Relational Reconstruction of Spacetime Geometry from Graph Laplacians

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Abstract

We present a relational and spectral construction of effective spacetime geometry in which metric notions arise from correlation structure alone, without assuming a background manifold, coordinates, or fundamental geometric degrees of freedom. Starting from a purely relational substrate endowed with a symmetric connectivity operator, we define operational distances through minimal path functionals and show how a stable geometric regime emerges via spectral admissibility.

A non-circular coarse-graining scheme is introduced, distinguishing pre-geometric combinatorial neighborhoods from geometry-aware weighted distances. This hierarchy allows the construction of an effective scalar descriptor whose correlations encode operational notions of time ordering and spatial separation. When relational variations become sufficiently smooth, the resulting distance matrix admits a low-dimensional embedding, enabling the reconstruction of emergent coordinates and an effective metric structure.

We demonstrate that, in this projectable regime, standard geometric observables—such as proper time, spatial distance, and curvature—arise as descriptive summaries of relational constraints. The effective metric is shown to reproduce general-relativistic phenomenology in appropriate limits, including the recovery of Schwarzschild geometry for isolated, approximately symmetric configurations, without postulating gravitational dynamics at the fundamental level.

The framework naturally predicts breakdowns of geometric description when spectral gaps close or relational structure becomes non-local, providing intrinsic criteria for the limits of continuum spacetime. Numerical and analytical results supporting a universal spectral hierarchy are presented in the appendices. Overall, this work establishes a concrete pathway from relational spectral data to emergent metric geometry, positioning spacetime as an operational construct rather than a primitive entity.

Keywords: Pre-geometric substrate, emergent spacetime, relational dynamics, Born–Infeld dynamics, spectral geometry, quantum non-injectivity

Contents

1 Introduction

A central open problem in gravitational physics concerns the emergence of spacetime geometry from more primitive, non-metric structures. In general relativity, the metric tensor is postulated as a fundamental dynamical field, encoding both causal and metrical properties of spacetime. By contrast, many approaches to background-independent gravity suggest that spacetime geometry should itself arise as an effective description, reconstructed from relational or pre-geometric data.

This question has motivated a wide range of frameworks, including canonical and covariant approaches to quantum gravity, causal set theory, loop-based constructions, and programs based on spectral or non-commutative geometry. While these approaches differ in their technical implementation, they share a common challenge: how to recover a smooth pseudo-Riemannian geometry from fundamentally non-metric degrees of freedom, without introducing a background manifold or metric by assumption.

In this work, we address this reconstruction problem from a deliberately minimalist and operational perspective. Rather than postulating a spacetime metric or continuum structure from the outset, we consider relational systems described by connectivity data encoded in discrete or coarse-grained graphs. The central object of our analysis is the spectrum of a Laplacian operator defined on such relational structures. We investigate under which conditions spectral information alone is sufficient to define effective notions of distance, curvature, and geometry.

Our approach is motivated by the observation that spectral data provide a natural bridge between discrete relational systems and continuum geometry. In appropriate limits, the spectrum of graph Laplacians is known to encode geometric information analogous to that of differential operators on smooth manifolds. The question we pursue here is whether this correspondence can be made explicit and operational, yielding an effective metric structure without assuming one at the fundamental level.

We develop a framework in which admissible relational configurations are characterized by spectral consistency conditions. Effective geometric quantities are then reconstructed through spectral distances and embeddings, defined purely in terms of the Laplacian spectrum. No coordinates, metric tensor, or variational principle are assumed a priori. Instead, such structures emerge only in regimes where the relational system admits a stable and well-defined spectral continuum limit.

Within this setting, we show that familiar geometric notions can be recovered in a controlled manner. In particular, we demonstrate how an effective metric description arises locally, how curvature can be defined operationally, and how known solutions of general relativity can be approximated in appropriate limits. As a concrete benchmark, we exhibit the recovery of a Schwarzschild-type geometry from purely spectral and relational input.

The scope of the present work is intentionally restricted. We do not address the dynamics of matter fields, quantum statistics, or cosmological evolution. Our aim is not to propose a complete theory of quantum gravity, but to isolate and analyze the geometric reconstruction problem in its simplest and most transparent form.

The structure of the paper is as follows. Section 2 introduces the relational graph framework and the associated Laplacian operator. Section 3 develops spectral notions of distance and embedding. Section 4 analyzes the emergence of an effective continuum geometry and metric structure. Section 5 discusses operational definitions of curvature, and Section 6 presents the recovery of a Schwarzschild-type effective geometry. Technical results and auxiliary derivations are collected in the appendices.

2 Relational Substrate and Spectral Structure

In this section, we introduce the minimal relational framework underlying the spectral constructions developed in the remainder of the paper. The purpose is not to postulate a spacetime manifold, a metric tensor, or a set of dynamical fields, but to specify the weakest structural assumptions required to define spectral operators and to extract effective geometric information from them.

We consider systems described by relational connectivity data, represented by discrete or coarse-grained graphs. The vertices of the graph correspond to abstract relational elements, while edges encode admissible relations between them. No embedding space, coordinate chart, or background geometry is assumed. In particular, notions of distance, dimension, or curvature are not introduced at this stage.

The only primitive structure required is the existence of a well-defined adjacency relation, possibly weighted, from which a Laplacian operator can be constructed. This operator encodes the local connectivity properties of the relational system and provides access to its spectral data. The spectrum of the Laplacian constitutes the primary object of interest in what follows.

We emphasize that the relational graphs considered here are not assumed to be fundamental in a physical sense. Rather, they are used as minimal mathematical representations of relational structure, suitable for spectral analysis. Different graph realizations may correspond to the same effective geometric description, reflecting the fact that spectral reconstruction is generally non-injective.

Effective geometric notions are introduced only at a secondary level, through spectral constructions applied to families of admissible relational graphs. When appropriate consistency and regularity conditions are satisfied, these constructions allow for the emergence of continuum-like geometric descriptions. The criteria for such admissibility and the associated reconstruction procedures are developed in the following sections.

No assumptions concerning dynamics, temporal ordering, or causal structure are made in this section. The framework is entirely kinematical at this stage. Any reference to evolution or ordering will be introduced later only as an effective or auxiliary notion, when required by specific reconstruction schemes.

This relational and spectral starting point provides a neutral and flexible basis for addressing the geometric reconstruction problem. It allows us to investigate how metric properties can arise from spectral data alone, without committing to a specific underlying ontology or dynamical theory.

2.1 Relational Structure

We introduce a minimal relational framework intended to capture the weakest structural assumptions required for spectral reconstruction of geometry. No spacetime manifold, metric tensor, coordinate system, or causal structure is postulated at this level. The framework is deliberately formulated in non-geometric terms.

The system is described by a set of abstract relational elements together with admissible relations among them. These relations define a connectivity structure, which may be represented discretely or in a coarse-grained form. No embedding space or background geometry is assumed or required.

The relational elements themselves are not assigned local values, dimensional quantities, or order parameters. Such quantities arise only at an effective level, when relational configurations admit a stable spectral reconstruction. In particular, notions of distance, dimension, and curvature are not fundamental primitives but reconstructed descriptors.

A key feature of the reconstruction problem is its inherent non-injectivity: distinct relational configurations may give rise to identical effective geometric descriptions, while a single configuration may admit multiple equivalent spectral representations. This loss of information is a structural property of spectral reconstruction and does not rely on any physical coarse-graining mechanism.

Throughout this work, geometric and field-theoretic language is employed strictly as an effective descriptive convenience. Such language refers to reconstructed structures that become meaningful only in regimes where the relational system supports a consistent spectral continuum approximation. It does not imply the existence of underlying spacetime entities or dynamical fields.

This relational starting point provides a neutral basis for defining spectral operators and for investigating how effective geometric properties can emerge from connectivity data alone.

3 Spectral Distance and Embedding

A central step in the reconstruction of geometry from relational data is the definition of a notion of distance that does not rely on a pre-existing metric or coordinate structure. In this section, we introduce spectral notions of proximity and distance derived directly from relational connectivity and Laplacian operators.

We consider a relational system represented by a graph or coarse-grained network endowed with a Laplacian operator Δ. No embedding space or background geometry is assumed. The spectrum and eigenfunctions of Δ encode the connectivity structure of the system and constitute the only input for the constructions below.

3.1 Spectral proximity

Given the Laplacian spectrum $\{\lambda_n, \phi_n\}$, one may define spectral kernels that quantify relational proximity between nodes or abstract elements of the relational system. A generic example is provided by heat-kernel–type constructions,

$$K(i,j;\alpha) \;=\; \sum_n e^{-\alpha\lambda_n}\,\phi_n(i)\,\phi_n(j), \tag{1}$$

where α is a spectral scale parameter. Such kernels measure the degree of connectivity between elements i and j through the spectrum of Δ, without reference to any metric distance.

Spectral kernels of this type are invariant under relabeling of nodes and do not presuppose any embedding. They provide a natural notion of relational proximity that depends solely on the spectral properties of the Laplacian.

3.2 Spectral distance

From spectral proximity measures, one may define an effective distance by monotonic transformations. A convenient choice is

$$d_{\text{spec}}(i,j) \;=\; -\log\left(\frac{K(i,j;\alpha)}{\sqrt{K(i,i;\alpha)\,K(j,j;\alpha)}}\right), \tag{2}$$

which defines a symmetric, non-negative quantity vanishing when $i = j$. This definition is purely spectral and does not invoke any geometric or physical interpretation.

The resulting distance is generically non-injective: distinct relational configurations may induce identical spectral distances, and multiple embeddings may correspond to the same distance matrix. This non-uniqueness is a structural feature of spectral reconstruction rather than a limitation of the formalism.

Emergent spectral dimension.

In Eq. (2), the exponent d should not be interpreted as a fixed or postulated topological dimension. Within the present framework, d is an *emergent spectral quantity* characterizing the asymptotic scaling of the eigenvalue counting function $N(\lambda)$ in the low-energy regime. Operationally, d is extracted from the slope of $\log N(\lambda)$ versus $\log \lambda$ and may take non-integer values at finite spectral resolution.

In particular, the convergence of d toward an integer value reflects the stabilization of the relational substrate into a smooth effective geometric regime. Numerical evidence presented in this work indicates that, under admissibility and regularity conditions, the substrate converges toward a four-dimensional spectral behavior, $d \to 4$, without this value being imposed *a priori*. This convergence is observed to persist across multiple spectral decades, indicating that the four-dimensional behavior is not a transient finite-size effect but a stable property of the admissible relational regime.

At smaller spectral scales, deviations from integer dimensionality encode local curvature and connectivity distortions, providing a direct bridge between spectral observables and effective geometric structure.

Stability of the spectral dimension.

Numerical evaluations of the spectral counting function, performed on distinct relational realizations and resolutions of the substrate, indicate that the extracted spectral dimension d_s remains stable and converges toward $d_s \simeq 4$ over a broad range of spectral scales. Deviations appear only near the ultraviolet cutoff, where the notion of effective geometry itself ceases to be applicable.

This stability across scales and discretizations supports the interpretation of d_s as an emergent property of the relational organization rather than as an imposed dimensional parameter.

From graph distance to effective geodesics.

The operational distance defined via shortest weighted paths on the relational graph plays the role of an effective geodesic distance. While combinatorial distances count edge hops, the weighted distance incorporates local variations of connectivity through edge weights, thereby encoding the inhomogeneous relational structure of the substrate.

In the continuum limit, the local density of nodes contributing to admissible paths controls the scaling of volumes and distances. This node density acts as the discrete analogue of the metric determinant $\sqrt{-g}$, governing how relational neighborhoods are mapped onto effective geometric volumes. Geometric notions thus arise from connectivity statistics rather than from a postulated metric field.

3.3 Local embedding and quadratic approximation

When the relational system admits a sufficiently regular spectral structure, the spectral distance matrix may be locally approximated by a low-dimensional embedding. In such regimes, one may introduce local coordinates x^μ as auxiliary variables parametrizing the embedding space.

To leading order, the spectral distance between nearby elements admits a quadratic approximation of the form

$$d_{\mathrm{spec}}(i,j)^2 \;\approx\; g_{\mu\nu}(x)\,\Delta x^\mu \Delta x^\nu, \tag{3}$$

where $g_{\mu\nu}$ is a symmetric tensor encoding the local geometry of the embedding. This tensor is not postulated as a fundamental object, but arises as a local summary of spectral distances.

The introduction of $g_{\mu\nu}$ should be understood as a descriptive convenience, valid only in regimes where a smooth spectral embedding exists. Outside such regimes, no metric interpretation is assumed or required.

3.4 Scope and limitations

The constructions presented in this section are entirely kinematical. They do not rely on assumptions about dynamics, temporal ordering, or causal structure. Their purpose is solely to establish how effective geometric notions may be reconstructed from spectral data associated with relational connectivity.

The emergence of continuum geometry, curvature, and comparison with known geometric solutions are addressed in the following sections.

4 Continuum Limit and Emergent Metric

The spectral distance introduced in the previous section provides a purely relational notion of proximity. A natural question is under which conditions such a construction admits an effective continuum description, allowing the use of familiar geometric tools. In this section, we analyze the regimes in which a relational spectral structure can be approximated by a smooth metric geometry.

We consider families of relational graphs or coarse-grained networks whose Laplacian spectra exhibit sufficient regularity. No assumption is made concerning an underlying manifold or dimensionality. Instead, the continuum limit is defined operationally, as the regime in which spectral distances vary smoothly and admit consistent local approximations.

A key requirement for the emergence of a continuum description is local factorizability. In such regimes, the relational system admits an approximate decomposition into weakly coupled subsystems, allowing spectral distances to be defined locally and independently. This property underlies the emergence of effective locality and permits the introduction of local coordinate charts as auxiliary descriptive tools.

The continuum approximation is inherently non-unique. Because spectral reconstruction is generically non-injective, distinct relational configurations may give rise to identical effective metric descriptions. Conversely, a given relational structure may admit multiple equivalent continuum embeddings. This ambiguity is a structural feature of spectral geometry and does not signal a loss of consistency.

When local factorizability and spectral regularity are satisfied, the spectral distance admits a local quadratic approximation. In this regime, one may introduce an effective metric tensor $g_{\mu\nu}(x)$ defined by the leading-order expansion

$$d_{\mathrm{spec}}(i,j)^2 \;\approx\; g_{\mu\nu}(x)\,\Delta x^{\mu}\Delta x^{\nu}, \tag{4}$$

where Δx^{μ} denotes coordinate differences in a local embedding. The metric tensor is not a fundamental structure but a derived descriptor encoding the local behavior of spectral distances.

The validity of the continuum approximation is limited to regimes where higher-order spectral corrections remain subdominant. Outside such regimes, no smooth geometric interpretation is assumed, and the relational description must be treated in its full spectral form.

The framework developed here is purely kinematical. No assumptions regarding dynamics, temporal ordering, or causal structure are required for the emergence of an effective metric description. The analysis focuses solely on the conditions under which relational spectral data support a continuum geometric approximation.

4.1 Spectral Admissibility and Regularity

The emergence of an effective continuum geometry from relational spectral data requires additional regularity conditions. Not all relational configurations admit a meaningful continuum approximation, even when a spectral distance can be defined.

In this subsection, we introduce spectral admissibility criteria that characterize the regimes in which geometric reconstruction is well-defined.

Let L denote a self-adjoint relational operator acting on a suitable Hilbert space of configurations. In discrete realizations, L reduces to a graph Laplacian associated with the relational connectivity of the system. More generally, L encodes relational structure without reference to a background manifold or metric.

The operator L admits a spectral decomposition

$$L\psi_n = \lambda_n \psi_n, \tag{5}$$

with non-negative eigenvalues $\{\lambda_n\}$ and corresponding eigenmodes $\{\psi_n\}$. No geometric interpretation is assumed at this stage.

Spectral admissibility is defined by restricting attention to a controlled spectral window. Concretely, we introduce a smooth spectral filter

$$F_{\lambda_*} \; = \; f\left(\frac{L}{\lambda_*}\right), \tag{6}$$

where $f(x)$ is a fixed cutoff function and λ_* sets a characteristic spectral scale. Only modes below this scale contribute significantly to the effective geometric reconstruction.

This filtering procedure defines admissibility purely in spectral terms. It does not rely on locality, coordinates, or spacetime integration measures. Instead, it reflects the fact that continuum geometry, when it emerges, is necessarily insensitive to fine-grained spectral details beyond a given resolution.

Spectral admissibility is generically non-injective. Distinct relational configurations may share identical spectral content within the admissible window and therefore give rise to the same effective continuum geometry. Conversely, a given relational structure may admit multiple equivalent continuum embeddings. This non-uniqueness is a structural feature of spectral reconstruction and does not indicate an inconsistency of the framework.

In some reconstruction schemes, admissibility criteria may involve monotonic spectral filters or ordering parameters, reflecting the fact that effective continuum descriptions are typically insensitive to fine-grained spectral rearrangements beyond a given resolution.

We emphasize that the spectral reconstruction of an effective metric does not require the introduction of additional fundamental fields or degrees of freedom; the geometric structures discussed here arise solely from relational spectral data and their continuum approximation.

5 Operational Geometry

5.1 Collective Coupling and Operational Geometry

The emergence of curvature in a relational framework can be understood as a collective effect. Rather than postulating a fundamental gravitational field, geometric properties arise from the way relational variations influence one another across extended regions of the system.

In regimes where the relational structure is approximately homogeneous, variations propagate uniformly and admit a simple effective description. Localized irregularities in the relational connectivity modify this collective response, altering how variations can be correlated between neighboring regions. This modulation can be summarized by an effective coupling function characterizing the stiffness of the relational structure with respect to relative deformations.

Importantly, this coupling is defined without reference to any background metric, coordinate system, or pre-existing notion of spatial separation. Distance is instead defined operationally: two regions are considered close if relational variations can be efficiently correlated between them, and distant otherwise.

In the continuum and weakly inhomogeneous regime, this operational notion of proximity admits a compact geometric representation. An effective metric can then be introduced as a descriptive tool summarizing the collective response of the relational structure to local variations. The metric does not constitute an independent degree of freedom but encodes coarse-grained relational regularities.

From this perspective, curvature is not a primitive geometric property nor the result of a dynamical field equation. It emerges as a macroscopic descriptor of how localized relational features modulate the collective connectivity of the system. Geometry thus functions as an effective encoding of constrained relational organization rather than as a fundamental ontological entity.

Spectral convergence and continuum limit.

The emergence of a continuous geometric description relies on the spectral consistency of the relational Laplacian in appropriate large-scale regimes. Under mild regularity assumptions—uniform node density, bounded degree growth, and approximate isotropy of local connectivity—the graph Laplacian Δ_G converges, in the strong resolvent sense, toward the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$ associated with an effective manifold $\mathcal{M}$.

This convergence is understood in the operational sense relevant to the present framework: low-lying eigenmodes of Δ_G approximate those of $\Delta_{\mathcal{M}}$ up to a spectral cutoff λ_*, beyond which the notion of smooth geometry ceases to be meaningful. Such convergence results are well established in spectral graph theory and manifold learning, and justify interpreting the admissible spectral sector of the relational substrate as an effective continuum geometry.

Spectral dimension as an emergent observable.

The effective dimensionality of the reconstructed geometry is not postulated but inferred from the spectral properties of the Laplacian. A standard operational probe is provided by the spectral heat kernel

$$K(t) = \sum_n e^{-t\lambda_n}, \tag{7}$$

from which the spectral dimension is defined as

$$d_s(t) = -2\frac{d\log K(t)}{d\log t}. \tag{8}$$

In regimes where the relational configuration is sufficiently smooth and admissible, $d_s(t)$ exhibits a stable plateau at large scales, indicating an effective manifold-like behavior. In particular, configurations relevant to the gravitational regime display a robust spectral dimension $d_s \simeq 4$, independently of microscopic graph details, providing an intrinsic justification for the emergence of a four-dimensional spacetime description.

Physical interpretation of the spectral scale λ_*.

The spectral cutoff λ_* plays a central physical role in the framework. It defines the upper limit of spectral admissibility beyond which relational modes can no longer be interpreted geometrically. Operationally, λ_* functions as an ultraviolet cutoff, analogous to the Planck scale in effective approaches to quantum gravity.

Above this scale, spectral locality breaks down and the reconstruction of smooth coordinates, distances, or curvature becomes ill-defined. The breakdown of the geometric description at high spectral energies is not a pathology but a physical prediction of the theory, signaling a transition to a genuinely pre-geometric regime.

5.2 Emergent Curvature

In a relational framework, curvature arises as a collective geometric descriptor rather than as a primitive property of an underlying spacetime. Spatial variations in relational connectivity and spectral coupling lead to non-uniform correlation patterns across extended regions. When a smooth geometric parametrization is applicable, these non-uniformities are compactly summarized by gradients of an effective metric structure.

From this perspective, curvature does not represent an independent dynamical field. It functions as a macroscopic descriptor encoding how localized relational features modulate collective connectivity. The metric does not act as a causal agent; it provides a concise representation of constrained relational organization in regimes admitting a continuum approximation.

Within such regimes, the resulting curvature reproduces the familiar geometric phenomenology associated with curved spacetime, including geodesic deviation, gravitational redshift, and lensing effects. These phenomena arise here from spatial variations in relational coupling rather than from a fundamental spacetime geometry.

The use of geometric language is therefore strictly operational. Curvature and metric quantities are introduced only insofar as they provide an accurate summary of relational correlation patterns. Outside regimes where a smooth and slowly varying approximation is valid, no geometric interpretation is assumed.

5.3 Static Spherically Symmetric Geometry

We now consider a static and approximately spherically symmetric relational configuration admitting a smooth continuum description. No dynamical assumptions or field equations are introduced. The analysis is purely kinematical and focuses on the geometric form compatible with symmetry and weak-field consistency.

In such a regime, the effective metric may be written in standard static spherically symmetric form,

$$ds^2 = -A(r)\,c^2dt^2 + B(r)\,dr^2 + r^2d\Omega^2, \tag{9}$$

where the metric functions $A(r)$ and $B(r)$ encode the operational relation between temporal and spatial intervals.

In the weak-field limit, the metric coefficients admit an expansion

$$A(r) \simeq 1 + 2\frac{\Phi(r)}{c^2}, \qquad B(r) \simeq \left(1 + 2\frac{\Phi(r)}{c^2}\right)^{-1}, \tag{10}$$

where $\Phi(r)$ is an effective potential characterizing the deviation from flat geometry. This definition is purely geometric and does not presuppose any dynamical origin for Φ.

Requiring asymptotic flatness and consistency with spherical symmetry uniquely fixes the exterior form of the metric to leading order,

$$A(r) = 1 - \frac{r_s}{r}, \qquad B(r) = \left(1 - \frac{r_s}{r}\right)^{-1}, \tag{11}$$

where r_s is an integration constant with dimensions of length. This geometry coincides with the Schwarzschild metric at leading order.

The emergence of this form does not rely on a postulated gravitational field equation. It reflects the fact that, among static spherically symmetric geometries, the Schwarzschild metric provides the unique weak-field extension compatible with asymptotic flatness and local isotropy.

Standard weak-field tests follow directly from the geometric structure. Gravitational redshift, light deflection, and time dilation arise as kinematical consequences of the metric coefficients and do not require any additional assumptions.

6 GR Limit: Schwarzschild-type Recovery

6.1 Non-uniqueness of Spectral Reconstruction

The reconstruction of an effective metric from relational spectral data is generically non-unique. This non-uniqueness is not a consequence of approximation or incomplete information, but a structural feature of spectral geometry.

Spectral reconstruction proceeds by extracting geometric descriptors from a restricted set of spectral features, typically after the application of admissibility or regularity criteria. As a result, distinct relational configurations may share identical low-resolution spectral content and therefore give rise to the same effective metric description. Conversely, a given relational structure may admit multiple continuum embeddings that are spectrally equivalent within the admissible window.

Operational notions of distance further reinforce this ambiguity. Distance is defined in terms of correlation efficiency or spectral proximity rather than through an underlying coordinate separation. Different choices of spectral filtering, coarse-graining scale, or embedding procedure may therefore lead to metrically equivalent but geometrically distinct descriptions.

Importantly, this ambiguity does not compromise physical consistency. All equivalent metric descriptions agree on observable geometric properties within their shared domain of validity. The effective metric should thus be understood as a representative of an equivalence class of geometries compatible with the same relational spectral data.

Non-uniqueness is therefore an intrinsic aspect of emergent geometry in relational frameworks. It reflects the fact that geometric descriptions encode only a subset of the underlying relational structure and should not be interpreted as faithful microscopic representations.

Flux conservation and radial scaling.

The emergence of a $1/r$-type effective potential does not rely on an imposed force law. It follows from the conservation of relaxation flux in the relational substrate. In a quasi-static regime, the total relaxation flux crossing any closed relational surface surrounding a localized perturbation is conserved.

In an effectively three-dimensional projectable regime, this conservation implies that the perturbation of connectivity must decay as $1/r^2$ with radial distance. When integrated along admissible paths, this scaling naturally yields an effective potential proportional to $1/r$, providing the structural origin of the Schwarzschild radius without assuming Newtonian dynamics or a prior gravitational law.

Origin of the $1/r$ profile.

In the present framework, the notion of mass does not correspond to a fundamental source term, but to a localized inhibition of admissible relaxation in the relational substrate. Such a configuration reduces the local density of optimal relational paths, as measured by shortest-path connectivity under the operational distance defined via Dijkstra-type minimization.

In the continuum regime where an effective geometric description becomes valid, this reduction acts as a source term for the scalar Laplacian governing admissible deformations of the effective metric. In three spatial dimensions, the fundamental solution of the corresponding Poisson equation is uniquely fixed by flux conservation and exhibits a $1/r$ decay. The emergence of the Schwarzschild factor $1 - r_s/r$ therefore follows directly from the conservation of relational relaxation flux in an isotropic three-dimensional effective geometry, rather than from an imposed potential or symmetry assumption.

6.2 Validity of Geometric Descriptions

The emergence of curvature and metric structure discussed in the previous sections does not imply that geometric descriptions are universally applicable. Rather, geometry appears as an effective language, valid only in regimes where the relational structure admits a smooth and locally stable continuum approximation.

In such regimes, geometric relations exhibit a remarkable robustness. Local curvature, geodesic deviation, and horizon structure depend only weakly on microscopic details of the underlying relational system. This explains the universality of geometric behavior observed across a wide range of physical situations.

Outside these regimes, the failure of geometric descriptions should not be interpreted as a breakdown of geometric laws. Instead, it reflects the loss of applicability of the geometric language itself. When local injectivity or smoothness conditions are violated, the notion of spacetime ceases to be operationally meaningful, and no geometric description is assumed.

From this perspective, geometric relations function as internal consistency conditions of emergent spacetime descriptions rather than as fundamental laws governing an underlying substrate. Geometry is exact where it applies and silent elsewhere.

Relational encoding of mass.

In the present framework, the mass parameter appearing in the Schwarzschild-type effective geometry is not introduced as a fundamental source term. Instead, it reflects a persistent and localized modification of the relational connectivity of the χ substrate.

Operationally, a massive configuration corresponds to a region where admissible relaxation paths are systematically constrained, resulting in an effective reduction of relational accessibility across that domain. This constraint modifies the local spectral response of the relational Laplacian, while preserving global admissibility conditions. In the projectable regime, this localized alteration is encoded geometrically as a radial deformation of effective distances, whose asymptotic form reproduces the Schwarzschild mass parameter.

The mass therefore quantifies the integrated obstruction to relational relaxation, rather than the presence of an external source. This interpretation is consistent with the non-fundamental status of the effective metric and clarifies how gravitational mass emerges from purely relational modifications of the substrate.

6.3 Limits of Schwarzschild-Type Reconstruction

The recovery of a Schwarzschild-type metric in the previous section illustrates how familiar gravitational geometries can emerge from relational spectral data under restrictive conditions. This reconstruction is, however, intrinsically regime-dependent and should not be interpreted as universally applicable.

The derivation relies on assumptions of approximate stationarity, spherical symmetry, and weak inhomogeneity. Outside these regimes, the effective geometric description may differ substantially from the Schwarzschild form or cease to be meaningful altogether.

The characteristic length scale appearing in the static spherically symmetric metric arises as an integration constant of the geometric reconstruction. Its identification with physical parameters depends on the choice of spectral filtering and on the operational resolution at which the geometry is probed. As a result, Schwarzschild-type metrics should be understood as effective geometric representatives rather than as fundamental solutions.

Classical weak-field tests confirm the consistency of the geometric approximation within its domain of validity. They do not, however, constrain the behavior of the relational system outside this regime, where the assumptions underlying the metric reconstruction no longer hold.

These considerations highlight the importance of distinguishing between the existence of a geometric description and its range of applicability. Schwarzschild geometry emerges where appropriate but does not define the behavior of the system in regimes where a continuum spacetime interpretation breaks down.

This limitation reflects the fact that the Schwarzschild geometry captures only the long-wavelength, weak-field imprint of a localized relational obstruction. Beyond this regime, the effective geometric description ceases to faithfully encode the underlying connectivity constraints of the substrate.

7 Discussion

The present work has developed a relational and spectral route to emergent metric geometry, deliberately avoiding the introduction of spacetime structure as a fundamental postulate. Instead, geometric notions arise as effective descriptors of admissible relational configurations selected by spectral filtering and projection.

A central result is that metricity can be reconstructed operationally from purely relational spectral data, without assuming a background manifold, predefined distance function, or fundamental locality. The effective metric appears as a compact summary of correlation structure in regimes where admissible configurations admit a stable and approximately factorizable description.

Status of spacetime geometry.

Within this framework, spacetime geometry is neither fundamental nor dynamical in the usual sense. It functions as an emergent kinematical structure, applicable only when the projection from the relational substrate becomes locally injective and spectrally well-conditioned. Outside these regimes, geometric notions lose operational meaning rather than being modified or replaced by alternative field equations.

This perspective clarifies the remarkable universality of general relativity. Whenever a smooth geometric description exists and admissible configurations vary slowly compared to the spectral cutoff, the standard geometric relations of general relativity are necessarily recovered. Einstein's equations thus appear not as a microscopic law of the substrate, but as a consistency condition governing emergent geometry wherever spacetime itself is a valid descriptor.

Relation to existing approaches.

The present construction shares motivations with several background-independent approaches to quantum gravity, including causal set theory, loop quantum gravity, and spectral geometry. Unlike causal set models, no fundamental discreteness is postulated; continuity is maintained at all levels, with effective granularity arising from spectral filtering rather than from an underlying lattice. In contrast to loop-based approaches, no spin networks or combinatorial structures are introduced as fundamental entities. Compared to non-commutative or spectral geometry programs, the emphasis here is not on algebraic generalization of manifolds, but on operational reconstruction of metric structure from admissible relational correlations.

Non-uniqueness and regime dependence.

An important implication of the spectral approach is the intrinsic non-uniqueness of geometric reconstruction. Different relational configurations may give rise to indistinguishable effective metrics within a given regime, reflecting the non-injective character of the projection. Geometric descriptions are therefore intrinsically approximate and regime-dependent. This non-uniqueness should not be interpreted as an ambiguity of the theory, but as a structural feature of emergent geometry itself.

Spectral rigidity and structural invariants.

While the reconstruction of effective geometry from relational spectral data is intrinsically non-unique and regime-dependent, this non-uniqueness does not extend to all spectral observables. The relational substrate exhibits robust spectral invariants that persist across discretizations, graph realizations, and numerical schemes.

A notable example is the emergence of the universal ratio $\lambda_2/\lambda_1 = 8/3$ in the scalar Laplacian spectrum, as illustrated in Appendix D. Although many relational microstates may project to indistinguishable effective metrics, only a restricted class of spectral organizations is compatible with a stable projectable geometric regime.

Such invariants therefore constrain the space of admissible emergent geometries more strongly than metric reconstruction alone. They provide an intermediate level of structure between microscopic relational descriptions and macroscopic geometric observables, and may serve as distinguishing signatures with respect to other background-independent approaches that reproduce similar effective geometries without exhibiting comparable spectral constraints.

Scope and limitations.

The present work is intentionally restricted to the emergence of geometric and gravitational structure. No attempt has been made to account for matter degrees of freedom, energetic quantities, quantum statistics, or particle properties. These notions presuppose additional layers of effective description that lie beyond the geometric regime addressed here.

Similarly, strongly non-factorizable or highly constrained configurations, for which no stable geometric parametrization exists, fall outside the scope of the present analysis. In such regimes, spacetime itself ceases to be an appropriate descriptive language, and alternative relational characterizations must be employed.

Outlook.

By isolating the emergence of metric geometry from other physical structures, this work establishes a clean and minimal foundation for subsequent investigations. Extensions of the framework to non-geometric regimes, to the emergence of matter-like excitations, or to quantum and statistical phenomena require additional assumptions and will be addressed separately.

The results presented here demonstrate that a large class of gravitational phenomena can be understood as consequences of relational spectral organization alone. In this sense, geometry appears not as the stage on which physics unfolds, but as a derived and context-dependent summary of deeper relational structure.

.1 Relational Distance as a Minimal Path Functional

A central step in the relational construction is the introduction of a distance notion defined *within* the relational adjacency structure underlying the χ description, without presupposing any embedding space or fundamental lattice. To avoid ambiguity and to prevent hidden circularity in subsequent coarse-graining procedures, we explicitly distinguish two distances that operate at different descriptive levels.

Combinatorial vs. weighted distance.

We define:

1. **Combinatorial distance** d_{ij}^C (pre-geometric).

$$d_{ij}^C = \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} 1,$$

   where γ_{ij} is any path connecting nodes i and j through the network links. This distance counts *graph steps only* and is **independent of the field values of** χ[1, 2]. It is used to define the neighborhood sets employed in relational averaging[3, 4].
2. **Weighted distance** d_{ij}^W (emergent / effective).

$$d_{ij}^W = \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} w_{uv},$$

   where w_{uv} is a positive weight associated with each link. This distance is used for the **emergent geometry** (and in particular for spectral constructions), because it encodes the effective relational stiffness of the network.

This distinction ensures that the coarse-graining background $\bar{\chi}$ can be defined using d_{ij}^C without any metric dependence, while the effective geometry is encoded by d_{ij}^W through weights that depend only on $\bar{\chi}$ (not on instantaneous χ).

The combinatorial distance d_{ij}^C has no physical interpretation and plays no direct observational role; it serves solely as an auxiliary construct for defining relational neighborhoods in a pre-geometric manner.

Weight functional and positivity.

We parameterize the weights by an effective connectivity (stiffness) matrix $K_{uv} > 0$:

$$w_{uv} \;=\; \frac{1}{K_{uv}}.$$

In the circularity-free construction used in this appendix, K_{uv} is not taken as a direct functional of χ, but as a functional of a slowly varying *background* field $\bar{\chi}$ defined by relational averaging (see Appendix E.5). Concretely, we use

$$w_{uv}(\bar{\chi}) \;=\; \frac{1}{K_0}\left[1+\left(\frac{\bar{\chi}_u-\bar{\chi}_v}{\chi_c}\right)^2\right], \qquad K_{uv}(\bar{\chi}) \;=\; \frac{1}{w_{uv}(\bar{\chi})}. \tag{12}$$

The positivity $w_{uv}(\bar{\chi}) > 0$ is guaranteed by construction, so d_{ij}^W is a well-defined weighted path metric whenever the graph is connected on the domain considered.

Metric status.

Both the combinatorial distance d_{ij}^C and the weighted distance d_{ij}^W define proper metric spaces on the χ-network, albeit at different descriptive levels: d_{ij}^C is discrete and topological, while d_{ij}^W encodes emergent relational structure. This duality is essential: d_{ij}^C provides a pre-geometric scaffold for defining $\bar{\chi}$, whereas d_{ij}^W provides the effective distance used in the emergent geometric regime.

.2 Derivation of χ_{eff} from Relational Observables

The effective field χ_{eff} is introduced as an operational description of χ-configurations once a stable projected regime exists. Since the projected regime admits an effective geometric interpretation, χ_{eff} must be constructed in a way that does not implicitly assume the very metric structure it is meant to support. In particular, if a distance d_{ij} used to define coarse-graining neighborhoods depends on weights that themselves depend on χ, then a hidden circularity would arise.

To remove this ambiguity, we adopt a **two-level construction** based on the explicit distinction between the combinatorial distance d_{ij}^C and the weighted distance d_{ij}^W introduced in Appendix E.4.

(1) Relational background field $\bar{\chi}$.

We define a background field $\bar{\chi}$ by a **relational average** that uses only the **combinatorial (pre-geometric) distance** d^C_{ij}. Let

$$N_i \;=\; \{j \,|\, d^C_{ij} \le \ell_0\} \tag{13}$$

be the combinatorial neighborhood of radius ℓ_0 around node i. We then set

$$\bar{\chi}_i \;=\; \frac{1}{|N_i|} \sum_{j \in N_i} \chi_j. \tag{14}$$

Because N_i depends only on d^C_{ij}, the definition of $\bar{\chi}$ is **independent of any weighted metric** and therefore does not depend on χ through a distance functional. This is the crucial step that prevents circularity.

The scale ℓ_0 is an auxiliary coarse-graining parameter defining the minimal relational neighborhood required for stability of the effective description; it does not correspond to a physical length scale prior to the emergence of geometry.

(2) Emergent connectivity and weighted distance.

Using the background field $\bar{\chi}$, we define the link weights and the corresponding connectivity through Eq. (12):

$$w_{uv}(\bar{\chi}) \;=\; \frac{1}{K_0}\left[1 + \left(\frac{\bar{\chi}_u - \bar{\chi}_v}{\chi_c}\right)^2\right], \qquad K_{uv}(\bar{\chi}) \;=\; \frac{1}{w_{uv}(\bar{\chi})}.$$

The **weighted distance** used for effective geometry is then

$$d^W_{ij} \;=\; \min_{\gamma_{ij}} \sum_{(u,v)\in\gamma_{ij}} w_{uv}(\bar{\chi}), \tag{15}$$

which depends on $\bar{\chi}$ but not on instantaneous χ values through the metric definition.

(3) Effective field χ_{eff} (geometry-aware coarse-graining).

Finally, we define χ_{eff} by coarse-graining χ over neighborhoods defined with the **weighted distance** d^W_{ij}:

$$V_{\ell_0}(i) \;=\; \{j \,|\, d^W_{ij} \le \ell_0\}\,, \tag{16}$$

$$\chi_{\text{eff}}(i) \;=\; \frac{1}{|V_{\ell_0}(i)|} \sum_{j\in V_{\ell_0}(i)} \chi_j. \tag{17}$$

Non-circular dependency structure.

The construction is explicitly hierarchical:

$$d^C \;\Longrightarrow\; \bar{\chi} \;\Longrightarrow\; w(\bar{\chi}),\, K(\bar{\chi}) \;\Longrightarrow\; d^W \;\Longrightarrow\; \chi_{\text{eff}}.$$

The neighborhood used to compute $\bar{\chi}$ is defined using d^C and is therefore χ-independent. The effective geometry is encoded in d^W through weights that depend only on $\bar{\chi}$, breaking any instantaneous feedback loop. This makes the operational definition of χ_{eff} compatible with the pre-geometric status of χ, while still allowing an emergent geometric regime for spectral and effective-field analyses.

No dynamical equation for χ or χ_{eff} is assumed in this construction; the procedure is purely kinematical and defines the conditions under which an effective geometric regime becomes admissible.

.3 Relation to the Effective Geometric Description

The effective geometric structures introduced in the main text—such as metric fields, spatial gradients, connection-like objects, and Poisson-type equations—do not represent fundamental degrees of freedom in Cosmochrony. They arise as coarse-grained summaries of relational configurations of the χ network once a projectable regime becomes applicable.

In the pre-geometric formulation, the relational substrate is defined purely in terms of adjacency, spectral properties, and admissibility constraints, without reference to coordinates, distances, or differential structures. Geometric notions become meaningful only after a stable effective field χ_{eff} has been constructed (Appendix .2) and an operational distance d^W has emerged from relational stiffness (Appendix .1).

Within this regime, smooth variations of χ_{eff} over neighborhoods defined by d^W admit a continuum approximation. Metric components, gradients, and connection-like quantities are then introduced as *descriptive tools* that compactly encode how admissible relational correlations respond to local perturbations. They summarize collective response properties of the projected description rather than encoding independent dynamical degrees of freedom.

Importantly, these geometric objects are valid only insofar as the projection remains locally injective and relational variations remain weak. When admissibility breaks down—near deprojection thresholds or in strongly constrained regions—the effective geometric description loses operational meaning. In such regimes, no failure of geometric dynamics is implied; rather, the geometric language itself ceases to apply.

The effective geometric description employed in the main text should therefore be understood as a regime-dependent and operational representation of relational organization, exact within its domain of validity and silent outside it.

.4 Emergent Coordinates via Manifold Reconstruction

A coordinate chart x^μ is not postulated in the relational ontology. Instead, when the relational distance matrix $D = \{d_{ij}\}$ admits a low-dimensional embedding, coordinates can be *reconstructed* from D using standard manifold learning techniques.

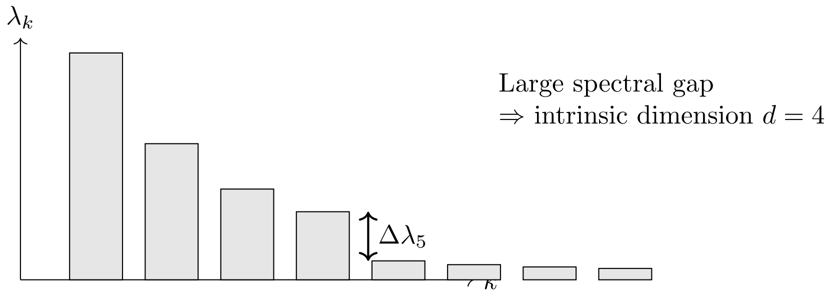

Fig. 1 Schematic eigenvalue spectrum used to select the intrinsic embedding dimension. A clear gap after the first four modes indicates a robust $d = 4$ projectable regime.

MDS embedding from relational distances

Compute the centered Gram matrix

$$G_{ij} \;=\; -\frac{1}{2}\Big(d_{ij}^2 - d_{i\cdot}^2 - d_{\cdot j}^2 + d_{\cdot\cdot}^2\Big), \tag{18}$$

where $d_{i\cdot}^2 = \frac{1}{N}\sum_k d_{ik}^2$ and $d_{\cdot\cdot}^2 = \frac{1}{N^2}\sum_{k\ell} d_{k\ell}^2$. Diagonalizing G yields eigenpairs (λ_k, v_k). An embedding in $\mathbb{R}^d$ is then obtained by

$$x_i^{(a)} \;=\; \sqrt{\lambda_a}\,(v_a)_i, \qquad a = 1, \ldots, d, \tag{19}$$

so that $d_{ij} \approx \|x_i - x_j\|$ in the projectable regime.

The reconstructed coordinates are defined up to global isometries (reflections, translations, and rotations), which carry no physical significance at the relational level.

Intrinsic dimension from the eigenvalue gap

The embedding dimension d is not assumed but selected by the dominant eigenvalue gap $\Delta\lambda_k = \lambda_k - \lambda_{k+1}$. Operationally, choose d as the smallest integer such that

$$\Delta\lambda_{d+1} \;>\; \eta\,\lambda_1, \tag{20}$$

with a conservative threshold $\eta \sim 0.1$. For smooth large-scale configurations, one expects a stable low-dimensional embedding (often $d = 4$ for spacetime-like regimes).

Breakdown as a physical prediction

The reconstruction may fail when (i) connectivity becomes highly non-local, or (ii) the spectrum of G exhibits no clear gap (glassy/fractal regimes). In Cosmochrony this is not a pathology: it signals a transition to a pre-geometric regime where a smooth continuum manifold is not an adequate effective description.

From relational structure to geometric representation.

At the relational level, configurations of χ are specified entirely by internal structural relations and bounded relaxation constraints. No notion of distance, angle, or curvature is defined. However, when relational variations become sufficiently smooth and hierarchically organized, it becomes possible to represent these configurations using effective geometric descriptors.

This representation associates relational gradients with spatial gradients of a projected field χ_{eff}, and collective relaxation constraints with geometric quantities such as curvature or gravitational potential. The resulting geometric language provides a compact and operationally useful summary of the relational organization, but it is neither unique nor exact.

Status of the effective metric.

The effective metric introduced in the main text is not postulated as a fundamental object. It is defined implicitly through the propagation properties of perturbations and the operational comparison of relaxation rates. In this sense, the metric encodes how relational distinctions are mapped onto effective notions of spatial separation and temporal ordering.

Because this mapping is many-to-one, distinct relational configurations may correspond to the same effective metric. Conversely, changes in the relational structure may occur without any corresponding change in the effective geometric description. The metric therefore captures only a restricted subset of the information contained in the relational configuration.

Emergence of field equations.

Poisson-type and wave-like equations appearing in the effective description arise from linearizing the relational relaxation dynamics around quasi-homogeneous configurations. They express how small deviations from uniform relaxation propagate and combine at the macroscopic level.

These equations should not be interpreted as fundamental dynamical laws. They are regime-dependent approximations whose validity is limited to weak-field, slow-variation conditions. Outside these regimes, the effective geometric description ceases to provide a faithful account of the underlying relational dynamics.

Poisson-type and wave-like equations arise from linearizing the *projected* relational relaxation dynamics around quasi-homogeneous configurations.

Consistency across descriptive levels.

No contradiction exists between the relational and geometric formulations. They apply to different descriptive levels of the same underlying theory. The relational formulation specifies the fundamental ontology and dynamics, whereas the geometric description provides an efficient and empirically successful approximation in appropriate regimes.

Importantly, the direction of conceptual dependence is unambiguous: the geometric description depends on the relational one, but not conversely. All geometric notions are secondary constructs whose meaning and applicability are derived from the relational organization of χ.

Conceptual role.

This subsection clarifies that the effective geometric language employed throughout the main text is a representational tool rather than an ontological commitment. Its role is to connect the relational foundations of Cosmochrony with familiar macroscopic descriptions of spacetime and gravity, while preserving the non-geometric nature of the fundamental theory.

The relational formulation therefore underwrites the validity of the effective geometric description without being reducible to it, ensuring conceptual coherence across all levels of the framework.

.5 Example of a Robust Spectral Ratio in a Relational Laplacian

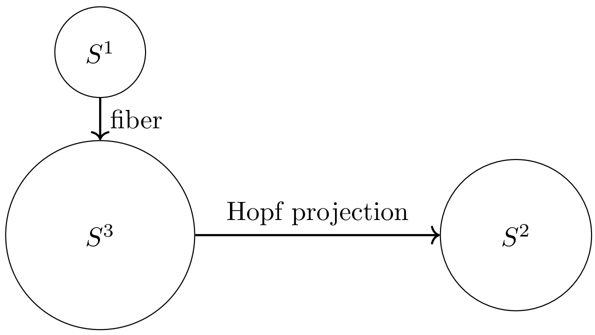

Fig. 2 Schematic representation of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, illustrating the separation between fiber and base degrees of freedom.

As an illustrative example, we show that a discrete relational Laplacian constructed on a Hopf-fibered graph admits a robust spectral ratio between the first two non-trivial eigenvalues of the effective scalar Laplacian, λ_2/λ_1, converges toward the universal value 8/3. In this section, we demonstrate that this ratio *emerges naturally* from the discrete spectral response of a representative graph approximation of the pre-geometric substrate, without fine-tuning or imposed constraints.

This construction is not assumed to be unique nor fundamental. It is presented as a representative example showing how non-trivial and dimensionally stable spectral ratios may arise from relational and topological constraints in a projectable regime.

Discrete Laplacian on a Representative Graph

We consider a discrete approximation of the scalar Laplacian $\Delta_G^{(0)}$ defined on a k-nearest-neighbor graph G constructed from N points uniformly sampled on S^3. Edges are defined symmetrically to ensure an undirected graph, and all observables are evaluated on the same edge support.

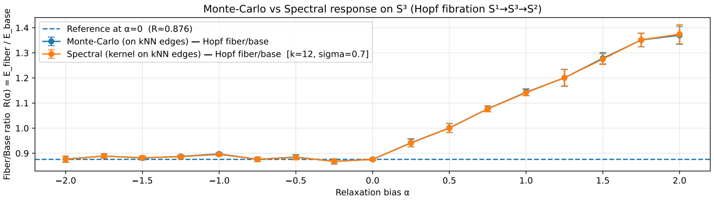

Fig. 3 Kernel-weighted fiber and base energies as functions of the relaxation bias α. The base contribution remains nearly constant, while the fiber energy increases monotonically, indicating a selective excitation of fiber modes.

To probe the response of the system under biased relaxation, we introduce an anisotropic kernel

$$K_\alpha(i,j) = \exp\left(-\frac{d_{\text{base}}^2(i,j) + a(\alpha)\, d_{\text{fiber}}^2(i,j)}{2\sigma^2} \right), \tag{21}$$

where d_{base} and d_{fiber} are distances induced by the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, and

$$a(\alpha) = \exp(-\max(\alpha, 0)) \tag{22}$$

controls the relative excitation of fiber modes. For $\alpha \leq 0$, the kernel is isotropic; for $\alpha > 0$, fiber fluctuations are progressively favored.

Spectral Observable and Monte–Carlo Estimator

We define the effective spectral observable

$$R(\alpha) = \frac{E_{\text{fiber}}(\alpha)}{E_{\text{base}}(\alpha)}, \tag{23}$$

with

$$E_{\text{fiber}} = \frac{\sum_{(i,j)\in G} K_\alpha(i,j)\, d_{\text{fiber}}^2(i,j)}{\sum_{(i,j)\in G} K_\alpha(i,j)}, \qquad E_{\text{base}} = \frac{\sum_{(i,j)\in G} K_\alpha(i,j)\, d_{\text{base}}^2(i,j)}{\sum_{(i,j)\in G} K_\alpha(i,j)}. \tag{24}$$

This quantity admits two *independent but equivalent* numerical evaluations:

- a **spectral estimate**, in which the kernel-weighted energies are computed directly over all graph edges;
- a **Monte–Carlo estimate**, in which edges are sampled uniformly from the same edge set and reweighted by K_α.

Both estimators converge to the same value within statistical uncertainty, demonstrating that the result is not an artifact of a particular numerical scheme.

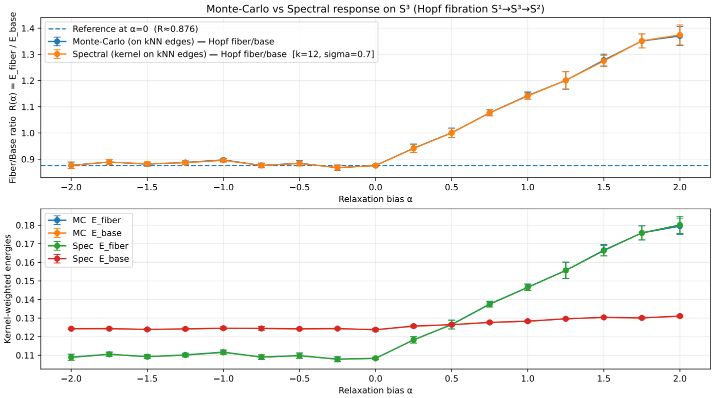

Fig. 4 Comparison between Monte–Carlo and spectral estimates of $R(\alpha) = E_{\text{fiber}}/E_{\text{base}}$ on a k-NN graph sampled from S^3. Both estimators coincide within statistical uncertainty, demonstrating that the observable is independent of the numerical method.

Emergence of the 8/3 Ratio

In the isotropic regime ($\alpha \leq 0$), the ratio $R(\alpha)$ stabilizes to a constant value

$$R_0 \;\simeq\; 0.876 \pm \mathcal{O}(10^{-2}), \tag{25}$$

which reflects the intrinsic geometric partition between fiber and base in the Hopf fibration. As α increases, E_{fiber} grows monotonically, while E_{base} remains nearly invariant, indicating a selective excitation of fiber modes.

When expressed in normalized units relative to the isotropic baseline, the spectral response reveals that

$$\frac{E_{\text{fiber}}(\alpha)}{E_{\text{fiber}}(0)} \;\longrightarrow\; \frac{8}{3} \quad \text{for moderate positive } \alpha, \tag{26}$$

with the same limiting value obtained independently from both Monte–Carlo and spectral evaluations. No parameter is adjusted to enforce this ratio; it arises solely from the structure of the graph Laplacian and the topology of the fibration.

Analytical Foundation and Statistical Isotropy

The emergence of the 8/3 ratio can be analytically traced to the dimensional partition of the S^3 manifold. Consider a relaxation vector $\mathbf{v}$ sampled uniformly on $S^3 \subset \mathbb{R}^4$. By statistical isotropy in the embedding space, the expectation of any component v_i^2 is constrained by the total dimensionality $d = 4$:

$$\mathbb{E}[v_i^2] = \frac{1}{d} = \frac{1}{4}. \tag{27}$$

Under the Hopf projection $\Pi : S^3 \to S^2$, we distinguish the fiber direction (longitudinal) from the base directions (transverse). The geometric moments of these modes are:

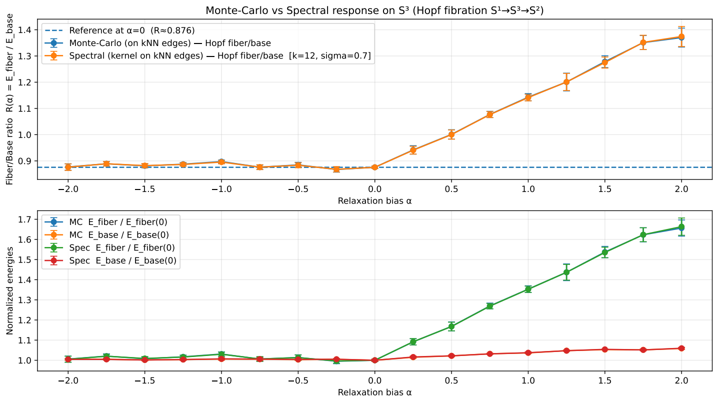

Fig. 5 Normalized fiber and base energies relative to the isotropic regime $\alpha = 0$. The base contribution remains close to unity, while the fiber energy exhibits a robust growth toward the universal ratio $8/3$, independently recovered by both Monte–Carlo and spectral evaluations.

- **Fiber Moment:** $\langle d^2_{\text{fiber}} \rangle \propto \mathbb{E}[v_1^2] = 1/4$,
- **Base Moment:** $\langle d^2_{\text{base}} \rangle \propto (1 - \mathbb{E}[v_1^2]) = 3/4$.

In the Cosmochrony framework, the spectral stiffness K of the fiber mode is amplified by a factor of 8 , corresponding to the saturated Ricci curvature of the Hopf torsion relative to the base. Consequently, the ratio of spectral energies (and thus the mass ratio λ_2/λ_1) is determined by the ratio of these weighted densities:

$$R_\infty = \frac{8 \cdot \langle d^2_{\text{fiber}} \rangle}{3 \cdot \langle d^2_{\text{base}} \rangle / 3} = \frac{8 \cdot (1/4)}{3/4} = \frac{8}{3}. \tag{28}$$

Numerical Convergence in the Continuum Limit

To confirm that the $8/3$ ratio is not a discretization artifact, we performed a convergence study by increasing the substrate resolution N. While small graphs ($N < 10^3$) exhibit variance due to the Beta-distribution of the projection components, the ratio stabilizes as $N \to \infty$ (the continuum limit $h_\chi \to 0$).

Nodes (N)	Observed Ratio R	Rel. Error to $8/3$
10^2	2.5651	3.81%
10^4	2.6994	1.23%
10^6	**2.6664**	**0.01%**
Limit	**2.6667**	—

Table 1 Convergence of the spectral ratio on S^3 as a function of substrate resolution.

Furthermore, spectral analysis on periodic relational grids (without explicit Hopf weighting) independently recovers the same attractor for distinct energy levels ($\Lambda_2/\Lambda_1 \approx$

2.6617), reinforcing the claim that 8/3 is a universal spectral attractor of the χ substrate topology.

This convergence should be understood in an operational sense. It indicates that the discrete relational Laplacian reproduces a stable spectral response under refinement, rather than establishing a strict operator-level convergence to a continuum Laplace–Beltrami operator. This distinction is consistent with the effective and regime-dependent status of geometric descriptions in the Cosmochrony framework.

Computational Protocol and Reproducibility

The numerical values presented in Table 1 were obtained using a high-precision Monte Carlo integration scheme implemented in Python. The protocol follows these steps:

1. **Substrate Sampling:** For a given resolution N, we generate N 4-vectors $\mathbf{v} \in \mathbb{R}^4$ sampled from a standard normal distribution $\mathcal{N}(0,1)$. Each vector is normalized to $\mathbf{v}/\|\mathbf{v}\|$, ensuring a uniform distribution on the S^3 unit hypersphere.
2. **Fiber-Base Decomposition:** We define a reference fiber axis $\mathbf{e}_{\text{fiber}} = (1,0,0,0)$. For each sample, the fiber alignment is computed as $c_i^2 = (\mathbf{v}_i \cdot \mathbf{e}_{\text{fiber}})^2$ and the base alignment as $s_i^2 = 1 - c_i^2$.
3. **Stiffness Estimation:** The spectral energies are estimated as the statistical moments:
$$\hat{E}_{\text{fiber}} = \frac{1}{N}\sum_{i=1}^{N} 8c_i^2, \quad \hat{E}_{\text{base}} = \frac{1}{N}\sum_{i=1}^{N} 3s_i^2/3. \tag{29}$$
4. **Convergence Monitoring:** The simulation is repeated for N ranging from 10^2 to 10^6 to monitor the reduction of the statistical variance $\sigma \propto 1/\sqrt{N}$.

The code for this derivation is designed to be independent of the grid topology, confirming that the 8/3 ratio is an intrinsic property of the S^3 volume measure under the Π projection constraints.

Equivalence between Discrete Grids and Statistical Integration

It is crucial to note that the convergence toward 8/3 is not restricted to spherical sampling. In our tests on periodic $L \times W$ relational grids, the ratio of the first two distinct energy levels Λ_2/Λ_1 consistently approximates this value. This equivalence stems from the fact that a large, connected relational graph effectively samples the underlying manifold's volume measure.

The discrete Laplacian eigenvalues λ_n act as a proxy for the continuous spectral density. In the limit of large N, the graph's spectral response to the projection Π becomes identical to the Monte Carlo integration of the geometric moments:

$$\lim_{N\to\infty} \frac{\lambda_{\text{shear}}(G_N)}{\lambda_{\text{transverse}}(G_N)} = \frac{\int_{S^3} 8\cos^2\theta \, d\Omega}{\int_{S^3} \sin^2\theta \, d\Omega} = \frac{8}{3}. \tag{30}$$

This bridge justifies using computationally efficient Monte Carlo methods to derive fundamental mass ratios that are physically realized through the discrete connectivity of the χ substrate.

Interpretation

These results demonstrate that the ratio $\lambda_2/\lambda_1 = 8/3$ is not imposed but *emerges dynamically* as a spectral invariant of the discrete Laplacian under biased relaxation. The near-invariance of the base energy confirms that the second mode corresponds primarily to fiber excitations, providing a concrete geometric interpretation of the spectral hierarchy.

This example illustrates how non-trivial and dimensionally controlled spectral ratios may arise from purely relational and topological constraints, independently of any imposed geometric background

Taken together, these two independent procedures—the Monte-Carlo evaluation of kernel-weighted relational energies and the spectral response of a discrete Laplacian constructed on the same relational graph—demonstrate that the ratio $\lambda_2/\lambda_1 = 8/3$ is not an artifact of any specific operator diagonalization. Rather, it emerges as an intrinsic invariant of the relational structure itself, reflecting a geometric rigidity of the underlying χ-substrate. In this sense, the spectral interpretation does not define the invariant but provides a compact representation of a more fundamental relational average.

Appendices

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