CHAPTER 13

GAMMA FUNCTION

The gamma function is probably the special function that occurs most frequently in the discussion of problems in physics. For integer values, as the factorial function, it appears in every Taylor expansion. As we shall later see, it also occurs frequently with half-integer arguments, and is needed for general nonintegral values in the expansion of many functions, e.g., Bessel functions of noninteger order.

It has been shown that the gamma function is one of a general class of functions that do not satisfy any differential equation with rational coefficients. Specifically, the gamma function is one of very few functions of mathematical physics that do not satisfy either the hypergeometric differential equation (Section 18.5) or the confluent hypergeometric equation (Section 18.6). Since most physical theories involve quantities governed by differential equations, the gamma function (by itself) does not usually describe a physical quantity of interest, but rather tends to appear as a factor in expansions of physically relevant quantities.

13.1 DEFINITIONS, PROPERTIES

At least three different convenient definitions of the gamma function are in common use. Our first task is to state these definitions, to develop some simple, direct consequences, and to show the equivalence of the three forms.

Infinite Limit (Euler)

The first definition, named after Euler, is

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \neq 0, -1, -2, -3, \ldots. \tag{13.1}$$

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This definition of $\Gamma(z)$ is useful in developing the Weierstrass infinite-product form of $\Gamma(z)$, Eq. (13.16), and in obtaining the derivative of $\ln\Gamma(z)$ (Section 13.2). Here and elsewhere in this chapter z may be either real or complex. Replacing z with $z+1$, we have

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)(z+3)\cdots(z+n+1)} n^{z+1}\\ &= \lim_{n\to\infty} \frac{nz}{z+n+1}\cdot\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^{z}\\ &= z\Gamma(z). \qquad (13.2)\end{aligned}$$

This is the basic functional relation for the gamma function. It should be noted that it is a **difference** equation.

Also, from the definition,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)} n = 1. \qquad (13.3)$$

Now, repeated application of Eq. (13.2) gives

$$\begin{aligned}&\Gamma(2)=1,\\ &\Gamma(3)=2\,\Gamma(2)=2,\\ &\Gamma(4)=3\,\Gamma(3)=2\cdot 3, \quad \text{etc.,}\end{aligned}$$

so

$$\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!. \qquad (13.4)$$

Definite Integral (Euler)

A second definition, also frequently called the Euler integral, and already presented in Table 1.2, is

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re e(z) > 0. \qquad (13.5)$$

The restriction on z is necessary to avoid divergence of the integral. When the gamma function does appear in physical problems, it is often in this form or some variation, such as

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt, \quad \Re e(z) > 0, \qquad (13.6)$$

or

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0. \qquad (13.7)$$

When $z = \frac{1}{2}$, Eq. (13.6) is just the Gauss error integral, and, cf. Eq. (1.148), we have the interesting result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{13.8}$$

Generalizations of Eq. (13.6), the Gaussian integrals, are considered in Exercise 13.1.10.

To show the equivalence of these two definitions, Eqs. (13.1) and (13.5), consider the function of two variables

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt, \quad \Re e(z) > 0, \tag{13.9}$$

with n a positive integer. This form was chosen because the exponential has the definition

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \equiv e^{-t}. \tag{13.10}$$

Inserting Eq. (13.10) into Eq. (13.9), we see that the infinite-n limit of $F(z,n)$ corresponds to $\Gamma(z)$ as given by Eq. (13.5):

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) = \int_0^\infty e^{-t} t^{z-1}\, dt \equiv \Gamma(z). \tag{13.11}$$

Our remaining task is to identify this limit also with Eq. (13.1).

Returning to $F(z,n)$, we evaluate it by carrying out successive integrations by parts. For convenience we make the substitution $u = t/n$. Then

$$F(z,n) = n^z \int_0^1 (1-u)^n u^{z-1}\, du. \tag{13.12}$$

The first integration by parts yields

$$\frac{F(z,n)}{n^z} = (1-u)^n \frac{u^z}{z}\bigg|_0^1 + \frac{n}{z}\int_0^1 (1-u)^{n-1} u^z\, du\,; \tag{13.13}$$

note that (because $z \neq 0$) the integrated part vanishes at both endpoints. Repeating this n times, with the integrated part vanishing at both endpoints each time, we finally get

$$\begin{aligned} F(z,n) &= n^z \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 u^{z+n-1}\, du \\ &= \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z. \end{aligned} \tag{13.14}$$

This is identical with the expression on the right side of Eq. (13.1). Hence

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) \equiv \Gamma(z),$$

where $\Gamma(z)$ is in the form given by Eq. (13.1), thereby completing the proof.

Infinite Product (Weierstrass)

The third definition (Weierstrass' form) is the infinite product

$$\frac{1}{\Gamma(z)} \equiv z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \tag{13.15}$$

where γ is the Euler-Mascheroni constant

$$\gamma = 0.5772156619 \cdots, \tag{13.16}$$

which was introduced as a limit in Eq. (1.13). Existence of the limit was the topic of Exercise 1.2.13.

This infinite-product form is useful for proving various properties of $\Gamma(z)$. It can be derived from the original definition, Eq. (13.1), by rewriting it as

$$\Gamma(z) = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)\cdots(z+n)} n^z = \lim_{n\to\infty} \frac{1}{z} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right)^{-1} n^z. \tag{13.17}$$

Taking the reciprocal of Eq. (13.17) and using

$$n^{-z} = e^{(-\ln n)z}, \tag{13.18}$$

we obtain

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} e^{(-\ln n)z} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right). \tag{13.19}$$

Multiplying and dividing the right-hand side of Eq. (13.19) by

$$\exp\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) z\right] = \prod_{m=1}^{n} e^{z/m}, \tag{13.20}$$

we get

$$\frac{1}{\Gamma(z)} = z \left\{ \lim_{n\to\infty} \exp\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) z\right]\right\}$$
$$\times \left[\lim_{n\to\infty} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right) e^{-z/m}\right]. \tag{13.21}$$

Comparing with Eq. (1.13), we see that the parenthesized quantity in the exponent approaches as a limit the Euler-Mascheroni constant, thereby confirming Eq. (13.15).

Functional Relations

In Eq. (13.2) we already obtained the most important functional relation for the gamma function,

$$\Gamma(z+1) = z\,\Gamma(z). \tag{13.22}$$

Viewed as a complex-valued function, this formula permits the extension to negative z of values obtained via numerical evaluation of the integral representation, Eq. (13.5). While the Euler limit formula already tells us that $\Gamma(z)$ is an analytic function for all z except 0, $-1, \ldots,$ stepwise extrapolation from the integral is a more efficient numerical approach.

The gamma function satisfies several other functional relations, of which one of the most interesting is the **reflection formula**,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}. \tag{13.23}$$

This relation connects (for nonintegral z) values of $\Gamma(z)$ that are related by reflection about the line $z = 1/2$.

One way to prove the reflection formula starts from the product of Euler integrals,

$$\begin{aligned}\Gamma(z+1)\Gamma(1-z) &= \int_0^\infty s^z e^{-s}\,ds \int_0^\infty t^{-z} e^{-t}\,dt \\ &= \int_0^\infty \frac{v^z\,dv}{(v+1)^2} \int_0^\infty u\,e^{-u}\,du.\end{aligned} \tag{13.24}$$

In obtaining the second line of Eq. (13.24) we transformed from the variables s, t to $u = s + t$, $v = s/t$, as suggested by combining the exponentials and the powers in the integrands. We also needed to insert the Jacobian of this transformation,

$$J^{-1} = -\begin{vmatrix} 1 & 1 \\ \dfrac{1}{t} & -\dfrac{s}{t^2} \end{vmatrix} = \frac{s+t}{t^2} = \frac{(v+1)^2}{u};$$

the final substitution becomes obvious if we note that $v + 1 = u/t$.

Returning to Eq. (13.24), the u integration is elementary, being equal to 1!, while the v integration can be evaluated by contour-integration methods; it was the topic of Exercise 11.8.20, and has the value

$$\int_0^\infty \frac{v^z\,dv}{(v+1)^2} = \frac{\pi z}{\sin \pi z}. \tag{13.25}$$

Using these results, and then replacing $\Gamma(z+1)$ in Eq. (13.24) by $z\Gamma(z)$ and canceling z from the two sides of the resulting equation, we complete the demonstration of Eq. (13.23).

A special case of Eq. (13.23) results if we set $z = 1/2$. Then (taking the positive square root), we get

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \tag{13.26}$$

in agreement with Eq. (13.8).

Another functional relation is **Legendre's duplication formula**,

$$\Gamma(1+z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\;\Gamma(2z+1), \tag{13.27}$$

which we prove for general z in Section 13.3. However, it is instructive to prove it now for integer values of z. Assuming z to be a nonnegative integer n, we start the proof by writing $\Gamma(n+1) = n!$, $\Gamma(2n+1) = (2n)!$, and

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\left[\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right] = \sqrt{\pi}\,\frac{1\cdot 3\cdots(2n-1)}{2^n} = \sqrt{\pi}\,\frac{(2n-1)!!}{2^n}, \tag{13.28}$$

where we have used Eq. (13.26) and the double factorial notation first introduced in Eqs. (1.75) and (1.76). The double factorial notation is used frequently enough in physics applications that a familiarity with it is essential, and will from here on be used without comment. Making the further observation that $n! = 2^{-n}(2n)!!$, Eq. (13.27) follows directly.

Incidentally, we call attention to the fact that gamma functions with half-integer arguments appear frequently in physics problems, and Eq. (13.28) shows how to write them in closed form.

Analytic Properties

The Weierstrass definition shows immediately that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \ldots$ and that $[\Gamma(z)]^{-1}$ has no poles in the finite complex plane, which means that $\Gamma(z)$ has no zeros. This behavior may also be seen in Eq. (13.23), if we note that $\pi/(\sin\pi z)$ is never equal to zero. A plot of $\Gamma(z)$ for real z is shown in Fig. 13.1. We note sign changes for each unit interval of negative z, that $\Gamma(1) = \Gamma(2) = 1$, and that the gamma function has a minimum between $z = 1$ and $z = 2$, at $z_0 = 0.46143\ldots$, with $\Gamma(z_0) = 0.88560\ldots$. The residues R_n at the poles $z = -n$ (n an integer ≥ 0) are

$$\begin{aligned} R_n &= \lim_{\varepsilon\to 0}\Big(\varepsilon\Gamma(-n+\varepsilon)\Big) = \lim_{\varepsilon\to 0}\frac{\varepsilon\,\Gamma(-n+1+\varepsilon)}{-n+\varepsilon} = \lim_{\varepsilon\to 0}\frac{\varepsilon\,\Gamma(-n+2+\varepsilon)}{(-n+\varepsilon)(-n+1+\varepsilon)} \\ &= \lim_{\varepsilon\to 0}\frac{\varepsilon\Gamma(1+\varepsilon)}{(-n+\varepsilon)\cdots(\varepsilon)} = \frac{(-1)^n}{n!}, \end{aligned} \tag{13.29}$$

showing that the residues alternate in sign, with that at $z = -n$ having magnitude $1/n!$.

Schlaefli Integral

A contour integral representation of the gamma function that we will find useful in developing asymptotic series for the Bessel functions is the **Schlaefli integral**

$$\int_C e^{-t}t^{\nu}\,dt = (e^{2\pi i\nu}-1)\,\Gamma(\nu+1), \tag{13.30}$$

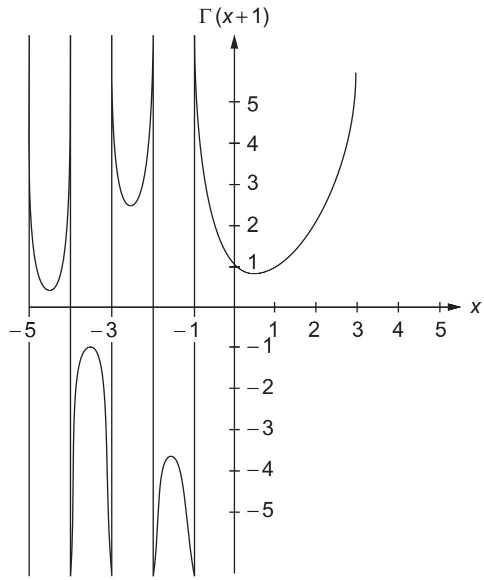

FIGURE 13.1 Gamma function $\Gamma(x+1)$ for real x.

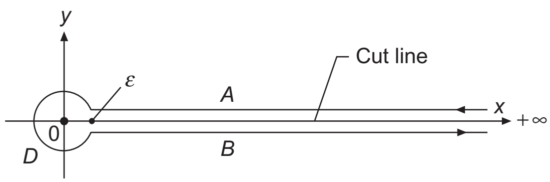

FIGURE 13.2 Gamma function contour.

where C is the contour shown in Fig. 13.2. This contour integral representation is only useful when ν is not an integer. For integer ν, the integrand is an entire function; both sides of Eq. (13.30) vanish and it yields no information. However, for noninteger ν, $t=0$ is a branch point of the integrand and the right-hand side of Eq. (13.30) then evaluates to a nonzero result. Note that, unlike the contour representations we considered in earlier chapters, the present contour is open; we cannot close it at $z=+\infty$ because of the branch cut, nor can we close it with a large circle, as e^{-t} becomes infinite in the limit of large negative t.

To verify Eq. (13.30), we proceed (for $\nu+1>0$) by evaluating the contributions from the various parts of the integration path. The integral from ∞ to $+\varepsilon$ on the real axis yields $-\Gamma(\nu+1)$, choosing $\arg(z)=0$. The integral $+\varepsilon$ to ∞ (in the fourth quadrant) then yields $e^{2\pi i\nu}\Gamma(\nu+1)$, the argument of z having increased to 2π. Since the circle around the origin contributes nothing when $\nu>-1$, Eq. (13.30) follows. Now that this equation is established, we can deform the contour as desired (providing that we avoid the branch point and cut), since there are no other singularities we must avoid.

It is often convenient to cast Eq. (13.30) into the more symmetrical form

$$\int_C e^{-t} t^\nu \, dt = 2i e^{i\nu\pi} \Gamma(\nu+1) \sin(\nu\pi), \tag{13.31}$$

where C can be the contour of Fig. 13.2 or any deformation thereof that encircles the origin, does not cross the branch cut, and begins and ends at any points respectively above and below the cut for which $x = +\infty$.

The above analysis establishes Eqs. (13.30) and (13.31) for $\nu > -1$. However, we note that the integral exists for $\nu < -1$ as long as we stay away from the origin, and therefore it remains valid for all nonintegral ν. What we have found is that this contour integral representation provides an analytic continuation of the Euler integral, Eq. (13.5), to all nonintegral ν.

Factorial Notation

Our discussion of the gamma function has been presented in terms of the classical notation, which was first introduced by Legendre. In an attempt to make a closer correspondence to the factorial notation (traditionally used for integers), and to simplify the Euler integral representation of the gamma function, Eq. (13.5), some authors have chosen to use the notation $z!$ as a synonym for $\Gamma(z+1)$ even when z has an arbitrary complex value. Occasionally one even encounters Gauss' notation, $\prod(z)$, for the factorial function:

$$\prod(z) = z! = \Gamma(z+1).$$

Neither the factorial (for nonintegral arguments) nor the Gauss notation are currently favored by most serious investigators, and we will not use them in this book.

Example 13.1.1 Maxwell-Boltzmann Distribution

In classical statistical mechanics, a state of energy E is occupied, according to the equation of Maxwell-Boltzmann statistics, with a probability proportional to $e^{-E/kT}$, where k is Boltzmann's constant and T is the absolute temperature; it is usual to define $\beta = 1/kT$ and to write the probability of occupancy of a state of energy E as $p(E) = Ce^{-\beta E}$. If the number of states in a small energy interval dE at energy E is given, using a density distribution function $n(E)$, as $n(E)\,dE$, then the total probability of states at energy E assumes the form $C\,n(E)\,e^{-\beta E}\,dE$. Under those conditions, the total probability of occupancy in **any** state (namely, unity) must be

$$1 = C \int n(E)\, e^{-\beta E}\, dE, \tag{13.32}$$

which enables us to set the **normalization constant** C, and the average energy $\langle E \rangle$ of such a classical system will be

$$\langle E \rangle = C \int E\, n(E)\, e^{-\beta E}\, dE. \tag{13.33}$$

For a structureless ideal gas, it can be shown that $n(E)$ is proportional to $E^{1/2}$, with E, the kinetic energy of a gas molecule, in the range $(0, \infty)$. Then we may find C from

$$1 = C \int_0^\infty E^{1/2} e^{-\beta E}\, dE = C\,\frac{\Gamma(\frac{3}{2})}{\beta^{3/2}} = C\,\frac{\sqrt{\pi}}{2\beta^{3/2}}, \quad \text{or} \;\; C = \frac{2\,\beta^{3/2}}{\sqrt{\pi}},$$

and

$$\langle E \rangle = C \int_0^\infty E^{3/2} e^{-\beta E}\, dE = C\,\frac{\Gamma(\frac{5}{2})}{\beta^{5/2}} = \left(\frac{2\,\beta^{3/2}}{\sqrt{\pi}}\right)\frac{\sqrt{\pi}}{\beta^{5/2}}\left(\frac{1}{2}\cdot\frac{3}{2}\right) = \frac{3}{2}\,kT,$$

the known value of the average kinetic energy per molecule for a structureless classical gas at temperature T.

In probability theory, the distribution used here is known as a **gamma distribution**; it is further discussed in Chapter 23. ■

Exercises

13.1.1 Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

13.1.2 In a power-series solution for the Legendre functions of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)},$$

in which s is a positive integer.

(a) Rewrite this expression in terms of factorials.

(b) Rewrite this expression using Pochhammer symbols; see Eq. (1.72).

13.1.3 Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt, \quad \Re e(z) > 0,$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0.$$

13.1.4 In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v + dv$ is

$$\frac{dN}{N} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^2\, dv,$$

where N is the total number of particles, k is Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $\langle v^n \rangle = N^{-1} \int v^n dN$. Show that

$$\langle v^n \rangle = \left(\frac{2kT}{m}\right)^{n/2} \frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{3}{2})}.$$

This is an extension of Example 13.1.1, in which the distribution was in kinetic energy $E = mv^2/2$, with $dE = mv\, dv$.

13.1.5 By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x\, dx = \frac{1}{(k+1)^2}, \quad k > -1.$$

13.1.6 Show that

$$\int_0^\infty e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right).$$

13.1.7 Show that

$$\lim_{x \to 0} \frac{\Gamma(ax)}{\Gamma(x)} = \frac{1}{a}.$$

13.1.8 Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

13.1.9 Show that the equation $\Gamma(x) = k$, $k \neq 0$, has an infinite number of real roots.

13.1.10 Show that, for integer s,

(a) $$\int_0^\infty x^{2s+1} \exp(-ax^2) dx = \frac{s!}{2a^{s+1}}.$$

(b) $$\int_0^\infty x^{2s} \exp(-ax^2) dx = \frac{\Gamma(s+\frac{1}{2})}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s} \sqrt{\frac{\pi}{a}}.$$

These Gaussian integrals are of major importance in statistical mechanics.

13.1.11 Express the coefficient of the nth term of the expansion of $(1+x)^{1/2}$ in powers of x

(a) in terms of factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^{n+1} \frac{(2n-3)!}{2^{2n-2} n!\,(n-2)!} = (-1)^{n+1} \frac{(2n-3)!!}{(2n)!!}, n = 2, 3, \ldots.$$

13.1.12 Express the coefficient of the nth term of the expansion of $(1+x)^{-1/2}$ in powers of x

(a) in terms of the factorials of integers,

(b) in terms of the double factorial (!!) functions.

$$ANS. \quad a_n = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad n = 1, 2, 3 \ldots.$$

13.1.13 The Legendre polynomial P_n may be written as

$$P_n(\cos\theta) = 2\frac{(2n-1)!!}{(2n)!!} \left\{ \cos n\theta + \frac{1}{1} \cdot \frac{n}{2n-1} \cos(n-2)\theta \right.$$
$$+ \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\theta$$
$$\left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\theta + \cdots \right\}.$$

Let $n = 2s+1$. Then the above can be written

$$P_n(\cos\theta) = P_{2s+1}(\cos\theta) = \sum_{m=0}^{s} a_m \cos(2m+1)\theta.$$

Find a_m in terms of factorials and double factorials.

13.1.14 (a) Show that $\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{1}{2} + n\right) = (-1)^n \pi$, where n is an integer.

(b) Express $\Gamma\left(\frac{1}{2} + n\right)$ and $\Gamma(\frac{1}{2} - n)$ separately in terms of $\pi^{1/2}$ and a double factorial function.

$$ANS. \quad \Gamma(\tfrac{1}{2} + n) = \frac{(2n-1)!!}{2^n} \pi^{1/2}.$$

13.1.15 Show that if $\Gamma(x+iy) = u+iv$, then $\Gamma(x-iy) = u-iv$.

This is a special case of the Schwarz reflection principle, Section 11.10.

13.1.16 Prove that $|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty} \left[1 + \frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}$.

This equation has been useful in calculations of beta decay theory.

13.1.17 Show that for n, a positive integer,

$$|\Gamma(n+ib+1)| = \left(\frac{\pi b}{\sinh \pi b}\right)^{1/2} \prod_{s=1}^{n} (s^2+b^2)^{1/2}.$$

13.1.18 Show that for all real values of x and y, $|\Gamma(x)| \geq |\Gamma(x+iy)|$.

13.1.19 Show that $|(\Gamma(\frac{1}{2}+iy)|^2 = \dfrac{\pi}{\cosh \pi y}$.

13.1.20 The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],$$

with $(-\infty, \infty)$ for the range of x. Show that

(a) $\langle x \rangle$, the mean value of x, is equal to μ,

(b) the standard deviation $(\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ is given by σ.

13.1.21 For the gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

show that

(a) $\langle x \rangle$, the mean value of x, is equal to $\alpha\beta$,

(b) σ^2, its variance, defined as $\langle x^2 \rangle - \langle x \rangle^2$, has the value $\alpha\beta^2$.

13.1.22 The wave function of a particle scattered by a Coulomb potential is $\psi(r, \theta)$. Given that at the origin the wave function becomes

$$\psi(0) = e^{-\pi\gamma/2}\,\Gamma(1 + i\gamma),$$

where $\gamma > 0$ is a dimensionless parameter, show that

$$|\psi(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1}.$$

13.1.23 Derive the contour integral representation of Eq. (13.31),

$$2i\Gamma(\nu + 1)\sin\nu\pi = \int_C e^{-t}(-t)^\nu\, dt.$$

13.2 Digamma and Polygamma Functions

Digamma Function

As may be noted from the three definitions in Section 13.1, it is inconvenient to deal with the derivatives of the gamma function directly. It is more productive to take the natural logarithm of the gamma function as given by Eq. (13.1), thereby converting the product to a sum, and then to differentiate. The most useful results are obtained if we start with $\Gamma(z + 1)$:

$$\Gamma(z+1) = z\Gamma(z) = \lim_{n\to\infty} \frac{n!}{(z+1)(z+2)\cdots(z+n)} n^z, \tag{13.34}$$

$$\ln\Gamma(z+1) = \lim_{n\to\infty}\Big[\ln(n!) + z\ln n - \ln(z+1) - \ln(z+2) - \cdots - \ln(z+n)\Big], \tag{13.35}$$

in which the logarithm of the limit is equal to the limit of the logarithm. Differentiating with respect to z, we obtain

$$\frac{d}{dz}\ln\Gamma(z+1) \equiv \psi(z+1) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n}\right), \tag{13.36}$$

which defines $\psi(z+1)$, the **digamma function**. Note that this definition also corresponds to

$$\psi(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}. \tag{13.37}$$

To bring Eq. (13.36) to a better form, we add and subtract the harmonic number

$$H_n = \sum_{m=1}^{n} \frac{1}{m},$$

thereby obtaining

$$\begin{aligned}\psi(z+1) &= \lim_{n\to\infty}\left[(\ln n - H_n) - \sum_{m=1}^{n}\left(\frac{1}{z+m} - \frac{1}{m}\right)\right] \\ &= -\gamma + \sum_{m=1}^{\infty}\frac{z}{m(m+z)}.\end{aligned} \tag{13.38}$$

We have now arranged the contributions in a way that causes each group of terms to approach a finite limit as $n \to \infty$: in that limit $\ln n - H_n$ became (minus) the Euler-Mascheroni constant, defined in Eq. (1.13), and the summation is convergent.

Setting $z = 0$, we find[1]

$$\psi(1) = -\gamma = -0.577\,215\,664\,901\cdots. \tag{13.39}$$

For integer $n > 0$, Eq. (13.38) reduces to a form that is good for revealing its structure but less desirable for actual computation:

$$\psi(n+1) = -\gamma + H_n = -\gamma + \sum_{m=1}^{n}\frac{1}{m}. \tag{13.40}$$

[1] γ has been computed to 1271 places by D. E. Knuth, *Math. Comput.* **16**: 275 (1962), and to 3566 decimal places by D. W. Sweeney, *ibid.* **17**: 170 (1963). It may be of interest that the fraction $228/395$ gives γ accurate to six places.

Polygamma Function

The digamma function may be differentiated repeatedly, giving rise to the polygamma function:

$$\psi^{(m)}(z+1) \equiv \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z+1)$$

$$= (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}, \quad m = 1, 2, 3, \ldots. \tag{13.41}$$

Plots of $\Gamma(x)$, $\psi(x)$, and $\psi'(x)$ are presented in Fig. 13.3.

If we set $z = 0$ in Eq. (13.41), the series in that equation is that defining the Riemann zeta function,[2]

$$\zeta(m) \equiv \sum_{n=1}^{\infty} \frac{1}{n^m}, \tag{13.42}$$

and we have

$$\psi^{(m)}(1) = (-1)^{m+1} m! \zeta(m+1), \quad m = 1, 2, 3, \ldots. \tag{13.43}$$

The values of polygamma functions of the positive integral argument, $\psi^{(m)}(n+1)$, may be calculated recursively; see Exercise 13.2.8.

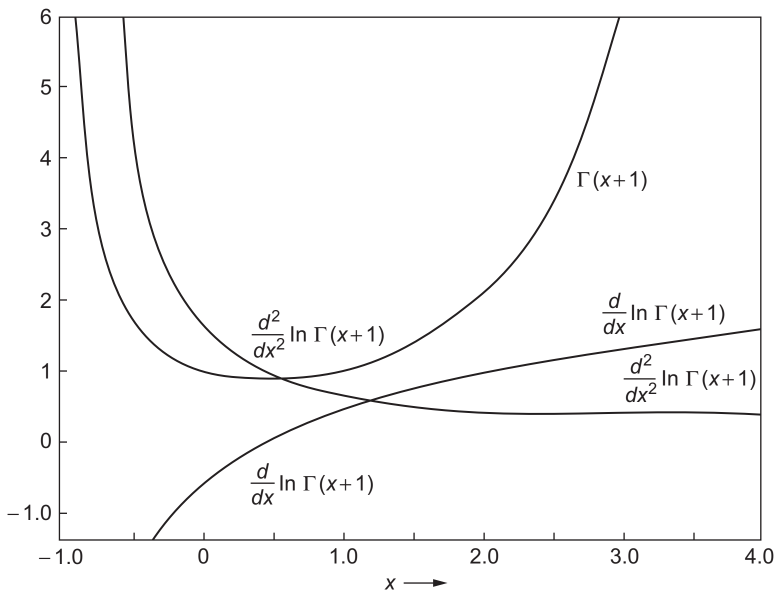

FIGURE 13.3 Gamma function and its first two logarithmic derivatives.

[2] For $z \neq 0$ this series has been used to define a generalization of $\zeta(m)$ known as the **Hurwitz zeta function**.

Maclaurin Expansion

It is now possible to write a Maclaurin expansion for $\ln\Gamma(z+1)$:

$$\ln\Gamma(z+1) = \sum_{n=1}^{\infty} \frac{z^n}{n!}\,\psi^{(n-1)}(1) = -\gamma z + \sum_{n=2}^{\infty}(-1)^n \frac{z^n}{n}\,\zeta(n). \qquad (13.44)$$

This expansion is convergent for $|z| < 1$; for $z = x$, the range is $-1 < x \le 1$. Alternate forms of this series appear in Exercise 13.2.2. Equation (13.44) is a possible means of computing $\Gamma(z+1)$ for real or complex z, but Stirling's series (Section 13.4) is usually better, and in addition, an excellent table of values of the gamma function for complex arguments based on the use of Stirling's series and the functional relation, Eq. (13.22), is now available.[3]

Series Summation

The digamma and polygamma functions may also be used in summing series. If the general term of the series has the form of a rational fraction (with the highest power of the index in the numerator at least two less than the highest power of the index in the denominator), it may be transformed by the method of partial fractions; see Eq. (1.83). This transformation permits the infinite series to be expressed as a finite sum of digamma and polygamma functions. The usefulness of this method depends on the availability of tables of digamma and polygamma functions. Such tables and examples of series summation are given in AMS-55, chapter 6 (see Additional Readings for the reference).

Example 13.2.1 Catalan's Constant

Catalan's constant, $\beta(2)$, Eq. (12.65), is given by

$$K = \beta(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Grouping the positive and negative terms separately and starting with the unit index, to match the form of $\psi^{(1)}$, Eq. (13.41), we obtain

$$K = 1 + \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} - \frac{1}{9} - \sum_{n=1}^{\infty} \frac{1}{(4n+3)^2}.$$

Now, identifying the summations in terms of $\psi^{(1)}$, we get

$$K = \frac{8}{9} + \frac{1}{16}\psi^{(1)}\left(1+\frac{1}{4}\right) - \frac{1}{16}\psi^{(1)}\left(1+\frac{3}{4}\right).$$

[3] *Table of the Gamma Function for Complex Arguments*, Applied Mathematics Series No. 34. Washington, DC: National Bureau of Standards (1954).

Using the values of $\psi^{(1)}$ from Table 6.1 of AMS-55 (see Additional Readings for the reference), we obtain

$$K = 0.91596559\ldots.$$

Compare this calculation of Catalan's constant with those carried out in earlier chapters (Exercises 1.1.12 and 12.4.4). ■

Exercises

13.2.1 For "small" values of x,

$$\ln\Gamma(x+1) = -\gamma x + \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} x^n,$$

where γ is the Euler-Mascheroni constant and $\zeta(n)$ the Riemann zeta function. For what values of x does this series converge?

ANS. $-1 < x \leq 1$.

Note that if $x = 1$, we obtain

$$\gamma = \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n},$$

a series for the Euler-Mascheroni constant. The convergence of this series is exceedingly slow. For actual computation of γ, other, indirect, approaches are far superior (see Exercise 12.3.2).

13.2.2 Show that the series expansion of $\ln\Gamma(x+1)$ (Exercise 13.2.1) may be written as

(a) $$\ln\Gamma(x+1) = \frac{1}{2}\ln\left(\frac{\pi x}{\sin\pi x}\right) - \gamma x - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}x^{2n+1},$$

(b) $$\ln\Gamma(x+1) = \frac{1}{2}\ln\left(\frac{\pi x}{\sin\pi x}\right) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + (1-\gamma)x$$
$$-\sum_{n=1}^{\infty}\left[\zeta(2n+1) - 1\right]\frac{x^{2n+1}}{2n+1}.$$

Determine the range of convergence of each of these expressions.

13.2.3 Verify that for n, a positive integer, the following two forms of the digamma function are equal to each other:

$$\psi(n+1) = \sum_{j=1}^{n}\frac{1}{j} - \gamma \quad \text{and} \quad \psi(n+1) = \sum_{j=1}^{\infty}\frac{n}{j(n+j)} - \gamma.$$

13.2.4 Show that $\psi(z+1)$ has the series expansion

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n)\, z^{n-1}.$$

13.2.5 For a power-series expansion of $\ln\Gamma(z+1)$, AMS-55 (see Additional Readings for the reference) lists

$$\ln\Gamma(z+1) = -\ln(1+z) + z(1-\gamma) + \sum_{n=2}^{\infty} (-1)^n \Big[\zeta(n) - 1\Big] \frac{z^n}{n}.$$

(a) Show that this agrees with Eq. (13.44) for $|z| < 1$.

(b) What is the range of convergence of this new expression?

13.2.6 Show that

$$\frac{1}{2}\ln\left(\frac{\pi z}{\sin\pi z}\right) = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \qquad |z| < 1.$$

Hint. Use Eqs. (13.23) and (13.35).

13.2.7 Write out a Weierstrass infinite-product definition of $\ln\Gamma(z+1)$. Without differentiating, show that this leads directly to the Maclaurin expansion of $\ln\Gamma(z+1)$, Eq. (13.44).

13.2.8 Derive the difference relation for the polygamma function,

$$\psi^{(m)}(z+2) = \psi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}}, \qquad m = 0, 1, 2, \ldots.$$

13.2.9 The Pochhammer symbol $(a)_n$ is defined (for integral n) as

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1.$$

(a) Express $(a)_n$ in terms of factorials.

(b) Find $(d/da)(a)_n$ in terms of $(a)_n$ and digamma functions.

$$ANS. \quad \frac{d}{da}(a)_n = (a)_n[\psi(a+n) - \psi(a)].$$

(c) Show that

$$(a)_{n+k} = (a+n)_k \cdot (a)_n.$$

13.2.10 Verify the following special values of the ψ form of the digamma and polygamma functions:

$$\psi(1) = -\gamma, \quad \psi^{(1)}(1) = \zeta(2), \quad \psi^{(2)}(1) = -2\zeta(3).$$

13.2.11 Verify:

(a) $$\int_0^\infty e^{-r} \ln r \, dr = -\gamma.$$

(b) $$\int_0^\infty re^{-r} \ln r \, dr = 1 - \gamma.$$

(c) $$\int_0^\infty r^n e^{-r} \ln r \, dr = (n-1)! + n \int_0^\infty r^{n-1} e^{-r} \ln r \, dr, \quad n = 1, 2, 3, \ldots.$$

Hint. These may be verified by integration by parts, or by differentiating the Euler integral formula for $\Gamma(n+1)$ with respect to n.

13.2.12 Dirac relativistic wave functions for hydrogen involve factors such as $\Gamma[2(1 - \alpha^2 Z^2)^{1/2} + 1]$ where α, the fine structure constant, is 1/137 and Z is the atomic number. Expand $\Gamma[2(1 - \alpha^2 Z^2)^{1/2} + 1]$ in a series of powers of $\alpha^2 Z^2$.

13.2.13 The quantum mechanical description of a particle in a Coulomb field requires a knowledge of the argument of $\Gamma(z)$ when z is complex. Determine the argument of $\Gamma(1 + ib)$ for small, real b.

13.2.14 Using digamma and polygamma functions, sum the series

(a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n(n+1)}$, (b) $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2 - 1}$.

Note. You can use Exercise 13.2.8 to calculate the needed digamma functions.

13.2.15 Show that

$$\sum_{n=1}^\infty \frac{1}{(n+a)(n+b)} = \frac{1}{(b-a)} \Big[\psi(1+b) - \psi(1+a) \Big],$$

where $a \neq b$, and neither a nor b is a negative integer. It is of some interest to compare this summation with the corresponding integral,

$$\int_1^\infty \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a} \Big[\ln(1+b) - \ln(1+a) \Big].$$

The relation between $\psi(x)$ and $\ln x$ is made explicit in the analysis leading to Stirling's formula.

13.3 The Beta Function

Products of gamma functions can be identified as describing an important class of definite integrals involving powers of sine and cosine functions, and these integrals, in turn, can be further manipulated to evaluate a large number of algebraic definite integrals. These properties make it useful to define the **beta function**, defined as

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,. \tag{13.45}$$

For whatever it is worth, note that the B in Eq. (13.45) is an upper-case beta.

To understand the virtue of this definition, let us write the product $\Gamma(p)\Gamma(q)$ using the integral representation given as Eq. (13.6), valid for $\Re e(p)$, $\Re e(q) > 0$:

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty s^{2p-1}e^{-s^2}\,ds \int_0^\infty t^{2q-1}e^{-t^2}\,dt. \tag{13.46}$$

The reason for using this integral representation is that the quadratic terms in the exponent, s^2 and t^2, combine in a convenient way if we change the integration variables from s,t to polar coordinates r,θ, with $s = r\cos\theta$, $t = r\sin\theta$, $r^2 = s^2 + t^2$, and $ds\,dt = r\,dr\,d\theta$. Equation (13.46) becomes

$$\begin{aligned}\Gamma(p)\,\Gamma(q) &= 4\int_0^\infty r^{2p+2q-1}e^{-r^2}\,dr \int_0^{\pi/2} \cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta\\ &= 2\,\Gamma(p+q)\int_0^{\pi/2} \cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta,\end{aligned}$$

where we have used Eq. (13.6) to recognize the r integration as $\Gamma(p+q)$. This gives us our first integral evaluation based on the beta function:

$$B(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta. \tag{13.47}$$

Because Eq. (13.47) is often used when p and q are integers, we rewrite for the case $p = m+1$, $q = n+1$,

$$\frac{m!\,n!}{(m+n+1)!} = 2\int_0^{\pi/2} \cos^{2m+1}\theta\,\sin^{2n+1}\theta\,d\theta. \tag{13.48}$$

Because gamma functions of a half-integral argument are available in closed form, Eq. (13.47) also provides a route to these trigonometric integrals for even powers of the sine and/or cosine. Note also that from its definition it is obvious that $B(p,q) = B(q,p)$, showing that the integral in Eq. (13.47) does not change in value if the powers of the sine and cosine are interchanged.

Alternate Forms, Definite Integrals

The substitution $t = \cos^2\theta$ converts Eq. (13.47) to

$$B(p+1, q+1) = \int_0^1 t^p (1-t)^q \, dt. \tag{13.49}$$

Replacing t by x^2, we obtain

$$B(p=1, q+1) = 2\int_0^1 x^{2p+1}(1-x^2)^q \, dx. \tag{13.50}$$

The substitution $t = u/(1+u)$ in Eq. (13.49) yields still another useful form,

$$B(p+1, q+1) = \int_0^\infty \frac{u^p}{(1+u)^{p+q+2}} \, du. \tag{13.51}$$

The beta function as a definite integral is useful in establishing integral representations of the Bessel function (Exercise 14.1.17) and the hypergeometric function (Exercise 18.5.12).

Derivation of Legendre Duplication Formula

The Legendre duplication formula involves products of gamma functions, which suggests that the beta function may provide a useful route to its proof. We start by using Eq. (13.49) for $B\left(z+\frac{1}{2}, z+\frac{1}{2}\right)$:

$$B\left(z+\frac{1}{2}, z+\frac{1}{2}\right) = \int_0^1 t^{z-1/2}(1-t)^{z-1/2} \, dt. \tag{13.52}$$

Making the substitution $t = (1+s)/2$, we have

$$\begin{aligned} B\left(z+\frac{1}{2}, z+\frac{1}{2}\right) &= 2^{-2z}\int_{-1}^1 (1-s^2)^{z-1/2} \, ds \\ &= 2^{-2z+1}\int_0^1 (1-s^2)^{z-1/2} \, ds = 2^{-2z} B\left(\frac{1}{2}, z+\frac{1}{2}\right), \end{aligned} \tag{13.53}$$

where we used the fact that the s integrand was even to change the integration range to $(0, 1)$, and then used Eq. (13.50) to evaluate the resulting integral. Now, inserting the definition, Eq. (13.45), for both instances of B in Eq. (13.53), we reach

$$\frac{\Gamma(z+\frac{1}{2})\,\Gamma(z+\frac{1}{2})}{\Gamma(2z+1)} = 2^{-2z}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(z+\frac{1}{2})}{\Gamma(z+1)},$$

which is easily rearranged into

$$\Gamma(z+1)\Gamma\left(z+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2z}}\,\Gamma(2z+1), \tag{13.54}$$

the Legendre duplication formula, originally introduced as Eq. (13.27), but proved at that time only for integer values of z.

Although the integrals used in this derivation are defined only for $\Re e(z)>-1$, the result, Eq. (13.54), holds, by analytic continuation, for all z where the gamma functions are analytic.

Exercises

13.3.1 Verify the following beta function identities:

(a) $B(a,b)=B(a+1,b)+B(a,b+1)$,

(b) $B(a,b)=\dfrac{a+b}{b}B(a,b+1)$,

(c) $B(a,b)=\dfrac{b-1}{a}B(a+1,b-1)$,

(d) $B(a,b)B(a+b,c)=B(b,c)B(a,b+c)$.

13.3.2 (a) Show that

$$\int_{-1}^{1}(1-x^2)^{1/2}x^{2n}\,dx=\begin{cases}\pi/2, & n=0\\ \pi\dfrac{(2n-1)!!}{(2n+2)!!}, & n=1,2,3,\ldots.\end{cases}$$

(b) Show that

$$\int_{-1}^{1}(1-x^2)^{-1/2}x^{2n}\,dx=\begin{cases}\pi, & n=0,\\ \pi\dfrac{(2n-1)!!}{(2n)!!}, & n=1,2,3,\ldots.\end{cases}$$

13.3.3 Show that

$$\int_{-1}^{1}(1-x^2)^{n}\,dx=\frac{2\,(2n)!!}{(2n+1)!!},\qquad n=0,1,2,\ldots.$$

13.3.4 Evaluate $\displaystyle\int_{-1}^{1}(1+x)^a(1-x)^b\,dx$ in terms of the beta function.

ANS. $2^{a+b+1}B(a+1,b+1)$.

13.3.5 Show, by means of the beta function, that

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{\alpha}} = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \alpha < 1.$$

13.3.6 Show that the Dirichlet integral

$$\int\int x^p y^q \, dx\, dy = \frac{p!\,q!}{(p+q+2)!} = \frac{B(p+1, q+1)}{p+q+2},$$

where the range of integration is the triangle bounded by the positive x- and y-axes and the line $x + y = 1$.

13.3.7 Show that

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2+2xy\cos\theta)} \, dx\, dy = \frac{\theta}{2\sin\theta}.$$

What are the limits on θ?

Hint. Consider oblique xy-coordinates.

ANS. $-\pi < \theta < \pi$.

13.3.8 Evaluate (using the beta function)

(a) $$\int_0^{\pi/2} \cos^{1/2}\theta \, d\theta = \frac{(2\pi)^{3/2}}{16[\Gamma(5/4)]^2},$$

(b) $$\int_0^{\pi/2} \cos^n \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{\sqrt{\pi}[(n-1)/2]!}{2(n/2)!}$$

$$= \begin{cases} \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ odd,} \\ \dfrac{\pi}{2} \cdot \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ even.} \end{cases}$$

13.3.9 Evaluate $\int_0^1 (1-x^4)^{-1/2} dx$ as a beta function.

ANS. $\dfrac{[\Gamma(5/4)]^2 \cdot 4}{(2\pi)^{1/2}} = 1.311028777.$

13.3.10 Using beta functions, show that the integral representation

$$J_\nu(z) = \frac{2}{\pi^{1/2}\,\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \sin^{2\nu}\theta \cos(z\cos\theta)\, d\theta, \quad \Re(\nu) > -\tfrac{1}{2},$$

reduces to the Bessel series

$$J_\nu(z) = \sum_{s=0}^{\infty} (-1)^s \frac{1}{s!\,\Gamma(s+\nu+1)} \left(\frac{z}{2}\right)^{2s+\nu},$$

thereby confirming its validity.

13.3.11 Given the associated Legendre function, defined in Chapter 15,

$$P_m^m(x) = (2m-1)!!\,(1-x^2)^{m/2},$$

show that

(a) $$\int_{-1}^{1} [P_m^m(x)]^2\, dx = \frac{2}{2m+1}\,(2m)!, \quad m = 0, 1, 2, \ldots,$$

(b) $$\int_{-1}^{1} [P_m^m(x)]^2 \frac{dx}{1-x^2} = 2 \cdot (2m-1)!, \quad m = 1, 2, 3, \ldots.$$

13.3.12 Show that, for integers p and q,

(a) $$\int_{0}^{1} x^{2p+1}(1-x^2)^{-1/2}\, dx = \frac{(2p)!!}{(2p+1)!!},$$

(b) $$\int_{0}^{1} x^{2p}\,(1-x^2)^{q}\, dx = \frac{(2p-1)!!\,(2q)!!}{(2p+2q+1)!!}.$$

13.3.13 A particle of mass m moving in a symmetric potential that is well described by $V(x) = A|x|^n$ has a total energy $\frac{1}{2}m(dx/dt)^2 + V(x) = E$. Solving for dx/dt and integrating we find that the period of motion is

$$\tau = 2\sqrt{2m} \int_{0}^{x_{\max}} \frac{dx}{(E - Ax^n)^{1/2}},$$

where $x_{\max}$ is a classical turning point given by $Ax_{\max}^n = E$. Show that

$$\tau = \frac{2}{n}\sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A}\right)^{1/n} \frac{\Gamma(1/n)}{\Gamma(1/n + \frac{1}{2})}.$$

13.3.14 Referring to Exercise 13.3.13,

(a) Determine the limit as $n \to \infty$ of

$$\frac{2}{n}\sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A}\right)^{1/n} \frac{\Gamma(1/n)}{\Gamma(1/n + \frac{1}{2})}.$$

(b) Find $\lim_{n\to\infty} \tau$ from the behavior of the integrand $(E - Ax^n)^{-1/2}$.

(c) Investigate the behavior of the physical system (potential well) as $n \to \infty$. Obtain the period from inspection of this limiting physical system.

13.3.15 Show that

$$\int_0^\infty \frac{\sinh^\alpha x}{\cosh^\beta x} dx = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \quad \frac{\beta-\alpha}{2}\right), \quad -1 < \alpha < \beta.$$

Hint. Let $\sinh^2 x = u$.

13.3.16 The beta distribution of probability theory has a probability density

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1},$$

with x restricted to the interval (0, 1). Show that

(a) $\langle x \rangle$, the mean value, is $\dfrac{\alpha}{\alpha+\beta}$.

(b) σ^2, its variance, is $\langle x^2\rangle - \langle x\rangle^2 = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

13.3.17 From

$$\lim_{n\to\infty} \frac{\displaystyle\int_0^{\pi/2} \sin^{2n}\theta\, d\theta}{\displaystyle\int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta} = 1,$$

derive the Wallis formula for π:

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots.$$

13.4 Stirling's Series

In statistical mechanics we encounter the need to evaluate $\ln(n!)$ for very large values of n, and we occasionally need $\ln\Gamma(z)$ for nonintegral z when $|z|$ is large enough that it is inconvenient or impractical to use the Maclaurin series, Eq. (13.44), possibly followed by repeated use of the functional relation $\Gamma(z+1) = z\Gamma(z)$. These needs can be met by the asymptotic expansion for $\ln\Gamma(z)$ known as **Stirling's series** or **Stirling's formula**. While it is in principle possible to develop such an asymptotic formula by the method of steepest descents (and in fact we have already obtained the leading term of the expansion in this way; see Example 12.7.1), a relatively simple way of obtaining the full asymptotic expansion is by use of the Euler-Maclaurin integration formula in Section 12.3.

Derivation from Euler-Maclaurin Integration Formula

The Euler-Maclaurin formula for evaluating a definite integral on the range $(0, \infty)$, obtained by specializing Eq. (12.57) and ignoring the remainder, is

$$\int_0^\infty f(x)\,dx = \frac{1}{2} f(0) + f(1) + f(2) + f(3) + \cdots$$
$$+ \frac{B_2}{2!} f'(0) + \frac{B_4}{4!} f^{(3)}(0) + \frac{B_6}{6!} f^{(5)}(x) + \cdots, \qquad (13.55)$$

where B_n are Bernoulli numbers:

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad \cdots .$$

We proceed by applying Eq. (13.55) to the definite integral

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z}$$

(for z not on the negative real axis). We note, by comparing with Eq. (13.41), that

$$f(1) + f(2) + \cdots = \sum_{n=1}^\infty \frac{1}{(z+n)^2} = \psi^{(1)}(z+1);$$

this makes a connection to the gamma function and is the reason for our current strategy. We also note that

$$f^{(2n-1)}(0) = \left(\frac{d}{dx}\right)^{2n-1} \frac{1}{(z+x)^2}\bigg|_{x=0} = -\frac{(2n)!}{z^{2n+1}},$$

so the expansion yields

$$\frac{1}{z} = \int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{2z^2} + \psi^{(1)}(z+1) - \frac{B_2}{z^3} - \frac{B_4}{z^5} - \cdots .$$

Solving for $\psi^{(1)}(z+1)$, we have

$$\psi^{(1)}(z+1) = \frac{d}{dz}\psi(z+1) = \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \cdots$$
$$= \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^\infty \frac{B_{2n}}{z^{2n+1}}. \qquad (13.56)$$

Since the Bernoulli numbers diverge strongly, this series does not converge. It is a semi-convergent, or asymptotic, series, useful if one retains a small number of terms (compare with Section 12.6).

Integrating once, we get the digamma function

$$\begin{aligned}\psi(z+1) &= C_1 + \ln z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \cdots \\ &= C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}},\end{aligned} \tag{13.57}$$

where C_1 has a value still to be determined. In the next subsection we will show that $C_1 = 0$. Equation (13.57), then, gives us another expression for the digamma function, often more useful than Eq. (13.38) or Eq. (13.44).

Stirling's Formula

The indefinite integral of the digamma function, obtained by integrating Eq. (13.57), is

$$\ln \Gamma(z+1) = C_2 + \left(z + \frac{1}{2}\right) \ln z + (C_1 - 1)z + \frac{B_2}{2z} + \cdots + \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \cdots, \tag{13.58}$$

in which C_2 is another constant of integration. We are now ready to determine C_1 and C_2, which we can do by requiring that the asymptotic expansion be consistent with the Legendre duplication formula, Eq. (13.54). Substituting Eq. (13.58) into the logarithm of the duplication formula, we find that satisfaction of that formula dictates that $C_1 = 0$ and that C_2 must have the value

$$C_2 = \tfrac{1}{2} \ln 2\pi. \tag{13.59}$$

Thus, inserting also values of the B_{2n}, our final result is

$$\ln \Gamma(z+1) = \frac{1}{2} \ln 2\pi + \left(z + \frac{1}{2}\right) \ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots. \tag{13.60}$$

This is Stirling's series, an asymptotic expansion. The absolute value of the error is less than the absolute value of the first term neglected.

The leading term in the asymptotic behavior of the gamma function was one of the examples used to illustrate the method of steepest descents. In Example 12.7.1, we found that

$$\Gamma(z+1) \sim \sqrt{2\pi}\, z^{z+1/2} e^{-z},$$

corresponding to

$$\ln \Gamma(z+1) \sim \frac{1}{2} \ln 2\pi + \left(z + \frac{1}{2}\right) \ln z - z,$$

yielding all the terms of Eq. (13.60) that do not vanish in the limit of large $|z|$.

To help convey a feeling of the remarkable precision of Stirling's series for $\Gamma(s+1)$, the ratio of the first term of Stirling's approximation to $\Gamma(s+1)$ is plotted in Fig. 13.4. In Table 13.1 we give the ratio of the first term in the expansion to $\Gamma(s+1)$ and a similar ratio when two terms are kept in the expansion to $\Gamma(s+1)$. The derivation of these forms is Exercise 13.4.1.

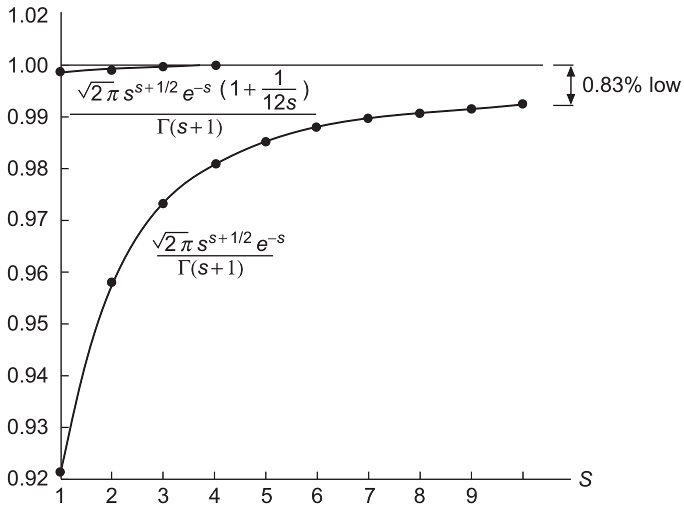

FIGURE 13.4 Accuracy of Stirling's formula.

Table 13.1 Ratios of One- and Two-Term Stirling Series to Exact Values of $\Gamma(s+1)$

s	$\dfrac{1}{\Gamma(s+1)}\sqrt{2\pi}s^{s+1/2}e^{-s}$	$\dfrac{1}{\Gamma(s+1)}\sqrt{2\pi}s^{s+1/2}e^{-s}\left(1+\dfrac{1}{12s}\right)$
1	0.92213	0.99898
2	0.95950	0.99949
3	0.97270	0.99972
4	0.97942	0.99983
5	0.98349	0.99988
6	0.98621	0.99992
7	0.98817	0.99994
8	0.98964	0.99995
9	0.99078	0.99996
10	0.99170	0.99998

Exercises

13.4.1 Rewrite Stirling's series to give $\Gamma(z+1)$ instead of $\ln\Gamma(z+1)$.

ANS. $$\Gamma(z+1) = \sqrt{2\pi}\, z^{z+1/2}e^{-z}\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51{,}840z^3}+\cdots\right).$$

13.4.2 Use Stirling's formula to estimate 52!, the number of possible rearrangements of cards in a standard deck of playing cards.

13.4.3 Show that the constants C_1 and C_2 in Stirling's formula have the respective values zero and $\frac{1}{2}\ln 2\pi$ by using the logarithm of the Legendre duplication formula (see Fig. 3.4).

13.4.4 Without using Stirling's series show that

$$\text{(a)}\quad \ln(n!) < \int_1^{n+1} \ln x\,dx, \quad \text{(b)}\quad \ln(n!) > \int_1^{n} \ln x\,dx;\ n \text{ is an integer} \geq 2.$$

Note that the arithmetic mean of these two integrals gives a good approximation for Stirling's series.

13.4.5 Test for convergence

$$\sum_{p=0}^{\infty}\left[\frac{\Gamma(p+\frac{1}{2})}{p!}\right]^2 \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!\,(2p+1)!!}{(2p)!!\,(2p+2)!!}.$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

13.4.6 Show that $\lim_{x\to\infty} x^{b-a}\dfrac{\Gamma(x+a+1)}{\Gamma(x+b+1)} = 1.$

13.4.7 Show that $\lim_{n\to\infty} \dfrac{(2n-1)!!}{(2n)!!}\, n^{1/2} = \pi^{-1/2}.$

13.4.8 A set of N distinguishable particles is assigned to states ψ_i, $i = 1, 2, \ldots, M$. If the numbers of particles in the various states are $n_1, n_2, \ldots, n_M$ (with $M \ll N$), the number of ways this can be done is

$$W = \frac{N!}{n_1!\,n_2!\cdots n_M!}.$$

The entropy associated with this assignment is $S = k \ln W$, where k is Boltzmann's constant. In the limit $N \to \infty$, with $n_i = p_i N$ (so p_i is the fraction of the particles in state i), find S as a function of N and the p_i.

(a) In the limit of large N, find the entropy associated with an arbitrary set of n_i. Is the entropy an extensive function of the system size (i.e., is it proportional to N)?

(b) Find the set of p_i that maximize S.

Hint. Remember that $\sum_i p_i = 1$ and that this is a constrained maximization (see Section 22.3).

Note. These formulas correspond to **classical**, or **Boltzmann**, statistics.

13.5 RIEMANN ZETA FUNCTION

We are now in a position to broaden our earlier survey of $\zeta(z)$, the Riemann zeta function. In so doing, we note an interesting degree of parallelism between some of the properties of $\zeta(z)$ and corresponding properties of the gamma function.

We open this section by repeating the definition of $\zeta(z)$, which is valid when the series converges:

$$\zeta(z) \equiv \sum_{n=1}^{\infty} n^{-z}. \tag{13.61}$$

The values of $\zeta(n)$ for integral n from 2 to 10 were listed in Table 1.1 on page 17.

We now want to consider the possibility of analytically continuing $\zeta(z)$ beyond the range of convergence of Eq. (13.61). As a first step toward doing so, we prove the integral representation that was given in Table 1.1:

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}\,dt}{e^t - 1}. \tag{13.62}$$

Equation (13.62) has a range of validity that is limited by the behavior of its integrand at small t; since the denominator then approaches t, the overall small-t dependence is t^{z-2}. Writing $z = x + iy$ and $t^{z-2} = t^{x-2} e^{iy \ln t}$, we see that, like Eq. (13.61), Eq. (13.62) will only converge when $\Re e\, z > 1$.

We start from the right-hand side of Eq. (13.62), denoted I, by multiplying the numerator and denominator of its integrand by e^{-t} and expanding the denominator in powers of e^{-t}, reaching

$$I = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1} e^{-t}\,dt}{1 - e^{-t}} = \frac{1}{\Gamma(z)} \int_0^{\infty} \sum_{m=1}^{\infty} t^{z-1} e^{-mt}\,dt.$$

We next change the variable of integration for the individual terms so that all terms contain an identical factor e^{-t}:

$$I = \frac{1}{\Gamma(z)} \int_0^{\infty} \sum_{m=1}^{\infty} \left(\frac{t}{m}\right)^{z-1} e^{-t} \left(\frac{dt}{m}\right) = \frac{1}{\Gamma(z)} \left(\sum_{m=1}^{\infty} \frac{1}{m^z}\right) \int_0^{\infty} t^{z-1} e^{-t}\,dt$$

$$= \zeta(z)\, \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-t}\,dt = \zeta(z). \tag{13.63}$$

In the second line of Eq. (13.63) we recognize the summation as a zeta function and the integral as the Euler integral representation of $\Gamma(z)$, Eq. (13.5). It then cancels against the initial factor $1/\Gamma(z)$, leaving the desired final result, Eq. (13.62). In passing, we note that the only difference between the integral of Eq. (13.62) and the Euler integral for the gamma function is that we now have a denominator $e^t - 1$ instead of simply e^t.

The next step toward the analytic continuation we seek is to introduce a contour integral with the same integrand as Eq. (13.62), using the same open contour that was found useful for the gamma function, shown in Fig. 13.2. Just as for the gamma function, we do not wish to restrict z to integral values, so the integrand will in general have a branch point at $t = 0$, and again we have placed the branch cut on the positive real axis. Restricting consideration for now to z with $\Re e\, z > 1$, we evaluate the contour integral, denoted I, as the sum of its contributions from the sections of the contour, respectively, labeled A, B, and D in Fig. 13.2. For $\Re e\, z > 1$, the small circle D makes no contribution to the integral, while

$$I_A = \frac{1}{\Gamma(z)} \int_{\infty}^{\varepsilon} \frac{t^{z-1}\,dt}{e^t - 1} = -\zeta(z),$$

$$I_B = \frac{1}{\Gamma(z)} \int_{\varepsilon}^{\infty} \frac{t^{z-1} e^{2\pi i(z-1)}\,dt}{e^t - 1} = e^{2\pi i(z-1)}\zeta(z) = e^{2\pi i z}\zeta(z).$$

Combining the above, we get

$$I = \frac{1}{\Gamma(z)} \int_C \frac{t^{z-1}\,dt}{e^t - 1} = \left(e^{2\pi i z} - 1\right)\zeta(z). \tag{13.64}$$

Note that Eq. (13.64) is useful as a relation involving $\zeta(z)$ only if z is not an integer.

We now wish to deform the contour of Eq. (13.64) in a way that will remove the restriction $\Re e\, z > 1$, which we originally needed to obtain that equation. The deformation corresponds to an analytic continuation of $\zeta(z)$ to a larger range of z, and will be effective because the deformation can avoid the divergence in the neighborhood of $t = 0$. When we consider possible deformations, we need to make the observation that, unlike the gamma function, the integrand of Eq. (13.64) has simple poles at the points $t = 2n\pi i$, $n = \pm 1, \pm 2, \ldots$, so that if we deform the contour in a way that encloses any of these poles, we must allow for the change thereby produced in the value of the contour integral.

If we initially deform the contour by expanding the circle D to some finite radius less than $2\pi i$, we do not change the value of the integral I but extend its range of validity to negative z. If, for $z < 0$, we further expand D until it becomes an open circle of infinite radius (but not through any of the poles), the value of the contour integral is reduced to zero, with the change caused by the inclusion of the contribution from the poles that are then encircled. We therefore have the interesting result that the original contour integral had a value that was the negative of $2\pi i$ times the sum of the residues that were newly enclosed. Thus,

$$I = \left(e^{2\pi i z} - 1\right)\zeta(z) = -\frac{2\pi i}{\Gamma(z)} \sum_{n=1}^{\infty} (\text{residues of } t^{z-1}/(e^t - 1) \text{ at } t = \pm 2n\pi i).$$

At the pole $t = +2\pi n i$, the residue is $\left(2n\pi e^{\pi i/2}\right)^{z-1}$, while at $t = -2\pi n i$ it is $\left(2n\pi e^{3\pi i/2}\right)^{z-1}$. Note that we must evaluate the residues taking cognizance of the branch cut. Inserting these values and rearranging a bit,

$$\begin{aligned} \left(e^{2\pi i z} - 1\right)\zeta(z) &= -\left(\sum_{n=1}^{\infty} \frac{1}{n^{-z+1}}\right) \frac{(2\pi)^z i}{\Gamma(z)} \left(e^{\pi i(z-1)/2} + e^{3\pi i(z-1)/2}\right) \\ &= \zeta(1-z)\frac{(2\pi)^z}{\Gamma(z)} \left(e^{3\pi i z/2} - e^{\pi i z/2}\right). \end{aligned} \tag{13.65}$$

Note that because $z < 0$, the summation over n converges and can be identified as $\zeta(1-z)$. Equation (13.65) can be simplified, but we already see its essential feature, namely that it provides a functional relation connecting $\zeta(z)$ and $\zeta(1-z)$, parallel to but more complicated than the reflection formula for the gamma function, Eq. (13.23). The derivation of Eq. (13.65) was carried out for $z < 0$, but now that we have obtained it, we can, appealing to analytic continuation, assert its validity for all z such that its constituent factors are nonsingular. This formula, in the simplified form we shall shortly obtain, was first found by Riemann.

The simplification of Eq. (13.65) can be accomplished by recognizing, with the aid of the gamma-function reflection formula, Eq. (13.23), that

$$\frac{e^{3\pi iz/2} - e^{\pi iz/2}}{e^{2\pi iz} - 1} = \frac{\sin(\pi z/2)}{\sin \pi z} = \frac{\Gamma(z)\,\Gamma(1-z)}{\Gamma(z/2)\,\Gamma(1-z/2)},$$

so

$$\zeta(z) = \zeta(1-z)\,\frac{\pi^z\, 2^z\, \Gamma(1-z)}{\Gamma(z/2)\,\Gamma(1-z/2)} = \zeta(1-z)\,\frac{\pi^{z-1/2}\Gamma((1-z)/2)}{\Gamma(z/2)}, \tag{13.66}$$

where the final member of Eq. (13.66) was obtained by using the duplication formula, Eq. (13.27), with the value of z in the duplication formula set to the present $-z/2$. Equation (13.66) can now be rearranged to the more symmetrical form

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-(1-z)/2}\zeta(1-z)\,. \tag{13.67}$$

Equation (13.67), the **zeta-function reflection formula**, enables generation of $\zeta(z)$ in the half-plane $\Re e\ z < 0$ from values in the region $\Re e\ z > 1$, where the series definition converges.

It is possible to show that $\zeta(z)$ has no zeros in the region where the series definition converges, and, from Eq. (13.67), this implies that $\zeta(z)$ is also nonzero for all z in the half-plane $\Re e\ z < 0$ except at points where $\Gamma(z/2)$ is singular, namely $z = -2, -4, \ldots, -2n, \ldots$. $\Gamma(z/2)$ is also singular at $z = 0$ but, as we shall see shortly, the singularity at $\zeta(1)$ compensates the singularity at $\Gamma(0)$, with the result that $\zeta(0)$ is nonzero.

The zeros of $\zeta(z)$ at the negative even integers are called its **trivial zeros**, as they arise from the singularities of the gamma function. Any other zeros of $\zeta(z)$ (and there are an infinite number of them) must lie in the region $0 \le \Re e\ z \le 1$, which has been called the **critical strip** of the Riemann zeta function.

To obtain values of $\zeta(z)$ in the critical strip, we proceed by analytically continuing toward $\Re e\ z = 0$ the formula from Eq. (12.62) that defines the Dirichlet series $\eta(z)$ (clearly valid for $\Re e\ z > 1$),

$$\zeta(z) = \frac{\eta(z)}{1 - 2^{1-z}} = \frac{1}{1 - 2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}. \tag{13.68}$$

This alternating series converges for all $\Re e\ z > 0$, thereby providing a formula for $\zeta(z)$ throughout the critical strip, but it is best used where the convergence is relatively rapid, namely for $\Re e\ z \ge \frac{1}{2}$. Values of $\zeta(z)$ for $\Re e\ z < \frac{1}{2}$ may be more conveniently obtained from those for $\Re e\ z \ge \frac{1}{2}$ using the reflection formula, Eq. (13.67).

Equation (13.68) may be used to verify that the singularity of $\zeta(z)$ at $z = 1$ is a simple pole and to find its residue. We proceed as follows:

$$\begin{aligned}(\text{Residue at } z = 1) &= \lim_{z\to 1}(z-1)\zeta(z) = \lim_{z\to 1}\left(\frac{z-1}{1-2^{1-z}}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \\ &= \left(\frac{1}{\ln 2}\right)(\ln 2) = 1, \end{aligned} \tag{13.69}$$

where we used l'Hôpital's rule, recognized that $d\,2^{1-z}/dz = -2^{1-z}\ln 2$, and identified the summation as that of Eq. (1.53). Returning now to Eq. (13.67), noting that

$$\lim_{z\to 0}\frac{\zeta(1-z)}{\Gamma(z/2)} = \frac{-\text{residue of } \zeta(s) \text{ at } s = 1}{2\,(\text{residue of } \Gamma(s) \text{ at } s = 0)} = -\frac{1}{2},$$

we obtain the nonzero result

$$\zeta(0) = \Gamma(1/2)\pi^{-1/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}. \tag{13.70}$$

In addition to the practical utility we have already noted for the Riemann zeta function, it plays a major role in current developments in analytic number theory. A starting point for such investigations is the celebrated Euler prime number product formula, which can be developed by forming

$$\zeta(s)(1-2^{-s}) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right), \tag{13.71}$$

eliminating all the n^{-s}, where n is a multiple of 2. Then we write

$$\begin{aligned}\zeta(s)(1-2^{-s})(1-3^{-s}) = 1 &+ \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots \\ &- \left(\frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \cdots\right),\end{aligned}$$

eliminating all the remaining terms in which n is a multiple of 3. Continuing, we have $\zeta(s)(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-P^{-s})$, where P is a prime number, and all terms n^{-s}, in which n is a multiple of any integer up through P, are canceled out. In the limit $P \to \infty$, we reach

$$\zeta(s)(1-2^{-s})(1-3^{-s})\cdots(1-P^{-s}) \longrightarrow \zeta(s)\prod_{P(\text{prime})=2}^{\infty}(1-P^{-s}) = 1.$$

Therefore

$$\zeta(s) = \prod_{P(\text{prime})=2}^{\infty}(1-P^{-s})^{-1}, \tag{13.72}$$

giving $\zeta(s)$ as an infinite product.[4] Incidentally, the cancellation procedure in the above derivation has a clear application in numerical computation. For example, Eq. (13.71) will give $\zeta(s)(1-2^{-s})$ to the same accuracy as Eq. (13.61) gives $\zeta(s)$, but with only half as many terms.

The asymptotic distribution of prime numbers can be related to the poles of ζ'/ζ, and in particular to the nontrivial zeros of the zeta function. Riemann conjectured that all the nontrivial zeros were on the **critical line** $\Re e\, z = \frac{1}{2}$, and there are potentially important results that can be proved if Riemann's conjecture is correct. Numerical work has verified that the first 300×10^9 nontrivial zeros of $\zeta(z)$ are simple and indeed fall on the critical line. See J. Van de Lune, H. J. J. Te Riele, and D. T. Winter, "On the zeros of the Riemann zeta function in the critical strip. IV," *Math. Comput.* **47**, 667 (1986).

Although many gifted mathematicians have attempted to establish what has come to be known as the **Riemann hypothesis**, it has for about 150 years remained unproven and is considered one of the premier unsolved problems in modern mathematics. Popular accounts of this fascinating problem can be found in M. du Santoy, *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*, New York: HarperCollins (2003); J. Derbyshire, *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003); and K. Sabbagh, *The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics*, New York: Farrar, Straus and Giroux (2003).

Exercises

13.5.1 Show that the symmetrical functional relation

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-(1-z)/2}\zeta(1-z)$$

follows from the equation

$$\left(e^{2\pi i z}-1\right)\zeta(z) = \zeta(1-z)\frac{(2\pi)^z}{\Gamma(z)}\left(e^{3\pi i z/2}-e^{\pi i z/2}\right).$$

13.5.2 Prove that

$$\int_0^\infty \frac{x^n e^x\,dx}{(e^x-1)^2} = n!\,\zeta(n).$$

Assuming n to be real, show that each side of the equation diverges if $n = 1$. Hence the preceding equation carries the condition $n > 1$. Integrals such as this appear in the quantum theory of transport effects: thermal and electrical conductivity.

[4] For further discussion, the reader is referred to the works by Edwards, Ivíc, Patterson, and Titchmarsh in Additional Readings.

13.5.3 The Bloch-Grüneisen approximation for the resistance in a monovalent metal at absolute temperature T is

$$\rho = C\frac{T^5}{\Theta^6}\int\limits_0^{\Theta/T}\frac{x^5\,dx}{(e^x-1)(1-e^{-x})},$$

where Θ is the Debye temperature characteristic of the metal.

(a) For $T\to\infty$, show that

$$\rho \approx \frac{C}{4}\cdot\frac{T}{\Theta^2}.$$

(b) For $T\to 0$, show that

$$\rho \approx 5!\,\zeta(5)\,C\frac{T^5}{\Theta^6}.$$

13.5.4 Derive the following expansion of the Debye function for $n\ge 1$:

$$\int\limits_0^x\frac{t^n\,dt}{e^t-1} = x^n\left[\frac{1}{n}-\frac{x}{2(n+1)}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k+n)(2k)!}\right],\ |x|<2\pi.$$

The complete integral $(0,\infty)$ equals $n!\,\zeta(n+1)$ (Exercise 13.5.6).

13.5.5 The total energy radiated by a blackbody is given by

$$u = \frac{8\pi k^4T^4}{c^3h^3}\int\limits_0^{\infty}\frac{x^3}{e^x-1}\,dx.$$

Show that the integral in this expression is equal to $3!\,\zeta(4)$. The final result is the Stefan-Boltzmann law.

13.5.6 As a generalization of the result in Exercise 13.5.5, show that

$$\int\limits_0^{\infty}\frac{x^s\,dx}{e^x-1} = s!\,\zeta(s+1),\qquad \Re e(s)>0.$$

13.5.7 Prove that

$$\int\limits_0^{\infty}\frac{x^s\,dx}{e^x+1} = s!\,(1-2^{-s})\,\zeta(s+1),\qquad \Re e(s)>0.$$

Exercises 13.5.6 and 13.5.7 give the Mellin integral transform of $1/(e^x\pm 1)$; this transform is defined in Eq. (20.9).

13.5.8 The neutrino energy density (Fermi distribution) in the early history of the universe is given by

$$\rho_\nu = \frac{4\pi}{h^3}\int_0^\infty \frac{x^3}{\exp(x/kT)+1}\,dx.$$

Show that

$$\rho_\nu = \frac{7\pi^5}{30h^3}(kT)^4.$$

13.5.9 Prove that

$$\psi^{(n)}(z) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}\,dt, \quad \Re e(z) > 0.$$

13.5.10 Show that $\zeta(s)$ is analytic in the entire finite complex plane except at $s = 1$, where it has a simple pole with a residue of $+1$.

Hint. The contour integral representation will be useful.

13.6 OTHER RELATED FUNCTIONS

Incomplete Gamma Functions

Generalizing the Euler-integral definition of the gamma function, Eq. (13.5), we define **incomplete gamma functions** by the variable-limit integrals

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}\,dt, \quad \Re(a) > 0,$$
$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt. \tag{13.73}$$

Clearly, these two functions are related, for

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a). \tag{13.74}$$

The choice of employing $\gamma(a,x)$ or $\Gamma(a,x)$ is purely a matter of convenience. If the parameter a is a positive integer, Eqs. (13.73) may be integrated completely to yield

$$\gamma(n,x) = (n-1)!\left(1 - e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}\right),$$
$$\Gamma(n,x) = (n-1)!\,e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}. \tag{13.75}$$

While the above expressions are valid only for positive integer n, the function $\Gamma(n,x)$ is well defined (providing $x > 0$) for $n = 0$ and corresponds to an exponential integral (see later subsection).

For nonintegral a, a power-series expansion of $\gamma(a,x)$ for small x and an asymptotic expansion of $\Gamma(a,x)$ are developed in Exercises 1.3.3 and 13.6.4:

$$\gamma(a,x) = x^a \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!\,(a+n)}, \quad \text{small } x,$$

$$\begin{aligned} \Gamma(a,x) &\sim x^{a-1} e^{-x} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-n)} \cdot \frac{1}{x^n} \\ &\sim x^{a-1} e^{-x} \sum_{n=0}^{\infty} (a-n)_n \frac{1}{x^n}, \quad \text{large } x, \end{aligned} \tag{13.76}$$

where $(a-n)_n$ is a Pochhammer symbol. The final expression in Eq. (13.76) makes it clear how to obtain an asymptotic expansion for $\Gamma(0,x)$. Noting that $(-n)_n = (-1)^n n!$, we have

$$\Gamma(0,x) \sim \frac{e^{-x}}{x} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^n}. \tag{13.77}$$

These incomplete gamma functions may also be expressed quite elegantly in terms of confluent hypergeometric functions (compare Section 18.6).

Incomplete Beta Function

Just as there are incomplete gamma functions, there is also an incomplete beta function, customarily defined for $0 \le x \le 1$, $p > 0$ (and, if $x = 1$, also $q > 0$) as

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt. \tag{13.78}$$

Clearly, $B_{x=1}(p,q)$ becomes the regular (complete) beta function, Eq. (13.49). A power-series expansion of $B_x(p,q)$ is the subject of Exercise 13.6.5. The relation to hypergeometric functions appears in Section 18.5.

The incomplete beta function makes an appearance in probability theory in calculating the probability of at most k successes in n independent trials.[5]

Exponential Integral

Although the incomplete gamma function $\Gamma(a,x)$ in its general form, Eq. (13.73), is only infrequently encountered in physical problems, a special case is quite common and very

[5] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. New York: Wiley (1968), Section VI.10.

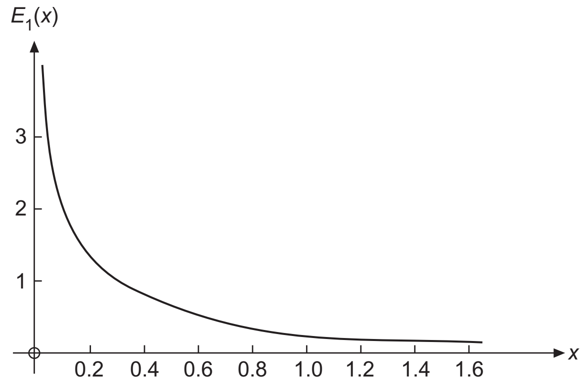

FIGURE 13.5 The exponential integral, $E_1(x) = -\mathrm{Ei}(-x)$.

useful. We define the **exponential integral** by[6]

$$-\mathrm{Ei}(-x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt \equiv E_1(x). \tag{13.79}$$

For a graph of this function, see Fig. 13.5. To obtain a series expansion of $E_1(x)$ for small x, we will need to proceed with caution, because the integral in Eq. (13.78) diverges logarithmically as $x \to 0$. We start from

$$E_1(x) = \Gamma(0, x) = \lim_{a\to 0}\Big[\Gamma(a) - \gamma(a, x)\Big]. \tag{13.80}$$

Setting $a = 0$ in the convergent terms (those with $n \geq 1$) in the expansion of $\gamma(a, x)$ and moving them outside the scope of the limiting process, we rearrange Eq. (13.80) to

$$E_1(x) = \lim_{a\to 0}\left[\frac{a\Gamma(a) - x^a}{a}\right] - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n \cdot n!}. \tag{13.81}$$

Using l'Hôpital's rule, Eq. (1.58), writing $a\Gamma(a) = \Gamma(a+1)$, and noting that $d\,x^a/da = x^a \ln x$, the limit in Eq. (13.81) reduces to

$$\left[\frac{d}{da}\,\Gamma(a+1) - \frac{d}{da}\,x^a\right]_{a=1} = \Gamma(1)\psi(1) - \ln x = -\gamma - \ln x, \tag{13.82}$$

where γ (without arguments) is the Euler-Mascheroni constant.[7] From Eqs. (13.81) and (13.82) we obtain the rapidly converging series

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n \cdot n!}. \tag{13.83}$$

[6]The appearance of the two minus signs in $-\mathrm{Ei}(-x)$ is a historical monstrosity. AMS-55, chapter 5, denotes this integral as $E_1(x)$. See Additional Readings for the reference.

[7]Having the notations $\gamma(a, x)$ and γ in the same discussion and with different meanings may seem unfortunate, but these are the traditional notations and should not lead to confusion if the reader is alert.

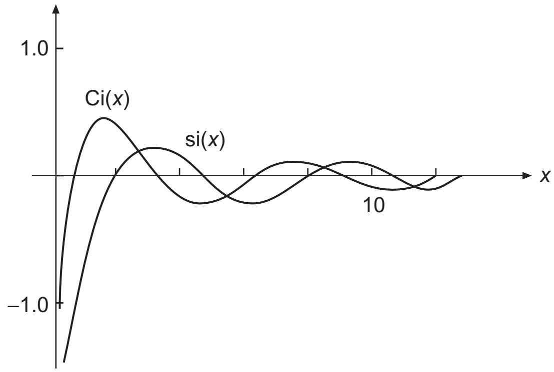

FIGURE 13.6 Sine and cosine integrals.

The asymptotic expansion for $E_1(x)$ is simply that given in Eq. (13.77) for $\Gamma(0,x)$. We repeat it here:

$$E_1(x) \sim e^{-x}\left[\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right]. \tag{13.84}$$

Further special forms related to the exponential integral are the sine integral, cosine integral (for both see Fig. 13.6), and the logarithmic integral, defined by[8]

$$\begin{aligned}
\operatorname{si}(x) &= -\int_x^\infty \frac{\sin t}{t}\,dt,\\
\operatorname{Ci}(x) &= -\int_x^\infty \frac{\cos t}{t}\,dt,\\
\operatorname{li}(x) &= \int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x).
\end{aligned} \tag{13.85}$$

Viewed as functions of a complex variable, $\operatorname{Ci}(z)$ and $\operatorname{li}(z)$ are multivalued, with a branch cut conventionally chosen to be along the negative real axis from the branch point at $z=0$. By transforming from real to imaginary argument, we can show that

$$\operatorname{si}(x) = \frac{1}{2i}\Big[\operatorname{Ei}(ix) - \operatorname{Ei}(-ix)\Big] = \frac{1}{2i}\Big[E_1(ix) - E_1(-ix)\Big], \tag{13.86}$$

whereas

$$\operatorname{Ci}(x) = \frac{1}{2}\Big[\operatorname{Ei}(ix) + \operatorname{Ei}(-ix)\Big] = -\frac{1}{2}\Big[E_1(ix) + E_1(-ix)\Big], \quad |\arg x| < \frac{\pi}{2}. \tag{13.87}$$

Adding these two relations, we obtain

$$\operatorname{Ei}(ix) = \operatorname{Ci}(x) + i\operatorname{si}(x), \tag{13.88}$$

[8] Another sine integral is denoted $\operatorname{Si}(x) = \operatorname{si}(x) + \pi/2$.

showing that the relation among these integrals is exactly analogous to that among e^{ix}, $\cos x$, and $\sin x$. In terms of E_1,

$$E_1(ix) = -\mathrm{Ci}(x) + i\ \mathrm{si}(x). \tag{13.89}$$

Asymptotic expansions of $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ were developed in Section 12.6, with explicit formulas in Eqs. (12.93) and (12.94). Power-series expansions about the origin for $\mathrm{Ci}(x)$, $\mathrm{si}(x)$, and $\mathrm{li}(x)$ may be obtained from those for the exponential integral, $E_1(x)$, or by direct integration, Exercise 13.6.13. The exponential, sine, and cosine integrals are tabulated in AMS-55, chapter 5 (see Additional Readings for the reference), and can also be accessed by symbolic software such as Mathematica, Maple, Mathcad, and Reduce.

Error Function

The **error function** $\mathrm{erf}(z)$ and the **complementary error function** $\mathrm{erfc}(z)$ are defined by the integrals

$$\mathrm{erf}\, z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \quad \mathrm{erfc}\, z = 1 - \mathrm{erf}\, z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt. \tag{13.90}$$

The factors $2/\sqrt{\pi}$ cause these functions to be scaled so that $\mathrm{erf}\,\infty = 1$. For a plot of $\mathrm{erf}\, x$, see Fig. 13.7.

The power-series expansion of $\mathrm{erf}\, x$ follows directly from the expansion of the exponential in the integrand:

$$\mathrm{erf}\, x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}. \tag{13.91}$$

Its asymptotic expansion, the subject of Exercise 12.6.3, is

$$\mathrm{erf}\, x \approx 1 - \frac{e^{-x^2}}{\sqrt{\pi}\, x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{2^2 x^4} - \frac{1\cdot 3\cdot 5}{2^3 x^6} + \cdots + (-1)^n \frac{(2n-1)!!}{2^n x^{2n}}\right). \tag{13.92}$$

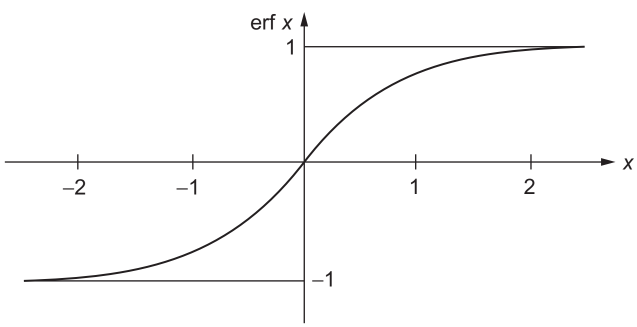

FIGURE 13.7 Error function, $\mathrm{erf}\, x$.

From the general form of the integrands and Eq. (13.6) we expect that erf z and erfc z may be written as incomplete gamma functions with $a = \frac{1}{2}$. The relations are

$$\operatorname{erf} z = \pi^{-1/2}\gamma(\tfrac{1}{2}, z^2), \quad \operatorname{erfc} z = \pi^{-1/2}\Gamma(\tfrac{1}{2}, z^2). \tag{13.93}$$

Exercises

13.6.1 Show that $\gamma(a,x) = e^{-x}\sum_{n=0}^{\infty}\frac{(a-1)!}{(a+n)!}\,x^{a+n}$

(a) by repeatedly integrating by parts,

(b) by transforming it into Eq. (13.76).

13.6.2 Show that

(a) $\frac{d^m}{dx^m}[x^{-a}\gamma(a,x)] = (-1)^m x^{-a-m}\gamma(a+m,x)$,

(b) $\frac{d^m}{dx^m}[e^x\gamma(a,x)] = e^x\frac{\Gamma(a)}{\Gamma(a-m)}\gamma(a-m,x)$.

13.6.3 Show that $\gamma(a,x)$ and $\Gamma(a,x)$ satisfy the recurrence relations

(a) $\gamma(a+1,x) = a\,\gamma(a,x) - x^a e^{-x}$,

(b) $\Gamma(a+1,x) = a\,\Gamma(a,x) + x^a e^{-x}$.

13.6.4 Show that the asymptotic expansion (for large x) of the incomplete gamma function $\Gamma(a,x)$ has the form

$$\Gamma(a,x) \sim x^{a-1}e^{-x}\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-n)}\cdot\frac{1}{x^n},$$

and that the above expression is equivalent to

$$\Gamma(a,x) \sim x^{a-1}e^{-x}\sum_{n=0}^{\infty}(a-n)_n\,\frac{1}{x^n}.$$

13.6.5 A series expansion of the incomplete beta function yields

$$B_x(p,q) = x^p\left\{\frac{1}{p} + \frac{1-q}{p+1}x + \frac{(1-q)(2-q)}{2!(p+2)}x^2 + \cdots \right.$$
$$\left. + \frac{(1-q)(2-q)\cdots(n-q)}{n!(p+n)}x^n + \cdots\right\}.$$

Given that $0 \le x \le 1$, $p > 0$, and $q > 0$, test this series for convergence. What happens at $x = 1$?

13.6.6 Using the definitions of the various functions, show that

(a) $\mathrm{si}(x) = \frac{1}{2i}[E_1(ix) - E_1(-ix)]$,

(b) $\mathrm{Ci}(x) = -\frac{1}{2}[E_1(ix) + E_1(-ix)]$,

(c) $E_1(ix) = -\mathrm{Ci}(x) + i\,\mathrm{si}(x)$.

13.6.7 The potential produced by a 1s hydrogen electron is given by

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left[\frac{1}{2r}\gamma(3, 2r) + \Gamma(2, 2r)\right].$$

(a) For $r \ll 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left[1 - \frac{2}{3}r^2 + \cdots\right].$$

(b) For $r \gg 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0} \cdot \frac{1}{r}.$$

Here r is expressed in units of a_0, the Bohr radius.

Note. $V(r)$ is illustrated in Fig. 13.8.

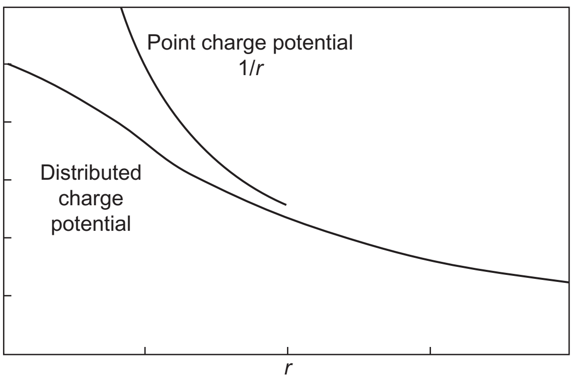

FIGURE 13.8 Distributed charge potential produced by a 1s hydrogen electron, Exercise 13.6.7.

13.6.8 The potential produced by a $2p$ hydrogen electron can be shown to be

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{24a_0}\left[\frac{1}{r}\gamma(5,r) + \Gamma(4,r)\right] - \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{120a_0}\left[\frac{1}{r^3}\gamma(7,r) + r^2\Gamma(2,r)\right]P_2(\cos\theta).$$

Here r is expressed in units of a_0, the Bohr radius. $P_2(\cos\theta)$ is a Legendre polynomial (Section 15.1).

(a) For $r \ll 1$, show that

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{a_0}\left[\frac{1}{4} - \frac{1}{120}r^2 P_2(\cos\theta) + \cdots\right].$$

(b) For $r \gg 1$, show that

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{a_0 r}\left[1 - \frac{6}{r^2}P_2(\cos\theta) + \cdots\right].$$

13.6.9 Prove that the exponential integral has the expansion

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n\cdot n!},$$

where γ is the Euler-Mascheroni constant.

13.6.10 Show that $E_1(z)$ may be written as

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-zt}}{1+t}\,dt.$$

Show also that we must impose the condition $|\arg z| \le \pi/2$.

13.6.11 Related to the exponential integral by a simple change of variable is the function

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\,dt.$$

Show that $E_n(x)$ satisfies the recurrence relation

$$E_{n+1}(x) = \frac{1}{n}e^{-x} - \frac{x}{n}E_n(x), \qquad n = 1, 2, 3, \cdots.$$

13.6.12 With $E_n(x)$ as defined in Exercise 13.6.11, show that for $n > 1$,

$E_n(0) = 1/(n-1)$.

13.6.13 Develop the following power-series expansions:

(a) $\mathrm{si}(x) = -\dfrac{\pi}{2} + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$,

(b) $\mathrm{Ci}(x) = \gamma + \ln x + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n)!}$.

13.6.14 An analysis of a center-fed linear antenna leads to the expression

$$\int_0^x \frac{1-\cos t}{t}\,dt.$$

Show that this is equal to $\gamma + \ln x - \mathrm{Ci}(x)$.

13.6.15 Using the relation

$$\Gamma(a) = \gamma(a,x) + \Gamma(a,x),$$

show that if $\gamma(a,x)$ satisfies the relations of Exercise 13.6.2, then $\Gamma(a,x)$ must satisfy the same relations.

13.6.16 For $x > 0$, show that

$$\int_x^{\infty} \frac{t^n\,dt}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kx}\left[\frac{x^n}{k} + \frac{nx^{n-1}}{k^2} + \frac{n(n-1)x^{n-2}}{k^3} + \cdots + \frac{n!}{k^{n+1}}\right].$$

Additional Readings

Abramowitz, M., and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). Washington, DC: National Bureau of Standards (1972), reprinted, Dover (1974). Contains a wealth of information about gamma functions, incomplete gamma functions, exponential integrals, error functions, and related functions in chapters 4 to 6.

Artin, E., *The Gamma Function* (translated by M. Butler). New York: Holt, Rinehart and Winston (1964). Demonstrates that if a function $f(x)$ is smooth (log convex) and equal to $(n-1)!$ when $x = n =$ integer, it is the gamma function.

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