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Cosmoglobe: Joint estimation of cosmological parameters and CMB signal without likelihood approximations

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ABSTRACT

We sample parameters

Key words. ISM: general – Cosmology: observations, polarization, cosmic microwave background, diffuse radiation – Galaxy: general

Contents

1 Introduction 1

2 Joint sampling of cosmological parameters and sky signal 1
2.1 Joint sampling 1

3 Results 2

4 Conclusions 2

A Analytic expression for a constant latitude mask 4

1. Introduction

2. Joint sampling of cosmological parameters and sky signal

In this section, we will introduce the algorithm proposed in Racine et al. (2016) to jointly sample the cosmological parameters of the CMB and its sky signal without using any likelihood approximations.

We define the data model as

$$d_{\ell m} = A_\ell s_{\ell m} + n_{\ell m}, \tag{1}$$

in spherical harmonics space where $d_{\ell m}$ is the observed data, A_ℓ is the diagonal beam and pixel window functions. $s_{\ell m}$ is the Gaussian CMB signal and $n_{\ell m}$ is the Gaussian instrumental noise. Eq. (1) is equivalent in pixel space where $d_p = \sum_{\ell m} d_{\ell m} Y_{\ell m}$. p denotes the pixel and $Y_{\ell m}$ are the spherical harmonics.

The covariance matrices are given as $\langle s_{\ell m} s^*_{\ell' m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$ and $\langle n_{\ell m} n^*_{\ell' m'} \rangle = N_{\ell m, \ell' m'}$. With isotropic noise and no mask, $N_{\ell m, \ell' m'}$ becomes diagonal, i.e. $N_{\ell m, \ell' m'} = N_\ell \delta_{\ell\ell'} \delta_{mm'}$. But for a realistic setup with anisotropic instrumental noise and partial sky coverage, it will be non-diagonal and can be dense.

The goal is to sample the joint density $P(\theta, s_{\ell m}|d_{\ell m})$ where θ is the set of cosmological parameters. In this work, we sample the 6 ΛCDM parameters, i.e. $\theta = (\Omega_\mathrm{b} h^2, \Omega_\mathrm{CDM} h^2, H_0, \tau, A_s, n_s)$. Here, Ω_b and Ω_CDM are the current density of baryonic and cold dark matter, respectively. H_0 si the Hubble constant today, while h is the normalized Hubble constant, $h = \frac{H_0}{100\,\mathrm{km s^{-1} MPc^{-1}}}$. τ is the optical depth at reionization, A_s is the amplitude and n_s is the tilt of the scalar primordial power spectrum.

2.1. Joint sampling

In this section, we show how one could sample $P(\theta, s_{\ell m}|d_{\ell m})$ jointly. The idea is to first draw the cosmological parameters θ then sample $s_{\ell m}$ from the corresponding power spectra $C_\ell(\theta)$. In other words, one would perform

$$\theta^{i+1} \leftarrow w(\theta|\theta^i) \tag{2}$$

$$s^{i+1}_{\ell m} \leftarrow P(s_{\ell m}|\theta^{i+1}, d_{\ell m}), \tag{3}$$

for each Gibbs iteration, where the arrow denotes drawing a sample from the distribution on the right-hand side. $w(\theta|\theta^i)$ is the proposal distribution of the cosmological parameters. We use a multivariate Gaussian,

$$w(\theta|\theta^i) = e^{-\frac{1}{2}(\theta-\theta^i)^T C_\theta^{-1} (\theta-\theta^i)}. \tag{4}$$

For optimal chain convergence, the proposal variance C_θ should resemble the posterior distribution. The proposal variance we use is created from the samples of an earlier chain created with a diagonal proposal matrix.

To draw a sample of the sky given the power spectra given by θ^i, one has to solve the map-making equation. In matrix notation valid for both pixel and spherical harmonics space, this is explain this equation

$$\left[\mathbf{S}^{-1} + \mathbf{A}^t \mathbf{N}^{-1} \mathbf{A}\right] \mathbf{s} = \mathbf{A}\mathbf{N}^{-1}\mathbf{d} + \mathbf{S}^{-\frac{1}{2}} \mathbf{w}_0 + \mathbf{A}\mathbf{N}^{-\frac{1}{2}} \mathbf{w}_1, \tag{5}$$

where $\mathbf{w}_0$ and $\mathbf{w}_1$ are vectors where each element is a random number drawn from $\mathcal{N}(0, 1)$. In spherical harmonics space, these two vectors must be complex[1] with amplitude from $\mathcal{N}(0, 1)$. And the signal covariance matrix is $\mathbf{S} = \mathbf{S}(\theta^i) = C_\ell\left(\theta^i\right) \delta_{\ell\ell'} \delta_{mm'}$.

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[1] Except for $m = 0$ where w_0 and w_1 are real.

We define the deterministic mean-field map

$$\hat{\mathbf{s}} \equiv \left[\mathbf{S}^{-1} + \mathbf{A}^t\mathbf{N}^{-1}\mathbf{A}\right]^{-1}\mathbf{A}\mathbf{N}^{-1}\mathbf{d} \tag{6}$$

and the fluctuation map

$$\hat{\mathbf{f}} \equiv \left[\mathbf{S}^{-1} + \mathbf{A}^t\mathbf{N}^{-1}\mathbf{A}\right]^{-1}\left(\mathbf{S}^{-\frac{1}{2}}\mathbf{w}_0 + \mathbf{A}\mathbf{N}^{-\frac{1}{2}}\mathbf{w}_1\right) \tag{7}$$

so that the sky signal decomposes into $s_{\ell m} = \hat{s}_{\ell m} + \hat{f}_{\ell m}$.

If we only analyze temperature fluctuations in a setup with isotropic noise and no mask, this can be solved exactly:

$$\hat{s}_{\ell m} = d_{\ell m}\frac{A_\ell C_\ell}{N_\ell + A_\ell^2 C_\ell}, \tag{8}$$

$$\hat{f}_{\ell m} = w_{0\ell m}\frac{N_\ell\sqrt{C_\ell}}{N_\ell + A_\ell^2 C_\ell} + w_{1\ell m}\frac{\sqrt{N_\ell}A_\ell C_\ell}{N_\ell + A_\ell^2 C_\ell}. \tag{9}$$

In a more realistic scenario with anisotropic noise and masking, Eq. (5) becomes a computationally hard equation to solve. In Appendix A, we show how we can solve this equation for a constant latitude mask with isotropic noise computationally fast. We will use this solution to verify our algorithm implementation in Commander later.

Following Racine et al. (2016) we make a slight altercation. Instead of generating a new fluctuation term that we enter into the acceptance rate, we scale the fluctuation map of the last accepted sample.

$$f_{\ell m}^{\mathrm{scaled},i+1} = \left(\mathbf{C}_\ell^{i+1}\right)^{1/2}\left(\mathbf{C}_\ell^{i}\right)^{-1/2} f_{\ell m}^{i}, \tag{10}$$

where

$$\mathbf{C}_\ell = \begin{bmatrix} C_\ell^{TT} & C_\ell^{TE} & 0 \\ C_\ell^{TE} & C_\ell^{EE} & 0 \\ 0 & 0 & C_\ell^{BB} \end{bmatrix}. \tag{11}$$

Here, we have set the parity-odd power spectra to zero, i.e. $C_\ell^{EB} = C_\ell^{TB} = 0$ as predicted by ΛCDM.

The acceptance rate for this proposal was calculated in Racine et al. (2016) to be

$$A = \min\left[1, \frac{\pi(\theta^{i+1})}{\pi(\theta^i)}\frac{w(\theta^i|\theta^{i+1})}{w(\theta^{i+1}|\theta^i)}\frac{P(\theta^{i+1})}{P(\theta^i)}\right]. \tag{12}$$

Here, $P(\theta)$ is the prior on θ and

$$\pi(\theta^i) = \exp\Big[-\frac{1}{2}\left(d - A\hat{s}^i\right)^T\mathbf{N}^{-1}\left(d - A\hat{s}^i\right) - \frac{1}{2}\hat{s}^{i,T}\mathbf{S}^{i,-1}\hat{s}^i - \frac{1}{2}\hat{f}^{i,T}A^T\mathbf{N}^{-1}A\hat{f}^i\Big], \tag{13}$$

where we have omitted the θ^i dependence of $\hat{s}^i$, $\hat{f}^i$ and S^i. Our proposal matrix for the cosmological parameters is symmetric, $w(\theta^i|\theta^{i+1}) = w(\theta^{i+1}|\theta^i)$, and hence we can remove them from Eq. (12).

As stated the acceptance rate will use the scaled fluctuation term, $f_{\ell m}^{\mathrm{scaled},i+1}$, for sample $i+1$ instead of the calculated fluctuation term in Eq. (5). Sample i however, will use the fluctuation solution to Eq. (5).

The algorithm can be broken down into these steps:

- 1. Start with an initial value θ^0. From that, calculate the power spectra S^0, the mean-field map $\hat{s}^0$ and the fluctuation map $\hat{f}^0$ from the cosmological parameters.
- 2. Pick a new cosmological parameter sample, θ^{i+1}. Calculate S^{i+1}, $\hat{s}^{i+1}$ and the *scaled* fluctuation term $f_{\ell m}^{\mathrm{scaled},i+1}$.
- 3. Calculate the acceptance term of Eq. 12, and accept/reject according to the regular Metropolis-Hasting rule.
- 4. If the sample is accepted, then calculate $f_{\ell m}^{i+1}$. This term will in the next iteration become $f_{\ell m}^{i}$ which will appear in the acceptance probability. The fluctuation map is not necessary to calculate if the sample is discarded.
- 5. Iterate 2-4.

3. Results

This work aimed to implement the algorithm into the Commander framework to work with a mask and anisotropic noise map. To verify the implementation, we also created a Python script that could sample the isotropic noise CMB simulation with a constant latitude mask. We show how we calculate the $N^{-1}_{\ell m,\ell' m'}$ in Appendix A.

We created 2 sets of CMB simulations with noise: One with isotropic noise and one with anisotropic noise. Both simulations are created using $N_{\mathrm{side}} = 512$ and the 70 GHz beams.

First, we run the isotropic noise simulations without a mask. In this case, the noise matrix $\mathbf{N}$ is diagonal and we can sample the cosmological parameters by using Cobaya. Say a couple of words about Cobaya

Figure 1 shows the results from Commander, Python and Cobaya, and we see that the three contours align perfectly.

Then we test the algorithm with a constant latitude mask. $\mathbf{N}$ is no longer diagonal, and Cobaya will not be able to capture all the information any longer. We, therefore, sample it with Python and Commander. This works.

We then turn to the realistic case: we use anisotropic noise and a realistic mask. Our Python script can not sample this as $\mathbf{N}$ is highly complex and so solving the map equation is computationally hard. To solve this equation, we, therefore, rely on the conjugate gradient method in Commander.

4. Conclusions

We have shown that the algorithm presented in Racine et al. (2016) works well in the Commander framework for anisotropic noise and mask.

References

Racine, B., Jewell, J. B., Eriksen, H. K., & Wehus, I. K. 2016, ApJ, 820, 31

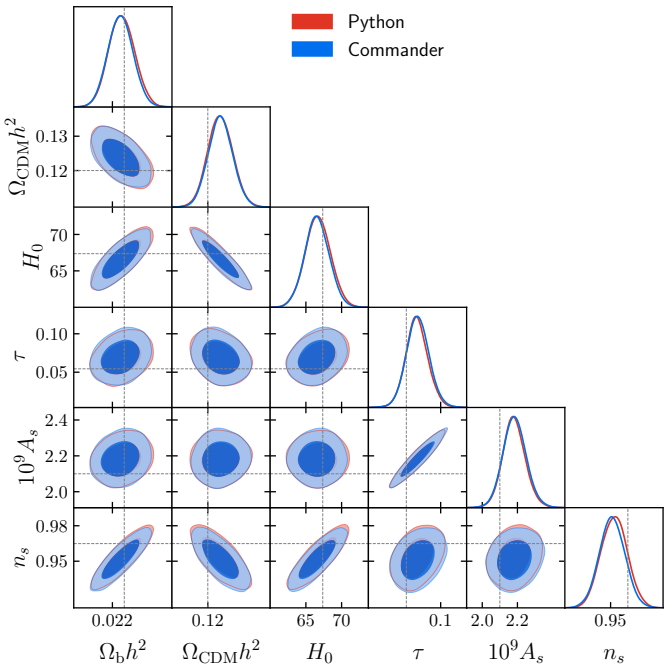

Fig. 1. Parameter estimation on a CMB realization with isotropic noise. We use $N_{\rm side} = 512$ and the beam convolve using the beam window functions of 70 GHz band of Planck. No mask is applied.

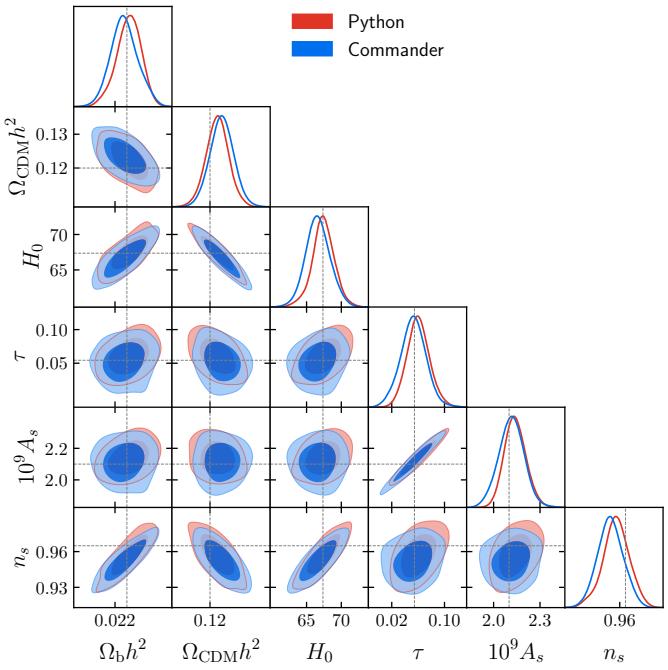

Fig. 2. Cosmological parameter estimation on same simulation as in Fig. 1 but a constant latitude mask is applied so that the sky fraction is 90%.

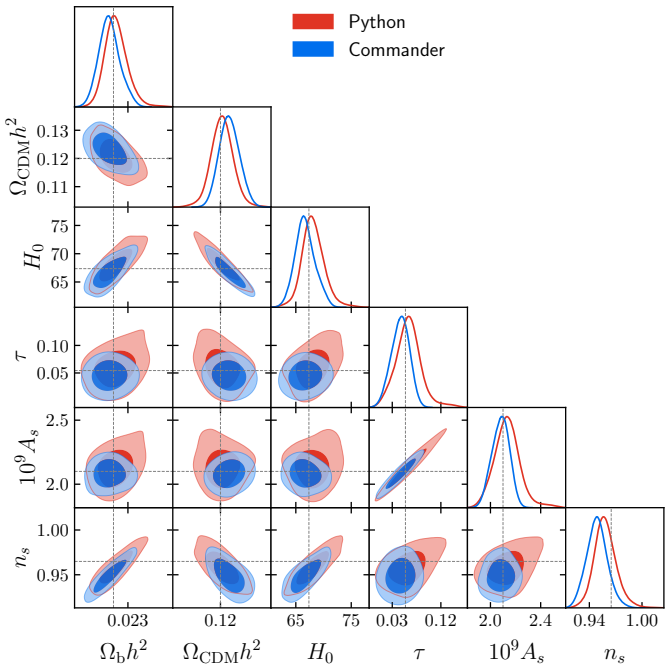

Fig. 3. Cosmological parameter estimation on same simulation as in Fig. 1 but a constant latitude mask is applied so that the sky fraction is 80%.

Appendix A: Analytic expression for a constant latitude mask

The goal of this appendix is to show how we can calculate $N^{-1}_{\ell m,\ell' m'}$ exactly for a constant latitude mask and uniform noise, and how this expression makes the map making equation computationally fast to solve. The noise can be written as this in pixel space:

$$\left(N^{-1}\right)_{pp'} = \frac{N_{\rm pix}}{\sigma^2}\delta_{pp'}H(|\theta(p) - \pi/2| - b).$$

where H is the Heavyside function, meaning that we mask every pixel p where $|\theta(p) - \pi/2| < b$ for some latitude b in radians. For a masked pixel, $\left(N^{-1}\right)_{pp} = 0$ which means that $N_{pp} = \infty$ for that pixel, as we want.

Using that

$$\left(N^{-1}\right)_{pp'} = \sum_{\ell m}\sum_{\ell' m'}\left(N^{-1}\right)_{\ell m,\ell' m'} Y_{\ell m}\,(p)\,Y^*_{\ell' m'}\,(p')\,,$$

where $Y_{\ell m}\,(p) = Y_{\ell m}\,(\hat{n}(p))$ and $\hat{n}(p)$ are the spherical coordinates for pixel p, we can transform to spherical harmonics space,

$$\begin{aligned}\left(N^{-1}\right)_{\ell m,\ell' m'} &= \sum_{pp'}\left(N^{-1}\right)_{pp'} Y^*_{\ell m}(p)Y_{\ell' m'}(p') \\ &= \frac{N_{\rm pix}}{\sigma^2}\sum_{pp'} Y^*_{\ell m}(p)Y_{\ell' m'}(p')\delta_{pp'}H(|\theta - \pi/2| - b) \\ &= \frac{N_{\rm pix}}{\sigma^2}\sum_{p} Y^*_{\ell m}(p)Y_{\ell' m'}(p)H(|\theta - \pi/2| - b) \\ &= \frac{N_{\rm pix}}{\sigma^2}\sum_{p} \tilde{Y}_{\ell m}\tilde{Y}_{\ell' m'}e^{-i(m-m')\phi}H(|\theta - \pi/2| - b). \quad \text{(A.1)}\end{aligned}$$

Here, we used that $Y_{\ell m}(p) = \tilde{Y}_{\ell m}e^{im\phi}$ where it is implied that $\tilde{Y}_{\ell m} = \tilde{Y}_{\ell m}(\theta)$, and θ and ϕ are functions of p.

We now take the limit where there is a large amount of pixels with equal surface area each. In this limit, we change the sum into an integration where we account for the number of pixels per area element

$$\sum_p \to \frac{N_{\rm pix}}{4\pi}\int d\Omega = \frac{N_{\rm pix}}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta). \quad \text{(A.2)}$$

This gives us in spherical coordinates

$$\begin{aligned}\left(N^{-1}\right)_{\ell m,\ell' m'} &= \frac{N^2_{\rm pix}}{4\pi\sigma^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\tilde{Y}_{\ell m}\tilde{Y}_{\ell' m'}e^{-i(m-m')\phi} \\ &\cdot H(|\theta - \pi/2| - b) \\ &= \frac{N^2_{\rm pix}}{2\sigma^2}\delta_{mm'}\int_0^{\pi} d\theta \sin(\theta)\tilde{Y}_{\ell m}\tilde{Y}_{\ell' m'}H(|\theta - \pi/2| - b) \\ &= \frac{N^2_{\rm pix}}{2\sigma^2}\delta_{mm'} \\ &\cdot\left(\int_0^{\pi/2-b} d\theta \sin(\theta)\tilde{Y}_{\ell m}\tilde{Y}_{\ell' m'} + \int_{\pi/2+b}^{\pi} d\theta \sin(\theta)\tilde{Y}_{\ell m}\tilde{Y}_{\ell' m'}\right). \quad \text{(A.3)}\end{aligned}$$

Writing the above equation in terms of Legendre polynomials $P_{\ell m}(\cos(\theta))$, we have $\tilde{Y}_{\ell m} = \Delta_{\ell m}P_{\ell m}(\cos(\theta)))$, where $\Delta_{\ell m} = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}$. Writing $x = \cos(\theta)$, we know that Legendre polynomials $P_{\ell m}(x)$ are either symmetric or antisymmetric in $x \to -x$. $P_{\ell m}(x)$ is symmetric in $x \to -x$ when $\ell + m =$ even and antisymmetric when $\ell + m =$ odd. Since $m = m'$, we note that the two integrals cancel each other if $\ell + \ell' =$ odd. We, therefore, only get non-zero elements when $\ell + \ell' =$ even, for which the two integrals are equal. Hence, for $\ell + \ell' =$ even, we get

$$\begin{aligned}\left(N^{-1}\right)_{\ell m,\ell' m'} &= \frac{N^2_{\rm pix}}{\sigma^2}\delta_{mm'}\int_0^{\pi/2-b} d\theta \sin(\theta)\tilde{Y}_{\ell m}(\theta)\tilde{Y}_{\ell' m'}(\theta) \\ &= \frac{N^2_{\rm pix}}{\sigma^2}\delta_{mm'}\int_{\sin(b)}^{1} dx\,\tilde{Y}_{\ell m}(\arccos(x))\tilde{Y}_{\ell' m}(\arccos(x)).\end{aligned} \quad \text{(A.4)}$$

This integral can be solved numerically by gridding x, and the spherical harmonics where the phase factor is removed can be calculated in Python using a library like ...

Since the masks used for this work are $\sin(b) \geq 0.2$, we require much fewer grid points for x if we use the following identity for $\ell + \ell' =$ even:

$$\delta_{mm'}\int_{\sin(b)}^{1} dx\,\tilde{Y}_{\ell m}(\arccos(x))\tilde{Y}_{\ell' m}(\arccos(x)) = \quad \text{(A.5)}$$

$$\frac{1}{4\pi}\delta_{\ell\ell'}\delta_{mm'} - \delta_{mm'}\int_0^{\sin(b)} dx\,\tilde{Y}_{\ell m}(\arccos(x))\tilde{Y}_{\ell' m}(\arccos(x)), \quad \text{(A.6)}$$

which comes from the orthonormality condition for spherical harmonics. We now only need to grid x in the interval $0 \leq x \leq \sin(b) \leq 0.2$ for this work.

Since our expression for $N^{-1}_{\ell m,\ell' m'}$ has a $\delta_{mm'}$, we can create an $\ell_{\rm max}$ number of matrices, one for each $\ell_{\rm max} \geq m \geq 0$. And then solve the map-making equation for each matrix equation m.

To show how this works in matrix multiplication notation, imagine multiplying the matrix $\mathbf{N}^{-1} = N^{-1}_{\ell m,\ell' m'}$ with the vector $\mathbf{b} = b_{\ell m}$

$$\mathbf{N}^{-1}\cdot\mathbf{b} = \sum_{\ell' m'}\left(N^{-1}\right)_{\ell m,\ell' m'} b_{\ell' m'} = \sum_{\ell'}\left(N^{-1}\right)_{\ell m,\ell' m} b_{\ell' m} \quad \text{(A.7)}$$

$$= \sum_{\ell'}\left(N^{-1}\right)^{(m)}_{\ell,\ell'} b^{(m)}_{\ell'}. \quad \text{(A.8)}$$

So to solve the map-making equation, we get a matrix equation for each $\ell_{\rm max} \geq m \geq 0$. This gives us $\ell_{\rm max}$ number of matrix equations where the dimensions of the matrices are maximal $\ell_{\rm max} \times \ell_{\rm max}$. This is numerically much quicker than inverting the full $(\ell_{\rm max})^2 \times (\ell_{\rm max})^2$ matrix once.