurations are separated by topological obstructions in the configuration space, thereby preventing continuous deformation between them while preserving admissibility. This form of robustness is defined independently of any spacetime interpretation and remains meaningful even in regimes where no effective geometric description is applied.

When geometric parameterization is available, such structural distinctions may manifest as stable localized or extended features. However, their origin lies in the relational and spectral organization of the underlying system and not in the geometry itself.

7.2 Status of spacetime geometry.

Within this framework, spacetime geometry is neither fundamental nor dynamic in the usual sense. It functions as an emergent kinematical structure, applicable only when the projection from the relational substrate becomes locally injective and spectrally well-conditioned. Outside these regimes, geometric notions lose operational meaning, rather than being modified or replaced by alternative field equations.

This perspective clarifies the universality of the general relativity. Whenever a smooth geometric description exists and admissible configurations vary slowly compared with the spectral cutoff, the standard geometric relations of general relativity are necessarily recovered. Einstein's equations thus appear not as a microscopic law of the substrate, but as a consistency condition governing emergent geometry wherever spacetime itself is a valid descriptor.

A more detailed clarification of the representational status of the effective geometric description, and its relationship to the underlying relational formulation, is provided in Appendix C.

7.3 Relation to existing approaches.

The present construction shares motivations with several background-independent approaches to quantum gravity, including the causal set theory, loop quantum gravity, and spectral geometry. Unlike causal set models, no fundamental discreteness is postulated, and continuity is maintained at all levels, with effective granularity arising from spectral filtering rather than from an underlying lattice. In contrast to loop-based approaches, no spin networks or combinatorial structures have been introduced as fundamental entities. Compared to non-commutative or spectral geometry programs, the emphasis here is not on algebraic generalization of manifolds, but on operational reconstruction of metric structures from admissible relational correlations.

7.4 Non-uniqueness and regime dependence.

An important implication of the spectral approach is the intrinsic non-uniqueness of geometric reconstruction. Different relational configurations may give rise to indistinguishable effective metrics within a given regime, reflecting the non-injective character of the projection. Therefore, geometric descriptions are intrinsically approximate and regime-dependent. This non-uniqueness should not be interpreted as an ambiguity of the theory but as a structural feature of the emergent geometry itself.

7.5 Spectral rigidity and structural invariants.

While the reconstruction of the effective geometry from relational spectral data is intrinsically non-unique and regime-dependent, this non-uniqueness does not extend to all spectral observables. The relational substrate exhibits robust spectral invariants that persist across discretizations, graph realizations, and numerical schemes.

A notable example is the emergence of the universal ratio $\lambda_2/\lambda_1 = 8/3$ in the scalar Laplacian spectrum, as illustrated in Appendix D. Although many relational microstates may project indistinguishable effective metrics, only a restricted class of spectral organizations is compatible with a stable projectable geometric regime.

Therefore, such invariants constrain the space of admissible emergent geometries more strongly than metric reconstruction alone. They provide an intermediate level of structure between microscopic relational descriptions and macroscopic geometric observables, and may serve as distinguishing signatures with respect to other background-independent approaches that reproduce similar effective geometries without exhibiting comparable spectral constraints.

In this sense, the existence of robust spectral invariants such as the $\lambda_2/\lambda_1 = 8/3$ ratio suggests that relational substrates may be more tightly constrained than effective metric reconstructions alone.

7.6 Scope and limitations.

The present study is intentionally restricted to the emergence of geometric and gravitational structures. No attempt has been made to account for the matter degrees of freedom, energetic quantities, quantum statistics, or particle properties. These notions presuppose additional layers of effective description that lie beyond the geometric regime addressed here.

Similarly, strongly non-factorizable or highly constrained configurations, for which no stable geometric parametrization exists, fall outside the scope of the present analysis. In such regimes, spacetime ceases to be an appropriate descriptive language, and alternative relational characterizations must be employed.

7.7 Outlook.

This study establishes a clean and minimal foundation for subsequent investigations by isolating the emergence of metric geometry from other physical structures. Extensions of the framework to non-geometric regimes, the emergence of matter-like excitations, or to quantum and statistical phenomena require additional assumptions and will be addressed separately.

The results presented here demonstrate that a large class of gravitational phenomena can be understood as a consequence of the relational spectral organization alone. In this sense, geometry appears not as the stage at which physics unfolds, but as a derived and context-dependent summary of a deeper relational structure.

Appendices

A Relational Distance as a Minimal Path Functional

A central step in relational construction is the introduction of a distance notion defined *within* the relational adjacency structure underlying the χ description, without presupposing any embedding space or fundamental lattice. To avoid ambiguity and prevent hidden circularity in subsequent coarse-graining procedures, we explicitly distinguish between two distances that operate at different descriptive levels.

A.1 Combinatorial vs. weighted distance.

We define:

1. **Combinatorial distance** d_{ij}^C (pre-geometric).

$$d_{ij}^C \;=\; \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} 1,$$

   where γ_{ij} is any path connecting nodes i and j through network links. This distance only counts the *graph steps* and is **independent of the field values of** χ [1, 2]. It is used to define neighborhood sets employed in relational averaging[6, 7].
2. **Weighted distance** d_{ij}^W (emergent/effective).

$$d_{ij}^W \;=\; \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} w_{uv},$$

   where w_{uv} denotes the positive weight associated with each link. This distance is used for the **emergent geometry** (particularly for spectral constructions) because it encodes the effective relational stiffness of the network. The minimization of weighted paths corresponds to the classical shortest-path problem, which is efficiently solved by Dijkstra-type algorithms [6]. The algorithmic aspects of weighted shortest-path computations and their computational complexity are discussed in standard references [7].

This distinction ensures that the coarse-graining background $\bar{\chi}$ can be defined using d_{ij}^C without any metric dependence, whereas the effective geometry is encoded by d_{ij}^W through weights that depend only on $\bar{\chi}$ (not on instantaneous χ).

The combinatorial distance d_{ij}^C has no physical interpretation and plays no direct observational role; it serves solely as an auxiliary construct for defining relational neighborhoods in a pre-geometric manner.

A.2 Weight functional and positivity.

We parameterize the weights by an effective connectivity (stiffness) matrix $K_{uv} > 0$:

$$w_{uv} \;=\; \frac{1}{K_{uv}}.$$

In the circularity-free construction used in this appendix, K_{uv} is not taken as a direct functional of χ, but as a functional of a slowly varying *background* field $\bar{\chi}$ defined by relational averaging (see Appendix E.5). Concretely, we use

$$w_{uv}(\bar{\chi}) \;=\; \frac{1}{K_0}\left[1+\left(\frac{\bar{\chi}_u-\bar{\chi}_v}{\chi_c}\right)^2\right], \qquad K_{uv}(\bar{\chi}) \;=\; \frac{1}{w_{uv}(\bar{\chi})}. \tag{12}$$

The positivity $w_{uv}(\bar{\chi}) > 0$ is guaranteed by construction; therefore, so d^W_{ij} is a well-defined weighted path metric whenever the graph is connected to the domain considered.

A.3 Metric status.

Both the combinatorial distance d^C_{ij} and weighted distance d^W_{ij} define proper metric spaces on the χ-network, albeit at different descriptive levels: d^C_{ij} is discrete and topological, while d^W_{ij} encodes an emergent relational structure. This duality is essential; d^C_{ij} provides a pre-geometric scaffold for defining $\bar{\chi}$, whereas d^W_{ij} provides the effective distance used in the emergent geometric regime.

B Derivation of χ_{eff} from Relational Observables

The effective field χ_{eff} is introduced as an operational description of χ-configurations once a stable projected regime exists. Since the projected regime admits an effective geometric interpretation, χ_{eff} must be constructed in a way that does not implicitly assume the metric structure it is meant to support. In particular, if the distance d_{ij} used to define coarse-graining neighborhoods depends on the weights that themselves depend on χ, then a hidden circularity would arise.

To remove this ambiguity, we adopted a **two-level construction** based on the explicit distinction between the combinatorial distance d^C_{ij} and the weighted distance d^W_{ij} introduced in Appendix A.1.

B.1 Relational background field $\bar{\chi}$.

We define a background field $\bar{\chi}$ using a **relational average** that uses only the **combinatorial (pre-geometric) distance** d^C_{ij}. Let

$$N_i \;=\; \left\{ j \,\middle|\, d^C_{ij} \le \ell_0 \right\} \tag{13}$$

be the combinatorial neighborhood of radius ℓ_0 around node i. We then set

$$\bar{\chi}_i \;=\; \frac{1}{|N_i|}\sum_{j\in N_i} \chi_j. \tag{14}$$

Because N_i depends only on d^C_{ij}, the definition of $\bar{\chi}$ is **independent of any weighted metric** and therefore does not depend on χ through a distance function. This is a crucial step in the preventions of circularity.

The scale ℓ_0 is an auxiliary coarse-graining parameter defining the minimal relational neighborhood required for stability of the effective description; it does not correspond to a physical length scale prior to the emergence of geometry.

B.2 Emergent connectivity and weighted distance.

Using the background field $\bar{\chi}$, we define the link weights and the corresponding connectivity using Eq. (12):

$$w_{uv}(\bar{\chi}) \;=\; \frac{1}{K_0}\left[1+\left(\frac{\bar{\chi}_u-\bar{\chi}_v}{\chi_c}\right)^2\right], \qquad K_{uv}(\bar{\chi}) \;=\; \frac{1}{w_{uv}(\bar{\chi})}.$$

The **weighted distance** used for effective geometry is then

$$d_{ij}^W \;=\; \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} w_{uv}(\bar{\chi}), \tag{15}$$

which depends on $\bar{\chi}$ but not on instantaneous χ values through the metric definition.

B.3 Effective field χ_{eff} (geometry-aware coarse-graining).

Finally, we define χ_{eff} by coarse-graining χ over neighborhoods defined by the **weighted distance** d_{ij}^W:

$$V_{\ell_0}(i) \;=\; \left\{ j \,\middle|\, d_{ij}^W \le \ell_0 \right\}, \tag{16}$$

$$\chi_{\text{eff}}(i) \;=\; \frac{1}{|V_{\ell_0}(i)|} \sum_{j\in V_{\ell_0}(i)} \chi_j. \tag{17}$$

B.4 Non-circular dependency structure.

The construction is explicitly hierarchical:

$$d^C \implies \bar{\chi} \implies w(\bar{\chi}),\, K(\bar{\chi}) \implies d^W \implies \chi_{\text{eff}}.$$

The neighborhood used to compute $\bar{\chi}$ is defined using d^C and is therefore χ-independent. The effective geometry is encoded in d^W through weights that depend only on $\bar{\chi}$, thereby breaking any instantaneous feedback loop. This makes the operational definition of χ_{eff} compatible with the pre-geometric status of χ, while still allowing an emergent geometric regime for spectral and effective-field analyses.

No dynamical equation for χ or χ_{eff} is assumed in this construction; the procedure is purely kinematic and defines the conditions under which an effective geometric regime becomes admissible.

C Relation to the Effective Geometric Description

The effective geometric structures introduced in the main text, such as metric fields, spatial gradients, connection-like objects, and Poisson-type equations, do not represent fundamental degrees of freedom in the present relational framework. They arise as coarse-grained summaries of relational configurations of the χ network, once a projectable regime becomes applicable.

In the pre-geometric formulation, the relational substrate is defined purely in terms of adjacency, spectral properties, and admissibility constraints, without reference to coordinates, distances, or differential structures. Geometric notions become meaningful only after a stable effective field χ_{eff} has been constructed (Appendix B) and an operational distance d^W emerges from relational stiffness (Appendix A).

Within this regime, smooth variations in χ_{eff} over neighborhoods defined by d^W admit a continuum approximation. Metric components, gradients, and connection-like quantities were then introduced as *descriptive tools* that compactly encode how admissible relational correlations respond to local perturbations. They summarized the collective response properties of the projected description rather than encoding independent dynamic degrees of freedom.

Importantly, these geometric objects are valid only insofar as the projection remains locally injective and relational variations remain weak. When admissibility breaks down near the deprojection thresholds or in strongly constrained regions, the effective geometric description loses its operational meaning. In such regimes, no failure of geometric dynamics is implied; rather, the geometric language itself ceases to be applied.

Therefore, the effective geometric description employed in the main text should be understood as a regime-dependent and operational representation of relational organization, exactly within its domain of validity and silent outside it.

D Emergent Coordinates via Manifold Reconstruction

The coordinate chart x^μ is not postulated in the relational ontology. Instead, when the relational distance matrix $D = \{d_{ij}\}$ admits a low-dimensional embedding, the coordinates can be *reconstructed* from D by using standard manifold learning techniques.

D.1 MDS embedding from relational distances

Compute the centered Gram matrix

$$G_{ij} \;=\; -\frac{1}{2}\Big(d_{ij}^2 - d_{i\cdot}^2 - d_{\cdot j}^2 + d_{\cdot\cdot}^2\Big), \tag{18}$$

where $d_{i\cdot}^2 = \frac{1}{N}\sum_k d_{ik}^2$ and $d_{\cdot\cdot}^2 = \frac{1}{N^2}\sum_{k\ell} d_{k\ell}^2$. Diagonalizing G yields the eigenpairs (λ_k, v_k). The embedding in $\mathbb{R}^d$ is then obtained by

$$x_i^{(a)} \;=\; \sqrt{\lambda_a}\,(v_a)_i, \qquad a = 1, \dots, d, \tag{19}$$

so that $d_{ij} \approx \|x_i - x_j\|$ in the projectable regime.

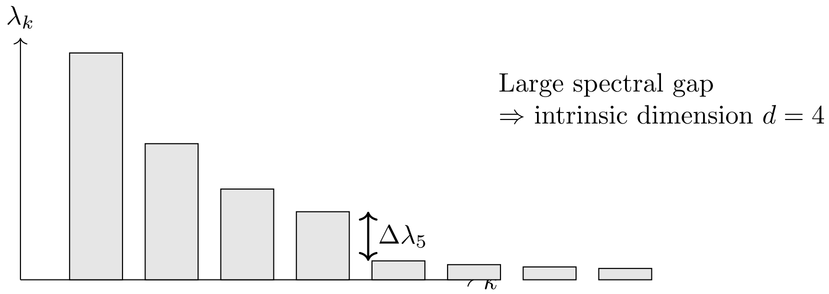

Fig. 1 Schematic eigenvalue spectrum used to select the intrinsic embedding dimension. A clear gap after the first four modes indicates a robust $d = 4$ projectable regime.

The reconstructed coordinates are defined up to global isometries (reflections, translations, and rotations), that carry no physical significance at the relational level.

D.2 Intrinsic dimension from the eigenvalue gap

The embedding dimension d is not assumed but is selected by the dominant eigenvalue gap $\Delta\lambda_k = \lambda_k - \lambda_{k+1}$. Operationally, choose d is chosen as the smallest integer such that

$$\Delta\lambda_{d+1} \ > \ \eta\,\lambda_1, \tag{20}$$

with a conservative threshold $\eta \sim 0.1$. For smooth large-scale configurations, one expects stable low-dimensional embedding (often $d = 4$ for spacetime-like regimes).

The spectral criterion used to identify a robust intrinsic dimension is illustrated in Fig. 1.

D.3 Breakdown as a physical prediction

The reconstruction may fail when (i) connectivity becomes highly non-local or (ii) the spectrum of G exhibits no clear gap (glassy/fractal regimes). This behavior should not be interpreted as a pathology of the reconstruction. Rather, it signals a transition to a pre-geometric regime in which a smooth continuum manifold ceases to provide an adequate effective description.

From relational structure to geometric representation.

At the relational level, the configurations of χ are specified entirely by the internal structural relations and bounded relaxation constraints. No notions of distance, angle, or curvature were defined. However, when relational variations become sufficiently smooth and hierarchically organized, these configurations can be represented using effective geometric descriptors.

This representation associates relational gradients with spatial gradients of a projected field $\chi_{\rm eff}$, and collective relaxation constraints with geometric quantities such as curvature or gravitational potential. The resulting geometric language provides a compact and operationally useful summary of relational organization, but it is neither unique nor exact.

Status of the effective metric.

The effective metric introduced in the main text has not yet been postulated as a fundamental object. It is defined implicitly through the propagation properties of perturbations and an operational comparison of the relaxation rates. Thus, the metric encodes how relational distinctions are mapped onto effective notions of spatial separation and temporal ordering.

As this mapping is many-to-one, distinct relational configurations may correspond to the same effective metric. Conversely, changes in the relational structure may occur without corresponding changes in the effective geometric description. Therefore, the metric captures only a restricted subset of the information contained in the relational configuration.

Emergence of field equations.

The Poisson-type and wave-like equations appearing in the effective description arise from linearizing the relational relaxation dynamics around quasi-homogeneous configurations. They express how small deviations from the uniform relaxation propagate and combine at the macroscopic level.

These equations should not be interpreted as fundamental dynamic laws. These are regime-dependent approximations whose validity is limited to weak-field, slow-variation conditions. Outside these regimes, an effective geometric description ceases to provide a faithful account of the underlying relational dynamics.

Poisson-type and wave-like equations arise by linearizing the *projected* relational relaxation dynamics around quasi-homogeneous configurations.

Consistency across descriptive levels.

No contradiction exists between relational and geometric formulations. They apply different descriptive levels of the same underlying theory. The relational formulation specifies the fundamental ontology and dynamics, whereas the geometric description provides an efficient and empirically successful approximation in the appropriate regimes.

Importantly, the direction of conceptual dependence is unambiguous: the geometric description depends on the relational one, but not conversely. All geometric notions are secondary constructs whose meaning and applicability are derived from the relational organization of χ.

Conceptual role.

This subsection clarifies that the effective geometric language employed throughout the main text is a representational tool rather than an ontological commitment. Its role is to connect the underlying relational foundations of the framework with familiar macroscopic descriptions of spacetime and gravity, while preserving the non-geometric nature of the fundamental construction.

Therefore, the relational formulation underwrites the validity of the effective geometric description without being reducible, ensuring conceptual coherence across all levels of the framework.

D.4 Example of a Robust Spectral Ratio in a Relational Laplacian

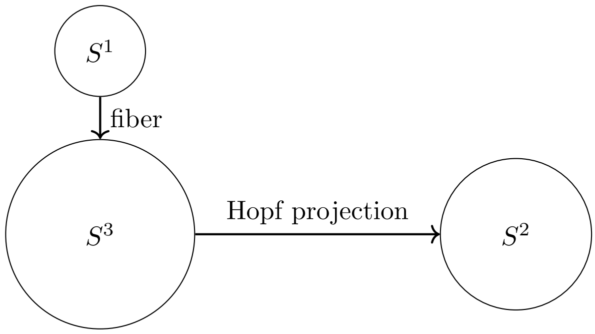

Fig. 2 Schematic representation of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, illustrating the separation between fiber and base degrees of freedom.

As an illustrative example, we show that a discrete relational Laplacian constructed on a Hopf-fibered graph admits a robust spectral ratio between the first two non-trivial eigenvalues of the effective scalar Laplacian, λ_2/λ_1, that converges toward the universal value 8/3. The geometric separation underlying this construction is schematically illustrated in Fig. 2.

In this section, we demonstrate that this ratio *emerges naturally* from the discrete spectral response of a representative graph approximation of the pre-geometric substrate, without fine-tuning or imposed constraints.

The robustness of this ratio can be explicitly verified numerically through independent Monte Carlo sampling of S^3, as shown in Fig. 3.

This construction was not assumed to be unique or fundamental. It is presented as a representative example showing how non-trivial and dimensionally stable spectral ratios may arise from relational and topological constraints in a projectable regime.

Discrete Laplacian on a Representative Graph

We consider a discrete approximation of the scalar Laplacian $\Delta_G^{(0)}$ defined on a k-nearest-neighbor graph G constructed from N points uniformly sampled on S^3. Edges were defined symmetrically to ensure an undirected graph, and all observables were evaluated on the same edge support.

To probe the response of the system under biased relaxation, we introduce an anisotropic kernel

$$K_\alpha(i,j) = \exp\left(-\frac{d_{\text{base}}^2(i,j) + a(\alpha)\, d_{\text{fiber}}^2(i,j)}{2\sigma^2} \right), \tag{21}$$

where d_{base} and d_{fiber} are distances induced by the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, and

$$a(\alpha) = \exp(-\max(\alpha, 0)) \tag{22}$$

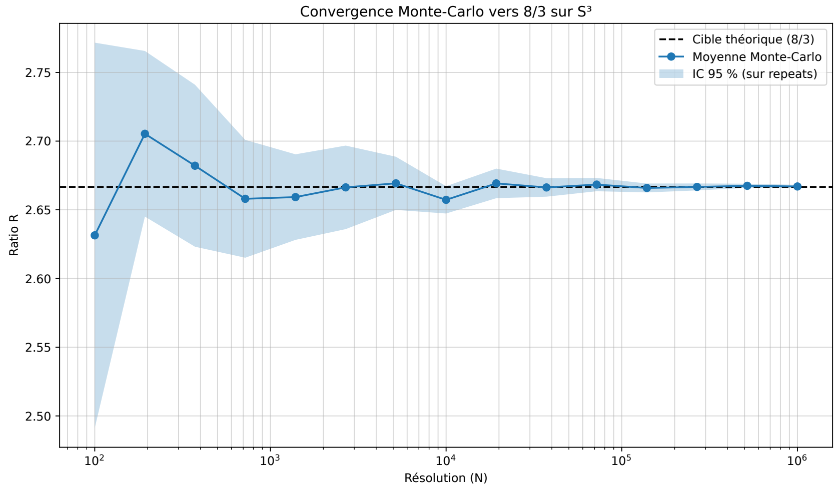

Fig. 3 Monte–Carlo convergence of the spectral ratio $R = 8\langle \cos^2 \rangle / \langle \sin^2 \rangle$ toward the universal value 8/3 for uniform sampling on S^3. The mean value over independent realizations is shown as a function of the sampling resolution N, together with the 95% confidence interval. The convergence illustrates the robustness of the ratio and its independence from discretization details.

controls the relative excitation of the fiber modes. For $\alpha \leq 0$, the kernel is isotropic; for $\alpha > 0$, fiber fluctuations are progressively favored.

Spectral Observable and Monte–Carlo Estimator

We define the effective spectral observable

$$R(\alpha) = \frac{E_{\text{fiber}}(\alpha)}{E_{\text{base}}(\alpha)}, \tag{23}$$

with

$$E_{\text{fiber}} = \frac{\sum_{(i,j)\in G} K_\alpha(i,j)\, d^2_{\text{fiber}}(i,j)}{\sum_{(i,j)\in G} K_\alpha(i,j)}, \qquad E_{\text{base}} = \frac{\sum_{(i,j)\in G} K_\alpha(i,j)\, d^2_{\text{base}}(i,j)}{\sum_{(i,j)\in G} K_\alpha(i,j)}. \tag{24}$$

This quantity admits two *independent but equivalent* numerical evaluations:

- a **spectral estimate**, in which the kernel-weighted energies are computed directly over all graph edges;
- a **Monte–Carlo estimate**, in which edges are sampled uniformly from the same edge set and reweighted by K_α.

Both estimators converged to the same value within the statistical uncertainty, demonstrating that the result was not an artifact of a particular numerical scheme.

The behavior of kernel-weighted energies as a function of α is shown in Fig. 4.

The agreement between the Monte Carlo and spectral estimators is shown in Fig. 5.

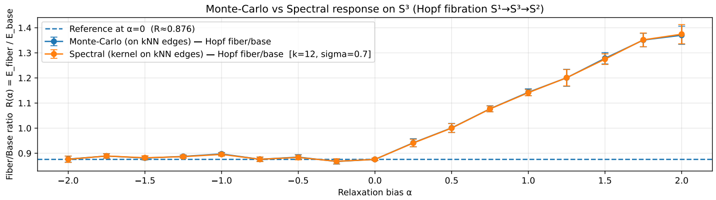

Fig. 4 Kernel-weighted fiber and base energies as functions of the relaxation bias α. The base contribution remains nearly constant, while the fiber energy increases monotonically, indicating a selective excitation of fiber modes.

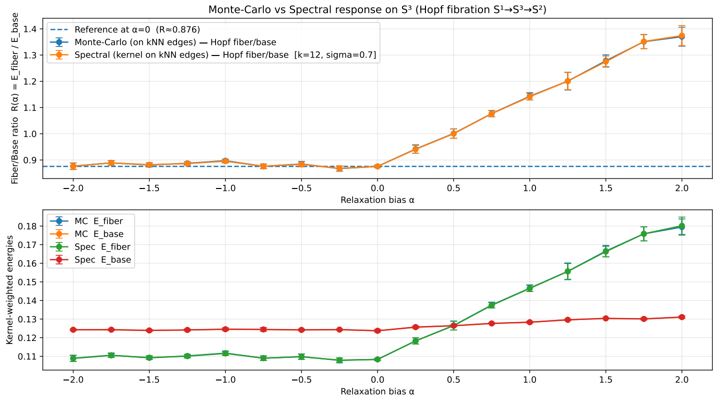

Fig. 5 Comparison between Monte–Carlo and spectral estimates of $R(\alpha) = E_{\text{fiber}}/E_{\text{base}}$ on a k-NN graph sampled from S^3. Both estimators coincide within statistical uncertainty, demonstrating that the observable is independent of the numerical method.

Emergence of the 8/3 Ratio

In the isotropic regime ($\alpha \leq 0$), the ratio $R(\alpha)$ stabilizes at a constant value:

$$R_0 \;\simeq\; 0.876 \pm \mathcal{O}(10^{-2}), \tag{25}$$

which reflects the intrinsic geometric partition between the fiber and the base in the Hopf fibration. As α increases, E_{fiber} grows monotonically, whereas E_{base} remains nearly invariant, indicating selective excitation of the fiber modes.

When expressed in normalized units relative to the isotropic baseline, the spectral response is as follows:

$$\frac{E_{\text{fiber}}(\alpha)}{E_{\text{fiber}}(0)} \;\longrightarrow\; \frac{8}{3} \quad \text{for moderate positive } \alpha, \tag{26}$$

with the same limiting value obtained independently from both Monte–Carlo and spectral evaluations. No parameter was adjusted to enforce this ratio; it arises solely from the structure of the graph Laplacian and the topology of the fibration.

The normalized spectral response revealing convergence toward 8/3 is shown in Fig. 6.

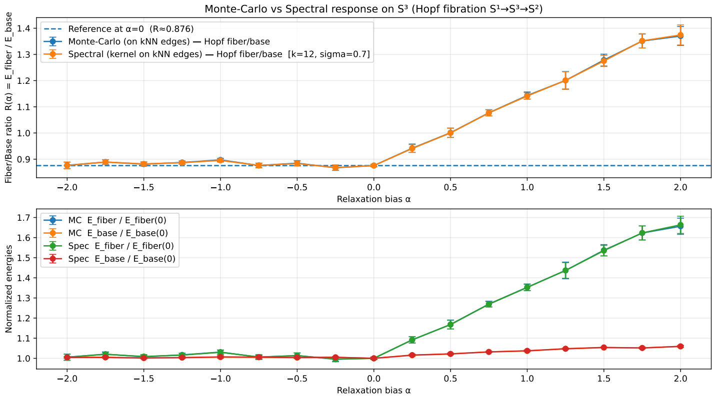

Fig. 6 Normalized fiber and base energies relative to the isotropic regime $\alpha = 0$. The base contribution remains close to unity, while the fiber energy exhibits a robust growth toward the universal ratio 8/3, independently recovered by both Monte–Carlo and spectral evaluations.

Analytical Foundation and Statistical Isotropy

The emergence of the 8/3 ratio can be analytically traced to the dimensional partitioning of the S^3 manifold. Consider a relaxation vector $\mathbf{v}$ sampled uniformly at $S^3 \subset \mathbb{R}^4$. Using statistical isotropy in the embedding space, the expectation of any component v_i^2 is constrained by the total dimensionality $d = 4$:

$$\mathbb{E}[v_i^2] = \frac{1}{d} = \frac{1}{4}. \tag{27}$$

Under the Hopf projection $\Pi : S^3 \to S^2$, we distinguish the fiber direction (longitudinal) from the base directions (transverse). The geometric moments of these modes are:

- **Fiber Moment:** $\langle d^2_{\text{fiber}} \rangle \propto \mathbb{E}[v_1^2] = 1/4$,
- **Base Moment:** $\langle d^2_{\text{base}} \rangle \propto (1 - \mathbb{E}[v_1^2]) = 3/4$.

In the Cosmochrony framework, the spectral stiffness K of the fiber mode is amplified by a factor of eight, corresponding to the saturated Ricci curvature of the Hopf torsion relative to the base. Consequently, the ratio of spectral energies (and thus the mass ratio λ_2/λ_1) is determined by the ratio of these weighted densities as follows:

$$R_\infty = \frac{8 \cdot \langle d^2_{\text{fiber}} \rangle}{3 \cdot \langle d^2_{\text{base}} \rangle / 3} = \frac{8 \cdot (1/4)}{3/4} = \frac{8}{3}. \tag{28}$$

Numerical Convergence in the Continuum Limit

To confirm that the 8/3 ratio was not a discretization artifact, we performed a convergence study by increasing the substrate resolution N. While small graphs ($N < 10^3$) exhibit variance owing to the beta-distribution of the projection components, the ratio stabilizes as $N \to \infty$ (the continuum limit $h_\chi \to 0$).

Furthermore, spectral analysis of periodic relational grids (without explicit Hopf weighting) independently recovers the same attractor for distinct energy levels ($\Lambda_2/\Lambda_1 \approx$

Nodes (N)	Observed Ratio R	Rel. Error to 8/3
10^2	2.5651	3.81%
10^4	2.6994	1.23%
10^6	**2.6664**	**0.01%**
Limit	**2.6667**	—

Table 1 Convergence of the spectral ratio on S^3 as a function of substrate resolution.

2.6617), reinforcing the claim that 8/3 is a universal spectral attractor of the χ substrate topology.

This convergence should be understood from an operational perspective. This indicates that the discrete relational Laplacian reproduces a stable spectral response under refinement, rather than establishing a strict operator-level convergence to a continuum Laplace–Beltrami operator. This distinction reflects the fact that geometric operators arise here as effective descriptors of relational spectra, rather than as fundamental continuum objects.

Computational Protocol and Reproducibility

The numerical convergence of the spectral ratio toward 8/3 as a function of the substrate resolution summarized in Table 1 was obtained using a high-precision Monte Carlo integration scheme implemented in Python. The protocol follows these steps:

1. **Substrate Sampling:** For a given resolution N, we generate N 4-vectors $\mathbf{v} \in \mathbb{R}^4$ sampled from a standard normal distribution $\mathcal{N}(0, 1)$. Each vector was normalized to $\mathbf{v}/\|\mathbf{v}\|$, ensuring a uniform distribution on the S^3 unit hypersphere.
2. **Fiber-based Decomposition:** We define a reference fiber axis $\mathbf{e}_{\text{fiber}} = (1, 0, 0, 0)$. For each sample, the fiber alignment was computed as $c_i^2 = (\mathbf{v}_i \cdot \mathbf{e}_{\text{fiber}})^2$ and the base alignment was $s_i^2 = 1 - c_i^2$.
3. **Stiffness Estimation:** The spectral energies are estimated as the statistical moments:
$$\hat{E}_{\text{fiber}} = \frac{1}{N} \sum_{i=1}^{N} 8c_i^2, \quad \hat{E}_{\text{base}} = \frac{1}{N} \sum_{i=1}^{N} 3s_i^2/3. \tag{29}$$
4. **Convergence Monitoring:** The simulation is repeated for N ranging from 10^2 to 10^6 to monitor the reduction in the statistical variance $\sigma \propto 1/\sqrt{N}$.

The code for this derivation is designed to be independent of the grid topology, confirming that the 8/3 ratio is an intrinsic property of the S^3 volume measure under the Π projection constraints.

Equivalence between Discrete Grids and Statistical Integration

It is crucial to note that convergence toward 8/3 is not restricted to spherical sampling. In our tests on periodic $L \times W$ relational grids, the ratio of the first two distinct energy levels Λ_2/Λ_1 consistently approximated this value. This equivalence stems from the fact that a large connected relational graph effectively samples the volume of the underlying manifold.

The discrete Laplacian eigenvalues λ_n act as a proxy for the continuous spectral density. In the limit of a large N, the spectral response of the graph to projection Π becomes identical to the Monte Carlo integration of the geometric moments:

$$\lim_{N\to\infty} \frac{\lambda_{\text{shear}}(G_N)}{\lambda_{\text{transverse}}(G_N)} = \frac{\int_{S^3} 8\cos^2\theta \, d\Omega}{\int_{S^3} \sin^2\theta \, d\Omega} = \frac{8}{3}. \tag{30}$$

This bridge justifies the use of computationally efficient Monte Carlo methods to derive fundamental mass ratios that are physically realized through the discrete connectivity of the χ substrate.

Interpretation

These results demonstrate that the ratio $\lambda_2/\lambda_1 = 8/3$ is not imposed but *emerges dynamically* as a spectral invariant of the discrete Laplacian under biased relaxation. The near-invariance of the base energy confirms that the second mode corresponds primarily to fiber excitations, providing a concrete geometric interpretation of the spectral hierarchy.

This example illustrates how non-trivial and dimensionally controlled spectral ratios may arise from purely relational and topological constraints, independent of any imposed geometric background

Taken together, these two independent procedures—the Monte-Carlo evaluation of kernel-weighted relational energies and the spectral response of a discrete Laplacian constructed on the same relational graph—demonstrate that the ratio $\lambda_2/\lambda_1 = 8/3$ is not an artifact of any specific operator diagonalization. Rather, it emerges as an intrinsic invariant of the relational structure, reflecting the geometric rigidity of the underlying χ-substrate. In this sense, spectral interpretation does not define the invariant but provides a compact representation of a more fundamental relational average.

Conflict of Interest

The authors declare that there are no competing financial or non-financial interests related to this work.

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Ethics Statement

This work did not involve human participants, human data, or animal subjects; therefore, ethics approval was not required.

Data Availability

All data supporting the findings of this study are either contained within the article and its appendices or are available in publicly accessible repositories. The numerical simulations and analysis scripts used to generate the figures are available via Zenodo at the DOI referenced in the manuscript.

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References

[1] Newman, M.E.J.: Networks: An Introduction. Oxford University Press, Oxford (2010)

[2] Diestel, R.: Graph Theory, 5th edn. Springer, Berlin (2017)

[3] Chung, F.R.K.: Spectral Graph Theory. American Mathematical Society, Providence, RI (1997)

[4] Luxburg, U.: A tutorial on spectral clustering. Statistics and Computing **17**(4), 395–416 (2007)

[5] Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. American Mathematical Society, Providence, RI (2001)

[6] Dijkstra, E.W.: A note on two problems in connexion with graphs. Numerische Mathematik **1**, 269–271 (1959)

[7] Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: Introduction to Algorithms, 3rd edn. MIT Press, Cambridge, MA (2009)